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The sets of divergence points of self-similar measures are residual

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ABSTRACT

Let μ be a self-similar measure supported on a self-similar set K with the open set condition. For $x \in K$, let $A(D(x))$ be the set of accumulation points of $D_r(x) = \frac{\log \mu(B(x,r))}{\log r}$ as $r \searrow 0$. In this paper, we show that for any closed non-singleton subinterval $I \subset \mathbb{R}$, the set of points x for which the set $A(D(x))$ equals I is either empty or residual.

1. Introduction and statement of results

Let $S_i : \mathbb{R}^d \to \mathbb{R}^d$ $(i = 1, 2, \dots, N)$ be the contracting similarities with contraction ratios $r_i \in (0, 1)$ and let $(p_1, \dots, p_N)$ be a probability vector (i.e. $0 < p_i < 1$ for all i and $\sum_{i=1}^N p_i = 1$). Using the framework of [11] we say that K is a self-similar set and μ is a self-similar measure if K is the unique non-empty compact subset of $\mathbb{R}^d$ such that

$$K = \bigcup_i S_i(K),$$

and μ is the unique Borel probability measure on $\mathbb{R}^d$ such that

$$\mu = \sum_i p_i \mu \circ S_i^{-1}.$$

It is well known that the support of μ equals K. We say that the list $(S_1, \dots, S_N)$ satisfies the open set condition (OSC) (sometimes we also say that the self-similar measure μ satisfies the OSC) if there exists a non-empty, bounded and open set U such that $S_i(U) \subset U$ for all i and $S_i(U) \cap S_j(U) = \varnothing$ for all $i \neq j$.

Multifractal analysis of the self-similar measure μ refers to the study of the fractal geometry of the sets of the points x for which the measure $\mu(B(x, r))$ behaves like r^α for small r. Here $B(x, r)$ is the closed ball of radius r centered at x. That is, we study the sets

$$K_\alpha = \left\{ x \in \mathbb{R}^d : \lim_{r\to 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha \right\}, \quad \alpha \geq 0.$$

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The support of the self-similar measure μ has the following natural decomposition:

$$K = \bigcup_{\alpha} K_\alpha \cup \widehat{K},$$

where

$$\widehat{K} = \left\{ x \in K : \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} < \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \right\}.$$

The set $\widehat{K}$ is called the set of divergence points (or irregular set) of the self-similar measure and the elements in the set $\widehat{K}$ are called divergence points. It is well known that the set $\widehat{K}$ has μ-measure zero, see [21]. In other words, the set of divergence points of the self-similar measure is negligible from the measure-theoretical point of view. However, there is an extensive literature showing that the set of divergence points of the self-similar measure and other type irregular sets can be large from the point of view of dimension theory, see [4,5,7,9,14,13,16,18,20,22,23] and references therein. In particular, Barreira and Schmeling [4] and Chen and Xiong [5] showed that the set $\widehat{K}$ has full Hausdorff dimension for $(p_1, \ldots, p_N) \neq (r_1^s, \ldots, r_N^s)$, where s denotes the Hausdorff dimension of K (that is, s is the unique solution of the equation $\sum_i r_i^s = 1$). We remark that Chen and Xiong obtained the above result under the assumption that $(S_1, \ldots, S_N)$ satisfies the strong separation condition (SSC), that is, $S_i(K) \cap S_j(K) = \varnothing$ for all i, j with $i \neq j$. Very recently, using the technique in [8], Xiao, Wu and Gao [23] proved that Chen and Xiong's result remained valid under the OSC. Li and Wu [14] studied the Hausdorff dimensions of some refined subsets of the set $\widehat{K}$ under the OSC, and their results unified the above mentioned results as well as some classical results on the multifractal analysis of the self-similar measure.

The notion of residual set is usually used to describe a set being "large" in a topological sense. Recall that in a metric space X, a set R is called residual if its complement is of the first category. Moreover, in a complete metric space a set is residual if it contains a dense G_δ set, see [19]. We say that a set is large from the topological point of view if it is residual. Recently, some results show that the sets of some kinds of divergence points (or irregular sets) can also be large from the topological point of view. For example, Albeverio, Pratsiovytyi and Torbin [1], Hyde et al. [12] and Olsen [17] proved that the sets of some kinds of divergence points associated with integer expansion are residual. Baek and Olsen [2] discussed the set of extremely non-normal points of self-similar set from the topological point of view. Barreira, Li and Valls [3] proved that the set of divergence points of the Birkhoff averages of a continuous function is residual. Motivated by these results, we show in this paper that the set $\widehat{K}$ is large from the topological point of view. Namely, we prove that $\widehat{K}$ is either empty or residual. In fact, we show that the set of points for which the function $D_r(x)$ has a prescribed set of accumulation points is also residual.

To state our result, we first introduce some notations. For $x \in K$, let $A(D(x))$ denote the set of accumulation points of $D_r(x) := \frac{\log \mu(B(x,r))}{\log r}$ as $r \searrow 0$, that is

$$A(D(x)) = \{y \in (0, +\infty) : \lim_{k\to\infty} D_{r_k}(x) = y \text{ for some } \{r_k\}_k \searrow 0\}.$$

Write $\alpha_{\min} = \min_i \frac{\log p_i}{\log r_i}$ and $\alpha_{\max} = \max_i \frac{\log p_i}{\log r_i}$. It was shown in [14] that the set $A(D(x))$ is either a singleton or a closed subinterval for any $x \in \operatorname{supp} \mu$, and $K_I = \varnothing$ if I is not a closed subinterval of $[\alpha_{\min}, \alpha_{\max}]$.

The following are our main results.

Theorem 1.1. *Assume that $\{S_i\}_{i=1}^N$ satisfies the OSC and $I \subset [\alpha_{\min}, \alpha_{\max}]$ is a closed non-singleton subinterval. Then the set*

$$K_I = \{x \in K : A(D(x)) = I\}$$

is residual if it is not empty.

The following result follows immediately from the above theorem.

Corollary 1.2. *Assume that $\{S_i\}_{i=1}^N$ satisfies the OSC. Then the set $\widehat{K}$ is residual if it is not empty.*

2. Preliminaries

For $n \in \mathbb{N}$, let

$$\Sigma^n = \{1, \ldots, N\}^n.$$

Let $\Sigma^* = \bigcup_n \Sigma^n$ and $\Sigma = \{1, \ldots, N\}^{\mathbb{N}}$. We equip Σ with the distance defined by

$$d(\omega, \omega') = 2^{-n}, \quad \omega = (\omega_i)_{i\in\mathbb{N}},\ \omega' = (\omega'_i)_{i\in\mathbb{N}},$$

where n is the smallest integer such that $\omega_n \neq \omega'_n$. It is well known that (Σ, d) is a compact metric space. For $\omega = (\omega_1 \ldots \omega_n) \in \Sigma^n$, we denote by $|\omega| = n$ the length of ω. For $\omega = (\omega_1 \ldots \omega_n) \in \Sigma^n$ and a positive integer m with $m \leq n$, or for $\omega = (\omega_1, \omega_2, \ldots) \in \Sigma$ and a positive integer m, let $\omega|m = (\omega_1 \ldots \omega_m)$. For $\omega = (\omega_1 \ldots \omega_n) \in \Sigma^n$ and $\omega' = (\omega'_1 \ldots \omega'_m) \in \Sigma^m$, we let $\omega\omega' = (\omega_1 \ldots \omega_n\omega'_1 \ldots \omega'_m) \in \Sigma^{n+m}$. Analogously, for $\omega = (\omega_1 \ldots \omega_n) \in \Sigma^n$ and $\omega' = (\omega'_1, \omega'_2, \ldots) \in \Sigma$, we let $\omega\omega' = (\omega_1 \dots \omega_n \omega'_1 \omega'_2 \dots) \in \Sigma$. Moreover, if $\omega \in \Sigma^*$ we define the cylinder $[\omega]$ generated by ω by

$$[\omega] = \{\omega\sigma : \sigma \in \Sigma\}.$$

Furthermore, for $\omega \in \Sigma^*, X, X_1, \dots, X_m \subset \Sigma^*$ and a positive integer n, we write

$$\omega X = \{\omega\eta : \eta \in X\}$$

and

$$X_1 \dots X_m = \{\omega_1 \dots \omega_m : \omega_i \in X_i\}, \qquad X^n = \{\omega_1 \dots \omega_n : \omega_i \in X\}.$$

We denote the cardinality of a set A by $\#A$. Define $\Pi : \Sigma^* \to \mathbb{R}^N$ by

$$\begin{aligned}\Pi(\omega) &= (\Pi_1(\omega), \dots, \Pi_N(\omega)) \\ &= \left(\frac{\#\{1 \le i \le n : \omega_i = 1\}}{n}, \dots, \frac{\#\{1 \le i \le n : \omega_i = N\}}{n}\right).\end{aligned} \tag{1}$$

Let

$$\Delta = \left\{\mathbf{q} = (q_1, \dots, q_N) \in \mathbb{R}^N : q_i \ge 0, \sum_i q_i = 1\right\}.$$

The following simple lemma is taken directly from [18]. We reproduce the proof for the reader's convenience.

Lemma 2.1. *Let $\omega_1, \omega_2 \in \Sigma^*$ and $\mathbf{q}_1, \mathbf{q}_2 \in \Delta$. Then*

$$|\Pi(\omega_1\omega_2) - \mathbf{q}_2| \le |\Pi(\omega_1) - \mathbf{q}_1| + |\Pi(\omega_2) - \mathbf{q}_2| + |\mathbf{q}_1 - \mathbf{q}_2|.$$

Proof. Write

$$\mathbf{q} = \frac{|\omega_1|}{|\omega_1| + |\omega_2|}\mathbf{q}_1 + \frac{|\omega_2|}{|\omega_1| + |\omega_2|}\mathbf{q}_2.$$

It is easy to check that

$$\Pi(\omega_1\omega_2) = \frac{|\omega_1|}{|\omega_1| + |\omega_2|}\Pi(\omega_1) + \frac{|\omega_2|}{|\omega_1| + |\omega_2|}\Pi(\omega_2).$$

Hence,

$$\begin{aligned}|\Pi(\omega_1\omega_2) - \mathbf{q}_2| &\le |\Pi(\omega_1\omega_2) - \mathbf{q}| + |\mathbf{q} - \mathbf{q}_2| \\ &\le \frac{|\omega_1|}{|\omega_1| + |\omega_2|}|\Pi(\omega_1) - \mathbf{q}_1| + \frac{|\omega_2|}{|\omega_1| + |\omega_2|}|\Pi(\omega_2) - \mathbf{q}_2| + \frac{|\omega_1|}{|\omega_1| + |\omega_2|}|\mathbf{q}_1 - \mathbf{q}_2| \\ &\le |\Pi(\omega_1) - \mathbf{q}_1| + |\Pi(\omega_2) - \mathbf{q}_2| + |\mathbf{q}_1 - \mathbf{q}_2|.\end{aligned}$$

The proof of the lemma is completed. □

Also, for $\omega \in \Sigma$ let $A(\Pi(\omega|n))$ denote the set of accumulation points of the sequence $\{\Pi(\omega|n)\}_n$.

Theorem 2.2. *Assume that $\{S_i\}_{i=1}^N$ satisfies the OSC and $C \subset \Delta$ is a closed and connected subset that is not a singleton. The set*

$$\Sigma_C := \{\omega \in \Sigma : A(\Pi(\omega|n)) = C\}$$

is residual if it is not empty.

Proof. We will construct a dense G_δ set $E \subset \Sigma$ such that $E \subset \Sigma_C$.

For each $\mathbf{q} \in \Delta$, $n \in \mathbb{N}$ and $\varepsilon > 0$, define

$$F(\mathbf{q}, n, \varepsilon) = \left\{\omega|n : \omega \in \Sigma^*, |\Pi(\omega) - \mathbf{q}| < \varepsilon \text{ and } r_\omega \le \frac{1}{N^n} < r_{\omega|\omega|-1}\right\}. \tag{2}$$

Note that $r_{\min} \le 1/N$ since $\{S_i\}_{i=1}^N$ satisfies the OSC. Therefore, $|\omega| \ge n$ if $r_\omega \le \frac{1}{N^n} < r_{\omega|\omega|-1}$. This implies that the above definition is reasonable. Given $\varepsilon > 0$, we have $F(\mathbf{q}, n, \varepsilon) \ne \varnothing$ for each $\mathbf{q} \in \Delta$ and any sufficiently large n, see [18].

Now for each $k \in \mathbb{N}$ choose numbers $\mathbf{q}_{k,1}, \dots, \mathbf{q}_{k,\ell_k} \in C$ such that

$$C \subset \bigcup_{i=1}^{\ell_k} B\big(\mathbf{q}_{k,i}, 1/k\big) \tag{3}$$

and

$$|\mathbf{q}_{k,i+1} - \mathbf{q}_{k,i}| < \frac{1}{k} \quad \text{for } i = 0, \ldots, \ell_k - 1, \qquad |\mathbf{q}_{k,\ell_k} - \mathbf{q}_{k+1,1}| < \frac{1}{k}. \tag{4}$$

Moreover, let $\varepsilon_1 > \varepsilon_2 > \cdots$ be a sequence of positive numbers decreasing to zero with $\varepsilon_k < 1/k$ and let

$$n_{1,1} < n_{1,2} < \cdots < n_{1,\ell_1} < n_{2,1} < n_{2,2} < \cdots < n_{2,\ell_2} < \cdots$$

be positive integers such that

$$F(\mathbf{q}_{k,i}, n_{k,i}, \varepsilon_k) \neq \varnothing \quad \text{for } k \in \mathbb{N},\ 1 \le i \le \ell_k.$$

Let $\Omega_0 = \Sigma^*$. For each $\omega \in \Omega_0$, we take rapidly increasing integers $\{N_{k,i}(\omega)\}_{k\in\mathbb{N}, i=1,\ldots,\ell_k}$ such that if

$$\begin{aligned} &\eta \in \omega F(\mathbf{q}_{1,1}, n_{1,1}, \varepsilon_1)^{N_{1,1}(\omega)} \cdots F(\mathbf{q}_{1,\ell_1}, n_{1,\ell_1}, \varepsilon_1)^{N_{1,\ell_1}(\omega)} \\ &F(\mathbf{q}_{2,1}, n_{2,1}, \varepsilon_2)^{N_{2,1}(\omega)} \cdots F(\mathbf{q}_{2,\ell_2}, n_{2,\ell_2}, \varepsilon_2)^{N_{2,\ell_2}(\omega)} \\ &\vdots \\ &F(\mathbf{q}_{k,1}, n_{k,1}, \varepsilon_k)^{N_{k,1}(\omega)} \cdots F(\mathbf{q}_{k,i}, n_{k,i}, \varepsilon_k)^{N_{k,i}(\omega)} \end{aligned}$$

then

$$|\Pi(\eta) - \mathbf{q}_{k,i}| \le \varepsilon_k \tag{5}$$

and

$$\frac{n_{k+1,\ell_{k+1}}}{|\omega| + \sum_{i=1}^{\ell_1} N_{1,i}(\omega) n_{1,i} + \cdots + \sum_{i=1}^{\ell_k} N_{k,i}(\omega) n_{k,i}} \to 0 \quad \text{as } k \to \infty. \tag{6}$$

Now we define recursively the sets $\Omega_{k,i} \subset \Sigma^*$ for $k \in \mathbb{N}$ and $i = 1, \ldots, \ell_k$ by

$$\begin{aligned} \Omega_{1,1} &= \bigcup_{\omega\in\Omega_0} \omega F(\mathbf{q}_{1,1}, n_{1,1}, \varepsilon_1)^{N_{1,1}(\omega)}, \\ \Omega_{1,2} &= \bigcup_{\eta\in\Omega_{1,1}} \eta F(\mathbf{q}_{1,2}, n_{1,2}, \varepsilon_1)^{N_{1,2}(\omega)}, \\ &\vdots \\ \Omega_{1,\ell_1} &= \bigcup_{\eta\in\Omega_{1,\ell_1-1}} \eta F(\mathbf{q}_{1,\ell_1}, n_{1,\ell_1}, \varepsilon_1)^{N_{1,\ell_1}(\omega)}, \\ \Omega_{2,1} &= \bigcup_{\eta\in\Omega_{1,\ell_1}} \eta F(\mathbf{q}_{2,1}, n_{2,1}, \varepsilon_2)^{N_{2,1}(\omega)}, \end{aligned}$$

and so on.

Finally, let

$$E_{k,i} = \bigcup_{\omega\in\Omega_{k,i}} [\omega]$$

and

$$E = \bigcap_{k=1}^{\infty} \bigcap_{i=1}^{\ell_k} E_{k,i}.$$

Clearly, E is a G_δ set since each cylinder set $[\omega]$ is open.

Next we show that E has the desired properties.

Claim 1. *E is dense in Σ.*

Proof of Claim 1. It suffices to show that $E \cap B(\omega, r) \neq \varnothing$ for every $\omega \in \Sigma$ and $r > 0$, where $B(\omega, r)$ is the ball of radius r centered at ω. Given $\omega \in \Sigma$ and $r > 0$, clearly there exist $\omega' \in \Sigma^*$ and $n \in \mathbb{N}$ such that $[\omega'|n] \subset B(\omega, r)$. Write $\eta = \omega'|n$.

Clearly, $\eta \in \Omega_0$. We take

$$\eta_{1,1} \in \eta F(\mathbf{q}_{1,1}, n_{1,1}, \varepsilon_1)^{N_{1,1}(\eta)},$$
$$\eta_{1,2} \in \eta_{1,1} F(\mathbf{q}_{1,2}, n_{1,2}, \varepsilon_1)^{N_{1,2}(\eta)},$$
$$\vdots$$
$$\eta_{1,\ell_1} \in \eta_{1,\ell_1-1} F(\mathbf{q}_{1,\ell_1}, n_{1,\ell_1}, \varepsilon_1)^{N_{1,\ell_1}(\eta)},$$
$$\eta_{2,1} \in \eta_{1,\ell_1} F(\mathbf{q}_{2,1}, n_{2,1}, \varepsilon_2)^{N_{2,1}(\eta)},$$

and so on.

It is easy to check that $([\eta_{k,i}])_{k\in\mathbb{N},\ i=1,\dots,\ell_k}$ is a decreasing sequence of non-empty compact subsets of Σ. Thus,

$$S := \bigcap_{k,i} [\eta_{k,i}] \cap [\eta] \neq \varnothing.$$

Let $\rho \in S$. We claim that $\rho \in E \cap B(\omega, r)$. Indeed, it follows from the inclusion $S \subset \bigcap_{k,i} E_{k,i}$ that $\rho \in E$. On the other hand,

$$S \subset [\eta] \subset B(\omega, r).$$

Hence, $\rho \in B(\omega, r)$. $\square$

Claim 2. $E \subset \Sigma_C$.

Proof of Claim 2. In order to prove that $E \subset \Sigma_C$, we must show that $A(\Pi(\omega|n)) = C$ for $\omega \in E$. We recall that for each $\omega \in E$, there exists $\omega^0 \in \Omega_0$ such that

$$\omega \in \omega^0 F(\mathbf{q}_{1,1}, n_{1,1}, \varepsilon_1)^{N_{1,1}(\omega^0)} \cdots. \tag{7}$$

We first show that

$$C \subset A(\Pi(\omega|n)). \tag{8}$$

Given

$$\mathbf{q} \in C \subset \bigcup_{i=1}^{\ell_k} B\left(\mathbf{q}_{k,i}, 1/k\right),$$

take an integer $i_k \in \{1, \dots, \ell_k\}$ such that $\mathbf{q} \in B(\mathbf{q}_{k,i_k}, 1/k)$. In order to avoid tedious notation we assume that $i_k \notin \{1, \ell_k\}$ although the argument is identical when $i_k \in \{1, \ell_k\}$. Let

$$n_k = |\omega^0| + \sum_{j=1}^{\ell_1} N_{1,j}(\omega^0) n_{1,j} + \cdots + \sum_{j=1}^{\ell_{k-1}} N_{k-1,j}(\omega^0) n_{k-1,j} + \sum_{j=1}^{i_k} N_{k,j}(\omega^0) n_{k,j}.$$

It follows from (5) that

$$|\Pi(\omega|n_k) - \mathbf{q}| \le |\Pi(\omega|n_k) - \mathbf{q}_{k,i_k}| + |\mathbf{q}_{k,i_k} - \mathbf{q}| \le \varepsilon_k + \frac{1}{k} \to 0$$

as $k \to \infty$. Therefore, $\mathbf{q} \in A(\Pi(\omega|n))$ and the proof of (8) is completed.

Now we show that

$$A(\Pi(\omega|n)) \subset C. \tag{9}$$

For each positive integer $n > |\omega^0|$ there exist $k \in \mathbb{N}$, $0 \le j < \ell_{k+1}$, $0 \le N < N_{k+1,j+1}(\omega^0)$ and $0 \le p \le n_{k+1,j+1}$ such that

$$n = |\omega^0| + s + t + p, \tag{10}$$

where

$$s = \sum_{i=1}^{\ell_1} N_{1,i}(\omega^0) n_{1,i} + \cdots + \sum_{i=1}^{\ell_k} N_{k,i}(\omega^0) n_{k,i} + \sum_{i=1}^{j} N_{k+1,i}(\omega^0) n_{k+1,i}$$

and

$$t = N n_{k+1,j+1}.$$

We claim that

$$\operatorname{dist}(\Pi(\omega|n), C) \to 0 \quad \text{when } n \to \infty. \tag{11}$$

Note that C is closed, this implies that $A(\Pi(\omega|n)) \subset C$ and therefore the proof of (9) is completed.

Next we establish property (11). Write $\omega = \omega^0\eta_1\eta_2\eta_3\eta$ where $|\eta_1| = s$, $|\eta_2| = t$, $|\eta_3| = p$ and $\eta \in \Sigma$. We divide the proof of (11) into two cases.

Case 1: $j \neq 0$. It follows from Lemma 2.1 that

$$\begin{aligned}
\operatorname{dist}(\Pi(\omega|n), C) &\le |\Pi(\omega|n) - \Pi(\omega^0\eta_1\eta_2)| + |\Pi(\omega^0\eta_1\eta_2) - \mathbf{q}_{k+1,j+1}| + \operatorname{dist}(\mathbf{q}_{k+1,j+1}, C) \\
&\le |\Pi(\omega|n) - \Pi(\omega^0\eta_1\eta_2)| + |\Pi(\omega^0\eta_1) - \mathbf{q}_{k+1,j}| + |\Pi(\eta_2) - \mathbf{q}_{k+1,j+1}| \\
&\quad + |\mathbf{q}_{k+1,j} - \mathbf{q}_{k+1,j+1}| + \frac{1}{k+1} \\
&\le |\Pi(\omega|n) - \Pi(\omega^0\eta_1\eta_2)| + \varepsilon_{k+1} + \varepsilon_{k+1} + \frac{1}{k+1} + \frac{1}{k+1} \quad \text{(by (5), (2) and (4))} \\
&\le |\Pi(\omega|n) - \Pi(\omega^0\eta_1\eta_2)| + \frac{4}{k}. \quad (\text{since } \varepsilon_{k+1} < 1/(k+1)).
\end{aligned} \tag{12}$$

Case 2: $j = 0$. Also, it follows from Lemma 2.1 that

$$\begin{aligned}
\operatorname{dist}(\Pi(\omega|n), C) &\le |\Pi(\omega|n) - \Pi(\omega^0\eta_1\eta_2)| + |\Pi(\omega^0\eta_1\eta_2) - \mathbf{q}_{k+1,1}| + \operatorname{dist}(\mathbf{q}_{k+1,1}, C) \\
&\le |\Pi(\omega|n) - \Pi(\omega^0\eta_1\eta_2)| + |\Pi(\omega^0\eta_1) - \mathbf{q}_{k,\ell_k}| + |\Pi(\eta_2) - \mathbf{q}_{k+1,1}| \\
&\quad + |\mathbf{q}_{k,\ell_k} - \mathbf{q}_{k+1,1}| + \frac{1}{k+1} \\
&\le |\Pi(\omega|n) - \Pi(\omega^0\eta_1\eta_2)| + \varepsilon_k + \varepsilon_{k+1} + \frac{1}{k} + \frac{1}{k+1} \quad \text{(by (5), (2) and (4))} \\
&\le |\Pi(\omega|n) - \Pi(\omega^0\eta_1\eta_2)| + \frac{4}{k}. \quad (\text{since } \varepsilon_k < 1/k).
\end{aligned} \tag{13}$$

Next we estimate the term $|\Pi(\omega|n) - \Pi(\omega^0\eta_1\eta_2)|$. It is easy to check that

$$\Pi(\omega|n) = \Pi(\omega^0\eta_1\eta_2\eta_3) = \frac{|\omega^0\eta_1\eta_2|}{|\omega^0\eta_1\eta_2| + |\eta_3|}\Pi(\omega^0\eta_1\eta_2) + \frac{|\eta_3|}{|\omega^0\eta_1\eta_2| + |\eta_3|}\Pi(\eta_3).$$

Therefore,

$$\begin{aligned}
|\Pi(\omega|n) - \Pi(\omega^0\eta_1\eta_2)| &\le \frac{|\eta_3|}{|\omega^0\eta_1\eta_2| + |\eta_3|}|\Pi(\eta_3) - \Pi(\omega^0\eta_1\eta_2)| \\
&\le \frac{|\eta_3|}{|\omega^0\eta_1\eta_2|}\big|\Pi(\eta_3) + \Pi(\omega^0\eta_1\eta_2)\big| \\
&\le \frac{2|\eta_3|}{|\omega^0\eta_1\eta_2|} \\
&\le \frac{2n_{k+1,\ell_{k+1}}}{|\omega^0| + \sum_{i=1}^{\ell_1} N_{1,i}(\omega^0)n_{1,i} + \cdots + \sum_{i=1}^{\ell_k} N_{k,i}(\omega^0)n_{k,i}}.
\end{aligned} \tag{14}$$

By (12)–(14) we have

$$\operatorname{dist}(\Pi(\omega|n), C) \le \frac{2n_{k+1,\ell_{k+1}}}{|\omega^0| + \sum_{i=1}^{\ell_1} N_{1,i}(\omega^0)n_{1,i} + \cdots + \sum_{i=1}^{\ell_k} N_{k,i}(\omega^0)n_{k,i}} + \frac{4}{k}. \tag{15}$$

Finally, property (11) follows from (6) and (15). Therefore, Claim 2 follows from (8) and (9). □

The proof of Theorem 2.2 is completed. □

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

Let us first introduce some notations. For $\omega = (\omega_1 \dots \omega_n) \in \Sigma^n$, write $p_\omega = p_{\omega_1} \dots p_{\omega_n}$, $r_\omega = r_{\omega_1} \dots r_{\omega_n}$, $S_\omega = S_{\omega_1} \circ \cdots \circ S_{\omega_n}$ and $K_\omega = S_\omega(K)$. Furthermore, define $\pi : \Sigma \to \mathbb{R}^d$ by

$$\{\pi(\omega)\} = \bigcap_{k=1}^{\infty} K_{\omega|k}.$$

It is well known that the map is continuous and onto, see [6].

For $\omega \in \Sigma$ and $n \in \mathbb{N}$ define

$$L(\omega|n) = \frac{\log \mu(K_{\omega|n})}{\log \operatorname{diam} K_{\omega|n}}.$$

It follows from [10] that $\mu(K_{\omega|n}) = p_{\omega|n}$ under the OSC. Without loss of generality, we can assume that $\operatorname{diam} K = 1$. Hence,

$$L(\omega|n) = \frac{\log p_{\omega|n}}{\log r_{\omega|n}} = \frac{\sum_i \Pi_i(\omega|n) \log p_i}{\sum_i \Pi_i(\omega|n) \log r_i}. \tag{16}$$

Given $I \subset [\alpha_{\min}, \alpha_{\max}]$, write

$$S_I = \{\omega \in \Sigma : A(L(\omega|n)) = I\},$$

where $A(L(\omega|n))$ denotes the set of accumulation points of the sequence $\{L(\omega|n)\}_n$. It is worth pointing out that $\pi(S_I) = K_I$ if the SSC is satisfied, see [6,18]. However, due to the overlap structure the two sets do not coincide under the OSC.

The following result appeared in [18] but we present its proof for the reader's convenience.

Proposition 3.1. *For any closed non-singleton subinterval $I \subset [\alpha_{\min}, \alpha_{\max}]$, there exists a compact connected non-singleton subset $C \subset \Delta$ such that $\Sigma_C \subset S_I$.*

Proof. Let $I = [a, b]$. There exist $q_1, q_2 \in \mathbb{R}$ with $q_1 \le q_2$ such that $\alpha(q_1) = b$, $\alpha(q_2) = a$ and $\alpha([q_1, q_2]) = [a, b]$. Here the function $\alpha(q)$ is defined by

$$\alpha(q) = -\beta'(q) = \frac{\sum_i p_i^q r_i^{\beta(q)} \log p_i}{\sum_i p_i^q r_i^{\beta(q)} \log r_i},$$

and $\beta : \mathbb{R} \to \mathbb{R}$ is defined by $\sum_i p_i^q r_i^{\beta(q)} = 1$. For $q \in \mathbb{R}$, define the probability vector $v_q = (p_1^q r_1^{\beta(q)}, \ldots, p_N^q r_N^{\beta(q)})$ and let $C = \{v_q : q_1 \le q \le q_2\}$. Clearly, the set C is not a singleton if I is not a singleton. Moreover, the set C is compact and connected since $q \to \Delta_q$ is continuous.

Next we show that

$$\Sigma_C \subset S_I.$$

Let $\omega \in \Sigma_C$. We must prove that

$$A(L(\omega|n)) \subset I \tag{17}$$

and

$$I \subset A(L(\omega|n)). \tag{18}$$

Let $x \in A(L(\omega|n))$. Then there exists a subsequence $(L(\omega|n_k))_k$ such that $L(\omega|n_k) \to x$ as $k \to \infty$. Since $(L(\omega|n_k))_k \subset \Delta$ and Δ is compact, there exists a probability vector $v = (v_i)_i \in \Delta$ and a subsequence $(\Pi(\omega|n_{k_j}))_j$ such that $\Pi(\omega|n_{k_j}) \to v$ as $j \to \infty$. Therefore, $v \in A(\Pi(\omega|n)) = C$ and $v = v_q$ for some $q_1 \le q \le q_2$. It follows from (16) that

$$\begin{aligned} x &= \lim_{j\to\infty} L(\omega|n_{k_j}) = \lim_{j\to\infty} \frac{\sum_i \Pi_i(\omega|n_{k_j}) \log p_i}{\sum_i \Pi_i(\omega|n_{k_j}) \log r_i} \\ &= \frac{\sum_i v_i \log p_i}{\sum_i v_i \log r_i} = \frac{\sum_i p_i^q r_i^{\beta(q)} \log p_i}{\sum_i p_i^q r_i^{\beta(q)} \log r_i} = \alpha(q) \in [a, b] = I. \end{aligned}$$

The proof of (17) is completed.

On the other hand, let $x \in I$. We can find $q \in [q_1, q_2]$ such that $x = \alpha(q)$. Since $v_q \in C = A(\Pi(\omega|n))$, there exists a subsequence $(\Pi(\omega|n_k))_k$ such that $\Pi(\omega|n_k) \to v_q$ as $k \to \infty$. Therefore,

$$L(\omega|n_k) = \frac{\sum_i \Pi_i(\omega|n_k) \log p_i}{\sum_i \Pi_i(\omega|n_k) \log r_i} \to \frac{\sum_i p_i^q r_i^{\beta(q)} \log p_i}{\sum_i p_i^q r_i^{\beta(q)} \log r_i} = \alpha(q) = x.$$

This implies that $x \in A(L(\omega|n))$ and the proof of (18) is completed. $\square$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Put

$$W=\left(\bigcup_{n=1}^{\infty}\bigcup_{\omega\in\Sigma^*}S_\omega(V)\right)\bigcup V,$$

where

$$V=\bigcup_{i\neq i}S_i(K)\cap S_j(K).$$

Write

$$\widetilde{\Sigma}=\Sigma\setminus\pi^{-1}(W),\qquad \widetilde{K}=K\setminus W.$$

Then, $\pi:\widetilde{\Sigma}\to\widetilde{K}$ is bijective and $\sigma(\widetilde{\Sigma})=\widetilde{\Sigma}$, where σ is the shift operator on Σ, see [15]. Clearly, $\widetilde{\Sigma}$ is a G_δ set since $\pi^{-1}(W)$ is an F_σ set. Moreover, since any invariant set of one-sided shift is dense, $\widetilde{\Sigma}$ is a dense G_δ set.

Let $I\subset[\alpha_{\min},\alpha_{\max}]$ be a closed non-singleton subinterval. It follows from Proposition 3.1 and Theorem 2.2 that there exists a dense G_δ set $E\subset S_I$. To complete the proof, it suffices to show that the set $F=\pi(E\cap\widetilde{\Sigma})\subset\widetilde{K}$ satisfies the following properties:

(1) $F\subset K_I$;
(2) F is dense in K;
(3) F is a G_δ set.

It is easy to check that

$$F=\pi(E\cap\widetilde{\Sigma})\subset\pi(S_I\cap\widetilde{\Sigma})=K_I\cap\widetilde{K}\subset K_I.$$

Moreover, $E\cap\widetilde{\Sigma}$ is a dense G_δ set since both E and $\widetilde{\Sigma}$ are dense G_δ sets. In particular,

$$K=\pi(\Sigma)=\pi(\overline{E\cap\widetilde{\Sigma}})\subset\overline{\pi(E\cap\widetilde{\Sigma})}=\overline{F}$$

and therefore F is dense in K.

For the last property, we observe that

$$\begin{aligned}K\setminus F&=(W\cup\widetilde{K})\setminus F=W\cup(\widetilde{K}\setminus F)\quad(\text{since } W\cap F=\varnothing)\\&=W\cup\big(\pi(\widetilde{\Sigma})\setminus\pi(E\cap\widetilde{\Sigma})\big)\\&=W\cup\pi\big(\widetilde{\Sigma}\setminus(E\cap\widetilde{\Sigma})\big)\quad(\text{since }\pi\text{ is bijective on }\widetilde{\Sigma})\\&=\pi(\Sigma\setminus\widetilde{\Sigma})\cup\pi\big(\widetilde{\Sigma}\setminus(E\cap\widetilde{\Sigma})\big)\\&=\pi\big((\Sigma\setminus\widetilde{\Sigma})\cup(\widetilde{\Sigma}\setminus(E\cap\widetilde{\Sigma}))\big)\\&=\pi\big(\Sigma\setminus(E\cap\widetilde{\Sigma})\big).\end{aligned}$$

Finally, $\Sigma\setminus(E\cap\widetilde{\Sigma})$ is an F_σ set (since $E\cap\widetilde{\Sigma}$ is a G_δ set). Writing $\Sigma\setminus(E\cap\widetilde{\Sigma})=\bigcup_i F_i$ as a countable union of closed sets $F_i\subset\Sigma$, we obtain

$$K\setminus F=\pi\big(\Sigma\setminus(E\cap\widetilde{\Sigma})\big)=\bigcup_i\pi(F_i),$$

where $\pi(F_i)$ is a closed set (since π is continuous and K is compact). This shows that F is a G_δ set and the proof of Theorem 1.1 is complete. $\square$

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