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A representation of solutions to a scalar conservation law in several dimensions[☆]

Sergio Albeverio[a,b,c,d,e], Olga Rozanova[f,*]

[a] Universität Bonn, Institut für Angewandte Mathematik, Abteilung für Stochastik, Endenicher Allee 60, D-53115 Bonn, Germany
[b] HCM, SFB611, Bonn, Germany
[c] IZKS, Bonn, Germany
[d] BiBoS, Bielefeld–Bonn, Germany
[e] CERFIM, Locarno, Switzerland
[f] Department of Mechanics and Mathematics, Moscow State University, Moscow 119991, Russia

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ABSTRACT

We find a representation of smooth solutions to the Cauchy problem for a scalar multidimensional conservation law as a small diffusion limit of a stochastic perturbation along characteristics. It helps, in particular, to study the process of singularity formation. Further, we introduce an associated system of balance laws that can be interpreted as describing the motion of a continuum with some specific pressure term. This term arises only after the instant when the solution to the initial Cauchy problem loses its smoothness. Before this instant the system coincides partly with the one known as pressure free gas dynamics.

0. Introduction

The theory of scalar balance laws is relatively well developed. In particular, the method of characteristics allows us to determine explicitly the life span of solutions with Lipschitz continuous initial data and demonstrate that in general this life span is finite. After this moment one can introduce a weak solution. There exists a number of ways for the construction of the admissible weak solution: the method of vanishing viscosity, the theory of L^1-contraction semigroups, the layering method, a relaxation method and an approach motivated by the kinetic theory [7].

Moreover, before the moment of the singularity formation the method of characteristics allows us to construct the smooth solution. Nevertheless, this solution cannot generally be obtained explicitly. In this paper we propose a method for finding an asymptotical representation of the smooth solution before the moment of the singularity formation.

Namely, we consider the initial value problem

$$u_t + \sum_{i=1}^{n} a_i(t,x,u)\, u_{x_i} = 0, \qquad u(0,x) = u_0(x), \quad u_0(x) \in C_b^1(\mathbb{R}^n;\mathbb{R}), \tag{1}$$

where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, $a_i(t,x,u)$, $i = 1,\dots,n$, is a real-valued C^1 function defined on some open subset of $(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R})$. For a technical reason the functions $a_i(t,x,u)$ are assumed to grow at infinity not quicker than a linear function in u.

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* Corresponding author.
E-mail address: rozanova@mech.math.msu.su (O. Rozanova).

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A particular important special case is given by the scalar conservation law in the form

$$u_t + \mathrm{div} F(t, u) = 0,$$

where $F(t, \cdot) = (F_1(t, \cdot), \dots, F_n(t, \cdot))$ is a C^2 vector-function defined on some open subset of $\mathbb{R}$, for any $t \in \mathbb{R}_+$, and $a_i(t, u) = \frac{\partial F_i(t,u)}{\partial u}$, $i = 1, \dots, n$.

The main aim of this paper is to obtain an asymptotic formula for the solution to the Cauchy problem (1) for the case of a scalar conservation law. The formula is obtained by the limit for vanishing perturbation of the corresponding stochastically modified equation (small diffusion limit).

The introduction of a small perturbation of deterministic equation to study then the original equation in the limit of vanishing noise has appeared in several contexts, particularly for equations of the reaction–diffusion type, see, e.g. [1,9] and references therein. In [12] large deviations principles for a family of scalar conservation laws with both stochastic and viscous perturbations, in the limit of jointly vanishing noise and viscosity.

Our asymptotical representation satisfies a special system of balance laws with special integral terms. Nevertheless, after the moment of the loss of smoothness our asymptotical representation does not corresponds to the admissible weak solution as small perturbation limit.

In our previous works we considered the non-viscous Burgers equation including multidimensional case. The 1D case the Burgers equation is a very particular case of the 1D scalar conservation laws, therefore our previous results intersect with the results presented in this paper.

The paper is organized as follows. In Section 1 we introduce a stochastic differential equation that corresponds to the stochastic perturbation of the quasilinear equation (1) along characteristics. We consider the relative Fokker–Planck equation and introduce several integral values that pretend to represent the solution of (1). In Section 2 we show, for the case of conservation law, how these integral values are connected with the solution to the Cauchy problem (1). In Section 3 we prove that as a small perturbation limit the integrals introduced in Section 1 represent the smooth solution to the Cauchy problem (1). In Section 4 by the example of 1D conservation law we show how it is possible to describe the formation of singularity in solution, e.g. the formation of unbounded derivative. At last, in Section 5 we introduce a system of balance law associated with the Cauchy problem (1). Moreover, we describe a relationship between this system and the system of pressureless gas dynamics that arises in context of the non-viscous Burgers equation.

1. Stochastic differential equation associated with the equation of characteristics

Let us write the characteristic ODE of Eq. (1):

$$\frac{dx_i}{dt} = a_i(t, x, u), \qquad \frac{du}{dt} = 0, \quad i = 1, \dots, n.$$

Its stochastic analog is

$$\begin{aligned} &dX_i(t) = a_i(t, X(t), U(t))dt + \sigma_1 d(W_i^1)_t, \qquad dU(t) = \sigma_2 d(W^2)_t, \\ &X_i(0) = x_i, \qquad U(0) = u, \quad t > 0, \end{aligned} \tag{2}$$

$i = 1, \dots, n$, $X(t)$ and $U(t)$ are considered as random variables with given initial distributions, $(X(t), U(t))$ runs in the phase space $\mathbb{R}^n \times \mathbb{R}^1$, σ_1 and σ_2 are nonnegative constants such that $|\sigma| \neq 0$ ($\sigma = (\sigma_1, \sigma_2)$) and $((W^1)_t, (W^2)_t) = (W_1^1, \dots, W_n^1, W^2)_t$ is an $n + 1$-dimensional Brownian motion, i.e. the W_i^1, W^2, $i = 1, \dots, n$, are independent one-dimensional standard Brownian motions.

Let $P(t, dx, du)$, $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, be the joint probability distribution of the random variables (X, U), subject to the initial data

$$P_0(dx, du) = \delta_u(u_0(x))\, \rho_0(x) dx, \tag{3}$$

where ρ_0 is a bounded nonnegative function from $C(\mathbb{R}^n)$ and dx is Lebesgue measure on $\mathbb{R}^n$, δ_u is Dirac measure concentrated on u.

We look at $P = P(t, dx, du)$ as a generalized function (distribution) with respect to the variable u. It satisfies the Fokker–Planck equation

$$\frac{\partial P}{\partial t} = \left[-\sum_{k=1}^n \frac{\partial}{\partial x_k} a_k(t, x, u) + \sum_{k=1}^n \frac{1}{2}\sigma_1^2 \frac{\partial^2}{\partial x_k^2} + \sum_{k=1}^n \frac{1}{2}\sigma_2^2 \frac{\partial^2}{\partial u_k^2} \right] P, \tag{4}$$

subject to the initial data (3).

There is a standard procedure for finding the fundamental solution for (4) (see, e.g. [11]). This procedure consists in a reduction of the equation to a Fredholm integral equation, the solution of which can be found in the form of series. We are going to show that for $a(t, x, u) = a(t, u)$ one can also find an explicit solution to the Cauchy problem (4) and (3).

Let us introduce, still in the general case, the functions, for $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, depending on $\sigma = (\sigma_1, \sigma_2)$:

$$\rho_\sigma(t,x) = \int_{\mathbb{R}} P(t,x,du), \tag{5}$$

$$u_\sigma(t,x) = \frac{\int_{\mathbb{R}} uP(t,x,du)}{\int_{\mathbb{R}} P(t,x,du)}, \tag{6}$$

$$a_\sigma(t,x) = \frac{\int_{\mathbb{R}} a(t,x,u)P(t,x,du)}{\int_{\mathbb{R}} P(t,x,du)}. \tag{7}$$

We can consider these values if the integrals exist in the Lebesgue sense.

It will readily be observed that $u_\sigma(0,x) = u_0(x)$ and $a_\sigma(0,x) = a(0,x,u_0(x))$.

We denote

$$\bar\rho(t,x) = \lim_{\sigma\to 0} \rho_\sigma(t,x), \qquad \bar u(t,x) = \lim_{\sigma\to 0} u_\sigma(t,x), \qquad \bar a(t,x) = \lim_{\sigma\to 0} a_\sigma(t,x),$$

provided these limits exist.

2. Case of a conservation law

Now we dwell on the simpler case of a conservation law, where $a = a(t,u)$. Here the Eq. (4) can be solved explicitly. Moreover, for the sake of simplicity we set $\sigma_2 = 0$ and denote $\sigma_1 = \sigma$.

Proposition 1. *If $a = a(t,u)$, then problem (4), (3) has the following solution:*

$$P(t,x,du) = \frac{1}{(\sqrt{2\pi t}\sigma)^n} \int_{\mathbb{R}^n} \delta_u(u_0(y))\, \rho_0(y)\, e^{-\frac{\sum_{i=1}^n \left|\int_0^t a_i(\tau,u_0(y))d\tau + y_i - x_i\right|^2}{2\sigma^2 t}}\, dy, \tag{8}$$

$t \geq 0$, $x \in \mathbb{R}^n$, or, in other words,

$$\int_{\mathbb{R}} \phi(u)\, P(t,x,du) = \frac{1}{(\sqrt{2\pi t}\sigma)^n} \int_{\mathbb{R}^n} \phi(u_0(y))\, \rho_0(y)\, e^{-\frac{\sum_{i=1}^n \left|\int_0^t a_i(\tau,u_0(y))d\tau + y_i - x_i\right|^2}{2\sigma^2 t}}\, dy, \tag{9}$$

for all $\phi(u) \in C_0(\mathbb{R})$.

Proof. We act as in [5,2]. Namely, we apply the Fourier transform to $P(t,x,du)$ in (4), (3) with respect to the variable x and obtain the Cauchy problem for the Fourier transform $\tilde P = \tilde P(t,\lambda,du)$ of $P(t,x,du)$:

$$\frac{\partial \tilde P}{\partial t} = -\left(\frac{1}{2}\sigma^2|\lambda|^2 + i(\lambda, a(t,u))\right)\tilde P, \tag{10}$$

$$\tilde P(0,\lambda,du) = \int_{\mathbb{R}^n} e^{-i(\lambda,y)}\delta_u(u_0(y))\rho_0(y)dy, \quad \lambda \in \mathbb{R}^n. \tag{11}$$

Eq. (10) can easily be integrated and we obtain the solution given by the following formula:

$$\tilde P(t,\lambda,du) = \tilde P(0,\lambda,du) e^{-\frac{1}{2}\sigma^2|\lambda|^2 t + i\int_0^t (\lambda, a(\tau,u))d\tau}. \tag{12}$$

The inverse Fourier transform (in the distributional sense) allows us to find the density function $P(t,x,du)$, $t > 0$:

$$\begin{aligned}
P(t,x,du) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(\lambda,x)}\tilde P(t,\lambda,du)\, d\lambda \\
&= \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^n} e^{i(\lambda,x)}\left(\int_{\mathbb{R}^n} e^{-i(\lambda,y)} e^{-i\int_0^t(\lambda,a(\tau,u))d\tau}\, \delta_u(u_0(y))\, \rho_0(y)dy\right) e^{-\frac{1}{2}\sigma^2|\lambda|^2 t} d\lambda \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \delta_u(u_0(y))\, \rho_0(y) \int_{\mathbb{R}^n} e^{-\frac{1}{2}\sigma^2 t\left(\lambda - \frac{i|x - \int_0^t a(\tau,u)d\tau - y|}{\sigma^2 t}\right)^2 - \frac{\left|\int_0^t a(\tau,u)d\tau + y - x\right|^2}{2\sigma^2 t}} d\lambda dy \\
&= \frac{1}{(\sqrt{2\pi t}\sigma)^n}\int_{\mathbb{R}^n} \delta_u(u_0(y))\, \rho_0(y)\, e^{-\frac{\left|\int_0^t a(\tau,u_0(y))d\tau + y - x\right|^2}{2\sigma^2 t}} dy, \quad t \geq 0, x \in \mathbb{R}^n.
\end{aligned}$$

The third equality is satisfied by Fubini's theorem, which can be applied by the absolute integrability and the bound on the function involved. Thus, the proposition is proved. □

Remark 1. In the general case $\sigma_2 \neq 0$ an analogous formula can be obtained in a similar way.

Corollary 1. *The functions ρ_σ, u_σ and a_σ defined in (5)–(7) can be represented by the following formulas:*

$$\rho_\sigma(t,x) = \frac{1}{(\sqrt{2\pi t}\sigma)^n}\int_{\mathbb{R}^n}\rho_0(y)\, e^{-\frac{\sum_{i=1}^n\left|\int_0^t a_i(\tau,u_0(y))d\tau+y_i-x_i\right|^2}{2\sigma^2 t}}dy, \tag{13}$$

$$u_\sigma(t,x) = \frac{\int_{\mathbb{R}^n} u_0(y)\rho_0(y)\, e^{-\frac{\sum_{i=1}^n\left|\int_0^t a_i(\tau,u_0(y))d\tau+y_i-x_i\right|^2}{2\sigma^2 t}}dy}{\int_{\mathbb{R}^n}\rho_0(y)\, e^{-\frac{\sum_{i=1}^n\left|\int_0^t a_i(\tau,u_0(y))d\tau+y_i-x_i\right|^2}{2\sigma^2 t}}dy}, \tag{14}$$

$$a_\sigma(t,x) = \frac{\int_{\mathbb{R}^n} a(t,u_0(y))\rho_0(y)\, e^{-\frac{\sum_{i=1}^n\left|\int_0^t a_i(\tau,u_0(y))d\tau+y_i-x_i\right|^2}{2\sigma^2 t}}ds}{\int_{\mathbb{R}^n}\rho_0(y)\, e^{-\frac{\sum_{i=1}^n\left|\int_0^t a_i(\tau,u_0(y))d\tau+y_i-x_i\right|^2}{2\sigma^2 t}}ds}. \tag{15}$$

Proof. The result is obtained by substitution of $P(t,x,du)$ as given by (8) in (5)–(7). □

3. Asymptotic formula for smooth solutions

Let us define the following subset Λ of $\mathbb{R}$:

$$t\in\Lambda \quad \text{if} \inf_{y\in\mathbb{R}^n}\int_0^t\sum_{i=1}^n (a_i)_u(\tau,u_0(y))(u_0(y))_{y_i}d\tau > -1, \tag{16}$$

where $u_0\in C_b^1(\mathbb{R}^n)$. It is not difficult to show the Λ is an open set. Denote $t_*(u_0)=\sup\Lambda$.

The following theorem holds:

Theorem 1. *Let $u(t,x)$ be a solution to the Cauchy problem*

$$u_t+\sum_{i=1}^n a_i(t,u)\,u_{x_i}=0, \qquad u(0,x)=u_0(x), \tag{17}$$

where a_i, $i=1,\dots,n$, are C^1-functions defined on some open subset of $(\mathbb{R}_+\times\mathbb{R})$ and $u_0\in C_b^1(\mathbb{R}^n)$. Assume $t_*(u_0)=\sup\Lambda>0$, Λ being defined by (16). Then for $t\in[0,t_*(u_0))$,

$$u(t,x)=\bar u(t,x)=\lim_{\sigma\to 0}u_\sigma(t,x),$$

where $u_\sigma(t,x)$ is given by (6) and the limit exists pointwise.

Proof. The proof is similar to the one given in [2] for a related problem. According to the classical theory (see, e.g. [7, Theorem 5.1.1]), the solution u of (17) exists on some maximal interval $[0,T)$, $T\le\infty$ and is a C^1-smooth function. Since u is constant along characteristics, its value at any point (t,x), with $x\in\mathbb{R}^n$, $t\in\mathbb{R}_+$, satisfies the implicit relation

$$u(x,t)=u_0\left(x-\int_0^t a(\tau,u)d\tau\right). \tag{18}$$

In particular, the range of u coincides with the range of u_0.

Differentiating (18) yields

$$\partial_{x_i}u(t,x)=\frac{\partial_{y_i}u_0(y)}{1+\int_0^t\sum_{i=1}^n(a_i)_u(\tau,u_0(y))(u_0(y))_{y_i}d\tau}, \quad y=x-\int_0^t a(\tau,u)d\tau. \tag{19}$$

This imply $T=t_*(u_0)$. If $0<t_*(u_0)<+\infty$, then the solution to the Cauchy problem blows up at the instant $t_*(u_0)$. Otherwise, the solution keeps its smoothness for all $t>0$.

The formula (6) implies, using the weak convergence of measures and the fact that ρ_0 and u_0 are continuous and bounded and independent of σ

$$\lim_{\sigma\to0}u_\sigma(t,x) = \frac{\int_{\mathbb{R}^n}u_0(y)\rho_0(y)\lim_{\sigma\to0}\frac{1}{(\sqrt{2\pi t}\sigma)^n}e^{-\frac{\left|\int_0^t a(\tau,u_0(y))d\tau+y-x\right|^2}{2\sigma^2 t}}dy}{\int_{\mathbb{R}^n}\rho_0(y)\lim_{\sigma\to0}\frac{1}{(\sqrt{2\pi t}\sigma)^n}e^{-\frac{\left|\int_0^t a(\tau,u_0(y))d\tau+y-x\right|^2}{2\sigma^2 t}}dy}$$
$$= \frac{\int_{\mathbb{R}^n}u_0(y)\rho_0(y)\delta_{p(t,x,y)}dy}{\int_{\mathbb{R}^n}\rho_0(y)\delta_{p(t,x,y)}dy},$$

with

$$p(t,x,y) := \int_0^t a(\tau, u_0(y))d\tau + y - x, \tag{20}$$

where δ_p is the Dirac measure at $p \in \mathbb{R}^n$. We can use locally the implicit function theorem and find $y = y_{t,x}(p)$ from $p(t,x,y)$. The condition for existence of this function is the invertibility of the matrix

$$C_{ij}(t,y) = \frac{\partial p_i(t,x,y)}{\partial y_j}, \quad i,j = 1,\dots,n.$$

This matrix fails to be invertible for $t = t_*(u_0)$. For $t < t_*(u_0)$

$$\begin{aligned}\bar{u}(t,x) &= \lim_{\sigma\to 0} u_\sigma(t,x)\\ &= \frac{\int_{\mathbb{R}^n} u_0(y_{t,x}(p))\rho_0(y_{t,x}(p))\ \det(C(t,y_{t,x}(p)))^{-1}\,\delta_p\,(dy_{t,x})}{\int_{\mathbb{R}^n} \rho_0(y_{t,x}(p))\ \det(C(t,y_{t,x}(p)))^{-1}\,\delta_p\,(dy_{t,x})} = u_0(y_{t,x}(0)).\end{aligned}$$

Let us introduce the new notation $y_0(t,x) \equiv y_{t,x}(0)$. Then (20) implies the following vectorial equation:

$$\int_0^t a(\tau, u_0(y_0(\tau,x)))d\tau + y_0(t,x) - x = 0, \quad t \ge 0, x \in \mathbb{R}^n. \tag{21}$$

Let us show that $u(t,x) = u_0(y_0(t,x))$ satisfies Eq. (1), that is

$$\sum_{j=1}^n \partial_j(u_0)(y_{0,j})_t + \sum_{j,k=1}^n a_j(t,u_0)\partial_k(u_0)(y_{0,k})_{x_j} = 0 \tag{22}$$

and $u_0(y_0(0,x)) = u_0(x)$. Here we denote by $y_{0,i}$ the i-th components of the vector y_0.

For $t < t_*(u_0)$ we can differentiate (21) with respect to t and x_j to get the matrix equations:

$$\sum_{j=1}^n C_{ij}\,(y_{0,j})_t + u_{0,i} = 0, \quad i = 1,\dots,n,$$

and

$$\sum_{k=1}^n C_{ik}\,(y_{0,k})_{x_j} + \delta_{ij} = 0, \quad i,j = 1,\dots,n,$$

where δ_{ij} is the Kronecker symbol. The equations imply

$$(y_{0,j})_t = -\sum_{i=1}^n (C^{-1})_{ij}\,u_{0,i}, \qquad (y_{0,k})_{x_j} = -(C^{-1})_{jk}. \tag{23}$$

It remains now only to substitute (23) into (22) to see that $\bar{u}(t,x)$ satisfies the first equation in (17).

Further, (21) implies $u_0(y_0(0,x)) = u_0(x)$, thus Theorem 1 is proved. □

Remark 2. For $a_i = a_i(u)$ (i.e. a_i is independent of variable t) we have the Conway criterion [6]:

$$t_*(u_0) = \inf_{y\in\mathbb{R}^n}\left(-\frac{1}{\sum\limits_{i=1}^n (a_i)_u(u_0(y))(u_0(y))_{y_i}}\right). \tag{24}$$

Note that if the denominator vanishes, then $t_*(u_0) = \infty$ and the solution does not blow up. If $t_*(u_0) < 0$, then the solution is globally smooth for $t \ge 0$, as well.

Proposition 2. *Under the assumptions of Theorem 1 the vector*

$$\bar{a}(t,x) = \lim_{\sigma\to 0} a_\sigma(t,x),$$

where $a_\sigma(t,x)$ is given by (7), solves the multidimensional Burgers equation

$$(\bar{a}(t,u))_t + (\bar{a}(t,u),\nabla)\bar{a}(t,u) = 0$$

with initial data $\bar{a}(0,u(0,x)) = a(0,u_0(x))$.

Proof. This fact follows directly from Proposition 2.1 of [2]. □

4. Asymptotics of the singularity formation

In this section we show how the formulas from Corollary allows us to describe the formation of the singularity of solution to conservation laws. For the sake of simplicity we restrict ourselves to the case of one space variable and $a = a(u)$.

First of all we make several assumptions on the structure of initial data $u_0(x)$.

- (A) *The moment* $t_* = t_*(u_0)$ *given by* (24) *is such that* $0 < t_*(u_0) < \infty$ *and* $y^* \in \mathbb{R}$ *is the infimum point in* (24);
- (B) *There exists* $x^* \in \mathbb{R}$ *such that equation*

$$a(u_0(y))t_* + y - x^* = 0$$

has a unique solution y^*;
- (C) *The function* $a(u) \in C^k(\mathbb{R})$, $k > 2$ *and there exist* $m \in \mathbb{N}, k \geq m > 2$, *such that*

$$\left.\frac{\partial^k a(u_0(y))}{\partial^k y}\right|_{y=y^*} = 0, \quad k = 2, \dots, m-1, \qquad \left.\frac{\partial^m a(u_0(y))}{\partial^m y}\right|_{y=y^*} \neq 0. \tag{25}$$

Conditions (A)–(C) mean that y^* is a point of inflection of the graph of function $z = a(u_0(y))$ and x^* is a point, where the tangent line to the graph at the point $y = y^*$ intersects the abscissa axis. If $z = a(u_0(y))$ is bounded, then (A) implies (B) and (C).

Theorem 2. *Let the initial data* (ρ_0, u_0) *be at least* C^m*-smooth and bounded,* $m \geq 2$, *and the assumptions* (A)–(C) *hold. Then at the moment* $t = t_*$ *the singularity arises at the point* x_* *and the following asymptotics takes place as* $\sigma \to 0$:

$$\rho_\sigma(t_*, x_*) \sim B(x_*, t_*)\, \rho_0(y_*)\, \sigma^{-\frac{m-1}{m}}, \tag{26}$$

where

$$B(x_*, t_*) = K_m \frac{\left|\frac{\partial a(u_0(y_*))}{\partial y}\right|^{\frac{m+1}{2m}}}{\left|\frac{\partial^m a(u_0(y_*))}{\partial^m y}\right|^{\frac{1}{m}}}, \qquad K_m = \frac{2^{\frac{m+1}{2m}}}{m\sqrt{\pi}} (m!)^{\frac{1}{m}}\, \Gamma\left(\frac{1}{2m}\right),$$

$$\left.\frac{\partial u_\sigma(t_*, x)}{\partial x}\right|_{x=x^*} \sim C(x_*, t_*) \left.\frac{\partial u_0((x))}{\partial x}\right|_{x=y^*} \sigma^{-\frac{m-1}{m}}, \tag{27}$$

where

$$C(x_*, t_*) = \frac{\left(2(m!)^2\left|\frac{\partial a(u_0(y_*))}{\partial y}\right|\right)^{\frac{m+1}{2m}}}{\left|\frac{\partial^m a(u_0(y_*))}{\partial^m y}\right|^{\frac{m+1}{m}}} \left(\Gamma\left(\frac{m+2}{2m}\right)\Gamma\left(\frac{1}{2m}\right) - \Gamma\left(\frac{m+1}{2m}\right)\Gamma\left(\frac{1}{m}\right)\right).$$

Moreover, at the moment $t = t_*$

$$\rho_\sigma(t_*, x) \to \rho_0(y_0(t_*, x)), \quad \sigma \to 0, x \neq x_*, \tag{28}$$

$$u_\sigma(t_*, x) \to u_0(y_0(t_*, x)), \quad \sigma \to 0, x \in \mathbb{R}, \tag{29}$$

where the function $y_0(t, x)$ *has been introduced in the proof of Theorem* 1.

Proof. Proceeding as in the proof of Theorem 1 we can readily obtain properties (28), (29). Let us prove (26) analyzing formula (13) at the point (t_*, x_*).

For every $\varepsilon > 0$ we can consider

$$\sqrt{2\pi t_*}\sigma\, \rho_\sigma(t_*, x^*) = \int_{\mathbb{R}} \rho_0(y)\, e^{-\frac{\sum_{i=1}^n |a(u_0(y))\, t_* + y - x^*|^2}{2\sigma^2 t_*}}\, dy$$

given by (13) as a sum of two integrals, I_1 and I_2, over $U_\varepsilon(y^*)$ and $\mathbb{R}^n \setminus U_\varepsilon(y^*)$, respectively.

First we estimate I_2. Let us denote

$$S(y) = \frac{(a(u_0(y))\, t_* + y - x^*)^2}{2t_*}, \qquad M_1 = \sup_{y\in\mathbb{R}} |\rho(y)| > 0, \qquad M_2 = \inf_{y\in\mathbb{R}\setminus U_\varepsilon(y^*)} S(y) > 0.$$

Then for every $\sigma < \sigma_0$ we have

$$|I_2| \le M_1 e^{-\frac{M_2}{\sigma^2}} \int_{\mathbb{R}\setminus U_\varepsilon(y^*)} e^{-\frac{S(y)-M_2}{\sigma_0^2}} e^{-(S(y)-M_2)\left(\sigma^2-\frac{1}{\sigma_0^2}\right)} dy$$

$$\le 2M_1 e^{-M_2\left(\frac{1}{\sigma^2}-\frac{1}{\sigma_0^2}\right)} \int_{\mathbb{R}\setminus U_\varepsilon(y^*)} e^{-\frac{S(y)}{\sigma_0^2}} dy \le \operatorname{const} e^{-\frac{M_2}{\sigma^2}}, \tag{30}$$

since $\int_{\mathbb{R}\setminus U_\varepsilon(y^*)} e^{-\frac{S(y)}{\sigma_0^2}} dy < \infty$.

Inequality (30) implies $I_2 \to 0$ as $\sigma \to 0$.

Then,

$$I_1 = \rho_0(y^*) \int_{U_\varepsilon(y^*)} e^{-\frac{(a(u_0(y))\,t_*+y-x^*)^2}{2\sigma^2 t_*}} dy + \int_{U_\varepsilon(y^*)} (\rho_0(y)-\rho_0(y^*))\, e^{-\frac{(a(u_0(y))\,t_*+y-x^*)^2}{2\sigma^2 t_*}} dy = I_{11} + I_{12}.$$

Since $\rho_0(x)$ is continuous, then for an arbitrary small $\varepsilon > 0$ we have $|\rho_0(y) - \rho_0(y^*)| < \eta(\varepsilon) = O(\varepsilon)$, $y \in U_\varepsilon(y^*)$ Thus,

$$|I_{12}| \le \int_{U_{y^*}(\varepsilon)} |\rho_0(y) - \rho_0(y^*)|\, dy \le \operatorname{const} \varepsilon\, \eta(\varepsilon) \to 0, \quad \varepsilon \to 0. \tag{31}$$

To estimate I_{11} we expand $a(u_0(y))$ at the point $y = y^*$. Since $a(u_0(y_*))\, t_* = x^* - y^*$ and $t_* = -\frac{1}{(a'(u_0(y)))|_{y=y^*}}$, for y belonging to the ε-neighborhood $U_{y^*}(\varepsilon)$ of the point y^* we have

$$a(u_0(y))\, t_* + y - x = a\left(u_0(y^*) + a'(u_0(y))|_{y=y^*}\,(y-y^*) + \frac{1}{m!}\left.\frac{\partial^m a(u_0(y))}{\partial^m y}\right|_{y=y^{**}} (y-y^*)^m\right) t_* + y - x$$

$$= A(y^{**})\,(y-y^*)^m\, t_* - (x-x^*), \tag{32}$$

with $A(y^{**}) = \frac{1}{m!}\left.\frac{\partial^m a(u_0(y))}{\partial^m y}\right|_{y=y^{**}}$ and $y^{**} \in U_{y^*}(\varepsilon)$.

Taking into account (32), we have

$$\int_{U_{y^*}(\varepsilon)} e^{-\frac{(a(u_0(y))\,t_*+y-x^*)^2}{2\sigma^2 t_*}} dy = \int_{-\varepsilon}^{\varepsilon} e^{-\frac{A^2(y^*)\,(y-y^*)^{2m}\,t_*}{2\sigma^2}} dy + \int_{U_{y^*}(\varepsilon)} \left(e^{-\frac{A^2(y^{**})\,(y-y^*)^{2m}\,t_*}{2\sigma^2}} - e^{-\frac{A^2(y^*)\,(y-y^*)^{2m}\,t_*}{2\sigma^2}}\right) dy$$

$$= J_1 + J_2.$$

To find the asymptotics of J_1 we recall the Watson's Lemma [8,13], concerning the Laplace integral

$$F(\lambda) = \int_0^a x^{\beta-1} f(x) e^{-\lambda x^\alpha}\, dx, \quad a > 0, \alpha > 0, \beta > 0.$$

According to this lemma

$$F(\lambda) = \frac{1}{\alpha}\sum_{k=0}^{\infty} \lambda^{-\frac{k+\beta}{\alpha}}\, \Gamma\left(\frac{k+\beta}{\alpha}\right)\frac{f^{(k)}(0)}{k!}, \quad \lambda \to +\infty \tag{33}$$

for a suitably smooth $f(x)$.

By means of Watson's Lemma it can be readily shown that

$$J_1 \sim B(x_*, t_*)\, \sigma^{\frac{1}{m}}, \quad \sigma \to 0.$$

At last, $J_{12} \to 0$ as $\sigma \to 0$ analogously to (31), since $e^{-\frac{A^2(y)\,(y-y^*)^{2m}\,t_*}{2\sigma^2}}$ is continuous. Thus we obtain asymptotical formula (26).

Formula (27) is obtained from (14). First we differentiate (14) with respect to x, this is allowable due to properties of $u_0(x)$ and $\rho_0(x)$. Then we estimate the numerator and denominator as above. Here we have to take into account the second terms in expansion (33) from Watson's Lemma in the numerator, since the first terms are canceled. □

5. Associated system of balance laws

Now we consider the following question: what system of equations do the triple $(\rho_\sigma, u_\sigma, a_\sigma)$ and its limit $(\bar\rho, \bar u, \bar a)$ satisfy before and after the blow up time $t_*(u_0)$?

The following proposition holds:

Proposition 3. *The functions ρ_σ, u_σ and a_σ, given by (5)–(7), satisfy for $t \ge 0$ the following PDE system:*

$$\frac{\partial \rho_\sigma}{\partial t} + \mathrm{div}_x(\rho_\sigma a_\sigma) = \frac{1}{2}\sigma^2 \sum_{k=1}^n \frac{\partial^2 \rho_\sigma}{\partial x_k^2}, \tag{34}$$

$$\frac{\partial(\rho_\sigma u_\sigma)}{\partial t} + \mathrm{div}_x(\rho_\sigma\, u_\sigma\, a_\sigma) = \frac{1}{2}\sigma^2 \sum_{k=1}^n \frac{\partial^2(\rho_\sigma u_\sigma)}{\partial x_k^2} - I_\sigma^u, \tag{35}$$

where

$$I_\sigma^u = \int_{\mathbb{R}^n} (u - u_\sigma(t,x))((a(t,u) - a_\sigma(t,x)), \nabla_x P(t,x,du));$$

$$\frac{\partial(\rho_\sigma a_{\sigma,i})}{\partial t} + \mathrm{div}_x(\rho_\sigma a_{\sigma,i}\, a_\sigma) = \frac{1}{2}\sigma^2 \sum_{k=1}^n \frac{\partial^2(\rho_\sigma a_{\sigma,i})}{\partial x_k^2} - I_{\sigma,i}^a, \tag{36}$$

$i = 1, \ldots, n$, *where*

$$I_{\sigma,i}^a = \int_{\mathbb{R}^n} (a_i(t,u) - a_{\sigma,i}(t,x))((a(t,u) - a_\sigma(t,x)), \nabla_x P(t,x,du)) + \int_{\mathbb{R}^n} (a_i(t,u))_t P(t,x,du).$$

Proof. The Eq. (34) follows from the Fokker–Planck equation (4) directly.

Let us prove (36) (the derivation of (35) is analogous). We note hthat the definitions of $a_\sigma(t,x)$ and $\rho_\sigma(t,x)$ imply

$$\begin{aligned}\frac{\partial(\rho_\sigma a_\sigma)}{\partial t} &= \frac{\partial}{\partial t}\int_{\mathbb{R}^n} a(t,u)P(t,x,du) = \int_{\mathbb{R}^n} a(t,u)P_t(t,x,du)\\ &= -\int_{\mathbb{R}^n} a(t,u)(a(t,u), \nabla_x P(t,x,du)) + \frac{1}{2}\sigma^2 \sum_{k=1}^n \frac{\partial^2 a_\sigma \rho_\sigma}{\partial x_k^2},\end{aligned} \tag{37}$$

where $P_t \equiv \frac{\partial}{\partial t}P$.

Further, we have

$$\begin{aligned}\frac{\partial(\rho_\sigma\, a_{\sigma,k}\, a_{\sigma,i})}{\partial x_k} &= a_{\sigma,i}(t,x)\frac{\partial}{\partial x_k}\left(\int_{\mathbb{R}^n} a_k(t,u)\,P(t,x,du)\right)\\ &\quad + \int_{\mathbb{R}^n} a_k(t,u)\,P(t,x,du)\,\frac{\partial}{\partial x_k}\left(\frac{\int_{\mathbb{R}^n} a(t,u)\,P(t,x,du)}{\int_{\mathbb{R}^n} P(t,x,du)}\right)\\ &= \int_{\mathbb{R}^n} a_{\sigma,i}(t,x)\,a_k(t,u)\,P_{x_k}(t,x,du) + \int_{\mathbb{R}^n} a_k(t,u)\,P(t,x,du)\\ &\quad\times \frac{\int_{\mathbb{R}^n} a(t,u)\,P_{x_k}(t,x,du)\,\int_{\mathbb{R}^n} P(t,x,du) - \int_{\mathbb{R}^n} a(t,u)\,P(t,x,du)\,\int_{\mathbb{R}^n} P_{x_k}(t,x,du)}{\left(\int_{\mathbb{R}^n} P(t,x,du)\right)^2}\\ &= \int_{\mathbb{R}^n} (a_k(t,u)\,a_{\sigma,i}(t,x) + a(t,u)\,a_{\sigma,k}(t,x) - a_{\sigma,k}(t,x)\,a_{\sigma,i}(t,x))\,P_{x_k}(t,x,du),\end{aligned} \tag{38}$$

$i, k = 1, \ldots, n$, with $P_{x_k} \equiv \frac{\partial}{\partial x_k}P$.

Eq. (36) follows immediately from (37) and (39). Thus, Proposition 3 is proved. □

Corollary 2. *Before the instant $t_*(u_0)$, the blow up time of the solution to the Cauchy problem (17), the triple $(\bar\rho, \bar u, \bar a)$, which constitutes the limit as $|\sigma| \to 0$ of the triple $(\rho_\sigma, u_\sigma, a_\sigma)$, solves the following system:*

$$\partial_t \bar\rho + \mathrm{div}_x(\bar\rho \bar a) = 0, \tag{39}$$

$$\partial_t(\bar\rho \bar u) + \mathrm{div}_x(\bar\rho \bar u \bar a) = 0, \tag{40}$$

$$\partial_t(\bar\rho \bar a) + \mathrm{Div}_x(\bar\rho \bar a \otimes \bar a) = 0, \tag{41}$$

where Div *is the divergence of a tensor.*

Proof. Eq. (39) follows from the properties of parabolic differential equations with a small parameter in front of the derivatives of second order [10, Theorem 3.1], since until the instance $t_*(u_0)$ the coefficients of Eq. (34) are differentiable. Eq. (40) follows from (39) and Theorem 1, Proposition 2 implies (41). □

Remark 3. System (39) and (41) constitutes the so called pressureless gas dynamics system, the simplest model introduced to describe the formation of large structures in the Universe, see, e.g. [14].

Remark 4. As it has been shown in [2] on an example, for discontinuous solutions to (17) the limits as $\sigma \to 0$ of the terms I_σ^a and I_σ^u do not vanish as $\sigma \to 0$ and yield some specific pressure.

Remark 5. This paper is a continuation of our works concerning stochastic regularization of the non-viscous Burgers equation for the velocity of particles. In [4] we studied an influence of stochastic perturbation along characteristics with a variance which depends in a polynomial way on the velocity to the process of the singularity formation. We established a threshold effect: if the power in the above variance is less than 1, then the noise does not prevent the unbounded gradient of solution with a linear profile. In [5] we studied the decay rate of the initial particles distribution at infinity for the Langevin equation associated with the Burgers equation stochastically perturbed by uniform noise to the formation of unbounded gradient for initial data with a linear profile. In [2,3] we show that the balance laws associated with a stochastic perturbation of the Burgers equation form the pressureless gas dynamics system with special integral terms. We introduce a notion of generalized solution as a special small perturbation limit. On the example of Riemann data we show that this solution differs from the admissible weak solution and takes part of the solution to the gas dynamics equations with a specific pressure term as small perturbation limit.

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