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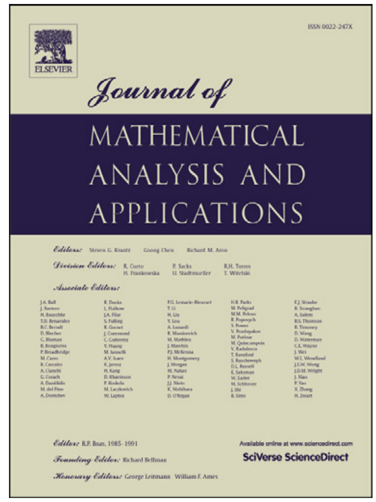

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CONTINUOUS RANDOM WALKS AND FRACTIONAL POWERS OF OPERATORS

MIRKO D'OVIDIO

ABSTRACT. We derive a probabilistic representation for the Fourier symbols of the generators of some stable processes. This short paper represents a bridge between probabilists and researchers working in PDE's.

1. Introduction

The connection between fractional operator in space and diffusion with long jumps has been pointed out by many researchers (see for example [1; 12; 19] and the references therein). It is well known that the compound Poisson process is a continuous time stochastic process with jumps which arrive, according to a Poisson process, with specific probability law for the size. Our aim is to characterize the jumps distribution in order to obtain singular limit measure characterizing fractional powers of operators.

2. Preliminaries

Let $N(t)$, $t > 0$ be a Poisson process with rate $\lambda > 0$. Let Y_j, $0 \leq j \leq n$ be $n+1$ independent and identically distributed (i.i.d.) random jumps such that $Y_j \sim Y$ for all j, where the symbol "$\sim$" stands for equality in law. That is, $X_1 \sim X_2$ if, for every Borel set $\mathcal{B}$, $P\{X_1 \in \mathcal{B}\} = P\{X_2 \in \mathcal{B}\}$ and therefore the random variables X_1 and X_2 have the same distribution or probability function. We note that, if probabilities are defined for a larger class of events, it is possible for two random variables to have the same distribution function but not the same probability for every event (see [5], Chapter 2). Furthermore, two random variables which are identically distributed are not necessarily equal.

It is well known that

$$Z_t = \sum_{j=0}^{N(t)} Y_j - \lambda t \mathbb{E} Y, \quad t > 0 \tag{2.1}$$

($\mathbb{E}$ is the mean operator and $\mathbb{E}Y = \int y P\{Y \in dy\}$) is the compensated compound Poisson process with generator

$$(\mathcal{A}f)(x) = \lambda \int_{\mathbb{R}} \left(f(x+y) - f(x) - y f'(x) \right) \nu_Y(dy) \tag{2.2}$$

where $\nu_Y : \Omega \subseteq \mathbb{R} \mapsto [0,1]$ is the density law of $Y \in \Omega$. The process (2.1) is a compensated process involving the compound Poisson process $\sum_{j=0}^{N(t)} Y_j$. Thus, we

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have a sequence of i.i.d. random jumps with common law ν_Y and a random number of jumps $1+N(t)$ with $P\{N(t)=n\}=e^{-\lambda t}(\lambda t)^n/n!$, $n=0,1,2,\dots$, $t>0$. It is worth to mention that ([17; 18]) the transport equation

$$\frac{\partial u}{\partial t} = \mathcal{A}u - \lambda u + \lambda K u = \mathcal{A}u - \lambda(I-K)u \tag{2.3}$$

where

$$\mathcal{A}u = -\sum_{k=1}^{n} \frac{\partial}{\partial x_k}\left(a(x)\,u\right)$$

(assume that a is sufficiently smooth) and K is the Frobenius-Perron operator corresponding to the transformation $T(x)=x-\tau(x)$ has a solution which is the law of the solution to the Poisson driven stochastic differential equation

$$d\mathbf{X}_t = a(\mathbf{X}_t)dt + \tau(\mathbf{X}_t)d\mathbf{N}_t. \tag{2.4}$$

Formula (2.4) says that $\mathbf{X}_t$ is a continuous-time stochastic process with jumps τ which arrive randomly according to the Poisson process $\mathbf{N}_t$. In particular, $d\mathbf{N}_t=1$ if a Poisson event arrives or $d\mathbf{N}_t=0$ otherwise. The compensated Poisson process $Z_t=N(t)-\lambda t$ (take $Y_j\equiv 1$ for all j in formula (2.1)) is therefore governed by the equation

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) &= \lambda\frac{\partial u}{\partial x}(x,t) - \lambda(I-K)u(x,t)\\ &= \lambda\frac{\partial u}{\partial x}(x,t) - \lambda\Big(u(x,t)-u(x-1,t)\Big)\end{aligned}$$

where the Frobenius-Perron operator is associated to the jump function $\tau\equiv 1$. The first derivative $\partial u/\partial x$ disappears if $Z_t=N(t)$ is the Poisson process. Equation (2.3) appears in such diverse areas as population dynamics (see for example [13; 14]) and in astrophysics ([4]).

Formula (2.2) is quite familiar in the representation of the fractional power of the Laplacian. Indeed, the fractional Laplace operator can be defined pointwise:

$$-(-\triangle)^\alpha f(\mathbf{x}) = \int_{\mathbb{R}^d}\Big(f(\mathbf{x}+\mathbf{y}) - f(\mathbf{x}) - \mathbf{y}\cdot\nabla f(\mathbf{x})\mathbf{1}_{(|\mathbf{y}|\le 1)}\Big)\frac{C_d(\alpha)\,\mathbf{dy}}{|\mathbf{y}|^{2\alpha+d}} \tag{2.5}$$

where $C_d(\alpha)$ is a constant depending on d and $\alpha\in(0,1)$, f is a suitable test function, C^2 function with bounded second derivative for instance. We note that ν_Y in (2.2) is a density defining a probability measure. Furthermore, the fractional Laplacian is the governing operator of symmetric stable processes.

Let us introduce the 1-dimensional β-stable process $\mathfrak{S}^\beta(t)$, $t>0$ with no drift. In this case we have the characteristic function $\mathbb{E}\exp i\xi\mathfrak{S}^\beta(t)=\exp -t\Psi(\xi)$ with Fourier symbol

$$\Psi(\xi) = |\xi|^\beta \exp\left(-i\frac{\pi\gamma}{2}\frac{\xi}{|\xi|}\right) = \sigma|\xi|^\beta\left(1-i\theta\frac{\xi}{|\xi|}\tan\frac{\pi\beta}{2}\right), \quad \beta\in(0,1)\cup(1,2]$$

where $\sigma=\cos\pi\gamma/2$, $\theta=\cot\left(\frac{\pi\beta}{2}\right)\tan\left(\frac{\pi\gamma}{2}\right)$ and γ must be determined in such a way that $\theta\in[-1,1]$ and $\sigma>0$. The parameter θ is the skewness parameter. In particular, $\mathfrak{S}^\beta(t)$ is a symmetric real-valued stable process for $\theta=0$ (and $\beta\in(0,1)\cup(1,2]$). If $\beta\in(0,1)$ and $\theta=-1$ then $\mathfrak{S}^\beta(t)$ is totally negatively skewed whereas, if $\beta\in(0,1)$ and $\theta=1$ (that is $\gamma=\beta$) then $\mathfrak{S}^\beta(t)$ is totally positively skewed. In the latter case, the stable process is also termed stable subordinator (see for example [2]) and we will denote such a process by $\mathfrak{H}^\alpha(t)$, $t>0$, $\alpha\in(0,1)$. We recall that a random variable is positively skewed if the right tail of its probability distribution is longer than the left tail. The converse holds for the negative case.

Compound Poisson and stable processes belong to the general class of Lévy processes whose characteristic function is written in terms of the following Fourier symbol (Lévy - Khintchine)

$$\Psi_L(\boldsymbol{\xi}) = i\mathbf{b}\cdot\boldsymbol{\xi} + \boldsymbol{\xi}\cdot M\boldsymbol{\xi} - \int_{\mathbb{R}^d-\{0\}} \left(e^{i\boldsymbol{\xi}\cdot\mathbf{y}} - 1 - i\boldsymbol{\xi}\cdot\mathbf{y}\mathbf{1}_{(|\mathbf{y}|\leq 1)}\right)\mu(d\mathbf{y}) \tag{2.6}$$

where $\mathbf{b} \in \mathbb{R}^d$, M is a positive definite symmetric $d \times d$ matrix and μ is a Lévy measure on $\mathbb{R}^d - \{0\}$, that is a Borel measure on $\mathbb{R}^d - \{0\}$ such that

$$\int (|\mathbf{y}|^2 \wedge 1)\mu(d\mathbf{y}) < \infty \quad \text{or equivalently} \quad \int \frac{|\mathbf{y}|^2}{1+|\mathbf{y}|^2}\mu(d\mathbf{y}) < \infty. \tag{2.7}$$

If D_t, $t > 0$ is a subordinator (not necessarily stable), then its Lévy symbol is written as

$$\eta(\xi) = ib\xi + \int_0^\infty \left(e^{i\xi y} - 1\right)\mu(dy) \tag{2.8}$$

where $b \geq 0$ and the Lévy measure μ satisfies the following requirements: $\mu(-\infty, 0) = 0$ and

$$\int (y \wedge 1)\mu(dy) < \infty \quad \text{or equivalently} \quad \int \frac{y}{1+y}\mu(dy) < \infty. \tag{2.9}$$

Throughout the paper, the symbol $\stackrel{d}{\to}$ stands for "converges in distribution" (or "converges weakly", or "converges in law", that is $X_n \stackrel{d}{\to} X$ iff $P\{X_n \in \mathcal{B}\} \to P\{X \in \mathcal{B}\}$ as $n \to \infty$ for every Borel set $\mathcal{B} \subset \mathbb{R}$). Furthermore,

$$P\{X \in \mathcal{B}\} = \int_{\mathcal{B}} \nu_X(dx) = \nu_X(\mathcal{B}).$$

3. Compensated Poisson and fractional Laplace operator

Let us consider the Lévy process $\mathbf{F}_t$, $t > 0$, with associated Feller semigroup $T_t f(\mathbf{x}) = \mathbb{E}f(\mathbf{F}_t - \mathbf{x})$ solving $\partial_t u = \mathcal{A}u$, $u_0 = f$. In particular, T_t is a positive contraction semigroup (i.e. $0 \leq f \leq 1 \Rightarrow 0 \leq T_t f \leq 1$ and $T_{t+s} = T_t T_s$) on $C_\infty(\mathbb{R}^d)$ such that (Feller semigroup)

- $T_t(C_\infty(\mathbb{R}^d)) \subset C_\infty(\mathbb{R}^d)$, $t > 0$ (T_t is invariant),
- $T_t f \to f$ as $t \to 0$ for all $f \in C_\infty(\mathbb{R}^d)$ under the sup-norm (T_t is a strongly continuous contraction semigroup on the Banach space $(C_\infty(\mathbb{R}^d), \|\cdot\|_\infty)$).

From the general theory of such semigroups we infer the existence of the generator

$$\mathcal{A}f = \lim_{t\to 0}\frac{T_t f - f}{t}$$

for all functions $f \in D(A)$, a linear space that is dense in $C_\infty(\mathbb{R}^d)$. $\mathcal{A}$ is a closed linear operator. Thus, we are able to compute the semigroup and its generator as pseudo-differential operators. We say that $\mathcal{A}$ is the infinitesimal generator of $\mathbf{F}_t$, $t > 0$ and the following representation holds

$$(\mathcal{A}f)(\mathbf{x}) = -\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}\Phi(\boldsymbol{\xi})\widehat{f}(\boldsymbol{\xi})\boldsymbol{d\xi} \tag{3.1}$$

for all functions in the domain

$$D(\mathcal{A}) = \left\{ f \in L^2(\mathbb{R}^d, \mathbf{dx}) \,:\, \int_{\mathbb{R}^d} \Phi(\boldsymbol{\xi}) |\widehat{f}(\boldsymbol{\xi})|^2 \boldsymbol{d\xi} < \infty \right\} \tag{3.2}$$

where $\widehat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} e^{i\boldsymbol{\xi}\cdot\mathbf{x}} f(\mathbf{x})\mathbf{dx}$ is the Fourier transform of f, $\Phi(\cdot)$ is continuous and negative definite. We say that T_t is a pseudo-differential operator with symbol $\exp(-t\Phi)$ and, $-\Phi$ is the Fourier multiplier (or Fourier symbol) of $\mathcal{A}$, $\widehat{(\mathcal{A}f)}(\boldsymbol{\xi}) = -\Phi(\boldsymbol{\xi})\widehat{f}(\boldsymbol{\xi})$. Furthermore (as in [9]), we write

$$- \partial_t \, \mathbb{E} e^{i\boldsymbol{\xi}\cdot\mathbf{F}_t}\Big|_{t=0} = \Phi(\boldsymbol{\xi}). \tag{3.3}$$

It is well known that, for $\Phi(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^\alpha$, formula (3.1) gives us the fractional power of the Laplace operator which can be also expressed as

$$\begin{aligned} -(-\triangle)^\alpha f(\mathbf{x}) =& C_d(\alpha)\, \text{p.v.} \int_{\mathbb{R}^d} \frac{f(\mathbf{y}) - f(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^{2\alpha+d}} \mathbf{dy} \\ =& C_d(\alpha)\, \text{p.v.} \int_{\mathbb{R}^d} \frac{f(\mathbf{x}+\mathbf{y}) - f(\mathbf{x})}{|\mathbf{y}|^{2\alpha+d}} \mathbf{dy} \end{aligned} \tag{3.4}$$

where "p.v." stands for the "principal value" of the singular integrals above near the origin. For $\alpha \in (0,1)$, the fractional Laplace operator can be defined, for $f \in \mathscr{S}$ (the space of rapidly decaying C_∞ functions), as follows

$$\begin{aligned} -(-\triangle)^\alpha f(\mathbf{x}) =& \frac{C_d(\alpha)}{2} \int_{\mathbb{R}^d} \frac{f(\mathbf{x}+\mathbf{y}) + f(\mathbf{x}-\mathbf{y}) - 2f(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^{2\alpha+d}} \mathbf{dy} \\ =& \frac{C_d(\alpha)}{2} \int_{\mathbb{R}^d} \frac{f(\mathbf{x}+\mathbf{y}) + f(\mathbf{x}-\mathbf{y}) - 2f(\mathbf{x})}{|\mathbf{y}|^{2\alpha+d}} \mathbf{dy}, \quad \forall \mathbf{x} \in \mathbb{R}^d. \end{aligned} \tag{3.5}$$

This representation comes out by considering straightforward calculations and removes the singularity at the origin ([6]). Indeed, from the second order Taylor expansion of the smooth function f ($f \in \mathscr{S}$) we obtain

$$\frac{f(\mathbf{x}+\mathbf{y}) + f(\mathbf{x}-\mathbf{y}) - 2f(\mathbf{x})}{|\mathbf{y}|^{2\alpha+d}} \leq \frac{\|D^2 f\|_{L^\infty}}{|\mathbf{y}|^{2\alpha+d-2}} \tag{3.6}$$

which is integrable near the origin provided that $\alpha \in (0,1)$. The constant $C_d(\alpha)$ must be considered in order to obtain $\widehat{(-\triangle)^\alpha f}(\cdot)(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^\alpha \widehat{f}(\boldsymbol{\xi})$.

Remark 1. *Let $\mathfrak{H}^\alpha_t$, $t > 0$ be a stable subordinator. The generator of $\mathbf{F}_{\mathfrak{H}^\alpha_t}$, $t > 0$ is given by the beautiful formula*

$$- (-\mathcal{A})^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \Big(T_s\, f(x) - f(x) \Big) \frac{ds}{s^{\alpha+1}} \tag{3.7}$$

for all $f \in \mathscr{S}$ ($T_s = e^{s\mathcal{A}}$ is the Feller semigroup of $\mathbf{F}_t$, $t > 0$). A generalization due to R.S Phillips allows the replacement of $\mathcal{A}$ with the generator of a general (not only associated with Lévy processes) contraction semigroup on a Banach space. The formula (3.7) comes from the representation of the Bernstein function $g(x) = x^\alpha$ for $\alpha \in (0,1)$,

$$x^\alpha = \int_0^\infty (1 - e^{-sx}) \mu(ds), \quad x > 0 \tag{3.8}$$

where μ is (in our view) a Lévy measure satisfying (2.9). In particular, $\mu(ds) = \frac{\alpha}{\Gamma(1-\alpha)} s^{-\alpha-1} ds$ is the Lévy measure of a stable subordinator with no drift (see formula (2.8)). By passing to the Fourier transform of (3.7) and considering that

$\widehat{T_s f} = e^{-s\Phi}\widehat{f}$, *the reader can immediately see that the Fourier multiplier of* (3.7) *is given by* $-(\Phi)^\alpha$. *If* $\mathbf{F}_t = \mathbf{B}_t$ *is the d-dimensional Brownian motion for instance, then* $\Phi(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^2$ *and therefore the associated Fourier symbol becomes* $-|\boldsymbol{\xi}|^{2\alpha}$. *The last result is known as the Bochner's subordination rule: the subordinate Brownian motion* $\mathbf{B}_{\mathfrak{H}_t^\alpha}$, $t > 0$ *possesses the same law of an n-dimensional isotropic stable process (its infinitesimal generator is therefore the fractional Laplacian* $-(-\triangle)^\alpha$*). The interested reader can consult the book by Schilling et al. [16] for a deep discussion on the role of Bernstein function in this context.*

Formula (2.2) can be obtained by considering the following characteristic function

$$\begin{aligned}\mathbb{E}e^{i\xi Z_t} =&\mathbb{E}\prod_{j=0}^{N(t)} e^{i\xi Y_j}\, e^{-i\xi\alpha t\mathbb{E}Y}\\ =&\mathbb{E}\left(\mathbb{E}e^{i\xi Y}\right)^{N(t)}\, e^{-i\xi\alpha t\mathbb{E}Y}\\ =&e^{-\lambda t(1-\mathbb{E}e^{i\xi Y})}\, e^{-i\xi\alpha t\mathbb{E}Y}\\ =&\exp\left(\lambda t\mathbb{E}\left(e^{i\xi Y}-1-i\xi Y\right)\right).\end{aligned}$$

Therefore, we get that

$$\partial_t\, \mathbb{E}e^{i\xi Z_t}\Big|_{t=0} = \lambda\mathbb{E}(e^{i\xi Y}-1-i\xi Y) = \lambda\int_{\mathbb{R}}(e^{i\xi y}-1-i\xi y)\nu_Y(dy) = -\Phi(\xi).$$

If $Y_j \sim Y$ are symmetric random variables such that $\mathbb{E}Y_j = \mathbb{E}Y = 0$ for all $j = 1, 2, \ldots$, then $\nu_Y(y) = \nu_Y(-y)$ and

$$\int_{\mathbb{R}} y f'(x)\,\nu_Y(dy) = f'(x)\int_{\mathbb{R}\setminus B_r} y\,\nu_Y(dy) + f'(x)\int_{B_r} y\,\nu_Y(dy) = 0 \tag{3.9}$$

where we also include those density law $\nu_Y(\cdot)$ for which (3.9) holds as principal value. If (3.9) holds true, then formula (2.2) takes the form

$$(\mathcal{A}f)(x) = \lambda\int_{\mathbb{R}}\left(f(x+y)-f(x)\right)\nu_Y(dy)$$

and the integral converges depending on $\nu_Y(\cdot)$. If we choose $\nu_Y(dy) = 2\alpha|y|^{-2\alpha-1}dy$ for instance, then the integral must be understood in the principal value sense and we get the fractional Laplace operator as formula (3.4) entails.

4. Main results

In this short paper, we construct continuous random walks with exponential and Gaussian jumps driven by pseudo-differential operators with Fourier multiplier $-\Phi_\gamma(\boldsymbol{\xi})$ which converges to $-|\boldsymbol{\xi}|^\beta$ with $\beta \in (0,2)$ as $\gamma \to 0$. In particular, we first consider the random jump $Y = \gamma e^X \in [\gamma, \infty)$ where $X \sim Exp(\alpha)$ and $\alpha, \gamma > 0$. We have that $P\{Y \in \mathcal{B}\} = \int_{\mathcal{B}} \nu_Y(dy)$ with $\nu_Y(y) = \alpha\gamma^\alpha y^{-\alpha-1}\mathbf{1}_{(y\geq\gamma)}$. By "symmetrizing", we get that

$$\nu_Y^*(y) = q\,\nu_Y(-y)\mathbf{1}_{(y\leq-\gamma)} + p\,\nu_Y(y)\mathbf{1}_{(y\geq\gamma)} \tag{4.1}$$

is the density of $Y^* = \epsilon\, Y$ with Rademacher law

$$P\{\epsilon = +1\} = p, \quad P\{\epsilon = -1\} = q$$

for the random variable ϵ where we obviously assume that $p+q=1$. For $p=q$, formula (4.1) takes the form

$$\nu_Y^*(y) = \frac{1}{2}\nu_Y(|y|) = \frac{\alpha\gamma^\alpha}{2}|y|^{-\alpha-1}\mathbf{1}_{(|y|\geq\gamma)} \tag{4.2}$$

and Y^* is therefore written as

$$Y^* = \begin{cases} \gamma e^X, & \text{with probability } 1/2, \\ -\gamma e^X, & \text{with probability } 1/2. \end{cases} \tag{4.3}$$

Now we write the corresponding compound Poisson process as follows

$$A(t) = \sum_{j=0}^{N(t)} Y_j^* = \sum_{j=0}^{N(t)} \epsilon_j\, Y_j = \sum_{j=0}^{N(t)} \epsilon_j\, \gamma_j\, e^{X_j}, \quad t>0 \tag{4.4}$$

with $X_j \sim X$, $\epsilon_j \sim \epsilon$ and $\gamma_j = \gamma$ for all $j=0,1,2,\ldots$. We also assume that all the random variables we are dealing with are taken to be independent, that is $\mathbb{E}\epsilon\epsilon_j = \mathbb{E}\epsilon_j\epsilon_{j'} = 0$, $\forall\, j,j'$ such that $j \neq j'$. Observe that, for all $\varepsilon > 0$, $P\{\gamma e^X < \varepsilon\} \to P\{e^X < +\infty\} = 1$ as $\gamma \to 0$.

We are now ready to state the following results

Theorem 1. *Let $\mathfrak{H}_j^\alpha(t)$, $t>0$, $j=1,2$ be two independent stable subordinators. For given $p,q \geq 0$ such that $p+q=1$, $\alpha \in (0,1)$,*

$$A(t/\gamma^\alpha) \xrightarrow[\gamma\to 0]{d} \mathfrak{H}_1^\alpha(pt^*) - \mathfrak{H}_2^\alpha(qt^*) \tag{4.5}$$

with $t^* : t \mapsto \lambda\Gamma(1-\alpha)t$ and generator

$$\mathcal{A}f(x) = -\lambda\Gamma(1-\alpha)\left(p\frac{d^\alpha}{dx^\alpha} + q\frac{d^\alpha}{d(-x)^\alpha}\right)f(x). \tag{4.6}$$

The Weyl's fractional derivatives appearing in (4.6) are defined as follows:

$$\begin{aligned}\frac{d^\alpha f}{dx^\alpha}(x) =& \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^\infty f(x-y)\frac{dy}{y^\alpha} \\ =& \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty (f(x)-f(x-y))\frac{dy}{y^{\alpha+1}} \\ =& \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \left(f(x)-e^{-y\partial_x}f(x)\right)\frac{dy}{y^{\alpha+1}} = \left(\frac{\partial}{\partial x}\right)^\alpha f(x);\end{aligned}$$

$$\begin{aligned}\frac{d^\alpha f}{d(-x)^\alpha}(x) =& \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^\infty f(x+y)\frac{dy}{y^\alpha} \\ =& \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty (f(x)-f(x+y))\frac{dy}{y^{\alpha+1}} \\ =& \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \left(f(x)-e^{y\partial_x}f(x)\right)\frac{dy}{y^{\alpha+1}} = \left(-\frac{\partial}{\partial x}\right)^\alpha f(x).\end{aligned}$$

Here we have considered "good" functions $f : \mathbb{R} \mapsto [0,1]$ whereas, we obtain the Riemann-Liouville derivatives (left-handed and right-handed respectively) by considering $f : \mathbb{R}_+ \mapsto [0,1]$ and $f : \mathbb{R}_- \mapsto [0,1]$ respectively ([8; 15]). In the integrals above, for example, one can write $f(z)\mathbf{1}_{(z\geq 0)}$ and $f(z)\mathbf{1}_{(z\leq 0)}$ and obtain the operator governing the (totally) positively and (totally) negatively skewed stable process, that is $q=0$ and $p=0$ respectively in (4.6). We notice that the definitions above are related to the representation (3.7) involving the translation semigroup $e^{\mp y\partial_x}$.

Remark 2. *The stable subordinator $\mathfrak{H}^\alpha_t$, $t>0$, $\alpha \in (0,1)$ has a probability law, say $h_\alpha = h_\alpha(x,t)$, $x \geq 0$, $t>0$, solving the fractional equation $\partial_t h_\alpha = -(\partial/\partial x)^\alpha h_\alpha$, $(x,t) \in [0,\infty) \times (0,\infty)$ subject to the initial condition $h_\alpha(x,0) = \delta(x)$, the Dirac delta function.*

Proof of Theorem 1. The characteristic function of (4.4) is written as follows

$$\begin{aligned}\mathbb{E}\exp\left(i\xi\sum_{j=0}^{N(t)}\epsilon_j\, Y_j\right) =& \mathbb{E}\left(\mathbb{E}e^{i\xi\epsilon Y}\right)^{N(t)}\\ =& \exp\left(\lambda t(\mathbb{E}e^{i\xi\epsilon Y}-1)\right)\\ =& \exp\left(\lambda t\Big(p\mathbb{E}e^{i\xi Y}+q\mathbb{E}e^{-i\xi Y}-(p+q)\Big)\right)\\ =& \exp\left(\lambda t\Big(p(\mathbb{E}e^{i\xi Y}-1)+q(\mathbb{E}e^{-i\xi Y}-1)\Big)\right).\end{aligned}$$

From this, we immediately get

$$\mathbb{E}\exp\left(i\xi A(t/\gamma^\alpha)\right) = \exp\left(\frac{\lambda t}{\gamma^\alpha}\Big(p(\mathbb{E}e^{i\xi Y}-1)+q(\mathbb{E}e^{-i\xi Y}-1)\Big)\right).$$

We recall that the Lévy symbol (see (2.8)) of a stable subordinator with no drift ($b=0$ in (2.8)) is a mapping from $\mathbb{R}\mapsto\mathbb{C}$ which takes the form

$$-(-i\xi)^\alpha = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \left(e^{i\xi y}-1\right)\frac{dy}{y^{\alpha+1}} \tag{4.7}$$

for $\alpha\in(0,1)$. The Fourier symbol (depending on γ) of the characteristic function of $A(t/\gamma^\alpha)$ is therefore given by

$$\begin{aligned}\Phi_\gamma(\xi) =& -\partial_t\,\mathbb{E}\exp\left(i\xi A(t/\gamma^\alpha)\right)\Big|_{t=0}\\ =& -\frac{\lambda}{\gamma^\alpha}\Big(p\int_0^\infty (e^{i\xi y}-1)\nu_Y(dy)+q\int_0^\infty(e^{-i\xi y}-1)\nu_Y(dy)\Big)\\ =& -\lambda p\int_\gamma^\infty (e^{i\xi y}-1)\frac{\alpha dy}{y^{\alpha+1}}-\lambda q\int_\gamma (e^{-i\xi y}-1)\frac{\alpha dy}{y^{\alpha+1}}.\end{aligned}$$

For $\gamma\to 0$ we obtain

$$\Phi_\gamma(\xi)\to\Phi(\xi) = \lambda\Gamma(1-\alpha)\Big(p(-i\xi)^\alpha+q(i\xi)^\alpha\Big), \quad \alpha\in(0,1) \tag{4.8}$$

and, from (3.1) we arrive at

$$\mathcal{A}f(x) = -\lambda\Gamma(1-\alpha)\left(p\frac{d^\alpha}{dx^\alpha}+q\frac{d^\alpha}{d(-x)^\alpha}\right)f(x). \tag{4.9}$$

The fact that the Lévy process (4.5) has infinitesimal generator (4.6) comes directly from the characteristic function

$$\mathbb{E}\exp\Big(i\xi\mathfrak{H}^\alpha_1(pt^*)-i\xi\mathfrak{H}^\alpha_2(qt^*)\Big) = \exp\left(-t^*p(-i\xi)^\alpha - t^*q(i\xi)^\alpha\right)$$

where $t^* = \lambda\Gamma(1-\alpha)t>0$. Thus, we get that

$$-\partial_t\mathbb{E}\exp\Big(i\xi\mathfrak{H}^\alpha_1(pt^*)-i\xi\mathfrak{H}^\alpha_2(qt^*)\Big)\Big|_{t=0} = \lambda\Gamma(1-\alpha)\Big(p(-i\xi)^\alpha+q(i\xi)^\alpha\Big)$$

which coincides with $\Phi(\xi)$ in (4.8).

In the last calculations we have used the fact that

$$\mathbb{E}e^{i\xi\mathfrak{H}^\alpha(t)} = \exp\left(-t(-i\xi)^\alpha\right) = \exp\left(-t|\xi|^\alpha e^{-i\frac{\pi\alpha}{2}\frac{\xi}{|\xi|}}\right), \quad \xi \in \mathbb{R},\ t \geq 0$$

and thus,

$$\mathbb{E}e^{-i\xi\mathfrak{H}^\alpha(t)} = \exp\left(-t|\xi|^\alpha e^{i\frac{\pi\alpha}{2}\frac{\xi}{|\xi|}}\right) = \exp\left(-t(i\xi)^\alpha\right).$$

The Fourier transforms of the Weyl's fractional derivatives, for $\alpha \in (0,1)$, are given by ([15])

$$\int_{\mathbb{R}} e^{i\xi x} \frac{d^\alpha}{d(\pm x)^\alpha} f(x)dx = (\mp i\xi)^\alpha \widehat{f}(\xi), \quad f \in L^1(\mathbb{R}). \tag{4.10}$$

□

Remark 3. *A stable subordinator is a one-dimensional non-decreasing Lévy process with $\mathbb{E}\exp(-\xi\mathfrak{H}^\alpha_t) = \exp(-t\xi^\alpha)$, $\alpha \in (0,1)$. We recall that $\mathfrak{H}^\alpha_t$ is a totally positively skewed stable process and therefore it is also non-negative. Its distribution has non-negative support, $P\{\mathfrak{H}^\alpha_t \in [0,\infty)\} = 1$. We observe that $\mathfrak{H}^\alpha_j(0) = 0$ for $j = 1,2$ and therefore the process (4.5) converges (in distribution) to a totally positively skewed (if $q = 0$, $P\{\epsilon_j = +1\} = 1$, $\forall j$) or totally negatively skewed (if $p = 0$, $P\{\epsilon_j = -1\} = 1$, $\forall j$) stable process with support for the distribution function respectively given by $[0,\infty)$ or $(-\infty, 0]$. Furthermore, for a given process X_t, $t > 0$ we notice that $X_{\theta t}$ runs slower than X_t if θ is less than 1 or faster than X_t if θ is greater than 1.*

Theorem 2. *Let $\mathfrak{S}^\beta(t)$, $t > 0$ be a symmetric stable process with $\beta \in (0,2)$. Then, for $p = q = 1/2$, $\alpha \in (0,2)$,*

$$A(t/\gamma^\alpha) \xrightarrow[\gamma\to 0]{d} \mathfrak{S}^\alpha(t^*) \tag{4.11}$$

with $t^* : t \mapsto \alpha\lambda C t$ and generator (Riesz operator)

$$\mathcal{A}f(x) = -\alpha\lambda C \frac{d^\alpha f}{d|x|^\alpha}(x) \tag{4.12}$$

where

$$C = \frac{1}{2}\int_{\mathbb{R}} \frac{1-\cos y}{|y|^{\alpha+1}} dy. \tag{4.13}$$

Proof. For $p = q = 1/2$ and $\alpha \in (0,2)$ we obtain that

$$\begin{aligned}
\Phi_\gamma(\xi) &= -\,\partial_t\, \mathbb{E}\exp\left(i\xi A(t/\gamma^\alpha)\right)\Big|_{t=0} \\
&= -\frac{\lambda}{2\gamma^\alpha}\left(\mathbb{E}e^{i\xi Y} + \mathbb{E}e^{-i\xi Y} - 2\right) \\
&= -\frac{\lambda}{2\gamma^\alpha}\int_0^\infty \left(e^{i\xi y} + e^{-i\xi y} - 2\right)\nu_Y(dy) \\
&= -\frac{\lambda}{\gamma^\alpha}\int_{\mathbb{R}} \left(\cos(\xi y) - 1\right)\nu^*_Y(dy)
\end{aligned}$$

where, we recall that

$$\nu^*_Y(y) = \frac{\alpha\gamma^\alpha}{2}|y|^{-\alpha-1}\mathbf{1}_{(|y|\geq\gamma)}.$$

We explicitly have that

$$\begin{aligned}\Phi_\gamma(\xi) &= -\frac{\lambda}{\gamma^\alpha}\int_{\mathbb{R}} (\cos(\xi y) - 1)\, \nu_Y^*(dy)\\ &= -\frac{\alpha\lambda}{2}\int_{\mathbb{R}\setminus B_\gamma} (\cos(\xi y) - 1)\, |y|^{-\alpha-1} dy.\end{aligned}$$

By taking the limit for $\gamma \to 0$, we obtain

$$\Phi_\gamma(\xi) \to \Phi(\xi) = -\frac{\alpha\lambda}{2}\int_{\mathbb{R}} (\cos(\xi y) - 1)\, |y|^{-\alpha-1} dy = \alpha\lambda C|\xi|^\alpha \tag{4.14}$$

where, due to the fact that $(\cos(y) - 1)\, |y|^{-\alpha-1} \leq y^2 |y|^{-\alpha-1}$ by Taylor expansion near the origin, we obtain that

$$0 < C = \frac{1}{2}\int_{\mathbb{R}} (1 - \cos(y))\, |y|^{-\alpha-1} dy < \infty$$

and $-|\xi|^\alpha$ is the Fourier multiplier of the infinitesimal generator of a stable symmetric process. Indeed, for the symmetric stable process $\mathfrak{S}^\beta(t)$, $t > 0$, $\beta \in (0, 2)$, we have that

$$-\partial_t\, \mathbb{E} e^{i\xi \mathfrak{S}^\beta(t)}\Big|_{t=0} = |\xi|^\beta$$

and (3.1) holds with

$$\mathcal{A} f(x) = -\frac{d^\beta f}{d|x|^\beta}(x) = -\frac{\sigma}{2}\left(\frac{d^\beta f}{dx^\beta}(x) + \frac{d^\beta f}{d(-x)^\beta}(x)\right) \tag{4.15}$$

where $\sigma = (\cos \pi\beta/2)^{-1}$. The Fourier symbol of the Riesz operator (4.15) is written as (see formula (4.10))

$$\int_{\mathbb{R}} e^{i\xi x}\frac{d^\beta f}{d|x|^\beta}(x) dx = \frac{\sigma}{2}\left((-i\xi)^\beta + (i\xi)^\beta\right) \widehat{f}(\xi) = |\xi|^\beta\, \widehat{f}(\xi).$$

Therefore, from (4.14), we conclude that

$$\sum_{j=0}^{N(t/\gamma^\alpha)} Y_j^* \xrightarrow{\gamma \to 0} \mathfrak{S}^\alpha(t^*)$$

in distribution, where

$$\mathbb{E} e^{i\xi \mathfrak{S}^\alpha(t^*)} = \exp\left(-t^* |\xi|^\alpha\right)$$

and $t^* = \alpha\lambda C t$, $t > 0$. Furthermore, the generator of $\mathfrak{S}^\alpha(t^*)$ with $\alpha \in (0, 2)$ is

$$-\alpha\lambda C \frac{d^\alpha f}{d|x|^\alpha}(x) = -\frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\xi x}\Phi(\xi)\widehat{f}(\xi) d\xi.$$

□

We now introduce the reciprocal gamma random variable E_α, $\alpha > 0$ with $P\{E_\alpha \in dx\}/dx = \frac{x^{-\alpha-1}}{\Gamma(\alpha)} e^{-1/x}$, $x > 0$ (the reciprocal gamma process has interesting connections with stable subordinators and Bessel processes, see for example [7]). The random variable E_α is termed reciprocal in the sense that $E_\alpha \sim 1/G_\alpha$ where G_α has Gamma density law. It is particularly interesting because it appears in many contexts. The density law of E_α appears as steady state solution to some diffusions ([10]), as density law for the reciprocal Bessel process at fixed time ([7]), as marginal density for diffusion models ([3]), as the density law of integrals involving geometric Brownian motions ([11]).

We also consider the (normal) random vector $\mathbf{Y} \sim N(\mathbf{0}, \sigma_\alpha^2)$, $\mathbf{Y} \in \mathbb{R}^d$, with random variance $\sigma_\alpha^2 \sim \frac{\gamma}{2} E_\alpha$ for some $\gamma > 0$ and define the process

$$\mathbf{A}(t) = \sum_{j=0}^{N(t)} \epsilon_j \mathbf{Y}_j \tag{4.16}$$

where $\epsilon_j \sim \epsilon \ \forall j$, ϵ has Rademacher law as above, $\mathbf{Y}_j \sim \mathbf{Y} \ \forall j$ and $\mathbb{E}\epsilon_j \epsilon_{j'} = 0$ for all j, j' such that $j \neq j'$. We notice that

$$P\{N(0, \sigma_\alpha^2) > x\} \leq \frac{1}{x} \int_x^\infty y \left(\int_0^\infty \frac{e^{-\frac{y^2}{2s}}}{\sqrt{2s}} P\{\sigma_\alpha^2 \in ds\} \right) dy \approx \gamma^\alpha x^{-2\alpha}$$

and therefore, for large x,

$$P\{N(0, \sigma_\alpha^2) \in dx\}/dx \approx 2\alpha \gamma^\alpha x^{-2\alpha - 1}.$$

After some calculations we explicitly write the law of $\mathbf{Y} \in \mathbb{R}^d$ as follows

$$\nu_{\mathbf{Y}}(\mathbf{y}) = \frac{\Gamma(\alpha + \frac{d}{2})}{\pi^{\frac{d}{2}} \Gamma(\alpha)} \frac{\gamma^\alpha}{(|\mathbf{y}|^2 + \gamma)^{\alpha + \frac{d}{2}}}, \quad \mathbf{y} \in \mathbb{R}^d. \tag{4.17}$$

We are now ready to present the next result.

Theorem 3. *Let $\mathfrak{S}^\beta(t) \in \mathbb{R}^d$, $t > 0$ be an isotropic stable process with $\beta \in (0, 2)$. Then, for $\alpha \in (0, 1)$,*

$$\mathbf{A}(t/\gamma^\alpha) \xrightarrow[\gamma \to 0]{d} \mathfrak{S}^{2\alpha}(t^*) \tag{4.18}$$

with $t^* : t \mapsto \lambda \frac{\Gamma(\alpha + \frac{d}{2})}{\pi^{\frac{d}{2}} \Gamma(\alpha)} C t$ and infinitesimal generator

$$\mathcal{A} f(\mathbf{x}) = -\lambda \frac{\Gamma(\alpha + \frac{d}{2})}{\pi^{\frac{d}{2}} \Gamma(\alpha)} C (-\triangle)^\alpha f(\mathbf{x}) \tag{4.19}$$

where

$$C = \int_{\mathbb{R}^d} \frac{1 - \cos y_1}{|\mathbf{y}|^{2\alpha + d}} \mathbf{dy}. \tag{4.20}$$

Proof. We have that

$$\mathbb{E} e^{i \boldsymbol{\xi} \cdot \mathbf{A}(t)} = \mathbb{E} \prod_{j=0}^{N(t)} \mathbb{E} e^{i \epsilon_j \boldsymbol{\xi} \cdot \mathbf{Y}_j} = \mathbb{E} \left(\mathbb{E} e^{i \epsilon \boldsymbol{\xi} \cdot \mathbf{Y}} \right)^{N(t)}$$

where we used the fact that $\mathbf{Y}_j \sim \mathbf{Y}$ for all j. Therefore, we obtain that

$$\begin{aligned} \mathbb{E} e^{i \boldsymbol{\xi} \cdot \mathbf{A}(t)} &= \exp\left(\lambda t (\mathbb{E} e^{i \epsilon \boldsymbol{\xi} \cdot \mathbf{Y}} - 1) \right) \\ &= \exp\left(\frac{\lambda t}{2} (\mathbb{E} e^{i \boldsymbol{\xi} \cdot \mathbf{Y}} + \mathbb{E} e^{-i \boldsymbol{\xi} \cdot \mathbf{Y}} - 2) \right) \end{aligned} \tag{4.21}$$

and $\mathbf{Y} \sim \nu_{\mathbf{Y}}$, see formula (4.17). The Fourier symbol corresponding to the characteristic function (4.21) is given by

$$\begin{aligned} -\partial_t \, \mathbb{E} e^{i \boldsymbol{\xi} \cdot \mathbf{A}(t)} \Big|_{t=0} &= -\frac{\lambda}{2} \int_{\mathbb{R}^d} (e^{i \boldsymbol{\xi} \cdot \mathbf{y}} + e^{-i \boldsymbol{\xi} \cdot \mathbf{y}} - 2) \nu_{\mathbf{Y}}(\mathbf{dy}) \\ &= -\lambda \int_{\mathbb{R}^d} \left(\cos \boldsymbol{\xi} \cdot \mathbf{y} - 1 \right) \frac{\gamma^\alpha \, \Gamma(\alpha + \frac{d}{2})}{\pi^{\frac{d}{2}} \Gamma(\alpha) \, (|\mathbf{y}|^2 + \gamma)^{\alpha + \frac{d}{2}}} \mathbf{dy}. \end{aligned}$$

and therefore, for the process (4.18), we get that

$$-\partial_t \,\mathbb{E}e^{i\boldsymbol{\xi}\cdot\mathbf{A}(t/\gamma^\alpha)}\Big|_{t=0} = -\lambda \int_{\mathbb{R}^d} \Big(\cos \boldsymbol{\xi}\cdot\mathbf{y} - 1\Big) \frac{\Gamma(\alpha+\frac{d}{2})}{\pi^{\frac{d}{2}}\,\Gamma(\alpha)\,(|\mathbf{y}|^2+\gamma)^{\alpha+\frac{d}{2}}}\mathbf{dy} = \Phi_\gamma(\boldsymbol{\xi}).$$

The limit for $\gamma \to 0$ leads to the Fourier symbol

$$\lim_{\gamma\to 0}\Phi_\gamma(\boldsymbol{\xi}) = -\lambda\frac{\Gamma(\alpha+\frac{d}{2})}{\pi^{\frac{d}{2}}\,\Gamma(\alpha)}\int_{\mathbb{R}^d}\Big(\cos\boldsymbol{\xi}\cdot\mathbf{y}-1\Big)\frac{\mathbf{dy}}{|\mathbf{y}|^{2\alpha+d}} = \lambda\frac{\Gamma(\alpha+\frac{d}{2})}{\pi^{\frac{d}{2}}\,\Gamma(\alpha)}C\,|\boldsymbol{\xi}|^{2\alpha} \tag{4.22}$$

where

$$C = \int_{\mathbb{R}^d}\frac{1-\cos y_1}{|\mathbf{y}|^{2\alpha+d}}\mathbf{dy}. \tag{4.23}$$

The interested reader can find in [6] a detailed computation of the integrals in (4.22) and (4.23). Finally, we observe that

$$-\partial_t \,\mathbb{E}e^{i\boldsymbol{\xi}\cdot\mathbf{A}(t/\gamma^\alpha)}\Big|_{t=0} = -\frac{\lambda}{2\gamma^\alpha}\int_{\mathbb{R}^d}(e^{i\boldsymbol{\xi}\cdot\mathbf{y}}+e^{-i\boldsymbol{\xi}\cdot\mathbf{y}}-2)\nu_{\mathbf{Y}}(\mathbf{dy}) = \Phi_\gamma(\boldsymbol{\xi})$$

converges, for $\gamma \to 0$, to the Fourier symbol

$$\Phi(\boldsymbol{\xi}) = -\frac{\lambda}{2}\frac{\Gamma(\alpha+\frac{d}{2})}{\pi^{\frac{d}{2}}\,\Gamma(\alpha)}\int_{\mathbb{R}^d}(e^{i\boldsymbol{\xi}\cdot\mathbf{y}}+e^{-i\boldsymbol{\xi}\cdot\mathbf{y}}-2)\frac{\mathbf{dy}}{|\mathbf{y}|^{2\alpha+d}}.$$

By applying formula (3.1), we get

$$\begin{aligned}\mathcal{A}f(\mathbf{x}) =& \lambda\frac{\Gamma(\alpha+\frac{d}{2})}{\pi^{\frac{d}{2}}\,\Gamma(\alpha)}C\frac{C_d(\alpha)}{2}\int_{\mathbb{R}^d}\Big(f(\mathbf{x}+\mathbf{y})+f(\mathbf{x}-\mathbf{y})-2f(\mathbf{x})\Big)\frac{\mathbf{dy}}{|\mathbf{y}|^{2\alpha+d}}\\ =& -\lambda\frac{\Gamma(\alpha+\frac{d}{2})}{\pi^{\frac{d}{2}}\,\Gamma(\alpha)}C(-\triangle)^\alpha f(\mathbf{x})\end{aligned}$$

where

$$C_d(\alpha) = \left(\int_{\mathbb{R}^d}\frac{1-\cos y_1}{|\mathbf{y}|^{2\alpha+d}}\mathbf{dy}\right)^{-1}$$

which is the generator of the isotropic stable process $\mathfrak{S}^{2\alpha}(t^*)$ with

$$t^* = \lambda\frac{\Gamma(\alpha+\frac{d}{2})}{\pi^{\frac{d}{2}}\,\Gamma(\alpha)}Ct, \quad t>0.$$

□

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