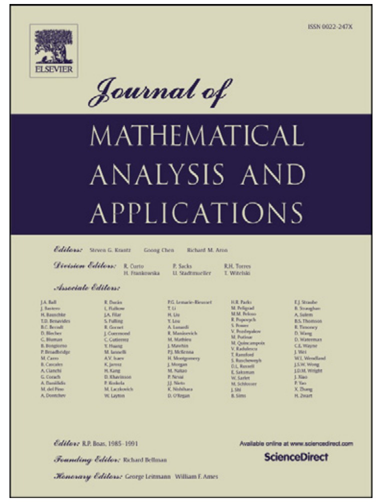

Parameter convexity and concavity of generalized trigonometric functions

D.B. Karp, E.G. Prilepkina

PII: S0022-247X(14)00660-X
DOI: 10.1016/j.jmaa.2014.07.017
Reference: YJMAA 18690

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 6 March 2014

Parameter convexity and concavity of generalized trigonometric functions

D.B. Karp[a,b*] and E.G. Prilepkina[a,b]

[a] *Far Eastern Federal University, 8 Sukhanova street, Vladivostok, 690950, Russia*

[b] *Institute of Applied Mathematics, FEBRAS, 7 Radio Street, Vladivostok, 690041, Russia*

We study the convexity properties of the generalized trigonometric functions viewed as functions of the parameter. We show that $p \to \sin_p(y)$ and $p \to \cos_p(y)$ are log-concave on the appropriate intervals while $p \to \tan_p(y)$ is log-convex. We also prove similar facts about the generalized hyperbolic functions. In particular, our results settle a major part of the conjecture recently put forward in [4].

Keywords: *generalized trigonometric functions, generalized hyperbolic functions, eigenfunctions of p-Laplacian, log-convexity, log-concavity, Turán type inequality*

MSC2010: 33B99, 33E30

1. Introduction and preliminaries. The symbol $\mathbb{R}_+$ will mean $[0,\infty)$. There are several ways in the literature to define generalized trigonometric functions (see, for instance, [11, 12, 17, 18, 24]). We will stick with the definition adopted in the book [16]. For $p>0$ define a differentiable function $F_p:[0,1)\to\mathbb{R}_+$ by

$$F_p(x)=\int_0^x(1-t^p)^{-1/p}dt. \tag{1}$$

Clearly, $F_2=\arcsin$ so that F_p can be viewed as generalized arcsine $F_p(x)=\arcsin_p(x)$. Since F_p is strictly increasing it has an inverse denoted by $\sin_p$. In all the references we could find the range of p is restricted to $(1,\infty)$ because only in this case $\sin_p(x)$ can be made periodic like usual sine. Nothing prohibits, however, defining $\sin_p(x)$ for all $p>0$, so we will be dealing with such generalized case here. If $p>1$ the function $\sin_p(x)$ is defined on the interval $[0,\pi_p/2]$, where

$$\pi_p=2\int_0^1(1-t^p)^{-1/p}dt=\frac{2\pi}{p\sin(\pi/p)}.$$

*Corresponding author. E-mail: D. Karp – *dimkrp@gmail.com*, E. Prilepkina – *pril-elena@yandex.ru*

It is convenient to extend the above definition by setting $\pi_p = +\infty$ for $0 < p \leq 1$. We will adopt this convention throughout the paper. In this way the function $y \to \sin_p(y)$ is strictly increasing on $[0, \pi_p/2]$ with $\sin_p(0) = 0$ and $\sin_p(\pi_p/2) = 1$ in analogy with the usual sine. It is easily seen that $p \to \pi_p$ is strictly decreasing on $(1, \infty)$ and maps this interval onto itself. For $p > 1$ the definition is extended to $[0, \pi_p]$ by

$$\sin_p(y) = \sin(\pi_p - y) \text{ for } y \in [\pi_p/2, \pi_p];$$

further extension to $[-\pi_p, \pi_p]$ is made by oddness; finally $\sin_p$ is extended to the whole $\mathbb{R}$ by $2\pi_p$ periodicity. If $p \in (0, 1]$ the inverse of $F_p(x)$ is defined on $\mathbb{R}_+$ and we just need oddness to extend the definition to the whole real line. The limiting cases are (see also [8]):

$$\sin_0(y) = 0 \text{ on } \mathbb{R}, \quad \sin_1(y) = 1 - e^{-y} \text{ on } \mathbb{R}_+, \quad \sin_\infty(y) = y \text{ on } [0, 1]. \tag{2}$$

Since

$$\frac{d}{dy}\sin_p(y) = \left(\frac{dF_p(x)}{dx}\right)^{-1}_{|x=\sin_p(y)} = (1 - [\sin_p(y)]^p)^{1/p}, \tag{3}$$

we get $\sin_p'(0) = 1$ and $\sin_p'(\pi_p/2) = 0$, which shows that $\sin_p(y)$ is continuously differentiable on $\mathbb{R}$ for all $p > 0$. The continuous derivative above is naturally called the generalized cosine:

$$\cos_p(y) = \frac{d}{dy}\sin_p(y), \quad y \in \mathbb{R}. \tag{4}$$

When $y \in [0, \pi_p/2]$ (for $p > 1$) and $y \in \mathbb{R}_+$ (for $0 < p \leq 1$) we can also define $\cos_p(y)$ by the right hand side of (3) which leads to an integral representation for $\arccos_p$:

$$\cos_p(y) = x = (1 - [\sin_p(y)]^p)^{1/p} \ \Rightarrow \ y = \arcsin_p\left((1 - x^p)^{1/p}\right) = \int\limits_0^{(1-x^p)^{\frac{1}{p}}} \frac{dt}{(1 - t^p)^{1/p}},$$

or, by substitution $s = (1 - t^p)^{1/p}$,

$$y = \arccos_p(x) = \int\limits_0^{(1-x^p)^{\frac{1}{p}}} \frac{dt}{(1 - t^p)^{1/p}} = \int\limits_x^1 \frac{s^{p-2}ds}{(1 - s^p)^{1-\frac{1}{p}}}, \quad 0 \leq x \leq 1. \tag{5}$$

The function $\cos_p$ can now be defined on $[0, \pi_p/2]$ as the inverse function to $\arccos_p$ and extended to $\mathbb{R}$ by evenness and $2\pi_p$ periodicity. The limiting values for $p = 0, 1, \infty$ can be obtained by differentiating (2). Pursuing an analogy with trigonometric functions further, the generalized tangent function is defined by

$$\tan_p(y) = \frac{\sin_p(y)}{\cos_p(y)}, \tag{6}$$

where $y \in \mathbb{R}\backslash\{(\mathbb{Z} + 1/2)\pi_p\}$ if $p > 1$. If $0 < p \leq 1$ the function $\tan_p(y)$ is continuous on $\mathbb{R}$. It is easy to show by differentiation that $\tan_p(y)$ is the inverse function to

$$\arctan_p(x) = \int_0^x \frac{dt}{1 + t^p}, \quad 0 \leq x < \infty, \tag{7}$$

extended from $[0, \pi_p/2]$ to $[-\pi_p/2, \pi_p/2]$ by oddity and further by periodicity [12, 16]. The limit cases are computed as:

$$\tan_0(y) = 2y \text{ on } \mathbb{R}, \quad \tan_1(y) = e^y - 1 \text{ on } \mathbb{R}_+, \quad \tan_\infty(y) = y \text{ on } [0, 1].$$

In a similar fashion one can define the generalized hyperbolic sine $\sinh_p(y)$ on $\mathbb{R}_+$ as the inverse function to the integral ($p > 0$)

$$y = \operatorname{arcsinh}_p(x) = \int_0^x \frac{dt}{(1 + t^p)^{1/p}}, \quad 0 \le x < \infty.$$

This definition is extended to negative values of y by $\sinh_p(y) = -\sinh_p(-y)$. Here the value $p = 1$ does not represent any additional difficulties. Further, we can define the hyperbolic cosine by

$$\cosh_p(y) = \frac{d}{dy} \sinh_p(y) = (1 + \sinh_p^p(y))^{1/p}, \quad 0 \le y < \infty,$$

which is extended to negative values of y by $\cosh_p(y) = \cosh_p(-y)$. This leads to the following identity

$$|\cosh_p(y)|^p - |\sinh_p(y)|^p = 1, \quad y \in \mathbb{R}. \tag{8}$$

Finally, the hyperbolic tangent is naturally defined by

$$\tanh_p(y) = \frac{\sinh_p(y)}{\cosh_p(y)}, \quad 0 \le y < \infty. \tag{9}$$

Differentiating (8) with respect to y we derive

$$\frac{d}{dy} \cosh_p(y) = \frac{\sinh_p^{p-1}(y)}{\cosh_p^{p-2}(y)} = \frac{\sinh_p^{p-1}(y)}{(1 + \sinh_p^p(y))^{(p-2)/p}}, \quad y \in \mathbb{R}_+,$$

and differentiating the definition of $\tanh_p(y)$ with respect to y we get after simplification

$$\frac{d}{dy} \tanh_p(y) = \frac{1}{\cosh_p^p(y)}$$

implying

$$\frac{d}{dx} \operatorname{arctanh}_p(x) = \frac{1}{1 - x^p}.$$

Hence, $\tanh_p(y)$ can be alternatively defined on $\mathbb{R}_+$ as the inverse function to the integral

$$y = \operatorname{arctanh}_p(x) = \int_0^x \frac{dt}{1 - t^p}, \quad 0 \le x < 1. \tag{10}$$

The definition given in (10) is correct because $\operatorname{arctanh}_p(x) \to \infty$ as $x \to 1$ for any $p > 0$.

Different variations of the functions $\sin_p$ and $\cos_p$ can be traced back to the 1879 paper of Lundberg [19, 20] which remained forgotten until Peetre found it in 1995 (see details in [17]). The next time related functions seem to appear in a paper by Shelupsky [24] in 1959 and some of their values were computed by Burgoyne in 1964 [7]. The generalized hyperbolic functions have been encountered by Peetre in [22] in connection with the study of K-functional. Later on, the function $\sin_p$ was found to be the eigenfunction of a boundary value problem for one-dimensional p-Laplacian [9, 10, 13, 17, 21].

More recently, monotonicity and convexity properties of the generalized trigonometric functions and various inequalities for these functions have been extensively studied by many authors, see [2, 3, 4, 5, 8, 11, 14, 15, 26, 27] and references in these papers. In particular, the monotonicity of $\sin_p(\pi_p x)$ as a function of p was required in [8] to fill a gap in the proof of basis properties of these functions given in [6]. Baricz, Bhayo and Vuorinen [4] investigated convexity properties of the functions $p \to \operatorname{arcsin}_p$, $p \to \operatorname{arctan}_p$ and their hyperbolic analogues and two-parameter generalizations. They also proposed the following conjecture.

Conjecture [4]. The following Turán type inequalities hold for all $p > 2$ and $y \in (0,1)$:

$$[\sin_p(y)]^2 > \sin_{p-1}(y)\sin_{p+1}(y),$$

$$[\cos_p(y)]^2 > \cos_{p-1}(y)\cos_{p+1}(y),$$

$$[\tan_p(y)]^2 < \tan_{p-1}(y)\tan_{p+1}(y),$$

$$[\sinh_p(y)]^2 < \sinh_{p-1}(y)\sinh_{p+1}(y),$$

$$[\tanh_p(y)]^2 > \tanh_{p-1}(y)\tanh_{p+1}(y).$$

The domain $(0,1)$ in this conjecture requires some explanation. As we have seen above the functions $\sin_p(y)$, $\cos_p(y)$ and $\tan_p(y)$ are defined on $[0,\pi_p/2]$ as the inverse functions of the corresponding integrals. Note that for $y \in (\pi_p/2, \pi_p]$ the functions $\cos_p(y)$ and $\tan_p(y)$ are negative and their logarithmic convexities are not well defined. For the purposes of the present investigation we will therefore restrict our attention to the intersection of the domains $[0,\pi_p/2]$ which is precisely $[0,1]$ since $\pi_\infty/2 = 1$. On the other hand, the hyperbolic functions are defined as the inverse functions of the appropriate integrals for all $y \in \mathbb{R}_+$ so it is natural to extend the part of the above conjecture pertaining to hyperbolic functions to $y \in \mathbb{R}_+$.

Let us remind some standard definitions. A positive function f defined on a finite or infinite interval I is said to be logarithmically convex, or log-convex, if its logarithm is convex, or equivalently,

$$f(\lambda x + (1-\lambda)y) \le [f(x)]^\lambda [f(y)]^{1-\lambda} \quad \text{for all } x, y \in I \quad \text{and } \lambda \in [0,1].$$

The function f is log-concave if the above inequality is reversed. If the inequality sign is strict for $\lambda \in (0,1)$ then the appropriate property is called strict. It is relatively straightforward to see from these definitions that log-convexity implies convexity and while log-concavity is implied by concavity but not vice versa.

The main results of the present paper are collected the next two theorems.

Theorem 1 *The following assertions hold true:*
(i) For each fixed $y \in (0,1)$ the function $p \to \sin_p(y)$ is strictly log-concave on $(0,\infty)$.
(ii) For each fixed $y \in (0,\log 2)$ the function $p \to \cos_p(y)$ is strictly log-concave on $(1,\infty)$.
(iii) For each fixed $y \in (0,\log 2)$ the function $p \to \tan_p(y)$ is strictly log-convex on $(1,\infty)$.

Theorem 2 *The following assertions hold true:*
(i) For each fixed $y \in (0,\infty)$ the function $p \to \sinh_p(y)$ is strictly log-convex on $(0,\infty)$.
(ii) For each fixed $y \in (0,\infty)$ the function $p \to \tanh_p(y)$ is strictly concave on $(0,\infty)$.
(iii) For each fixed $y \in (0,\infty)$ the function $p \to \cosh_p(y)$ is strictly log-convex on $(0,\infty)$.

These results confirm, strengthen and extend the above conjecture for $\sin_p$, $\sinh_p$ and $\tanh_p$. The conjecture for $\tan_p$ and $\cos_p$ is only confirmed by Theorem 1 under a stronger restriction $y \in (0, \log 2)$. However, we believe that it actually holds for all $y \in (0,1)$ although we were unable to come up with a proof.

2. Auxiliary results. The following lemma will be our key tool for the forgoing proof of Theorems 1 and 2.

Lemma 1 *Suppose I, J are finite or infinite open or closed subintervals of $\mathbb{R}$. Suppose $f(p,x) \in C^2(J \times I)$ is strictly monotone on I for each fixed $p \in J$ so that $y \to g(p,y) := f^{-1}(p,y)$ is well defined and monotone on $f(I)$ for each fixed $p \in J$. Then the following relations hold true*

$$\frac{\partial}{\partial p} g(p,y) = -f'_p/f'_x, \tag{11}$$

$$\frac{\partial^2}{\partial p^2} g(p,y) = (f'_x)^{-2}\left\{2f'_pf''_{xp} - f'_xf''_{pp} - \left(f'_p\right)^2 f''_{xx}/f'_x\right\}, \tag{12}$$

$$\frac{\partial^2}{\partial p^2} \log g(p,y) = (xf'_x)^{-2}\left\{2xf'_pf''_{px} - xf'_xf''_{pp} - x(f'_p)^2f''_{xx}/f'_x - (f'_p)^2\right\}, \tag{13}$$

where x on the right is related to y on the left by $y = f(p,x)$ or $x = g(p,y)$.

Remark. Formulas (12) and (13) can also be written in the following form:

$$\frac{\partial^2}{\partial p^2} g(p,y) = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{f'_p}{f'_x}\right)^2 - \frac{\partial}{\partial p}\left(\frac{f'_p}{f'_x}\right)$$

and

$$\frac{\partial^2}{\partial p^2} \log g(p,y) = \frac{1}{2x}\frac{\partial}{\partial x}\left(\frac{f'_p}{f'_x}\right)^2 - \frac{1}{x}\frac{\partial}{\partial p}\left(\frac{f'_p}{f'_x}\right) - \left(\frac{f'_p}{xf'_x}\right)^2.$$

Proof. By definition of the inverse function we have:

$$f(p, g(p,y)) = y \tag{14}$$

Differentiating (14) with respect to p while holding y fixed we get:

$$\frac{df}{dp} = \frac{\partial f}{\partial p} + \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} = 0 \;\Leftrightarrow\; \frac{\partial g}{\partial p} = -\frac{\partial f}{\partial p}\left(\frac{\partial f}{\partial x}\right)^{-1},$$

which proves (11). Further, differentiating $\partial g/\partial p$ once more with respect to p yields:

$$\frac{\partial^2 g}{\partial p^2} = -\left(\frac{\partial f}{\partial x}\right)^{-1}\frac{\partial}{\partial p}\left(\frac{\partial f}{\partial p}\right) - \frac{\partial f}{\partial p}\frac{\partial}{\partial p}\left(\frac{\partial f}{\partial x}\right)^{-1} =$$
$$= -\left(\frac{\partial f}{\partial x}\right)^{-1}\left(\frac{\partial^2 f}{\partial p^2} + \frac{\partial^2 f}{\partial x\partial p}\frac{\partial g}{\partial p}\right) + \frac{\partial f}{\partial p}\left(\frac{\partial f}{\partial x}\right)^{-2}\frac{\partial}{\partial p}\frac{\partial f}{\partial x}$$
$$= -\left(\frac{\partial f}{\partial x}\right)^{-1}\left(\frac{\partial^2 f}{\partial p^2} + \frac{\partial^2 f}{\partial x\partial p}\frac{\partial g}{\partial p}\right) + \frac{\partial f}{\partial p}\left(\frac{\partial f}{\partial x}\right)^{-2}\left(\frac{\partial^2 f}{\partial p\partial x} + \frac{\partial^2 f}{\partial x^2}\frac{\partial g}{\partial p}\right).$$

Substituting (11) for $\partial g/\partial p$ into the last formula we obtain:

$$\frac{\partial^2 g}{\partial p^2} = -\left(\frac{\partial f}{\partial x}\right)^{-1}\left(\frac{\partial^2 f}{\partial p^2} + \frac{\partial^2 f}{\partial x\partial p}\left[-\frac{\partial f}{\partial p}\left(\frac{\partial f}{\partial x}\right)^{-1}\right]\right)$$
$$+ \frac{\partial f}{\partial p}\left(\frac{\partial f}{\partial x}\right)^{-2}\left(\frac{\partial^2 f}{\partial p\partial x} + \frac{\partial^2 f}{\partial x^2}\left[-\frac{\partial f}{\partial p}\left(\frac{\partial f}{\partial x}\right)^{-1}\right]\right)$$
$$= -\left(\frac{\partial f}{\partial x}\right)^{-1}\frac{\partial^2 f}{\partial p^2} + 2\left(\frac{\partial f}{\partial x}\right)^{-2}\frac{\partial f}{\partial p}\frac{\partial^2 f}{\partial x\partial p} - \left(\frac{\partial f}{\partial x}\right)^{-3}\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial f}{\partial p}\right)^2$$
$$= -\left(\frac{\partial f}{\partial x}\right)^{-2}\left[\frac{\partial f}{\partial x}\frac{\partial^2 f}{\partial p^2} - 2\frac{\partial f}{\partial p}\frac{\partial^2 f}{\partial x\partial p} + \left(\frac{\partial f}{\partial x}\right)^{-1}\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial f}{\partial p}\right)^2\right]$$
$$= \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{f'_p}{f'_x}\right)^2 - \frac{\partial}{\partial p}\left(\frac{f'_p}{f'_x}\right).$$

To prove (13) compute

$$\frac{\partial^2}{\partial p^2}\log g(p,y) = \frac{g''_{pp}g - (g'_p)^2}{g^2}.$$

Substituting $g(p,y) = x$ and formulas for g'_p and g''_{pp} just derived into the above formula we obtain:

$$\frac{\partial^2}{\partial p^2}\log g(p,y) = \frac{1}{2x}\frac{\partial}{\partial x}\left(\frac{f'_p}{f'_x}\right)^2 - \frac{1}{x}\frac{\partial}{\partial p}\left(\frac{f'_p}{f'_x}\right) - \left(\frac{f'_p}{xf'_x}\right)^2.$$

Performing the differentiation we get (13). $\square$

The following corollaries specialize the formulas from Lemma 1 to those generalize trigonometric functions we will deal with in the present paper. Suppose $b > a > 0$, where b may equal ∞, are fixed. If p varies over (a,b) the common domain of definition for the families $\{\sin_p(y)\}_{p\in(a,b)}$, $\{\cos_p(y)\}_{p\in(a,b)}$ and $\{\tan_p(y)\}_{p\in(a,b)}$ is $[0,\pi_b/2]$, where $\pi_b = \infty$ for $0 < b \le 1$ and $\pi_\infty = 2$ (we restrict our attention to the primary definitions as the inverse functions of the corresponding integrals). The families $\{\sinh_p(y)\}_{p\in(a,b)}$, $\{\cosh_p(y)\}_{p\in(a,b)}$ and $\{\tanh_p(y)\}_{p\in(a,b)}$ are all defined on $[0,\infty)$.

Corollary 1 *The function $p \to \sin_p(y)$ is log-concave on the interval (a,b) for some $y \in [0,\pi_b/2]$ iff for all $p \in (a,b)$*

$$\frac{1}{x}\phi(p,x)^{p-1}\left(\int_0^x \phi'_p(p,t)dt\right)^2 - 2[\log\phi(p,x)]'_p\int_0^x \phi'_p(p,t)dt + \int_0^x \phi''_{pp}(p,t)dt \ge 0, \qquad (15)$$

where $x=\sin_p(y)$. It is concave on (a,b) for some $y\in[0,\pi_b/2]$ iff for all $p\in(a,b)$

$$\frac{x^p}{x}\phi(p,x)^{p-1}\left(\int_0^x \phi'_p(p,t)dt\right)^2-2[\log\phi(p,x)]'_p\int_0^x \phi'_p(p,t)dt+\int_0^x \phi''_{pp}(p,t)dt\geq 0. \qquad (16)$$

Here $\phi(p,t)=(1-t^p)^{-1/p}$ and

$$\frac{\phi'_p(p,t)}{\phi(p,t)}=\frac{1}{p^2}\log(1-t^p)+\frac{t^p\log t}{p(1-t^p)},\quad \phi''_{pp}(p,t)=\frac{\phi'_p(p,t)^2}{\phi(p,t)}-\frac{2}{p}\phi'_p(p,t)+\phi(p,t)\frac{t^p\log^2 t}{p(1-t^p)^2}.$$

The corresponding property is strict if and only if the inequality sign is strict.

Proof. Write $g(p,y)=\sin_p(y)$. A necessary and sufficient condition for log-concavity of the smooth function $p\to g(p,y)$ is $[\log(g)]''_{pp}\leq 0$. To compute $[\log(g)]''_{pp}$ substitute $f(p,x)=F_p(x)$ defined in (1) into (13) and notice that

$$f'_x(p,x)=(1-x^p)^{-1/p}=\phi(p,x),\quad f''_{xx}(p,x)=(1-x^p)^{-1-1/p}x^{p-1}=\frac{\phi(p,x)x^{p-1}}{1-x^p},$$

$$\frac{x}{f'_x}f''_{xx}+1=\frac{x(1-x^p)^{-1-1/p}x^{p-1}}{(1-x^p)^{-1/p}}+1=\frac{x^p}{1-x^p}+1=\frac{1}{1-x^p}=\phi(p,x)^p,$$

$$f''_{xp}=f''_{px}=\phi'_p(p,x),\quad f'_p=\int_0^x\phi'_p(t)dt,\quad f''_{pp}=\int_0^x\phi''_{pp}(t)dt.$$

Hence $[\log(g)]''_{pp}\leq 0$ reduces to (15). Similarly using (12) we get (16). Formulas for derivatives are obtained by straightforward differentiation. □

In precisely the same manner we can derive the next three corollaries whose proofs are omitted.

Corollary 2 *The function $p\to\tan_p(y)$ is log-convex on the interval (a,b) for some $y\in[0,\pi_b/2]$ iff for all $p\in(a,b)$*

$$(1/x-(p-1)x^{p-1})\left(\int_0^x\theta'_p(p,t)dt\right)^2+\frac{2x^p\log x}{1+x^p}\int_0^x\theta'_p(p,t)dt+\int_0^x\theta''_{pp}(p,t)dt\leq 0, \qquad (17)$$

where $x=\tan_p(y)$ and $\theta(p,x)=(1+x^p)^{-1}$. It is convex on (a,b) for some $y\in[0,\pi_b/2]$ iff for all $p\in(a,b)$

$$px^{p-1}\left(\int_0^x\theta'_p(p,t)dt\right)^2+\frac{2x^p\log x}{1+x^p}\int_0^x\theta'_p(p,t)dt+\int_0^x\theta''_{pp}(p,t)dt\leq 0. \qquad (18)$$

Here

$$\theta'_p(p,t)=\frac{-t^p\log t}{(1+t^p)^2},\qquad \theta''_{pp}(p,t)=\frac{-t^p(1-t^p)\log^2 t}{(1+t^p)^3}.$$

The corresponding property is strict if and only if the inequality sign is strict.

Corollary 3 *The function $p \to \sinh_p(y)$ is log-convex on the interval (a,b) for some $y \in [0,+\infty)$ iff for all $p \in (a,b)$*

$$\frac{1}{1+x^p}\left(\int_0^x \lambda_p'(p,t)dt\right)^2 - 2x\lambda_p'(p,x)\int_0^x \lambda_p'(p,t)dt + \frac{x}{(1+x^p)^{1/p}}\int_0^x \lambda_{pp}''(p,t)dt \le 0. \quad (19)$$

Here $x = \sinh_p(y)$, $\lambda(p,t) = (1+t^p)^{-1/p}$ and

$$\frac{\lambda_p'(p,t)}{\lambda(p,t)} = \frac{\log(1+t^p)}{p^2} - \frac{t^p \log t}{p(1+t^p)}, \quad \lambda_{pp}''(p,t) = \frac{\lambda_p'(p,t)^2}{\lambda(p,t)} - \frac{2}{p}\lambda_p'(p,t) - \lambda(p,t)\frac{t^p \log^2 t}{p(1+t^p)^2}.$$

The corresponding property is strict if and only if the inequality sign is strict.

Corollary 4 *The function $p \to \tanh_p(y)$ is concave on the interval (a,b) for some $y \in [0,\infty)$ iff for all $p \in (a,b)$*

$$\frac{px^{p-1}}{(1-x^p)}\left(\int_0^x \alpha_p'(p,t)dt\right)^2 - \frac{2x^p \log x}{(1-x^p)^2}\int_0^x \alpha_p'(p,t)dt + \frac{1}{(1-x^p)}\int_0^x \alpha_{pp}''(p,t)dt \ge 0, \quad (20)$$

where $x = \tanh_p(y)$. Here

$$\alpha(p,t) = \frac{1}{1-t^p}, \quad \alpha_p'(p,t) = \frac{t^p \log t}{(1-t^p)^2}, \quad \alpha_{pp}''(p,t) = \frac{t^p(t^p+1)\log^2 t}{(1-t^p)^3}.$$

The corresponding property is strict if and only if the inequality sign is strict.

The next lemma is a guise of the monotone L'Hôspital rule [1, 23]. As before we allow the value $b = \infty$.

Lemma 2 *Suppose r, s are continuously differentiable functions defined on a real interval (a,b), $r(a) = s(a) = 0$ and $ss' > 0$ on (a,b). If r'/s' is decreasing on (a,b) then $r/s > r'/s'$ on (a,b).*

Proof. According to monotone L'Hôspital rule the decrease of r'/s' implies that of r/s. On the other hand, by the quotient rule (see also [23, formula (1.1)])

$$s^2\left(\frac{r}{s}\right)' = \left(\frac{r'}{s'} - \frac{r}{s}\right)ss'$$

so that the expression in parentheses must be negative. $\square$

We will also need the following estimation.

Lemma 3 *Suppose $p > 1$, $0 < s < 1$. Then*

$$\frac{sp^3}{(p+1)^2}\left(\frac{1}{(p+1)^2} - \frac{\log^2 s}{p^2}\right) < \int_0^1 u^{1/p}\frac{(1-su)}{(1+su)^3}\log^2(su)du. \quad (21)$$

Proof. Define

$$t(s) = s\left(\frac{1}{(p+1)^2} - \frac{\log^2 s}{p^2}\right).$$

Taking derivative yields

$$t'(s) = \frac{1}{(p+1)^2} - \frac{\log^2 s}{p^2} - \frac{2\log s}{p^2}.$$

The equation $t'(s) = 0$ has two roots $s_{1,2} = \exp(-1 \pm \sqrt{1 + p^2/(p+1)^2})$. Hence, one of the roots lies in $(0, 1)$ and $t(s)$ is decreasing on $(0, s_*)$ and increasing on $(s_*, 1)$ for some $0 < s_* < 1$. Since $t(0) = 0$ and $t(1) > 0$ the maximum is attained at the point $s = 1$. On the other hand, the function $s \to (1 - su)\log^2(su)/(1 + su)^3$ decreases on $[0, 1]$ and attains its minimum at the point $s = 1$. This implies that we only need to prove the inequality

$$\frac{p^3}{(p+1)^4} < \int_0^1 u^{1/p}\frac{(1-u)}{(1+u)^3}\log^2 u du. \tag{22}$$

In view of

$$\int_0^1 u^{1/p}\log^2 u du = \frac{2p^3}{(1+p)^3}$$

we can rewrite (22) as

$$\int_0^1 u^{1/p}\log^2 u\left(\frac{(1-u)}{(1+u)^3} - \frac{1}{2(p+1)}\right) du > 0. \tag{23}$$

The derivative of the integrand with respect to p equals

$$\frac{u^{1/p}\log^2 u}{p^2}\left\{\frac{p^2}{2(p+1)^2} - \log u\left(\frac{(1-u)}{(1+u)^3} - \frac{1}{2(p+1)}\right)\right\}.$$

The expression in braces increases in p for each fixed $u \in (0, 1]$ and is easily seen to be positive for $p = 1$. Therefore, it is positive for all $p \geq 1$ and $u \in (0, 1]$. This shows that the left hand side of (23) increases in p for $p \geq 1$. When $p = 1$ computation gives

$$\int_0^1 u^{1/p}\log^2 u\left(\frac{(1-u)}{(1+u)^3} - \frac{1}{2(p+1)}\right) du = \frac{\pi^2}{3} - \log 4 - \frac{3}{2}\zeta(3) - \frac{1}{16} > 0,$$

where ζ is Riemann's zeta function. The last inequality completes the proof. $\square$

3. Proof of Theorem 1. Note first that assertion (ii) follows from (i) and (iii) via formula (6) which implies that

$$(\log\cos_p(y))''_{pp} = (\log\sin_p(y))''_{pp} - (\log\tan_p(y))''_{pp}.$$

Hence it remains to prove (i) and (iii). First we demonstrate (15) with strict sign which will prove (i). Consider the following quadratic

$$F(z) = \frac{\phi(p,x)^{p-1}}{x}z^2 - 2\eta(p,x)z + \int_0^x \phi''_{pp}(p,t)dt,$$

where $\eta(p,x) = [\log\phi(p,x)]'_p$. Explicit expressions for ϕ, η and ϕ''_{pp} are given in corollary 1. We need to show that $F(z) > 0$ for $z = \int_0^x \phi'_p(p,t)dt$. We will show that in fact this inequality holds for all real z. Indeed, $\phi(p,x)^{p-1}/x > 0$ and it remains to prove that

$$\frac{D}{4} = \eta^2 - \frac{\phi(p,x)^{p-1}}{x}\int_0^x \phi''_{pp}(p,t)dt < 0 \;\Leftrightarrow\; G(x) := \frac{x\eta^2}{\phi(p,x)^{p-1}} - \int_0^x \phi''_{pp}(p,t)dt < 0.$$

Here D denotes the discriminant of $F(z)$. Clearly, $G(0) = 0$. Further, elementary but long computation reveals

$$\begin{aligned} G'(x) &= \frac{2\eta(p,x)(1-x^p+px^p\log x)}{p(1-x^p)^{1+1/p}} - \frac{px^p\eta(p,x)^2}{(1-x^p)^{1/p}} - \frac{x^p\log^2 x}{p(1-x^p)^{2+1/p}} \\ &= (1-x^p)^{-1/p}\left\{\frac{2}{p}\eta(p,x) - px^p\eta(p,x)^2 + \frac{2\eta(p,x)x^p\log x}{1-x^p} - \frac{x^p\log^2 x}{p(1-x^p)^2}\right\}. \end{aligned}$$

Substituting

$$\eta(p,x) = \frac{1}{p^2}\log(1-x^p) + \frac{x^p\log x}{p(1-x^p)}$$

into the above formula we get after some more algebra:

$$G'(x) = (1-x^p)^{-1/p}\left\{\frac{2x^p\log x}{p^2(1-x^p)} + \frac{2}{p^3}\log(1-x^p) - \frac{x^p}{p}(\log(1-x^p)^{1/p} - \log x)^2\right\}.$$

Each term in braces is negative so that $G'(x) < 0$ which implies that $G(x) < 0$ completing the proof of (i).

We now turn to proving (iii). First we show that the claim made in (iii) is equivalent to validity of inequality (17) with strict sign for $x \in (0,1)$ and $p > 1$. Indeed, $y \to \tan_p(y)$ is increasing so that $\tan_1(y) < \tan_1(\log 2) = 1$ for $y \in (0, \log 2)$. Further, $\tan_p(y) < \tan_1(y) < 1$ because

$$\int_0^{\tan_p(y)} \frac{dt}{1+t^p} = y = \int_0^{\tan_1(y)} \frac{dt}{1+t} < \int_0^{\tan_1(y)} \frac{dt}{1+t^p}.$$

Next, since $1 + (1-p)x^p < 1$ it suffices to prove the inequality

$$\left(\int_0^x \theta'_p(p,t)dt\right)^2 + \frac{2x^{p+1}}{1+x^p}\log(x)\int_0^x \theta'_p(p,t)dt + x\int_0^x \theta''_{pp}(p,t)dt < 0. \tag{24}$$

Note that $\theta'_p(p,t) > 0$, $\theta''_{pp}(p,t) < 0$ for $0 < t < 1$. Further, by replacing the denominator in $\theta'_p(p,t)$ by 1 we get the estimate

$$0 < \int_0^x \theta'_p(p,t)dt < \psi := -\int_0^x t^p\log(t)dt = \frac{x^{p+1}}{p+1}\left(\frac{1}{p+1} - \log(x)\right). \tag{25}$$

Consider the following quadratic derived from (24):

$$Q(w) = w^2 + bw + c, \quad \text{where } b = \frac{2x^{p+1}\log(x)}{1+x^p}, \quad c = x\int_0^x \theta''_{pp}(p,t)dt.$$

We need to show that $Q(w) < 0$ for $w = \int_0^x \theta_p'(p,t)dt$. Since c is negative $Q(0) < 0$ and in view of (25) it suffices to demonstrate that $Q(\psi) < 0$ or

$$-c > \psi^2 + \psi b. \tag{26}$$

Denote $s = x^p$. Then after some rearrangement we get

$$\psi^2 + \psi b = \frac{s^2x^2}{(p+1)^2}\left(\frac{-\log(s)}{p} + \frac{1}{p+1}\right)\left(\frac{-\log(s)}{p} + \frac{1}{p+1} + \frac{2(p+1)\log(s)}{p(s+1)}\right). \tag{27}$$

Performing change of variable $t = ux$ in the integral representing c we obtain

$$-c = x\int_0^x \frac{t^p(1-t^p)\log^2(t)dt}{(1+t^p)^3} = x\int_0^1 \frac{\log^2(u^px^p)pu^{p-1}x^p(1-x^pu^p)xudu}{p^3(1+x^pu^p)^3}.$$

By writing $s = x^p$, $z = u^p$ the last integral transforms into

$$-c = \frac{x^2s}{p^3}\int_0^1 z^{1/p}\frac{(1-sz)}{(1+sz)^3}\log^2(sz)dz. \tag{28}$$

By dropping some terms in (27) we have

$$\psi^2 + \psi b < \frac{s^2x^2}{(p+1)^2}\left(\frac{1}{(p+1)^2} - \frac{\log^2(s)}{p^2}\right).$$

The right hand side of this inequality does not exceed $-c$ by Lemma 3 which completes the proof of Theorem 1. □

Remark. We remark here that log-convexity of $p \to \tan_p(y)$ does not hold for all $p > 0$ even if y is restricted to $(0, \log 2)$ as evidenced by numerical experiments.

Extensive numerical evidence supports the following assertion.

Conjecture. There exists $p_0 \in (0,1)$ such that the function $p \to \sin_p(y)$ is strictly concave on (p_0, ∞) for all $y \in (0,1)$. If $p \in (0, p_0)$ concavity is violated for some $y \in (0,1)$.

4. Proof of Theorem 2. First we note that claim (ii) follows from (i) and (iii) according to the formula

$$(\log\cosh_p(y))''_{pp} = (\log\sinh_p(y))''_{pp} - (\log\tanh_p(y))''_{pp}$$

which follows from (9) and the fact that concavity of $p \to \tanh_p(y)$ on $(0,\infty)$ implies its log-concavity on the same interval.

Next we demonstrate (i). By Corollary 3 we need to prove (19). After dividing (19) by $-x\lambda(p,x) < 0$ we get

$$H(p,x) = v(p,x)\mu(p,x)^2 + 2w(p,x)\mu(p,x) - \int_0^x \lambda''_{pp}(p,t)dt > 0, \tag{29}$$

where

$$v = v(p,x) = -\frac{(1+x^p)^{-1+1/p}}{x},$$
$$w = w(p,x) = \frac{1}{p^2}\log(1+x^p) - \frac{x^p\log(x)}{p(1+x^p)},$$
$$\mu = \mu(p,x) = \int_0^x \lambda_p'(p,t)dt.$$

Differentiation gives

$$H'_x(p,x) = v'_x\mu^2 + 2(v\lambda'_p + w'_x)\mu + (2w\lambda'_p - \lambda''_{pp}).$$

We will demonstrate that in fact $H'_x(x,p) > 0$ for all real μ. Indeed,

$$v'_x = \frac{(1+x^p)^{-2+1/p}(1+px^p)}{x^2} > 0$$

and it remains to prove that

$$\frac{D}{4} = (v\lambda'_p + w'_x)^2 - (2w\lambda'_p - \lambda''_{pp})v'_x < 0,$$

where D denotes the discriminant of the quadratic $H'_x(x,p)$ viewed as a function of μ. Straightforward calculations reveal:

$$v = -\frac{1}{\lambda(1+x^p)x}, \quad \lambda'_p = \lambda w, \quad w'_x = \frac{-x^{p-1}\log x}{(1+x^p)^2},$$

$$\lambda''_{pp} = \lambda\left(w^2 - \frac{2}{p}w - \frac{x^p\log^2(x)}{p(1+x^p)^2}\right), \quad v'_x = \frac{1+px^p}{\lambda x^2(1+x^p)^2}.$$

Substitution yields

$$\frac{D}{4} = \frac{1}{x^2(1+x^p)^2}\left\{-px^pw^2 - \frac{2w}{p}\left(1+px^p - \frac{x^p\log(x^p)}{1+x^p}\right) - \frac{x^p\log^2(x)}{p(1+x^p)^2}\right\}. \tag{30}$$

We only need to show that the middle term in braces is positive or

$$\frac{2w}{p}\left(1+px^p - \frac{x^p\log(x^p)}{1+x^p}\right) = \frac{2\left((1+z)\log(1+z) - z\log z\right)\left(1+pz - \frac{z\log z}{1+z}\right)}{p^3(1+z)} > 0,$$

where $z = x^p$. Indeed, this follows from the inequalities

$$(1+z)\log(1+z) - z\log z > 0,$$

$$1 + pz - \frac{z\log z}{1+z} > 1 + z - \frac{z\log(z)}{1+z} > 0$$

valid for any $z > 0$, $p > 1$. Hence, $H'_x(p,x) > 0$ implying $H(p,x) > H(p,0) = 0$ which completes the proof of log-concavity of $p \to \sinh_p(y)$.

It remains to show the validity of (iii). It suffices to demonstrate inequality (20). The trick with quadratic used in the previous item fails here, so we resort to another proof. The first term in (20) is positive so if we drop it the expression on the left becomes smaller. Hence, if we can prove it is still positive we are done. This amounts to showing that

$$\frac{r}{s} := \left(\int_0^x \frac{t^p(t^p+1)[\log(1/t)]^2}{(1-t^p)^3}dt\right)\Big/\left(\int_0^x \frac{t^p\log(1/t)dt}{(1-t^p)^2}\right) > \frac{2x^p\log(1/x)}{(1-x^p)}.$$

We have

$$\frac{r'}{s'} = \frac{(x^p+1)\log(1/x)}{1-x^p}.$$

It is easy to check by taking derivative that this function decreases on $(0,1)$. Clearly $r(0) = s(0) = 0$ and $ss' > 0$, so that we are in the position to apply Lemma 2 yielding

$$\left(\int_0^x \frac{t^p(t^p+1)[\log(t)]^2}{(1-t^p)^3}dt\right)\Big/\left(\int_0^x \frac{t^p\log(t)dt}{(1-t^p)^2}\right) > \frac{(x^p+1)\log(1/x)}{1-x^p} > \frac{2x^p\log(1/x)}{(1-x^p)}. \quad \square$$

5. Acknowledgements. We thank Sergei Sitnik for numerous useful discussions regarding the manuscript and an anonymous referee for careful reading of the paper and remarks that helped to improve the paper. The research of the first author was supported by the Ministry of Education of the Russian Federation (project 1398.2014), research of the second author was supported by Far Eastern Branch of the Russian Academy of Sciences (project 12-II-CO-01M-002) and Russian Foundation for Basic Research (project 13-01-12404-ofi_m2).

References

[1] G.D. Anderson, M.K. Vamanamurthy, and M. Vuorinen, Inequalities for quasiconformal mappings in space, Pacific J. Math. 160 (1993), 1–18.

[2] Á. Baricz, B.A. Bhayo, R. Klen, Convexity properties of generalized trigonometric and hyperbolic functions, Aequat. Math. 2013, DOI 10.1007/s00010-013-0222-x

[3] Á. Baricz, B.A. Bhayo, T.K. Pogány, Functional inequalities for generalized inverse trigonometric and hyperbolic functions, J. Math. Anal. Appl. 417 (2014), 244–259.

[4] Á. Baricz, B.A. Bhayo, M. Vuorinen, Turán type inequalities for generalized inverse trigonometric functions, 2013, arXiv:1305.0938v2. Available online at http://arxiv.org/abs/1305.0938v2

[5] B.A. Bhayo, M. Vuorinen, On generalized trigonometric functions with two parameters, J. Approx. Theory 164 (2012), 1415–1426.

[6] P. Binding, L. Boulton, J. Čepička, P. Drábek and P. Girg, Basis properties of eigenfunctions of the p-Laplacian, Proc. Amer. Math. Soc. 134(2006), 3487–3494.

[7] F.D. Burgoyne, Generalized trigonometric functions, Math. Comp. 18(1964), 314–316.

[8] P.J. Bushell, D.E. Edmunds, Remarks on generalised trigonometric functions, Rocky Mountain J. Math. 42 (2012) 25–57.

[9] M. del Pino, M. Elgueta and R. Manasevich, A homotopic deformation along p of a Leray-Schauder degree result and existence for $(|u'|^{p-2}u')'+f(t,u)=0$, $u(0)=u(T)=0$, $p>1$, J. of Diff. Equat., vol. 80, no. 1 (1989), 1–13.

[10] P. Drábek, R. Manásevich, On the closed solution to some nonhomogeneous eigenvalue problems with p-Laplacian, Diff. and Int. Equat., 12 (1999), no. 6, 773–788.

[11] D.E. Edmunds, P. Gurka, J. Lang, Properties of generalized trigonometric functions. J. Approx. Theory 164 (2012), 47–56.

[12] D.E. Edmunds, J. Lang, Generalizing trigonometric functions from different points of view, Progr. in Math., Phys and Astr. (Pokroky MFA), vol 4, 2009.

[13] A. Elbert, A half-linear second order differential equation, Colloq. Math. Soc. János Bolyai, 30 Qualitative theory of differential equations, Szeged (Hungary) (1979), 153–180.

[14] W-D. Jiang, M.-K. Wang, Y.-M. Chu, Y.-P. Jiang, F. Qi, Convexity of the generalized sine function and the generalized hyperbolic sine function, J. of Approx. Theory 174 (2013), 1–9.

[15] R. Klen, M. Vuorinen, X.-H. Zhang, Inequalities for the generalized trigonometric and hyperbolic functions, J. Math. Anal. Appl. 409 (2014) 521–529.

[16] J. Lang, D.E. Edmunds, Eigenvalues, Embeddings and Generalised Trigonometric Functions, Lecture Notes in Mathematics 2016, Springer, 2011.

[17] P. Lindqvist, Some remarkable sine and cosine functions, Ricerche di Mat. 44 (1995), 269–290.

[18] P. Lindqvist, J. Peetre, p-arclength of the q-circle, Math. Student 72 (2003), 139–145.

[19] P. Lindqvist, J. Peetre, Comments on Erik Lundberg's 1879 thesis, especially on the work of Göran Dillner and his infuence on Lundberg, Mem. dell'Istituto Lombardo, Accad. Sci. e Lett., Classe Sci. Mat. Nat. XXXI, Fasc. 1, Milano, 2004.

[20] E. Lundberg, Om hypergoniometriska funktioner af komplexa variabla, Stockholm, 1879. English translation: On hypergeometric functions of complex variables.

[21] M. Ôtani, A remark on certain nonlinear elliptic equations, Proc. of the Faculty of Science, Tokai University, 19 (1984), 23–28.

[22] J. Peetre, The best constant in some inequalities involving L_q norms of derivatives, Ricerche di Matematica XXI (1972), 176–183.

[23] I. Pinelis, L'Hospital type rules for monotonicity, J. of Ineq. in Pure and Appl. Math., volume 7, issue 2, article 40, 2006.

[24] D. Shelupsky, A Generalization of the Trigonometric Functions, Am. Math. Monthly, Vol. 66, No. 10(1959),879–884.

[25] S. Takeuchi, Generalized Jacobian elliptic functions and their application to bifurcation problems associated with p-Laplacian, J. Math. Anal. Appl. 385(2012), 24–35.

[26] M.-K. Wang, Y.-M. Chu, Y.-P. Jiang, Inequalities for generalized trigonometric and hyperbolic sine functions, ArXiv:1212.4681. Available online at http://arxiv.org/abs/1212.4681

[27] L. Yin and L.-G. Huang, Some inequalities for the generalized sine and the generalized hyperbolic sine, J. of Clas. Anal., Volume 3, Number 1 (2013), 85–90.