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Variational minimizing parabolic orbits for 2-fixed center problems

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ABSTRACT

Using variational minimizing methods, we prove the existence of an odd symmetric parabolic orbit for 2-fixed center problems with weak force type homogeneous potentials.

1. Introduction and main results

The circular restricted 3-body problem has attracted many researchers, e.g., Sitninkov [13], Moser [11], Mathlouthi [9], Souissi [14], and Zhang [16]. In this problem, it appears that two bodies with equal mass ($m_1 = m_2 = 1/2$) move in a circular orbit in a plane where their center of mass is at the origin. The motion of a third massless body is then considered under the attraction of the first two bodies. However, the circular motion of the first two bodies is not influenced by the third massless body. In particular, the massless body can move in a straight line perpendicular to the circular orbit plane and through the center of mass of the first two bodies.

Let $z(t)$ be the coordinate of the third body. Then, $z(t)$ satisfies

$$\ddot{z}(t) + \alpha \frac{z(t)}{(|z(t)|^2 + |r|^2)^{\alpha/2+1}} = 0. \tag{1}$$

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Zhang [16] used the variational minimizing method to prove the existence of an odd parabolic or hyperbolic orbit for Equation (1) with $0 < \alpha < 2$.

In this study, we consider the 2-fixed center problem, which is a classical problem with a long history [3–5,1,7,8,15]. For two masses, $1-\mu$ and μ fixed at $q_1 = (-\mu, 0)$ and $q_2 = (1-\mu, 0)$, respectively, the problem involves studying the motion $q(t) = (x(t), y(t))$ of a third body with mass $m_3 > 0$. In the present study, we consider that the motion of the third body is attracted by 2-fixed center masses with general homogeneous potentials, which satisfies the following equation:

$$\ddot{q}(t) + \frac{\partial V(q)}{\partial q} = 0, \tag{2}$$

$$V(q) = -\frac{1-\mu}{|q-q_1|^\alpha} - \frac{\mu}{|q-q_2|^\alpha}. \tag{3}$$

Definition 1.1. We refer to the solution $\tilde{q}_\alpha(t)$ of (2)–(3) as a parabolic type solution if

$$\max_{t\in R} |\tilde{q}_\alpha(t)| = +\infty,$$

$$\min_{t\in R} |\dot{\tilde{q}}_\alpha(t)| = 0$$

and the energy along the solution is zero:

$$\frac{1}{2}|\dot{\tilde{q}}_\alpha|^2 - \frac{1-\mu}{|\tilde{q}_\alpha - q_1|^\alpha} - \frac{\mu}{|\tilde{q}_\alpha - q_2|^\alpha} = h = 0. \tag{4}$$

For $\mu = 1/2$, we consider the existence of the motion $q(t) = (x(t), y(t))$ of the third body, which satisfies the odd symmetry $(x(-t), y(-t)) = (-x(t), -y(t))$. We use the variational minimizing method to prove the following.

Theorem 1.1. *For (2)–(3) with $\mu = \frac{1}{2}$ and $0 < \alpha < 2$, an odd symmetrical parabolic-type solution exists.*

Remark. Note that [16] studied the existence of the parabolic or hyperbolic solution for restricted 3-body problems, but the author only proved the energy $h \geq 0$ along the unbounded solution, whereas we provide a more detailed analysis to determine the existence of the parabolic solution for 2-fixed center problems, and thus we prove the energy $h = 0$ along the unbounded solution. We also note that the potentials for these two problems are similar but not the same.

2. Truncation functional and its minimizing critical points

In order to determine the parabolic-type orbit of (2)–(3), we first restrict $t \in [-n, n]$ and find solutions of (2)–(3), and we then let $n \to +\infty$ to obtain the parabolic-type orbit. By noting the symmetry of the equation, we can find the odd solutions of the following ODE:

$$\ddot{q}(t) = \frac{\partial U(q)}{\partial q}, \tag{5}$$

$$U(q) = \frac{1/2}{|q-q_1|^\alpha} + \frac{1/2}{|q-q_2|^\alpha}. \tag{6}$$

We define the functional:

$$f(q) = \int_{-n}^{n} (\frac{1}{2}|\dot{q}(t)|^2 + \frac{1/2}{|q-q_1|^\alpha} + \frac{1/2}{|q-q_2|^\alpha})dt, \tag{7}$$

where

$$q \in H_n = \{q(t) = (x(t), y(t)) : x, y \in W^{1,2}[-n, n]; q(-t) = -q(t), q(t) \neq q^i, t \in [-n, n]\}. \tag{8}$$

Since $\forall q \in H_n$, $q(0) = 0$, then according to the well-known Hardy–Littlewood–Polya inequality [1, inequality 256], for $\forall q \in H_n$, we have an equivalent norm:

$$\|q\|_n = (\int_{-n}^{n} |\dot{q}(t)|^2 dt)^{1/2}.$$

Remark. We do not assume that $q(-n) = q(n) = 0$ because we want to obtain the parabolic-type orbit that satisfies

$$\max_{t \in R} |q(t)| = +\infty,$$
$$\min_{t \in R} |\dot{q}(t)| = 0.$$

In addition, we do not assume the periodic property for $q(t)$ because we need a non-periodic odd test function in order to obtain Lemma 2.6.

Lemma 2.1. *$f(q)$ is weakly lower semi-continuous (w.l.s.c.) on the closure $\bar{H}_n$ of H_n.*

Proof. (i). It is well known that the norm and its square are w.l.s.c.

(ii). $\forall \{q_m\} \subset H_n$, if $q_m \rightharpoonup q \in H_n$ weakly, then by the compact embedding theorem, we have the uniform convergence:

$$\max_{-n \le t \le n} |q_m(t) - q(t)| \to 0,$$

as $m \to +\infty$, and thus

$$\int_{-n}^{n} \frac{1}{|q_m - q_i|^\alpha} dt \to \int_{-n}^{n} \frac{1}{|q - q_i|^\alpha} dt, i = 1, 2,$$

as $m \to +\infty$. Hence,

$$\varliminf_{m \to \infty} f(q_m) \ge f(q).$$

(iii). $\forall \{q_m\} \subset H_n$, if $q_m \rightharpoonup q \in \partial H_n$ weakly, let

$$S = \{t_0 \in [-n, n] : q(t_0) = q_1(t_0), \text{ or } q_2(t_0)\}.$$

(1) The Lebesgue measure of S is zero, so $U(q_m(t)) \to U(q(t))$ almost everywhere, and thus by Fatou's Lemma, $\int_{-n}^{n} U(q)dt$ is *w.l.s.c.*, and it is well known that the norm and its square are w.l.s.c. so $f(q)$ is also *w.l.s.c.*

(2) The Lebesgue measure of S: $L(S) > 0$, then

$$\int_{-n}^{n} U(q)dt = +\infty, f(q) = +\infty,$$

and thus by the compact embedding theorem, we have the uniform convergence on S:

$$\max_{-n\le t\le n} |q_m(t)-q(t)| \to 0,$$

as $m\to+\infty$, and so on S, we have the uniform convergence:

$$\int_{-n}^{n} \frac{1}{|q_m-q_i|^\alpha} dt \to +\infty, i=1 \text{ or } 2,$$

as $m\to+\infty$. Hence,

$$\int_{-n}^{n} U(q_m(t))dt \to +\infty$$

$$\underline{\lim}_{m\to\infty} f(q_m) = +\infty \ge f(q). \qquad \square$$

Lemma 2.2. *f is coercive on $\bar{H}_n$.*

Proof. From the definition of $f(q)$ and the Hardy–Littlewood inequality, it is clear that the coercivity holds $(f(q)\to+\infty, \|q\|\to+\infty)$. $\square$

Lemma 2.3 *(Tonelli). (See [2,10].) Let X be a reflexive Banach space, $M\subset X$ is a weakly closed subset, and $f: M\to R\cup\{+\infty\}$, but $f(x)$ is not always $+\infty$, and suppose that f is w.l.s.c. and coercive $(f(x)\to +\infty, \|x\|\to+\infty)$, then f attains its infimum on M.*

Lemma 2.4 *(Palais' Symmetry Principle). (See [12].) Let G be a finite or compact group, σ is an orthogonal representation of G, and let H be a real Hilbert space, $f: H\to R$, which satisfies*

$$f(\sigma\cdot x) = f(x), \forall\sigma\in G, \forall x\in H.$$

Let

$$F \triangleq \{x\in H \mid \sigma\cdot x = x, \forall \sigma\in G\}.$$

Then, the critical point of f in F is also a critical point of f in H.

Lemma 2.5. *$f(q)$ attains its infimum on $\bar{H}_n$, and thus the minimizer $\tilde{q}_{\alpha,n}(t)$ is an odd solution.*

Proof. We have proved Lemmas 2.1–2.2, so in order to apply Lemma 2.3, we need to apply Lemma 2.4 to prove that the critical point of $f(q)$ on H_n is the odd solution of (4)–(5). We define groups $G_1 = \{I_{2\times2}, -I_{2\times2}\}$, $G_2=\{1,-1\}$ and their actions:

$$\sigma_1\cdot q(t) = I_{2\times2}q(t),$$
$$\sigma_2\cdot q(t) = -I_{2\times2}q(t);$$
$$\tilde{\sigma}_1\cdot q(t) = q(t),$$
$$\tilde{\sigma}_2\cdot q(t) = q(-t).$$

Then, it is easy to prove that $f(q)$ is invariant under $\sigma_1, \sigma_2, \tilde{\sigma}_1, \tilde{\sigma}_2, \sigma_i \cdot \tilde{\sigma}_j, \tilde{\sigma}_j \cdot \sigma_i$ and the fixed point set of the group actions for $G_1 \times G_2$ is simply H_n, so we can apply Palais' Symmetrical Principle. □

In order to obtain the parabolic type solution, we need to prove that

$$\tilde{q}_{\alpha,n}(t) \to \tilde{q}_\alpha(t)$$

when $n \to \infty$, and $\tilde{q}_\alpha(t)$ has the following properties:

$$\max_{t\in R} |\tilde{q}_\alpha(t)| = +\infty,$$
$$\min_{t\in R} |\dot{\tilde{q}}_\alpha(t)| = 0.$$

To achieve this, we require some further lemmas, as follows.

Lemma 2.6. *Constants $c > 0$ and $0 < \theta < 1$ that are independent of n exist such that the variational minimizing value a_n for $f(q)$ on $\bar{H}_n$ satisfies $a_n \le cn^\theta$.*

Proof. (i). If $\tilde{q}(t) = (\tilde{x}, \tilde{y}) \in H_n$ is located on the y-axis, then we choose a special odd function defined by

$$\tilde{x} = 0,\ \tilde{y} = t^\beta,\ t \in [-n, n],$$

where we select β to satisfy $0 < \beta < 1$, $\frac{1}{2} < \beta = \frac{l}{m} < \frac{1}{\alpha}$, l, m are odd numbers, and $(l, m) = 1$. Then,

$$\begin{aligned} f(\tilde{q}(t)) &= \frac{1}{2} 2 \int_0^n \beta^2 t^{2(\beta-1)} dt + \int_{-n}^{n} [\frac{1/2}{|t^{2\beta} + \frac{1}{4}|^{\alpha/2}} + \frac{1/2}{|t^{2\beta} + \frac{1}{4}|^{\alpha/2}}] dt \\ &\le \frac{\beta^2}{2\beta - 1} n^{2\beta-1} + \frac{2}{1 - \alpha\beta} n^{1-\alpha\beta}. \end{aligned}$$

Now, we define

$$\theta = \max(2\beta - 1, 1 - \alpha\beta), \tag{9}$$

$$c = \frac{\beta^2}{2\beta - 1} + \frac{2}{1 - \alpha\beta} > 0. \tag{10}$$

When $0 < \beta < 1$ and

$$\frac{1}{2} < \beta = \frac{l}{m} < \frac{1}{\alpha},$$

then

$$1 > 2\beta - 1 > 0,\ 1 > 1 - \alpha\beta > 0$$

and $0 < \theta < 1$. Hence, we have

$$f(\tilde{q}) \le cn^\theta.$$

(ii). If $\tilde{q}(t) = (\tilde{x},\ \tilde{y})$ is not on the y-axis, we choose a special odd function on t defined by

$$\tilde{x}(t) = t^\beta, \tilde{y}(t) = 0,\ t \in [-n, n],$$

where $0 < \beta < 1$,

$$\frac{1}{2} < \beta = \frac{l}{m} < \frac{1}{\alpha},$$

l, m are odd numbers, and $(l, m) = 1$. Then, we have

$$\begin{aligned} f(\tilde{q}(t)) &\leq \int_0^n \beta^2 t^{2(\beta-1)} dt + \int_0^n [\frac{1}{|t^\beta + \frac{1}{2}|^\alpha} + \frac{1}{|t^\beta - \frac{1}{2}|^\alpha}] dt \\ &\leq \frac{\beta^2}{2\beta - 1} n^{2\beta-1} + [\frac{1}{1-\alpha\beta} n^{1-\alpha\beta} + \int_0^n \frac{1}{|t^\beta - \frac{1}{2}|^\alpha} dt]. \end{aligned}$$

Now, we estimate

$$\int_0^n \frac{1}{|t^\beta - \frac{1}{2}|^\alpha} dt.$$

Let

$$t^\beta - \frac{1}{2} = \tau^\beta,$$

then $t > \tau$ and

$$dt = (\frac{\tau}{t})^{\beta-1} d\tau,$$

and by changing the variables, we have

$$\begin{aligned} \int_0^n \frac{1}{|t^\beta - \frac{1}{2}|^\alpha} dt &< \int_{(-\frac{1}{2})^{\frac{1}{\beta}}}^{(n^\beta - \frac{1}{2})^{\frac{1}{\beta}}} \tau^{-\alpha\beta} d\tau \\ &< \frac{1}{1-\alpha\beta} [n^{1-\alpha\beta} - (-\frac{1}{2})^{-\frac{1}{\beta}(1-\alpha\beta)}]. \end{aligned}$$

Define

$$\theta = \max\{2\beta - 1, 1 - \alpha\beta\},$$
$$c = \frac{\beta^2}{2\beta - 1} + \frac{3}{1 - \alpha\beta} > 0.$$

When $0 < \beta < 1$ and

$$\frac{1}{2} < \beta = \frac{l}{m} < \frac{1}{\alpha},$$

then

$$1 > 2\beta - 1 > 0, 1 > 1 - \alpha\beta > 0$$

and $0 < \theta < 1$. Hence, we also have

$$f(\tilde{q}) \leq cn^{\theta}. \qquad \square$$

Furthermore, for our minimizer, we have

Lemma 2.7. *Let $\tilde{q}_{\alpha,n}$ be critical points that correspond to the minimizing critical values $a_n = \min_{H_n} f(q)$, then $\|\tilde{q}_{\alpha,n}\|_\infty \to +\infty$ when $n \to +\infty$.*

Proof. By the definition of $f(\tilde{q}_{\alpha,n})$ and Lemma 2.6, we have

$$\begin{aligned} cn^{\theta} &\geq f(\tilde{q}_{\alpha,n}) \\ &\geq \int_0^n [\frac{1}{|(x+\frac{1}{2})^2 + y^2|^{\alpha/2}} + \frac{1}{|(x-\frac{1}{2})^2 + y^2|^{\alpha/2}}]dt. \end{aligned}$$

We note that

$$\begin{aligned} (x+\frac{1}{2})^2 + y^2 &\leq 2(x^2+y^2) + \frac{5}{4}, \\ (x-\frac{1}{2})^2 + y^2 &\leq (x^2+y^2) + \frac{1}{4}, \end{aligned}$$

and thus

$$\begin{aligned} cn^{\theta} &\geq \int_0^n \frac{dt}{(2\|\tilde{q}_{\alpha,n}\|_\infty^2 + \frac{5}{4})^{\alpha/2}} + \frac{dt}{(\|\tilde{q}_{\alpha,n}\|_\infty^2 + \frac{1}{4})^{\alpha/2}} \\ &\geq \frac{2n}{(2\|\tilde{q}_{\alpha,n}\|_\infty^2 + \frac{5}{4})^{\alpha/2}}. \end{aligned}$$

Hence,

$$\|\tilde{q}_{\alpha,n}\|_\infty^2 \to +\infty, \tag{11}$$

as $n \to +\infty$. $\square$

Lemma 2.8. *$\int_a^b |\dot{\tilde{q}}_{\alpha,n}|^2 dt$ is uniformly bounded on any compact set $[a,b] \subset R$.*

Proof. The system is autonomous, so for any given α, n along the solution $\tilde{q}_{\alpha,n}(t)$, the energy $h(t)$ is conservative, i.e., a constant $h = h(\alpha, n)$:

$$\frac{1}{2}|\dot{\tilde{q}}_{\alpha,n}|^2 - \frac{1/2}{|\tilde{q}_{\alpha,n} - q_1|^\alpha} - \frac{1/2}{|\tilde{q}_{\alpha,n} - q_2|^\alpha} = h. \tag{12}$$

By the energy relationship above and the definition of the functional f, we have

$$
\begin{aligned}
f(\tilde{q}_{\alpha,n}) &= \int_{-n}^{n} \left(\frac{1}{2}|\dot{\tilde{q}}_{\alpha,n}|^2 + \frac{1/2}{|\tilde{q}_{\alpha,n}-q_1|^\alpha} + \frac{1/2}{|\tilde{q}_{\alpha,n}-q_2|^\alpha}\right)dt \\
&= \int_{-n}^{n} \left(\frac{1}{2}|\dot{\tilde{q}}_{\alpha,n}|^2 - \frac{1/2}{|\tilde{q}_{\alpha,n}-q_1|^\alpha} - \frac{1/2}{|\tilde{q}_{\alpha,n}-q_2|^\alpha}\right)dt \\
&+ 2\int_{-n}^{n} \frac{1/2}{|\tilde{q}_{\alpha,n}-q_1|^\alpha} + \frac{1/2}{|\tilde{q}_{\alpha,n}-q_2|^\alpha} dt \\
&= 2nh + 2\int_{-n}^{n} \frac{1/2}{|\tilde{q}_{\alpha,n}-q_1|^\alpha} + \frac{1/2}{|\tilde{q}_{\alpha,n}-q_2|^\alpha} dt.
\end{aligned}
$$

By Lemma 2.6, we have

$$
cn^\theta \geq 2nh + 2\int_{-n}^{n} \left(\frac{1/2}{|\tilde{q}_{\alpha,n}-q_1|^\alpha} + \frac{1/2}{|\tilde{q}_{\alpha,n}-q_2|^\alpha}\right)dt,
$$

and

$$
h \leq \frac{c}{2}n^{\theta-1} - \frac{1}{n}\int_{-n}^{n} \left(\frac{1/2}{|\tilde{q}_{\alpha,n}-q_1|^\alpha} + \frac{1/2}{|\tilde{q}_{\alpha,n}-q_2|^\alpha}\right)dt \leq \frac{c}{2}n^{\theta-1}. \tag{13}
$$

(1) When n is sufficiently large, $|\tilde{q}_{\alpha,n}(t) - q_i|$ has a uniformly positive lower bound, i.e., $\min_{a \leq t \leq b} |\tilde{q}_{\alpha,n}(t) - q_i| \geq c > 0$, and thus we have

$$
\begin{aligned}
\int_a^b \frac{1}{2}|\dot{\tilde{q}}_{\alpha,n}|^2 &= h(b-a) + \int_a^b \left[\frac{1/2}{|\tilde{q}_{\alpha,n}-q_1|^\alpha} + \frac{1/2}{|\tilde{q}_{\alpha,n}-q_2|^\alpha}\right]dt \\
&\leq \frac{c}{2}(b-a) + c^{-\alpha}(b-a).
\end{aligned}
$$

(2) $i_0 = 1$ or 2 and a sequence $t_n \subset [a,b]$ exist such that $\tilde{q}_{\alpha,n}(t_n) \to q_{i_0}$, and since $0 < \alpha < 2$, then there is a weak force potential; thus, when n is large, we have

$$
\int_a^b \left[\frac{1/2}{|\tilde{q}_{\alpha,n}-q_1|^\alpha} + \frac{1/2}{|\tilde{q}_{\alpha,n}-q_2|^\alpha}\right]dt \leq M,
$$

$$
\int_a^b \frac{1}{2}|\dot{\tilde{q}}_{\alpha,n}|^2 dt \leq \frac{c}{2}(b-a) + M. \qquad \square
$$

3. Proof of Theorem 1.1

By $\tilde{q}_{\alpha,n}(0) = 0$, the Cauchy–Schwarz inequality, and Lemma 2.8, we have

$$
|\tilde{q}_{\alpha,n}(t)| = \left|\int_0^t \dot{\tilde{q}}_{\alpha,n}(s)ds\right| \leq (b-a)^{1/2}\left[\int_a^b |\dot{\tilde{q}}_{\alpha,n}|^2 ds\right]^{1/2} \leq M_1,
$$

and thus we have the following:

(i) $\{\tilde{q}_{\alpha,n}\}$ is uniformly bounded on any compact set of R.

By the Cauchy–Schwarz inequality and Lemma 2.8, we have

$$|\tilde{q}_{\alpha,n}(t_2) - \tilde{q}_{\alpha,n}(t_1)| = |\int_{t_1}^{t_2} \dot{\tilde{q}}_{\alpha,n}(s)ds| \le [\int_a^b |\dot{\tilde{q}}_{\alpha,n}|^2 ds]^{1/2}(t_2 - t_1)^{1/2} \le M_2(t_2 - t_1)^{1/2},$$

and thus we have the following:

(ii) $\{\tilde{q}_{\alpha,n}\}$ is uniformly equi-continuous on any $[a, b] \subset R$.

Now, we can apply the Ascoli–Arzelà Theorem and we know that $\{\tilde{q}_{\alpha,n}\}$ has a sub-sequence that converges uniformly to a limit $\tilde{q}_\alpha(t)$ on any compact set of R, and $\tilde{q}_\alpha(t)$ is a solution of (5)–(6). By the energy conservation law and Lemma 2.7 and (13), we have

$$h = \frac{1}{2}|\dot{\tilde{q}}_\alpha|^2 - \frac{1}{2}(\frac{1}{|\tilde{q}_\alpha - q_1|^\alpha} + \frac{1}{|\tilde{q}_\alpha - q_2|^\alpha}) = 0.$$

Then, by Corollary 2.3 from [6], we have

$$\frac{1}{2}|\dot{\tilde{q}}_\alpha|^2 = \frac{1/2}{|\tilde{q}_\alpha - q_1|^\alpha} + \frac{1/2}{|\tilde{q}_\alpha - q_2|^\alpha} \ge [2^{\frac{\alpha+2}{2}}][2|\tilde{q}_\alpha|^2 + \frac{1}{2}]^{-\alpha/2}. \tag{14}$$

Now, we claim:

(a)

$$\max_{t\in R} |\tilde{q}_\alpha(t)| = +\infty. \tag{15}$$

In fact, if $\exists\beta > 0$ such that

$$|\tilde{q}_\alpha| < \beta, \forall t \in R,$$

then by (14), there exists $\gamma > 0$ such that

$$|\dot{\tilde{q}}_\alpha| > \gamma, \forall t \in R.$$

Therefore, when n is large, we have

$$|\dot{\tilde{q}}_{\alpha,n}| > \gamma, \forall t \in R,$$

$$cn^\theta \ge \int_{-n}^{n} |\dot{\tilde{q}}_{\alpha,n}|^2 > 2n\gamma^2,$$

which is a contradiction.

Now, by (14) we have

(b)

$$\min_{t\in R} |\dot{\tilde{q}}_\alpha(t)| = 0. \tag{16}$$

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