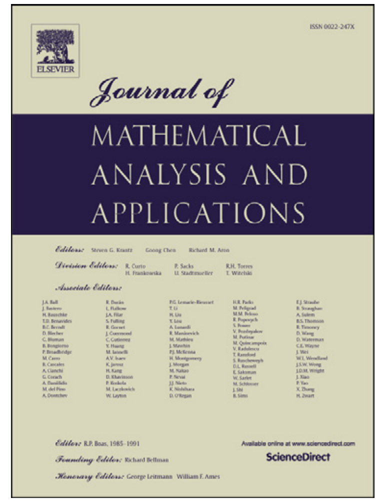

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Consequences of Universality Among Toeplitz Operators[1]

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Abstract

The *Invariant Subspace Problem* for Hilbert spaces is a long-standing question and the use of universal operators in the sense of Rota has been one tool for studying the problem. The best known universal operators have been adjoints of analytic Toeplitz operators or unitarily equivalent to them. We present many examples of Toeplitz operators whose adjoints are universal operators and exhibit some of their common properties. Some ways in which the invariant subspaces of these universal operators interact with operators in their commutants are given. Special attention is given to the closed subalgebra, not always the zero algebra, of compact operators in their commutants. Finally, three questions connecting shift invariant subspaces and invariant subspaces of analytic Toeplitz operators are raised. Positive answers for both of the first two imply the existence of non-trivial invariant subspaces for every bounded operator on separable Hilbert spaces of dimension two or more.

1. Introduction

The *Invariant Subspace Problem* is an important question in functional analysis but remains unsolved in the context of separable, infinite dimensional Hilbert spaces. Radjavi and Rosenthal's classic book [21] and the recent monograph by Chalendar and Partington [3] are excellent resources for both references and techniques developed in order to solve this and related problems.

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One approach to this problem has been the use of universal operators in the sense of Rota, a class of operators whose structure is rich enough to model every operator on a separable infinite dimensional Hilbert space. There are several well-known examples of universal operators in the literature, and most of these are adjoints of analytic Toeplitz operators or operators that have a reducing subspace on which the operator is universal and unitarily equivalent to the adjoint of an analytic Toeplitz.

In this work, we will examine the class of adjoints of analytic Toeplitz operators that are universal in the sense of Rota and greatly extend the class of known examples. We will then develop a strategy for attacking the *Invariant Subspace Problem* based on using operators that commute with the universal operators along with the structure of analytic Toeplitz operators. There are certain reductions in the problem that this makes possible and some of these results are stated as alternatives. Of course, the main difficulty with these results is that for most specific bounded linear operators, it is not clear which of the alternatives holds!

The invariant subspaces of the unilateral shift T_z acting on the classical Hardy space H^2, and therefore, of the backward shift are well known. These ideas associated with the universal operators that are adjoints of analytic Toeplitz operators lead us to consider the relationship between their invariant subspaces and the invariant subspaces of T_z^*, that is, proper closed invariant subspaces for the backward shift acting on H^2. If L and M are both closed subspaces of H^2, we say the subspace M has non-trivial intersection with L if $(0) \neq L \cap M \neq L$.

One of the most interesting consequences that follows from studying the class of adjoints of analytic Toeplitz operators that are universal in the sense of Rota is the following surprising result, a restatement of Corollary 27.

Theorem. *If every closed, infinite dimensional, invariant subspace for the adjoint of an analytic Toeplitz operator on the Hardy space H^2 that is universal in the sense of Rota has a non-trivial intersection with some invariant subspace of T_z^*, then every bounded linear operator on a separable Hilbert space of dimension two or more has a non-trivial closed invariant subspace.*

We observe that cyclic and non-cyclic vectors for the backward shift in the Hardy space were characterized by R. G. Douglas, H. S. Shapiro and A. L. Shields in a classic paper [11] from 1970 and other results for non-cyclic vectors may be found in work of Ahern and Clark [1] and Herrero and Sherman [15]. In addition to that, as Prof. N. Nikolski has kindly pointed out to us, there exist infinite dimensional closed subspaces consisting only of cyclic vectors for the backward shift. Of course, it is not known if any of these subspaces are invariant for the adjoint of an analytic Toeplitz operator that is universal.

We will close this work with the discussion of such issues along with some open questions that could lead to progress in the solution of the *Invariant Subspace Problem*, including two that get at the existence of *sharp vectors* (see Section 5) for the universal operators we study.

Notation and framework

We will reserve the word *subspace* for a linear manifold in a Hilbert space that is norm-closed. If T is a bounded operator on a Hilbert space, a subspace M is called *invariant for the operator* T if x in M implies Tx is also in M and we say M is a *proper invariant subspace* if M is not (0) and not the whole Hilbert space. A subspace M is said to be *hyperinvariant* for T if M is an invariant subspace for every bounded operator that commutes with T.

The major work in this paper is set in the Hardy Hilbert space, $H^2(\mathbb{D})$ (also written H^2). Of course, because any two separable, infinite dimensional complex Hilbert spaces are isometrically isomorphic, our choice of H^2 is not limiting in any way. There are two standard definitions for $H^2(\mathbb{D})$: the power series definition is

$$H^2(\mathbb{D}) = \{h \text{ analytic in } \mathbb{D} : h(z) = \sum_{n=0}^{\infty} a_n z^n \text{ where } \|h\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

If we regard the series for h as a Fourier series $\sum_{n=0}^{\infty} a_n e^{in\theta}$, then we see how $H^2(\mathbb{D})$ can be regarded as the closed subspace of $L^2(\partial\mathbb{D})$ consisting of those functions whose negative Fourier coefficients are all 0.

The second definition connects $H^2(\mathbb{D})$ with $L^2(\partial\mathbb{D})$ via integration:

$$H^2(\mathbb{D}) = \{h \text{ analytic in } \mathbb{D} : \sup_{0<r<1} \int_0^{2\pi} |h(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty\}$$

From this perspective, $\|h\|^2$ is the supremum in the above definition and the norm for H^2 is the same using either definition. $H^2(\mathbb{D})$ is a "functional Hilbert space of analytic functions" in the sense of [9]: in particular this means that for h in H^2, the map $h \mapsto h(\alpha)$ is a continuous linear functional for each α in the unit disk. It is well known that the kernel functions on H^2 are $K_\alpha(z) = (1 - \overline{\alpha}z)^{-1}$ for α in $\mathbb{D}$. This means for any h in H^2, $\langle h, K_\alpha \rangle = h(\alpha)$ where $\langle \cdot, \cdot \rangle$ is the inner product on H^2.

For f a bounded analytic function on the unit disk, that is, f is in $H^\infty(\mathbb{D})$, the *analytic Toeplitz operator*, T_f, on H^2 is the operator defined by $(T_f h)(z) = f(z)h(z)$ for h in H^2. For f in $H^\infty(\mathbb{D})$, the operator T_f is bounded on H^2 and it is easy to prove that $\|T_f\| = \|f\|_\infty$. More generally, if f is a function in $L^\infty(\partial\mathbb{D})$, the *Toeplitz operator* T_f is the operator on H^2 given by $T_f h = P_+ fh$ where P_+ is the orthogonal projection from $L^2(\partial\mathbb{D})$ onto H^2 and h is a function in H^2. Also when f is in $L^\infty(\partial\mathbb{D})$, the operator T_f is bounded on H^2 and $\|T_f\| = \|f\|_\infty$. In the case that f is in H^∞, the projection P_+ has no effect: for h in H^2 and f in H^∞, $P_+ fh = fh$. Douglas's book [10] can provide some background on properties of Toeplitz operators.

For J an analytic map of the unit disk into itself, the composition operator, C_J, on H^2 is the operator defined by $(C_J h)(z) = h(J(z))$. The boundedness of C_J for any analytic function J mapping the unit disk into itself is a consequence of the Littlewood Subordination Theorem [16] (or see [9, pp. 30 & 117]). For ψ in $H^\infty(\mathbb{D})$ and J an analytic map of the disk into itself, the weighted composition operator $W_{\psi,J} = T_\psi C_J$ is also a bounded operator on H^2. More information about composition operators can be found in the book of Cowen and MacCluer [9].

2. A Special Class of Operators

In 1960, Rota [22] introduced the idea of an operator whose lattice of invariant subspaces has a structure rich enough to model *every* Hilbert space operator and showed, perhaps surprisingly, that such operators exist.

Definition:[3, p. 213] Let $\mathcal{X}$ be a Banach space, let U be a bounded operator on $\mathcal{X}$, and let $\mathcal{B}(\mathcal{X})$ be the algebra of bounded operators on $\mathcal{X}$. We say U is *universal for* $\mathcal{X}$ if for each non-zero bounded operator A on $\mathcal{X}$, there is an invariant subspace M for U and a non-zero number λ such that λA is similar to $U|_M$, that is, there is a linear isomorphism X of $\mathcal{X}$ onto M such that $UX = \lambda XA$.

Now, A and λA have the same invariant subspaces and the similarity X takes invariant subspaces of λA to invariant subspaces of $U|_M$. Suppose U is a universal operator for a separable, infinite dimensional Hilbert space $\mathcal{H}$. Then every bounded operator on $\mathcal{H}$ has an invariant subspace if and only if every non-zero subspace M of $\mathcal{H}$ that is invariant for U has a non-zero, proper subspace M_0 such that M_0 is also invariant for U. In other words, understanding the invariant subspace problem on Hilbert spaces becomes a question of understanding the invariant subspaces of the single operator U.

In 1969 Caradus proved the following theorem that gives a prescription for finding universal operators on Hilbert spaces. The best known examples of universal operators, including the operator Rota used to introduce the concept, satisfy the hypotheses of Caradus' Theorem.

Theorem 1. (Caradus [2, p. 527] or see [3, p. 214]) *If $\mathcal{H}$ is a separable Hilbert space and U is a bounded operator on $\mathcal{H}$ such that:*

1. *The null space of U is infinite dimensional.*
2. *The range of U is $\mathcal{H}$.*

then U is universal for $\mathcal{H}$.

The best known example of a universal operator is the adjoint of a unilateral shift of infinite multiplicity: for example, suppose S is an analytic Toeplitz operator on the Hardy space H^2 whose symbol is an infinite Blaschke product or an inner function that has factor that is a non-trivial singular inner function. In this case, S is an isometric operator and S^* has infinite dimensional kernel and maps H^2 onto H^2, so S^* is a universal operator by the Caradus Theorem (Theorem 1). Defining $\mathcal{W}$ by $\mathcal{W} = H^2 \ominus SH^2$, the wandering subspace of S, using the Wold decomposition, $H^2 = \oplus_{k=0}^{\infty} S^k \mathcal{W}$, the operator S^* can be represented as an upper triangular block matrix that has the identity on the super-diagonal. As it was noted in [8], the only compact operator that commutes with the universal operator S^* is the zero operator.

Another widely known example of a universal operator was presented in the mid-1980's by Nordgren, Rosenthal and Wintrobe [19, 20] who proved that if φ is a hyperbolic automorphism of the unit disc with fixed points at ± 1 and μ is in the interior of the spectrum of the composition operator C_φ acting on the classical Hardy space H^2, then $C_\varphi - \mu I$ is a universal operator on H^2. The authors' paper [7] shows that for such a φ, the restriction of C_φ^* to its invariant subspace zH^2 is unitarily equivalent to an analytic Toeplitz operator whose symbol is the covering map of an annulus $\{\lambda : \rho^{-1} < |\lambda| < \rho\}$ for some $\rho > 1$. Nordgren, Rosenthal, and Wintrobe used the Caradus theorem to prove universality and it is also easily applied directly to the Toeplitz operator to get the result. The first author's papers [4, Thm. 10] and [5] show that this example cannot commute with a non-trivial compact, either.

More recently, in [8] and described in more detail below (Theorem 8), the authors gave an example of a universal operator, also the adjoint of an analytic Toeplitz operator, that commutes with an injective compact operator with dense range. The compact operator in that example is the adjoint of a weighted composition operator.

This paper is based on the fact that the use of universality in studying the invariant subspace problem in Hilbert spaces has mostly been based on universal operators that are adjoints of analytic Toeplitz operators acting on the Hardy Hilbert space H^2 or are operators that are universal and unitarily equivalent to such adjoints of Toeplitz operators.

Before beginning our main discussion, we give a version of the Caradus theorem with a broader conclusion.

Theorem 2. *If $\mathcal{H}$ is a separable Hilbert space and U is a bounded operator on $\mathcal{H}$ such that:*

1. *The null space of U is infinite dimensional.*
2. *The range of U is $\mathcal{H}$.*

then there is $\epsilon > 0$ so that for $|\mu| < \epsilon$, the operator $U + \mu I$ is universal. Moreover, for any complex number μ and any bounded operator A_0 on $\mathcal{H}$, there is an invariant subspace M for U and constants $\alpha \neq 0$ and β such that $A = (U + \mu I)|_M$ is similar to $\alpha A_0 + \beta I$. In particular, the lattices of invariant subspaces for A_0 and A are isomorphic as lattices.

Proof: We outline a version of Caradus' proof of Theorem 1. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$ and let $\{e'_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for $N = \{v : Uv = 0\}$, the kernel of U. Defining W on $\mathcal{H}$ by $We_n = e'_n$ for each positive integer n and extending linearly means W is isometric on $\mathcal{H}$ and $UW = 0$. Since the range of U is $\mathcal{H}$, the restriction of U to $N^\perp$ is an invertible operator and we let V be its inverse. That is, $V : \mathcal{H} \mapsto N^\perp$ and $UV = I$. Then to prove his theorem, Caradus shows that if T is an operator on $\mathcal{H}$ for which $\|T\| < \|V\|^{-1}$, then $X = \sum_{k=0}^{\infty} V^k W T^k$ is a bounded operator with closed range, the range of X is invariant for U, and $UX = XT$, that is, X gives a similarity between T and the restriction of U to its invariant subspace, range(X).

Using the notation above, to prove the first assertion, let $\epsilon = \|V\|^{-1}$. Given a complex number μ with $|\mu| < \epsilon$ and a bounded operator A_0 on $\mathcal{H}$, we want to find a constant $\lambda \neq 0$ so that there is an invariant subspace M for $U + \mu I$ so that $A = (U + \mu I)|_M$ is similar to λA_0, which will mean $U + \mu I$ is universal. By our choice of μ, we see $\epsilon - |\mu|$ is positive and we choose $\lambda \neq 0$ so that $\|\lambda A_0\| + |\mu| < \epsilon$. This means that $T = \lambda A_0 - \mu I$ satisfies $\|T\| \leq \|\lambda A_0\| + |\mu| < \|V\|^{-1}$. Thus, the construction of Caradus shows $X = \sum_{k=0}^{\infty} V^k W T^k$ is a bounded operator on $\mathcal{H}$ with closed range, the range of X is invariant for U, and X gives a similarity between the restriction of U to its invariant subspace $M = \text{range}(X)$ and the operator $T = \lambda A_0 - \mu I$. Now M is also invariant for $U + \mu I$ and

$$(U + \mu I)|_M \approx T + \mu I = \lambda A_0$$

Since A_0 was a arbitrary operator on $\mathcal{H}$ and we have shown that there is an invariant subspace M so that $(U + \mu I)|_M$ is similar to a multiple of A_0, we see that $U + \mu I$ is a universal operator, as we were to prove.

The final conclusion is an easy consequence of the universality of U. The invariant subspaces of U and $U + \mu I$ are the same, so if we are given a bounded operator A_0, then there is an invariant subspace M for U and a constant $\lambda \neq 0$ so that $U|_M$ is similar to λA_0. This means $(U + \mu I)|_M$ is similar to $\lambda A_0 + \mu I$, so the conclusion follows with $\alpha = \lambda$ and $\beta = \mu$. ■

Our goal is to study a large collection of universal operators that are adjoints of analytic Toeplitz operators and also the operators and especially the compact ones that commute with them in order to see how these operators interact with the invariant subspaces of the universal operators. Theorem 2 will be used, when convenient, to replace, by a more convenient translate of this operator, the adjoint of an analytic Toeplitz operator that satisfies the hypotheses of the Caradus Theorem and be assured, from the standpoint of the invariant subspace problem, to be able to obtain the same conclusions.

3. Toeplitz Operators as Universal Operators

For f in H^∞, the analytic Toeplitz operator T_f is invertible if and only if $1/f$ is also in H^∞ and in this case, $T_f^{-1} = T_{1/f}$. It is possible, of course, for the restriction of f to the unit circle to be invertible in $L^\infty(\partial\mathbb{D})$ without $1/f$ being in H^∞, for example, this is the case for any non-constant inner function. The following result describes this situation more fully and will be important in our work.

Lemma 3. *If f is a function in $H^\infty(\mathbb{D})$ and there is $\ell > 0$ so that $|f(e^{i\theta})| \geq \ell$ almost everywhere on the unit circle, then $1/f$ is in $L^\infty(\partial\mathbb{D})$ and the (non-analytic) Toeplitz operator $T_{1/f}$ is a left inverse for the analytic Toeplitz operator T_f.*

Proof: It is well known (for example, see [10]), that if g is in $L^\infty(\partial\mathbb{D})$ and f is in H^∞, then $T_g T_f = T_{gf}$. Since $1/f$ is in $L^\infty(\partial\mathbb{D})$ and f is H^∞, we have $T_{1/f} T_f = T_1 = I$ and T_f is left-invertible. ■

As a straightforward corollary, we have

Corollary 4. *If f satisfies the hypotheses of Lemma 3, the Toeplitz operator T_f^* has a right inverse and T_f^* maps $H^2(\mathbb{D})$ onto itself.*

Proof: We have $T_{1/f}T_f = I$, so $T_f^* T_{1/f}^* = (T_{1/f}T_f)^* = I$. This equality implies T_f^* maps $H^2(\mathbb{D})$ onto itself. ∎

Before going further, let us denote by $\mathcal{U}_0$ the set of adjoints of analytic Toeplitz operators that Lemma 3 implies are left invertible, that is

$$\mathcal{U}_0 = \{T_f^* : f \in H^\infty \text{ and on } \partial\mathbb{D},\ 1/f \in L^\infty(\partial\mathbb{D})\}.$$

Let us fix the following definition.

Definition. Let $\mathcal{U} = \{T_f^* \in \mathcal{U}_0 : \text{ kernel}(T_f^*) \text{ is infinite dimensional}\}$.

The following theorem, which begins our investigation, says that $\mathcal{U}$ is the set of adjoints of analytic Toeplitz operators that can be proved universal by using Lemma 3 and the Caradus theorem. It is not clear whether this is the same as the set of all adjoints of analytic Toeplitz operators that are universal, although in light of Theorem 2, it seems plausible that it is not true. Because there are operators on H^2 that have infinite dimensional kernel and a universal operator must be able to model these, it is easy to see that every universal operator must have infinite dimensional kernel, but it is not true that every universal operator must be left invertible: if Y is an operator that is not left invertible and U is a universal operator, then $Y \oplus U$ is universal without being left invertible.

Theorem 5. *If f is in H^∞ and T_f^* is in $\mathcal{U}$, then the Toeplitz operator T_f^* is universal for H^2.*

Proof: The fact that T_f^* is in $\mathcal{U}$ means its kernel is infinite dimensional and $\mathcal{U} \subset \mathcal{U}_0$ means Corollary 4 applies so T_f^* maps H^2 onto itself. Thus, the conclusion follows immediately from the Caradus theorem (Theorem 1). ∎

It now follows that the product of an operator in $\mathcal{U}_0$ and an operator in $\mathcal{U}$ is also universal. In particular, the set $\mathcal{U}$ is a multiplicative subset of $\mathcal{B}(H^2)$ so the set of universal adjoints of analytic Toeplitz operators is a very large set indeed! The set $\mathcal{U}$ is not an algebra and we should not expect that it would be because sums of operators with large kernels need not have kernels or even eigenvalues. We notice that no multiple of the identity is in $\mathcal{U}$, because the only multiple of the identity with infinite dimensional kernel is 0 and the zero operator is not left invertible. In particular, T_f^* in $\mathcal{U}$ implies f is non-constant.

Corollary 6. *If f and g are in H^∞ with T_f^* in $\mathcal{U}$ and T_g^* in $\mathcal{U}_0$, then $T_f^* T_g^* = T_{fg}^*$ is also in $\mathcal{U}$ and is a universal operator for H^2.*

Proof: It is easy to see that any two analytic Toeplitz operators, T_f and T_g commute and their product is the analytic Toeplitz operator T_{fg}. This means T^*_{fg} is also in $\mathcal{U}_0$ and since the kernel of $T^*_{fg} = T^*_g T^*_f$ contains the kernel of T^*_f, and T^*_{fg} is also in $\mathcal{U}$ and is universal for H^2. ∎

Theorem 2 motivates the following definition.

Definition. Let $\mathcal{V} = \{T^*_f - \mu I : \mu \in \mathbb{C} \text{ and } T^*_f \in \mathcal{U}\}$.

Thus, we have $\mathcal{U} \subset \mathcal{V}$ and because $\mathcal{V}$ includes invertible operators that cannot have infinite dimensional kernel and are not universal, the sets $\mathcal{U}$ and $\mathcal{V}$ are not the same. But, as Theorem 2 states, for application to the study of invariant subspaces of operators on H^2, the set $\mathcal{V}$ is a large set of adjoints of analytic Toeplitz operators each of which are still special enough to be able to model, by restrictions to invariant subspaces, all possible invariant subspace lattices for operators on H^2.

In [8], it was shown that there is a universal operator T that commutes with an injective compact operator W with dense range. At this point, we want to recall the specific descriptions of the operators $T = T^*_\phi$ and $W = W^*_{\psi,J}$. The following easy lemma, taken from [8], tells when such operators commute.

Lemma 7. *For ϕ and ψ in H^∞ and J an analytic map of the unit disk into itself, the analytic Toeplitz operator T_ϕ commutes with the composition operator C_J or the weighted composition operator $W_{\psi,J}$ if and only if $\phi \circ J = \phi$.*

To define the operators, let $\Omega = \{z \in \mathbb{C} : \operatorname{Im} z^2 < -1 \text{ and } \operatorname{Re} z < 0\}$, which is the region in the second quadrant of the complex plane above the branch of the hyperbola $2xy = -1$. Let σ be the Riemann map of $\mathbb{D}$ onto Ω defined by

$$\sigma(z) = \frac{-1+i}{\sqrt{z+1}} \tag{1}$$

where we choose the branch of $\sqrt{\cdot}$ on the halfplane $\{z : \operatorname{Re} z > 0\}$ satisfying $\sqrt{1} = 1$. Notice that $\sigma(1) = (-1+i)/\sqrt{2}$, $\sigma(0) = -1+i$, and $\sigma(-1) = \infty$. We define ϕ on the unit disk by

$$\phi(z) = e^{\sigma(z)} - e^{\sigma(0)} = e^{\sigma(z)} - e^{-1+i} \tag{2}$$

Since ϕ is an infinite-to-one map of the disk to a punctured disk that includes 0 in the range and the image of the unit circle, excluding -1, is completely contained in $\phi(\mathbb{D})$ and avoiding 0, it is not difficult to prove that T^*_ϕ is in $\mathcal{U}$.

Now, we let J be the analytic map of the unit disk into itself given by

$$J(z) = \sigma^{-1}(\sigma(z) + 2\pi i) \tag{3}$$

where σ is the map of the disk into the plane given by Equation (1). From this definition, an easy calculation shows that $\phi \circ J = \phi$ and letting $\psi(z) = (z+1)/2$ allows us to conclude that T_ϕ and $W_{\psi,J} = T_\psi C_J$ commute.

Theorem 8 (see [8]). *For ϕ as in Equation (2) and J as in Equation (3), the operator T_ϕ^* is a universal operator for H^2 and the operator $W_{\psi,J}^*$ is an injective compact operator that has dense range and commutes with T_ϕ^*.*

That $W_{\psi,J}^*$ is compact follows from general facts ($J(-1) = -1$ and $\psi(-1) = 0$) about weighted composition operators [13, p. 2896] or from [6], but the proof that $W_{\psi,J}^*$ is injective and has dense range is a much larger part of [8].

All of the universal operators noted above, backward shifts S^* of infinite multiplicity, the Toeplitz operator whose adjoint is unitarily equivalent to a translate of the restriction to zH^2 composition operator C_φ, and the operator T_ϕ^* just introduced, are in the class $\mathcal{U}$ defined above and, indeed, inspired the definitions above.

3.1. Commutants of Universal Toeplitz Operators Containing Compacts

Recall that if $\mathcal{S}$ is any set of bounded operators on H^2, the commutant of $\mathcal{S}$ is the set

$$\mathcal{S}' = \{G \in \mathcal{B}(H^2) : GF = FG \text{ for all } F \in \mathcal{S}\}.$$

If F is a bounded operator on H^2, the commutant of F is $\{F\}'$, the special case in which $\mathcal{S}$ is the set containing the single operator F. It is easy to see that for any set of operators $\mathcal{S}$, the commutant $\mathcal{S}'$ is a closed subalgebra of $\mathcal{B}(H^2)$. For f in H^∞, clearly $\{T_f^*\}'$ includes T_g^* for all g in H^∞.

We have seen that some universal operators commute with a compact operator and others do not. We want to consider this distinction and try to exploit the existence of a compact operator that commutes with a particular universal operator when there are some.

Definition. If f is a bounded analytic function on the unit disk such that T_f^* is in $\mathcal{V}$, let $\mathcal{C}_f$ be the set of compact operators in $\{T_f^*\}'$. That is,

$$\mathcal{C}_f = \{G \in \mathcal{B}(H^2) : G \text{ is compact, and } T_f^* G = G T_f^*\}$$

Notice that if T_f^* is in $\mathcal{V}$, then there is a complex number μ so that $g = f + \mu$ and T_g^* is in $\mathcal{U}$. For such f and g, $\{T_f^*\}' = \{T_g^*\}'$ and $\mathcal{C}_f = \mathcal{C}_g$, so the distinction between f in $\mathcal{U}$ or $\mathcal{V}$ is not important in this context.

In the examples noted above, if ζ is a inner function that is not a finite Blaschke product, so that T_ζ^* is a universal operator unitarily equivalent to the backward shift S^* of infinite multiplicity, or if η is an analytic map on the disk so that T_η^* is unitarily equivalent to the compression to zH^2 of the translated composition operator $C_\varphi - \mu I$, then $\mathcal{C}_\zeta = \mathcal{C}_\eta = \{0\}$, but for the map ϕ associated with the universal operator T_ϕ^*, then $W_{\psi,J}^*$ is in $\mathcal{C}_\phi$.

The following result gives further properties of $\mathcal{C}_f$ for T_f^* in $\mathcal{V}$.

Theorem 9. *Let f be a bounded analytic function for which T_f^* is in $\mathcal{V}$. The set $\mathcal{C}_f$ is a closed subalgebra of $\{T_f^*\}'$ that is an ideal in $\{T_f^*\}'$. In particular, if G is a compact operator in $\mathcal{C}_f$ and g and h are bounded analytic functions on the disk, then $T_g^* G$, $G T_h^*$, and $T_g^* G T_h^*$ are all in $\mathcal{C}_f$. Moreover, every operator G in $\mathcal{C}_f$ is quasi-nilpotent.*

Proof: Let f be a bounded analytic function for which T_f^* is in $\mathcal{V}$. Clearly, all of the conclusions of the theorem are correct if $\mathcal{C}_f = \{0\}$, so we assume that is not the case.

If G_n is a sequence in $\mathcal{C}_f$ such that $\lim_{n\to\infty} G_n = G$, then since each G_n is compact, G is compact and since each G_n is in the commutant of T_f^*, G is also. This means that G is in $\mathcal{C}_f$ also and $\mathcal{C}_f$ is closed.

Let g and h be in H^∞. It follows that T_g^* and T_h^* are in $\{T_f^*\}'$. Since G compact implies T_g^*G, GT_h^*, and $T_g^*GT_h^*$ are all compact, if G is in $\mathcal{C}_f$, then each of T_g^*G, GT_h^*, and $T_g^*GT_h^*$ are in $\{T_f^*\}'$ and we see that all are in $\mathcal{C}_f$ as well.

Every operator in $\mathcal{C}_f$ is a compact operator in $\{T_f^*\}'$, so if G is in $\mathcal{C}_f$, then G^* is a compact operator commuting with the analytic Toeplitz operator, T_f. Lemma A of [4, p. 26] says every compact operator commuting with an analytic Toeplitz operator that is not a multiple of the identity is quasi-nilpotent. This means G^* is quasi-nilpotent and therefore G is also. ∎

Theorem 10. *If f is a non-constant bounded analytic function for which T_f^* commutes with a non-zero compact operator, there is a backward shift invariant subspace, $L = (\eta H^2)^\perp$ for some inner function η, that is invariant for every operator in $\{T_f^*\}'$.*

Proof: Lomonosov's well-known theorem [17, 3] states that if an operator, not a multiple of the identity, commutes with a non-zero compact operator, then the operator has a hyperinvariant subspace. Our hypotheses are that T_f^* satisfies Lomonosov's hypotheses. Since T_z^* commutes with T_f^*, any subspace satisfying the conclusion of Lomonosov's theorem must be invariant for T_z^*, which is our conclusion. ∎

4. The Subspace Perspective

In many cases, the perspective on using universal operators to study the Invariant Subspace Problem has been to focus on the operators being modeled by a universal operator, U. In fact, however, as was clear in the papers [8, 12] for example, we can instead take the perspective of the invariant subspaces. The earlier perspective is that an operator A_0 (and the complex number λ) determine an invariant subspace M, and we study invariant subspaces for $\lambda A_0 \approx U|_M$. In fact, of course, there is not a one-to-one correspondence between operators A_0 (or operators and λ's) and invariant subspaces M because we are only dealing with a similarity of λA_0. The different perspective we take in this section is that the universal operator has certain invariant subspaces, M, and we study the invariant subspaces M_0 for U that are (properly) contained in M: these are exactly the invariant subspaces of the operator $A = U|_M$. On the other hand, Theorem 2 suggests we do not need to confine ourselves to universal operators, but can expand our horizons to translates of certain universal operators.

Let us formalize the notation for the perspective we will take. Let f be a bounded analytic function on the disk such that T_f^* is in the class $\mathcal{V}$. Suppose M is an infinite dimensional invariant subspace of T_f^*. The operator $T_f^*\big|_M$ is determined but from the perspective of the class $\mathcal{V}$, this operator is a stand-in for all operators that are similar to a non-zero multiple of a translate of this operator. We want to study the invariant subspaces of T_f^* that are contained in M.

It will be helpful to write H^2 as a direct sum of M and its orthogonal complement. The following result clarifies this situation somewhat.

Proposition 11. *If f is a non-constant bounded analytic function and M is a proper invariant subspace for T_f^*, then $M^\perp$ is infinite dimensional.*

Proof: We have assumed that $M \neq H^2$, so $M^\perp \neq (0)$. Since M is invariant for T_f^*, the subspace $M^\perp$ is invariant for $(T_f^*)^* = T_f$. Now T_f is an analytic Toeplitz operator with f non-constant, so T_f has no eigenvalues. This means the restriction of T_f to its invariant subspace $M^\perp$ also has no eigenvalues. But every operator on a finite dimensional space has eigenvalues, so $M^\perp$ must be infinite dimensional. ■

The following theorem provides the notation that we will often use to describe our results. It permits us to easily describe operators with invariant subspaces, compact operators, and relationships between commuting operators. In particular, operators commuting with T_f^* help us study its invariant subspaces. Note that Corollary 16 strengthens the conclusion of this result concerning invertibility.

Theorem 12. *Let T be a universal operator on H^2 or let $T = T_f^*$ for a bounded analytic function f for which T_f^* is in $\mathcal{V}$ and let W be a bounded operator on H^2.*

If M is an infinite dimensional invariant subspace for T, then there are block representations of T and W on $H^2 = M \oplus M^\perp$, say,

$$T \sim \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{and} \quad W \sim \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \tag{4}$$

If A and C are invertible on M and $M^\perp$, respectively, then T is invertible on H^2.

Moreover, W is compact if and only if P, Q, R, and S are compact and $TW = WT$ if and only if

$$\begin{aligned} AP + BR &= PA \quad \text{(5a)} & AQ + BS &= PB + QC \quad \text{(5b)} \\ CR &= RA \quad \text{(5c)} & CS &= RB + SC \quad \text{(5d)} \end{aligned}$$

Proof: If A and C are invertible, then it is easy to see that

$$\begin{pmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{pmatrix}$$

is the inverse of T.

The equivalence of the compactness of W to the compactness of P, Q, R, and S is a consequence of the finite block representation. The operator T commutes with the operator

W if and only if $T+\mu I$ commutes with W, and the Equations (5a), (5b), (5c), and (5d) are the expressions in the block multiplication form of the equality $TW = WT$. ■

If T is a universal operator satisfying the hypotheses of Theorem 1, Caradus' sufficient conditions for universality, more can be said.

Theorem 13. *Let $\mathcal{H}$ be a separable, infinite dimensional Hilbert space and suppose T is a bounded operator on $\mathcal{H}$ that satisfies the hypothesis of the Caradus Theorem. If M is an invariant subspace for T for which $M^\perp$ is infinite dimensional and T has the block representation in Equation (4) with respect to the representation $\mathcal{H} = M \oplus M^\perp$, then either the dimension of ker(A) is infinity or C also satisfies the hypotheses of Caradus Theorem and is universal.*

Proof: Suppose the dimension of $\ker(A)$ is finite. We want to prove that $\ker(C)$ is infinite dimensional and that the range of C is all of $M^\perp$.

Since T has infinite dimensional kernel, we may choose linearly independent non-zero vectors v_j for $j = 1, 2, \cdots$ for which $Tv_j = 0$. For each positive integer j, let x_j and y_j be the (unique) vectors in M and $M^\perp$ respectively such that $v_j = x_j + y_j$. Now $\text{span}\{y_j\}_{j=1}^\infty$ cannot be finite dimensional. Indeed, if it were, there is some integer n so that $\text{span}\{y_j\}_{j=1}^n$ is the same as $\text{span}\{y_j\}_{j=1}^\infty$. So for $j > n$ and $k = 1, \cdots, n$, we let $a_{j,k}$ be complex numbers so that $y_j = \sum_{k=1}^n a_{j,k} y_k$ and we let $\widetilde{v}_j = v_j - \sum_{k=1}^n a_{j,k} v_k$. Now, because the v_j are linearly independent, the set $\{\widetilde{v}_j\}_{j=n+1}^\infty$ is also linearly independent. Moreover,

$$T(\widetilde{v}_j) = Tv_j - \sum_{k=1}^n a_{j,k} T v_k = 0 \tag{6}$$

and (7)

$$\widetilde{v}_j = v_j - \sum_{k=1}^n a_{j,k} v_k = x_j - \sum_{k=1}^n a_{j,k} x_k + y_j - \sum_{k=1}^n a_{j,k} y_k \tag{8}$$

$$= x_j - \sum_{k=1}^n a_{j,k} x_k \tag{9}$$

That is, the set $\{\widetilde{v}_j\}_{j=n+1}^\infty$ is an infinite linearly independent set of vectors in the kernel of T and we see by Equation (9) they are, in fact, all in M. But on M, the operator T is A, so A has infinite dimensional kernel. This contradiction shows that $\text{span}\{y_j\}$ is infinite dimensional.

The choice of the v_j and the representation of T given in Equation (4) shows

$$0 = Tv_j = (Ax_j + By_j) + Cy_j$$

where $(Ax_j + By_j)$ is in M and Cy_j is in $M^\perp$. Because $M \cap M^\perp = (0)$, this means both $(Ax_j + By_j) = 0$ and $Cy_j = 0$, so in particular, the kernel of C includes the infinite dimensional subspace $\text{span}\{y_j\}$.

By hypothesis, the range of T is all of $\mathcal{H}$. In particular, if y is in $M^\perp$, then there is v in $\mathcal{H}$ so that $Tv = y$. Now $v = x_0 + y_0$ where x_0 is in M and y_0 is in $M^\perp$. Thus,

$$y = Tv = T(x_0 + y_0) = (Ax_0 + By_0) + Cy_0$$

Since y is in $M^\perp$ and $(Ax_0 + By_0)$ is in M and Cy_0 is in $M^\perp$, we must have $(Ax_0 + By_0) = 0$ and $Cy_0 = y$. In other words, the range of C includes the vector y, and since y was an arbitrary vector in $M^\perp$, the range of C is $M^\perp$.

This means that if the kernel of A is finite dimensional, then C satisfies the hypotheses of Theorem 1, which is the desired conclusion. ■

In fact, for operators in the class $\mathcal{U}$, the hypothesis that $M^\perp$ is infinite dimensional is *always* true! The following corollary is straightforward.

Corollary 14. *Let $\mathcal{H}$ be a separable, infinite dimensional Hilbert space and suppose T is a bounded operator on $\mathcal{H}$ that satisfies the hypothesis of Caradus Theorem. Suppose M is an invariant subspace for T for which $M^\perp$ is infinite dimensional and suppose $T|_M \neq 0$. Then, either there is a proper subspace of M that is invariant for T or the matrix C in the block representation of T in Equation (4) also satisfies the hypotheses of the Caradus Theorem and is universal.*

Proof: Since $A = T|_M \neq 0$, kernel$(A) \neq M$. Theorem 13 says either kernel(A) is infinite dimensional or the matrix C satisfies the hypotheses of Theorem 1 and is universal. Clearly, kernel(A) is a closed subspace of M that is invariant for T and is proper if it is infinite dimensional. ■

As noted in Section 1, $H^2(\mathbb{D})$ is a Hilbert space of analytic functions, so the linear functionals for evaluation at α in $\mathbb{D}$ are continuous and are given by inner products with functions in H^2. Because the analytic Toeplitz operators and composition operators on H^2 are directly connected to the values at points in the disk for functions in H^2, we can use ideas about these kernel functions in our work and we develop the notation and some of the ideas in this section.

For α in the disk, write K_α for the kernel function that satisfies $\langle f, K_\alpha\rangle = f(\alpha)$ for f in H^2. We let x_α and y_α be the projections of K_α onto M and $M^\perp$ respectively. Because they are subspaces of a Hilbert space of analytic functions, M and $M^\perp$ are also Hilbert spaces of analytic functions on the disk and we can identify their kernel functions. If f is in $H^2(\mathbb{D})$, we can write $f = f_0 + f_1$ where f_0 is in M and f_1 is in $M^\perp$ and, using $K_\alpha = x_\alpha + y_\alpha$, we have

$$f_0(\alpha) + f_1(\alpha) = f(\alpha) = \langle f, K_\alpha\rangle = \langle f_0 + f_1, x_\alpha + y_\alpha\rangle = \langle f_0, x_\alpha\rangle + \langle f_1, y_\alpha\rangle$$

because f_0 and x_α are orthogonal to f_1 and y_α.

As is well known, and clear from this calculation applied to functions f in M or $M^\perp$, the x_α and y_α are the kernel functions for evaluation at α for functions in M and $M^\perp$, respectively. Because the constant function 1 is in H^2, there is no α in the disk for which

$K_\alpha = 0$ and it follows from $K_\alpha = x_\alpha + y_\alpha$ that there is no α in the disk where *both* $x_\alpha = 0$ and $y_\alpha = 0$.

In addition, the assumption that M is neither $\{0\}$ nor $H^2(\mathbb{D})$ implies that for all but countably many α in $\mathbb{D}$, both x_α and y_α are non-zero because, for example, $x_\alpha = 0$ means every function in M vanishes at α and M non-zero means M contains functions that are not identically zero. Since functions in M are also in H^2, the usual properties of analytic functions show that the set of α in the disk for which f in H^2 satisfies $f(\alpha) = 0$ is countable and has no accumulation points in the open disk or else $f \equiv 0$, so $x_\alpha = 0$ for at most countably many points of the disk. The same argument applied to $M^\perp$ shows that $y_\alpha = 0$ only countably many times also.

In Theorem 3 of his paper [14], Domingo Herrero established some relationships between the spectrum of a bounded operator, T, on a Banach space, the spectrum of its restriction to a closed invariant subspace, and the spectrum of the operator obtained by composing T with the canonical projection of the Banach space onto the quotient of the space by the invariant subspace. Translating Herrero's theorem into the setting of this paper leads to the following theorem.

Theorem 15 (Herrero [14]). *Let T be a bounded operator on H^2. If M is a proper invariant subspace for T and A and C are as in Equation (4), then the spectra of A, C, and T satisfy*

$$\sigma(T) \cup \sigma(A) = \sigma(T) \cup \sigma(C) = \sigma(A) \cup \sigma(C)$$

From this general result, since our T has a special form, we get an interesting extension of Theorem 12.

Definition: If Δ is a compact subset of the plane, let Fill(Δ) denote the complement of the component of $\widehat{\mathbb{C}} \setminus \Delta$ that contains infinity.

Corollary 16. *Suppose f is a bounded analytic function for which T_f^* is in $\mathcal{V}$.*

If M is a proper invariant subspace for T_f^* and A and C are as in Equation (4), then the spectra of C and T_f^* satisfy $\sigma(T_f^*) \subset \sigma(C) \subset Fill(\sigma(T_f^*))$. Moreover, $\sigma(C) = \sigma(T_f^*) \cup \sigma(A)$.

Proof: For f in H^∞, it is well-known that $\sigma(T_f)$ is the closure of $f(\mathbb{D})$ and that the kernel functions are eigenvectors of T_f^* whose eigenvalues are the conjugates of the numbers in $f(\mathbb{D})$, the image of f.

Now the eigenvector K_α for T_f^* can be written $K_\alpha = (x_\alpha, y_\alpha)$. Then Equation (4) gives

$$\overline{f(\alpha)} \begin{pmatrix} x_\alpha \\ y_\alpha \end{pmatrix} = \overline{f(\alpha)} K_\alpha = T_f^* K_\alpha = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{pmatrix} x_\alpha \\ y_\alpha \end{pmatrix} = \begin{pmatrix} Ax_\alpha + By_\alpha \\ Cy_\alpha \end{pmatrix}$$

That is, $Cy_\alpha = \overline{f(\alpha)} y_\alpha$, which means either $\overline{f(\alpha)}$ is an eigenvalue of C or $y_\alpha = 0$. The set of eigenvalues of T_f^* is a non-empty open set and the set of α in the disk for which $y_\alpha = 0$ is countable and this set has no limit points in the disk. This means that the closure of the set of eigenvalues of C associated with the eigenvectors y_α is the same as the closure of the set of eigenvalues of T_f^* coming from the K_α, which is $\sigma(T_f^*)$. Since every eigenvalue of C is in the spectrum and $\sigma(C)$ is closed, we see $\sigma(T_f^*) \subset \sigma(C)$. Now, Herrero's theorem says $\sigma(C) = \sigma(T_f^*) \cup \sigma(C) = \sigma(T_f^*) \cup \sigma(A)$.

To see the other inclusion, we observe that the operator-valued function on the complement of $\mathrm{Fill}(\sigma(T_f^*))$ given by $\mu \mapsto (\mu I - T_f^*)^{-1}$ is analytic in μ. If we write the block representation

$$(\mu I - T_f^*)^{-1} = \begin{pmatrix} D(\mu) & E(\mu) \\ F(\mu) & G(\mu) \end{pmatrix}$$

then each entry is analytic in μ. But for $|\mu| > \|T_f^*\|$, the function $(\mu I - T_f^*)^{-1}$ is given as a convergent series in the powers of $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ which means $F(\mu) \equiv 0$ for $|\mu| > \|T_f^*\|$, but because F is an analytic function of μ, it must be that $F(\mu) \equiv 0$ on all of the complement of $\mathrm{Fill}(\sigma(T_f^*))$. This means $G(\mu) = (\mu I - C)^{-1}$ for all μ in the complement of $\mathrm{Fill}(\sigma(T_f^*))$, that is, $\sigma(C) \subset \mathrm{Fill}(\sigma(T_f^*))$. ■

Corollary 17. *If f, T_f^*, A, and C are as in Corollary 16 and $f(\mathbb{D})$ is simply connected, then $\sigma(C) = \sigma(T_f^*)$ and $\sigma(A) \subset \sigma(T_f^*)$.*

Proof: If $f(\mathbb{D})$ is simply connected, then $\sigma(T_f^*) = \mathrm{Fill}(\sigma(T_f^*))$, so $\sigma(T_f^*) = \sigma(C) = \sigma(T_f^*) \cup \sigma(A)$. ■

Of course, if Δ is a compact set in the plane, $\mathrm{Fill}(\Delta)$ is the set obtained by filling in the holes of Δ and Corollary 16 says that the spectrum of C is obtained by including some points from the holes of $\sigma(T_f^*)$ to that set. In fact, we can do better! Corollary 16 (along with part of the proof of Theorem 12) says that when $C - \mu I$ is invertible,

$$(T_f^* - \mu I)^{-1} = \begin{pmatrix} (A - \mu I)^{-1} & -(A - \mu I)^{-1} B (C - \mu I)^{-1} \\ 0 & (C - \mu I)^{-1} \end{pmatrix}$$

so both $\|(A - \mu I)^{-1}\|$ and $\|(C - \mu I)^{-1}\|$ are bounded by $\|(T_f^* - \mu I)^{-1}\|$.

Now, it is a standard result in operator theory that if X is a bounded operator and μ_n is sequence of numbers in $\rho(X) = \sigma(X)^c$ that converges to a point of $\sigma(X)$, then $\|(X - \mu_n I)^{-1}\| \to \infty$ and conversely, if $\mu_n \to \mu$ and $\|(X - \mu_n I)^{-1}\| \to \infty$, then μ is in $\sigma(X)$. It follows that if μ_n is sequence of numbers in $\rho(C)$ converging to a point, μ, of $\sigma(C)$, then $\lim_{n\to\infty} \|(T_f^* - \mu_n I)^{-1}\| = \infty$ and μ is in $\sigma(T_f^*)$. That is, we have the following result.

Corollary 18. *If f, T_f^*, and C are as in Corollary 16, then $\partial\sigma(C) \subset \partial\sigma(T_f^*)$.*

In other words, if the spectrum of C contains any point of a hole in $\sigma(T_f^*)$, then it must contain the entire hole. Moreover, in this case, because $\sigma(C) = \sigma(T_f^*) \cup \sigma(A)$, the hole must consist of points of the spectrum of A.

Corollary 19. *If f, T_f^*, A and C are as in Corollary 16, and the interior of $\sigma(A)$ is empty, then $\sigma(C) = \sigma(T_f^*)$ and $\sigma(A) \subset \sigma(T_f^*)$.*

Proof: The holes of $\sigma(T_f^*)$ are open sets, so if $\sigma(A)$ contains no open sets, then it cannot contain any holes of $\sigma(T_f^*)$, and it must lie completely in that set. The conclusion now follows from $\sigma(C) = \sigma(T_f^*) \cup \sigma(A)$. ∎

Of course, this Corollary applies to any translate of a quasi-nilpotent operator as well as other operators.

Corollary 20. *Suppose f is a bounded analytic function such that T_f^* is in $\mathcal{V}$ and $\sigma(T_f^*)$ has only finitely many holes. For any bounded operator A_0 on H^2, there is $\epsilon > 0$ so that for any λ with $|\lambda| < \epsilon$, there are invariant subspaces M for T_f^* and numbers β so that $\lambda A_0 + \beta I \approx A = T_f^*\big|_M$. Moreover, for any such M and A and C are as in Corollary 16, then $\sigma(C) = \sigma(T_f^*)$ and $\sigma(A) \subset \sigma(T_f^*)$.*

Proof: The proof of Theorem 2 shows that there is $\epsilon_1 > 0$ so that for $|\lambda| < \epsilon_1$, there are invariant subspaces M and numbers β so that $\lambda A_0 + \beta I \approx A = T_f^*\big|_M$. Now the holes in $\sigma(T_f^*)$ are open sets and because there are only finitely many, there is a number ϵ, with $0 < \|A_0\|\epsilon \leq \epsilon_1$ and each hole contains a closed disk of diameter $\|A_0\|\epsilon$.

Thus, if $|\lambda| < \epsilon$, there are invariant subspaces M and numbers β so that $\lambda A_0 + \beta I \approx A = T_f^*\big|_M$. Moreover, because the diameter of $\sigma(\lambda A_0)$ is too small to fill any hole in $\sigma(T_f^*)$, the spectrum of C must be $\sigma(T_f^*)$ and $\sigma(A) \subset \sigma(T_f^*)$. ∎

Corollary 16 and its corollaries refer to the spectra of T_f^*, A, and C but in so far as invariant subspaces can be connected to spectra, we hope that these relationships might lead to relationships among invariant subspaces as well.

It is well known and easy to prove that the kernel functions are linearly independent in H^2 and from this fact, we gain information about the independence of the y_α in $M^\perp$.

Theorem 21. *If T is in $\mathcal{U}$ and M is an infinite dimensional invariant subspace for T, then either $\{y_\alpha\}_{\alpha\in\mathbb{D}}$ is linearly independent in $M^\perp$ or there is a non-zero, finite dimensional subspace of M that is also invariant for T.*

Proof: Suppose $T = T_f^*$. Suppose $\{y_\alpha\}_{\alpha\in\mathbb{D}}$ is not linearly independent, that is, suppose for some $y_{\alpha_1}, \cdots, y_{\alpha_k}$, (with $k \geq 1$), we have $\sum_{j=1}^k a_j y_{\alpha_j} = 0$, but none of $a_1, a_2, \cdots, a_k$ are zero. Letting $L = \text{span}\{K_{\alpha_j}\}_{j=1}^k$, we see L is a k-dimensional subspace of H^2 and $v = \sum_{j=1}^k a_j K_{\alpha_j}$ is a non-zero vector in L because the $\{K_\alpha\}$ are linearly independent. This means

$$0 \neq v = \sum_{j=1}^k a_j K_{\alpha_j} = \sum_{j=1}^k a_j x_{\alpha_j} + \sum_{j=1}^k a_j y_{\alpha_j} = \sum_{j=1}^k a_j x_{\alpha_j} + 0 = \sum_{j=1}^k a_j x_{\alpha_j}$$

The vector on the right is clearly a vector in M, so $v = \sum_{j=1}^{k} a_j K_{\alpha_j}$ is a non-zero vector in M and, indeed, in $M \cap L$.

Now M is invariant for $T = T_f^*$ by hypothesis and L is invariant for T_f^* because each K_{α_j} is an eigenvector for T_f^*: $T_f^*(K_\alpha) = \overline{f(\alpha)} K_\alpha$.

In other words, we have proved that if $\{y_\alpha\}_{\alpha \in \mathbb{D}}$ is linearly dependent as above then the non-zero subspace $M \cap L$ is a finite-dimensional invariant subspace for T. This shows either $\{y_\alpha\}_{\alpha \in \mathbb{D}}$ is linearly independent or M contains a non-zero finite dimensional subspace that is invariant for T, as we wished to prove. ∎

Corollary 22. *Suppose T is in $\mathcal{U}$ and M is an infinite dimensional invariant subspace for T. If $y_\alpha = 0$ for any α in $\mathbb{D}$, then there is a non-zero, one-dimensional subspace of M that is also invariant for T.*

Proof: If the vector $y_\alpha = 0$, then the set $\{y_\alpha\}$ is a linearly dependent set and Theorem 21 gives the conclusion, after reviewing the proof to notice that $L = [x_\alpha] = [K_\alpha]$, the one-dimensional subspace spanned by x_α is invariant for T. ∎

5. Exploiting Commuting Operators

Our strategy, of course, is to obtain interesting consequences of the structure for operators described in Theorem 12 and we believe using operators that commute with operators in $\mathcal{V}$ may help. If f is a bounded analytic function with T_f^* in $\mathcal{V}$, we choose W to be an operator that commutes with T_f^* and depending on the application, we might choose W to have other properties, such as W being the adjoint of an analytic Toeplitz operator or a compact operator such as $W = W_{\psi,J}^*$, the injective, compact operator with dense range described in Theorem 8. Using the notation of Equation (4) in Theorem 12, the two most useful equalities coming from the commutativity $TW = WT$ are the equations relating the entries of the block matrices and their action on M:

$$AP + BR = PA \tag{10}$$

$$CR = RA \tag{11}$$

The operators in the blocks depend on the operators T and W, but also on M. We are given a choice of universal operator T and we seek to understand how T acts on each of its invariant subspaces. In particular, this means we must treat T and M as given, and we are free to choose W to exploit the features of T and M. In particular, in order to choose W, we need to understand how the properties of R depend on the nature of W.

Theorem 23. *Let T be a universal operator on H^2 that is in the class $\mathcal{U}$ and let M be an infinite dimensional, proper invariant subspace for T. Let W be an operator on H^2 that* *commutes with T for which $M \not\subset kernel(W)$. Using the notation from Equation (4), if $kernel(R) \neq (0)$ and $R \neq 0$, then $kernel(R)$ is a proper subspace of M that is invariant for T. Moreover, if $kernel(W) \cap M \neq (0)$ and $R = 0$, then $kernel(W) \cap M = kernel(P)$ is a proper subspace of M that is invariant for T.*

Proof: If $M \subset \text{kernel}(W)$, then for all $x \sim (x, 0)$ in M, using Equation (4), we have

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 = Wx = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} Px \\ Rx \end{pmatrix}$$

so both $Px = 0$ and $Rx = 0$ for all x in M and $P = 0$ and $R = 0$. Clearly the converse holds as well.

Thus, if $M \not\subset \text{kernel}(W)$, then either $R \neq 0$ or $P \neq 0$. Suppose $R \neq 0$ and $\text{kernel}(R) \neq (0)$. If x is in $\text{kernel}(R) \subset M$, then Equation (11) gives $R(Ax) = C(Rx) = C0 = 0$, and $Tx = Ax$ is also in $\text{kernel}(R)$. This shows that if $M \not\subset \text{kernel}(W)$, then either $R = 0$ or $\text{kernel}(R) = (0)$ or there is a proper invariant subspace of M that is invariant for T.

If $R = 0$, then $P \neq 0$, so $\text{kernel}(P) \neq M$. The calculation above shows

$$\text{kernel}(W) \cap M = \text{kernel}(P) \cap \text{kernel}(R) = \text{kernel}(P) \cap M = \text{kernel}(P)$$

Since $R = 0$, Equation (10) becomes $PA = AP$. This means that if x is in $\text{kernel}(P)$, then $P(Ax) = A(Px) = 0$ and $Tx = Ax$ is also in $\text{kernel}(P)$. ■

Observe that if T is a universal operator on H^2 that is in the class $\mathcal{U}$ then, in particular, T commutes with T_η^* for every inner function η. Since the kernels of the operators T_η^* are the vectors in H^2 that are not cyclic vectors for the backward shift, one might ask the question

> *Does every closed, infinite dimensional subspace of H^2 include a non-zero, non-cyclic vector for the backward shift?*

The cyclic and non-cyclic vectors for the backward shift acting on H^p, $1 < p < \infty$, were characterized by Douglas, Shapiro, and Shields [11] in terms of pseudocontinuation. Moreover, they showed that the set of non-cyclic vectors is a (not closed) linear manifold in H^p.

Prof. N. Nikolski pointed out to us that the answer to the question above is "No!" – it is possible to construct infinite dimensional, closed subspaces consisting only of cyclic vectors for the backward shift. We thank him for showing us the following example, referred to in his book [18, pp. 83], and included here for the sake of completeness.

Example. Let $S := \{n_j\}$ be a Hadamard lacunary sequence of positive integers, that is, $\inf_{j>k} n_j/n_k > 1$, and represent it as an infinite union of disjoint infinite subsequences, say $S = \bigcup_{k\geq 1} S_k$. For each $k \geq 1$, let $f_k \in H^2$ whose Fourier spectrum is exactly S_k, that is,

$$f_k(z) = \sum_{n=0}^{\infty} \hat{f}_k(n) z^n \in H^2$$

such that $\widehat{\sigma}(f_k) = \left\{ n \in \mathbb{Z}_+ : \ \hat{f}_k(n) \neq 0 \right\} = S_k$. Let us set

$$E = \overline{\text{span}\,\{f_k : k = 1, 2, ...\}}$$

Hence, since the f_k are pairwise orthogonal, if f in E is non-zero, it is not a polynomial, but it still has a lacunary Fourier spectrum. Therefore, it is backward cyclic.

Thus, we specialize our query to address the issue at hand:

Question 1: *Does every closed, infinite dimensional subspace of H^2 that is a proper invariant subspace for an operator in the class $\mathcal{U}$ include a non-zero vector that is not cyclic for the backward shift?*

Since non-cyclic vectors for the backward shift are related to inner functions as noted above, this question is easily seen to be equivalent to the question: Does every closed, infinite dimensional subspace of H^2 that is a proper invariant subspace for an operator in the class $\mathcal{U}$ have a non-zero intersection with $(\zeta H^2)^\perp = \text{kernel}\,(T_\zeta^*)$ for some non-constant inner function ζ?

We need some facts about the non-cyclic vectors for the backward shift, T_z^*. As we have noted, a vector in H^2 is not cyclic for the backward shift if and only if it is contained in the $(\zeta H^2)^\perp = \text{kernel}\,(T_\zeta^*)$ for some inner function ζ, that is, it is in an invariant subspace for the backward shift. It is easy to show that $\text{kernel}\,(T_{\zeta_2}^*) \subset \text{kernel}\,(T_{\zeta_1}^*)$ if and only if ζ_2 divides ζ_1. Herrero and Sherman [15, Thm. 3] show that a closed subspace consists only of non-cyclic vectors if and only if it is contained in one of these invariant subspaces for the backward shift. We give an easy sharper version of this result.

Proposition 24. *Suppose η is an inner function and $M \neq (0)$ is a closed subspace such that $M \subset kernel(T_\eta^*)$. If $\mathcal{J}$ is the set of inner functions ζ for which $M \subset kernel(T_\zeta^*)$, then there is an inner function ζ_0 such that*

$$kernel\,(T_{\zeta_0}^*) = \bigcap_{\zeta \in \mathcal{J}} kernel\,(T_\zeta^*) \tag{12}$$

The inner function ζ_0 is minimal in the sense that if $\zeta_1 \neq \zeta_0$ is any inner function that divides ζ_0, then $M \cap kernel(T_{\zeta_1}^*) \neq M$. Moreover, if ξ is any inner function, $M \cap kernel(T_\xi^*) = M \cap kernel(T_{\zeta_2}^*)$ where ζ_2 is an inner function that divides ζ_0.

Proof: Clearly the set on the right in Equation (12) is a closed subspace containing M because each set in the intersection is a closed subspace containing M. Moreover, because each of the subspaces in the intersection is an invariant subspace for T_z^*, the set on the right is also an invariant subspace for T_z^*. In particular, this implies that there is an inner function ζ_0 for which the equality in Equation (12) holds.

Now if $\zeta_1 \neq \zeta_0$ is any inner function that divides ζ_0, then $\text{kernel}\,(T_{\zeta_1}^*) \subset \text{kernel}\,(T_{\zeta_0}^*)$ but they are not equal, so ζ_1 is not in $\mathcal{J}$ because otherwise the intersection would be smaller. In particular, this means $\text{kernel}\,(T_{\zeta_1}^*) \not\supset M$ and $M \cap \text{kernel}\,(T_{\zeta_1}^*) \neq M$.

If ξ is any inner function, $M \cap \text{kernel}\,(T^*_\xi)$ is a closed subspace of M, so

$$\text{kernel}\,(T^*_{\zeta_0}) \cap \big(M \cap \text{kernel}\,(T^*_\xi)\big) = M \cap \text{kernel}\,(T^*_\xi)$$

because $\text{kernel}\,(T^*_{\zeta_0})$ contains M. On the other hand,

$$\text{kernel}\,(T^*_{\zeta_0}) \cap \big(M \cap \text{kernel}\,(T^*_\xi)\big) = M \cap \big(\text{kernel}\,(T^*_\xi) \cap \text{kernel}\,(T^*_{\zeta_0})\big)$$

Both $\text{kernel}\,(T^*_\xi)$ and $\text{kernel}\,(T^*_{\zeta_0})$ are invariant subspaces for the backward shift operator, so their intersection is as well, and it must be $\text{kernel}\,(T^*_{\zeta_2})$ for some inner function ζ_2. Indeed, because the intersection of the subspaces is a subspace of each of them, ζ_2 is the greatest common divisor of ζ_0 and ξ, so ζ_2 divides ζ_0 and, as desired, we see

$$M \cap \text{kernel}\,(T^*_\xi) = M \cap \text{kernel}\,(T^*_{\zeta_2})$$

■

Corollary 25. *If v is a non-zero vector that is not cyclic for T^*_z, then there is an inner function ζ_v such that $[v]$, the subspace spanned by v, satisfies $[v] \subset$ kernel$(T^*_{\zeta_v})$ and if $\zeta \neq \zeta_v$ is an inner function that divides ζ_v, then $[v] \cap$ kernel$(T^*_\zeta) = (0)$. That is, kernel$(T^*_{\zeta_v})$ is the smallest backward shift invariant subspace that includes v.*

Proof: The vector v is assumed to be a non-zero vector that is not cyclic for the backward shift, so v in in $\text{kernel}\,(T^*_\zeta)$ for some inner function ζ. Since $[v]$ is a subspace of $\text{kernel}\,(T^*_\zeta)$, it satifies the hypotheses of the Theorem above. We will write ζ_v for the inner function ζ_0 resulting from the application of the Theorem to the subspace $[v]$.

Now, if $\zeta \neq \zeta_v$ is an inner function that divides ζ_v, the theorem says $[v] \cap \text{kernel}\,(T^*_\zeta) \neq [v]$, but the only subspace of $[v]$ besides $[v]$ is (0), which is the conclusion of the Corollary. ■

Corollary 25 justifies the language "smallest invariant subspace" and allows us to make the following definition:

Definition: If M is a closed subspace of H^2 and v is a non-zero vector in M, we say *v is a sharp vector for M* if v is a not a cyclic vector for T^*_z and the smallest invariant subspace of T^*_z that contains v does not contain all of M.

In other words, v is a sharp vector for M if $M \cap \text{kernel}\,(T^*_{\zeta_v}) \neq M$.

Theorem 26. *Let T be a universal operator on H^2 that is in the class $\mathcal{U}$ and let M be an infinite dimensional, proper invariant subspace for T. If ζ is any inner function, then $M \cap$ kernel(T^*_ζ) is a subspace of M that is invariant for T. If there is a non-zero vector in $M \cap$ kernel(T^*_ζ) and M is not contained in kernel(T^*_ζ), then $M \cap$ kernel(T^*_ζ) is a proper subspace of M that is invariant for T.*

Proof: Let $T = T_f^*$ and suppose η is any inner function. Since f and η are both in H^∞, the adjoints of their Toeplitz operators, T_f^* and T_η^*, commute. We see that if v is a vector in the the kernel of T_η^*, then

$$T_\eta^*(T_f^* v) = T_f^*(T_\eta^* v) = T_f^* 0 = 0$$

so $T_f^* v$ is also in the kernel of T_η^* and kernel(T_η^*) is an invariant subspace for T_f^*. By hypothesis, M is an invariant subspace for T_f^*, so $M \cap$ kernel(T_η^*) is the intersection of two invariant subspaces for T_f^*, and is also invariant.

If v is a non-zero vector in $M \cap$ kernel(T_η^*), then this intersection is not (0). If $M \cap$ kernel(T_η^*) is also not equal to M, it is a proper subspace of M that is invariant for T_f^* as the Theorem asserts. ■

Corollary 27. *Let T be a universal operator on H^2 that is in the class $\mathcal{U}$ and let M be an infinite dimensional, invariant subspace for T. If v is a sharp vector for M, then there is a proper subspace of M that is invariant for T.*

Proof: Since v is a sharp vector for M, it is a non-zero vector in M that is not cyclic for the backward shift, T_z^*, that is in kernel$\,(T_{\zeta_v}^*)$ where ζ_v is the inner function of Corollary 25. In particular, $M \cap$ kernel$\,(T_{\zeta_v}^*) \neq (0)$ and, by hypothesis, $M \cap$ kernel$\,(T_{\zeta_v}^*) \neq M$, so it follows from Theorem 26 that $M \cap$ kernel$\,(T_{\zeta_v}^*)$ is a proper invariant subspace for T. ■

Corollary 28. *Let T be a universal operator on H^2 that is in the class $\mathcal{U}$ and let M be an infinite dimensional, proper invariant subspace for T. If M contains a non-zero vector that is not cyclic for the backward shift, T_z^*, then there is a non-zero subspace of M that is invariant for T.*

Proof: If v is a non-zero vector that is not cyclic for the backward shift, then ζ_v is an inner function so that $M \cap$ kernel$\,(T_{\zeta_v}^*)$ is a non-zero subspace of M that is invariant for T. ■

Note that in Corollary 28, the conclusion allows $M \cap$ kernel$\,(T_\zeta^*) = M$, which is not a proper invariant subspace of M.

Corollary 29. *Let T be a universal operator on H^2 that is in the class $\mathcal{U}$ and let M be an infinite dimensional, proper invariant subspace for T. If M contains a cyclic vector for the backward shift, T_z^*, and non-zero vector that is not cyclic for the backward shift, then there is a proper subspace of M that is invariant for T.*

Proof: Suppose u is a cyclic vector for the backward shift and suppose v is a non-zero vector that is not cyclic for the backward shift. By Corollary 28, $M \cap$ kernel$\,(T_{\zeta_v}^*)$ is a non-zero subspace of M that is invariant for T. On the other hand, since u is cyclic for the backward shift, u is a vector in M that is not in any backward shift invariant subspace, so $M \cap$ kernel$\,(T_{\zeta_v}^*)$ is also not all of M. Thus, this subspace is a proper subspace of M that is invariant for T. ■

Corollary 30. *Let T be a universal operator on H^2 that is in the class $\mathcal{U}$ and let M be an infinite dimensional, proper invariant subspace for T. If M is an invariant subspace for the backward shift, T_z^*, then there is a proper subspace of M that is invariant for T.*

Proof: Let $T = T_f^*$ and let $M = \text{kernel}(T_\eta^*)$ for some inner function, η. Since kernel (T_η^*) is an infinite dimensional space, there is an inner function ζ that divides η for which neither ζ nor η/ζ is constant. Since ζ is not constant, $\text{kernel}(T_\zeta^*) \neq (0)$ and since η/ζ is not constant, $\text{kernel}(T_\zeta^*) \neq M$. Thus, $\text{kernel}(T_\zeta^*)$ is a proper subspace of M that is invariant for T_f^*. ■

It seems possible that the hypotheses of some of the results above are more restrictive than necessary, although we have not been able to weaken them. If we are considering the case of an invariant subspace M for a universal operator T in the class $\mathcal{U}$ in which M contains a non-zero vector, v, that is not cyclic for the backward shift, we have no problem if kernel $(T_{\zeta_v}^*)$ does not contain M because in that case $M \cap \text{kernel}\,(T_{\zeta_v}^*)$ is a proper subspace of M that is invariant for T. On the other hand, if $M \subset \text{kernel}\,(T_{\zeta_v}^*)$, we have a problem because $M \cap \text{kernel}\,(T_{\zeta_v}^*) = M$, not a proper subspace of M. This leads to the following question about a possible reduction for this situation:

Question 2: *Suppose M is an infinite dimensional closed subspace that is invariant for T, a universal operator in the class $\mathcal{U}$, and suppose η is an inner function for which $M \subset \textit{kernel}\,(T_\eta^*)$. Is there always an inner function ζ so that $(0) \neq M \cap \textit{kernel}\,(T_\zeta^*) \neq M$?*

First, notice that we have restricted our question to the particular case in which we want to apply it, but it can be asked more generally for any infinite dimensional subspace M, without involving any universal operators. As a second observation, Proposition 24 allows us to restrict our attention to just the smallest backward shift invariant subspace containing M, if we wish, rather than looking at the invariant subspace given by the inner function η. The last conclusion of Proposition 24 allows us to only consider subspaces of the type $M \cap \text{kernel}\,(T_{\zeta_2}^*)$ where ζ_2 is an inner function that divides the inner function associated with the smallest backward shift invariant subspace containing M (which also divides η). Finally, we observe that the two questions raised here, Question 1 and Question 2, can be combined in the form:

> *If T is a universal operator in the class $\mathcal{U}$ and M is an infinite dimensional closed subspace that is invariant for T, does M always contain a vector v that is sharp for M?*

Notice that if M is contained in a backward shift invariant subspace, then every v in M is a non-cyclic vector for the backward shift and there is an inner function ζ_v such that $(\zeta_v H^2)^\perp$ is the smallest backward shift invariant subspace containing v. On the other hand, because M is contained in a backward shift invariant subspace, there is a smallest backward shift invariant subspace $(\zeta_M H^2)^\perp$ that contains M. If v is in M, the inner function ζ_v divides ζ_M and either $(\zeta_v H^2)^\perp = (\zeta_M H^2)^\perp$ or $M \cap (\zeta_v H^2)^\perp \neq M$. This suggests a general question and a question more pointed to our concerns:

Question 3: *Is there an an infinite dimensional closed subspace M of H^2 with $M \subset kernel(T_\eta^*)$ for some inner function η such that for each v in M we have $(\zeta_v H^2)^\perp = (\zeta_M H^2)^\perp$ where $(\zeta_v H^2)^\perp$ (respectively, $(\zeta_M H^2)^\perp$) is the smallest backward shift invariant subspace containing v (repsectively, M)?*

or, more pointedly:

Question 3a: *If T is a universal operator in the class $\mathcal{U}$, is there an infinite dimensional closed subspace M that is invariant for T with $M \subset kernel(T_\eta^*)$ for some inner function η such that for each v in M we have $(\zeta_v H^2)^\perp = (\zeta_M H^2)^\perp$?*

The next three propositions are observations that appear to be useful in considering properties of the subspaces of M. The first puts further limits on the kinds of operators that can be C in Equation (4).

Proposition 31. *For f a bounded analytic function for which T_f^* is in $\mathcal{V}$, using the notation from Equation (4), the operator C^* has no eigenvalues.*

Proof: Since M is an invariant subspace for T_f^*, the subspace $M^\perp$ is invariant for T_f and C^* is the restriction of T_f to $M^\perp$. Since T_f^* is in $\mathcal{V}$, the function f is non-constant. In particular, this means for every complex number μ, the operator $T_f - \mu I$ has trivial kernel and, as is well known, T_f has no eigenvalues. Because every eigenvector of C^* is also an eigenvector of T_f, we see C^* has no eigenvalues. ∎

The next two provide tools for using information about C to learn about the operator A of primary interest.

Proposition 32. *If N is a non-trivial closed, invariant subspace for C, we let $\widetilde{N} = \{x \in M : Rx \in N\}$, then the subspace $\widetilde{N}$ is a closed invariant subspace for A.*

Proof: Since R is a bounded operator, $\widetilde{N}$ is a closed subspace of M. Let x be in $\widetilde{N}$ so that Rx is in N. Because N is an invariant subspace for C, we see CRx is in N also. This means that $RAx = CRx$ is in N, so Ax is in $\widetilde{N}$. In other words, $\widetilde{N}$ is an invariant subspace for A.

∎

Proposition 33. *Since $RA = CR$, we have $R^*C^* = A^*R^*$ and if L is any invariant subspace for C^* then R^*L is an invariant linear manifold for A^*.*

Proof: Suppose v is a vector in R^*L, say $v = R^*w$ for w in L. Then $A^*v = A^*R^*w = R^*(C^*w)$ which is in R^*L because C^*w is in L. Thus, R^*L is an invariant linear manifold of A^*. ∎

Notice that Proposition 32 does not make any assertion about the size of $\widetilde{N}$. In particular, if $N \neq (0)$ but $\widetilde{N} = (0)$ or if $N \neq M^\perp$ but $\widetilde{N} = M$, this lemma is not helpful.

Similarly, Proposition 33 does not make any assertion about the size of R^*L, but if R^*L is dense or if $R^*L = (0)$, this lemma is not helpful. The subspace $R^*L = (0)$ if and only if $\mathrm{range}(R) \subset L^\perp$. Notice that if R^*L is not dense, then there is $v \neq 0$ in $(R^*L)^\perp$ which means Rv is in $L^\perp$. In this case, if $Rv = 0$, then Theorem 23 applies. Since L is invariant for C^*, we know $L^\perp$ is invariant for C. If $Rv \neq 0$, then $\widetilde{L^\perp}$ is a subspace of M that is invariant for A (and T) by Proposition 32 and it is non-trivial because $v \neq 0$ is in $\widetilde{L^\perp}$.

In this work, we have pointed out that there are many universal operators that are adjoints of analytic Toeplitz operators and exhibited some of their shared properties as well as pointing out consequences of differences among these operators. We believe that these results provide a wide variety of tools for the study of the Invariant Subspace Problem and that choices can be made between universal operators based on the types of questions under study.

While we have been unable to answer the questions raised above, or to exploit these results for using operators in the class $\mathcal{V}$ to show the existence of invariant subspaces for broader classes of operators, we believe these results can lead to more concrete answers than have been possible before.

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