J. Math. Anal. Appl. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

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Approximate analytical solutions of nonlinear differential equations using the Least Squares Homotopy Perturbation Method

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ARTICLE INFO

Article history:
Received 9 July 2016
Available online xxxx
Submitted by S.A. Fulling

Keywords:
Least Squares Homotopy Perturbation Method
Nonlinear differential equations
Approximate analytical solution
Homotopy Perturbation Method

ABSTRACT

In the present paper we introduce a new method to compute approximate analytical solutions for nonlinear differential equations, called the Least Squares Homotopy Perturbation Method. The method is based on the well-known Homotopy Perturbation Method and its main feature is an accelerated convergence compared to the regular Homotopy Perturbation Method. The comparison with previous results emphasizes the high accuracy of the method.

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0. Nomenclature

Symbol	Description
HAM	Homotopy Analysis Method
HPM	Homotopy Perturbation Method
DTM	Differential Transform Method
MDTM	Multi-Step Differential Transform Method
$\mathfrak{L}$	Linear operator of the HPM
$\mathfrak{N}$	Nonlinear operator of the HPM
For Application 2	
Re	Reynolds number
Ha	Hartman number
α	Angle of the inclined plane walls of the channel

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http://dx.doi.org/10.1016/j.jmaa.2016.11.031
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Please cite this article in press as: C. Bota, B. Căruntu, Approximate analytical solutions of nonlinear differential equations using the Least Squares Homotopy Perturbation Method, J. Math. Anal. Appl. (2017), http://dx.doi.org/10.1016/j.jmaa.2016.11.031

1. Introduction

Most of the phenomena encountered in nature and technology are modelled using nonlinear differential equations and, for this reason, it is very important to find solutions of these equations. In many cases, as the calculation of exact solutions is not possible, approximate solutions of the equations are computed.

Among the most well-known and most widely used methods to compute analytical approximate solutions for nonlinear differential equations we mention:

- The Homotopy Analysis Method (HAM), introduced by Liao [8] was one of the first analytic methods for nonlinear problems which avoids the need for a "small" parameter and thus is applicable to strong nonlinear problems. HAM has been successfully applied for many types of nonlinear problems from various fields of science and engineering such as, for example, [1,5,2].
- The Homotopy Perturbation Method (HPM), introduced by He [6,7] was also used to solve a wide range of strongly nonlinear problems (for example [12,9,11]).
- The Differential Transform Method (DTM), introduced by Zhou [13], is an iterative procedure for obtaining analytic Taylor series solutions of differential equations. DTM together with its modifications (such as the multi-step DTM) has been also successfully employed to solve many types of nonlinear problems [2,10,4].

In the following we will introduce a new approximation method named Least Squares Homotopy Perturbation Method (LSHPM). As the name suggests, the method is based on the Homotopy Perturbation Method and its main feature is an accelerated convergence compared to the regular Homotopy Perturbation Method.

The examples presented show that the approximate solutions obtained by using LSHPM require less iterations in comparison with other iterative methods for approximate solutions of differential equations. Moreover, for an equal number of iterations, the approximate solutions obtained by using LSHPM are more accurate than the ones obtained by other methods.

2. The Least Squares Homotopy Perturbation Method

We consider the following problem, consisting of a differential equation and some boundary conditions:

$$\mathfrak{L}(x(t)) + \mathfrak{N}(x(t)) - f(t) = 0, \quad B(x) = 0 \tag{1}$$

Here $\mathfrak{L}$ is a linear operator, $x(t)$ is the unknown function, $\mathfrak{N}$ is a nonlinear operator, $f(t)$ is a known, given function and B is a boundary operator.

If $\tilde{x}$ is an approximate solution of equation (1), we evaluate the error obtained by replacing the exact solution x with the approximate one $\tilde{x}$ as the remainder:

$$R(t, \tilde{x}) = \mathfrak{L}(\tilde{x}(t)) + \mathfrak{N}(\tilde{x}(t)) - f(t), \quad t \in \mathbb{R} \tag{2}$$

The first step in applying LSHPM is to attach to the problem (1) the family of equations (see [6,7]):

$$(1-p)[\mathfrak{L}(\Phi(t,p)) - f(t)] + p\,[\mathfrak{L}(\Phi(t,p)) + \mathfrak{N}(\Phi(t,p)) - f(t)] = 0 \tag{3}$$

where $p \in [0,1]$ is an embedding parameter, $\Phi(t,p)$ is an unknown function.

When $p = 0$, $\Phi(t,0) = x_0(t)$ and when $p = 1$, $\Phi(t,1) = x(t)$. Thus, as p increases from 0 to 1, the solution $\Phi(t,p)$ varies from $x_0(t)$ to the solution $x(t)$, where $x_0(t)$ is obtained from the equation:

$$\mathfrak{L}(x_0(t)) - f(t) = 0, \quad B(x_0) = 0 \tag{4}$$

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We consider the following expansion of $\Phi(t,p)$:

$$\Phi(t,p) = x_0(t) + \sum_{m\geq 1} x_m(t)\ p^m \tag{5}$$

Substituting the relation (5) in (3), collecting the same powers of p and equating each coefficient of the powers of p with zero we obtain:

$$\begin{gathered}\mathfrak{L}(x_m(t)) = -\mathfrak{N}_{m-1}(x_0(t), x_1(t), ..., x_{m-1}(t)) \\ m \geq 1, ...,\ B(x_m) = 0,\end{gathered} \tag{6}$$

where $\mathfrak{N}_i$, $i \geq 0$ are the coefficients of p^i in the nonlinear operator $\mathfrak{N}$:

$$\mathfrak{N}(x(t)) = \mathfrak{N}_0(x_0(t)) + p\mathfrak{N}_1(x_0(t), x_1(t)) + p^2\mathfrak{N}_2(x_0(t), x_1(t), x_2(t)) + ... \tag{7}$$

We remark that x_m, $m \geq 1$ are obtained from the linear equations (6), which are easily solved together with the boundary conditions.

We denote $f_m = x_0 + x_1 + ... + x_m$.

We consider the set S_m $(m = 0, 1, 2, ...)$ containing the functions $\varphi_{m0}, \varphi_{m1}, \varphi_{m2}, ..., \varphi_{mn_m}$, chosen as linearly independent functions in the vector space of the continuous functions on the real interval I such that $S_{m-1} \subseteq S_m$ and $x_0 + x_1 + ... + x_m$ is a real linear combination of these functions.

We remark that such a construction is always possible. For example we can choose $S_m = \{x_0, x_1, ..., x_m\}$, $m = 0, 1, 2, ...$ In this case $\varphi_{m0} = x_0, \varphi_{m1} = x_1, \varphi_{m2} = x_2, ..., \varphi_{mn_m} = x_m$.

Definition 1. We call an *HP-sequence* of the problem (1) a sequence of functions $\{s_m(t)\}_{m\in\mathbb{N}}$ of the form $s_m(t) = \sum_{k=0}^{n_m} \alpha_m^k \varphi_{mk}$, where $m \in \mathbb{N}$, $\alpha_m^k \in \mathbb{R}$.

A function of the sequence is called an *HP-function* of the problem (1).

We call the HP-sequence $\{s_m(t)\}_{m\in\mathbb{N}}$, *convergent to the solution of the problem (1)* if $\lim_{m\to\infty} R(t, s_m(t)) = 0$.

Definition 2. We call an ϵ-*approximate HP-solution* of the problem (1) on the real interval I an HP-function $\tilde{x}$ which satisfies the following condition:

$$|R(t, \tilde{x})| < \epsilon \tag{8}$$

together with the boundary conditions from (1).

Definition 3. We call a *weak* δ-*approximate HP-solution* of the problem (1) on the real interval I an HP-function $\tilde{x}$ satisfying the relation $\int_I R^2(t, \tilde{x})dt \leq \delta$, together with the boundary conditions from (1).

We will find a weak ϵ-approximate HP-solution of the type $\tilde{x} = \sum_{k=0}^{n_m} c_m^k \varphi_{mk}$ where $m \geq 0$ and the constants c_m^k are calculated using the following steps:

- We substitute the approximate solution $\tilde{x}$ in the equation (1) and obtain the expression:

$$\mathfrak{R}(t, c_m^k) = R(t, \tilde{x}) \tag{9}$$

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- We attach to the problem (1) the following real functional:

$$J(c_m^k) = \int_I \mathfrak{R}^2(t, c_m^k)dt \tag{10}$$

where, by imposing the boundary conditions, we can determine $l \in N$, $l \leq m$ such that $c_0^m, c_1^m, ..., c_l^m$ are computed as functions of $c_{l+1}^m, c_{l+2}^m, ..., c_n^m$.

- We compute the values of $\tilde{c}_{l+1}^m, \tilde{c}_{l+2}^m, ..., \tilde{c}_n^m$ as the values which give the minimum of the functional (10) and the values of $\tilde{c}_0^m, \tilde{c}_1^m, ..., \tilde{c}_l^m$ again as functions of $\tilde{c}_{l+1}^m, \tilde{c}_{l+2}^m, ..., \tilde{c}_n^m$ by using the boundary conditions.
- Using the constants $\tilde{c}_0^m, ..., \tilde{c}_n^m$ thus determined, we consider the HP-sequence:

$$s_m(t) = \sum_{k=0}^{n_m} \tilde{c}_m^k \varphi_{mk} \tag{11}$$

The following convergence theorem holds:

Theorem 1. *The HP-sequence $s_m(t)$ from (11) satisfies the property:*

$$\lim_{m\to\infty} \int_I R^2(t, s_m(t))dt = 0$$

Moreover, $\forall \epsilon > 0$, $\exists m_0 \in \mathbb{N}$ such that $\forall m \in \mathbb{N}$, $m > m_0$ it follows that $s_m(t)$ is a weak ϵ-approximate HP-solution of the problem (1).

Proof. Based on the way the HP-function $s_m(t)$ is computed, the following inequality holds:

$$0 \leq \int_I R^2(t, s_m(t))dt \leq \int_I R^2(t, f_m(t))dt, \quad \forall m \in \mathbb{N}.$$

It follows that:

$$0 \leq \lim_{m\to\infty} \int_I R^2(t, s_m(t))dt \leq \lim_{m\to\infty} \int_I R^2(t, f_m(t))dt = 0, \quad \forall m \in \mathbb{N}.$$

We obtain:

$$\lim_{m\to\infty} \int_I R^2(t, s_m(t))dt = 0.$$

From this limit we obtain that $\forall \epsilon > 0$, $\exists m_0 \in \mathbb{N}$ such that $\forall m \in \mathbb{N}$, $m > m_0$ it follows that $s_m(t)$ is a weak ϵ-approximate HP-solution of the problem (1). □

Remark 1. Any ϵ-approximate HP-solution of the problem (1) is also a weak approximate HP-solution, but the opposite is not always true. It follows that the set of weak approximate HP-solutions of the problem (1) also contains the approximate HP-solutions of the problem.

Taking into account the above remark, in order to find ϵ-approximate HP-solutions of the problem (1) by the LSHPM method we will first determine weak approximate HP-solutions, $\tilde{x}$. If $|R(t, \tilde{x})| < \epsilon$ then $\tilde{x}$ is also an ϵ-approximate HP-solution of the problem.

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3. Applications

3.1. Application 1 – fifth-order boundary value problem

Our first application is the following fifth-order boundary value test problem [12,3]:

$$\begin{cases} x^{(5)}(t) + x^{(4)}(t) + e^{-2t} \cdot x^2(t) - 2 \cdot e^t - 1 = 0 \\ x(0) = 1,\ x^{(1)}(0) = 1,\ x^{(2)}(0) = 1,\ x(1) = e,\ x^{(1)}(1) = e \end{cases} \tag{12}$$

The exact solution of this problem is $x_e(t) = e^t$. In [3] an approximate solution of the problem (12) is computed using the Sinc–Galerkin Method and in [12] an approximate solution of (12) is computed using the Homotopy Perturbation Method. The absolute errors, computed as the difference in absolute value between the exact solution and approximate solution, are of the order of 10^{-4} for both methods.

Using LSHPM, the following steps are performed:

- In the construction of the homotopy (2) we choose the following operators: $\mathfrak{L}(x(t)) = x^{(5)}(t)$, $\mathfrak{N}(x(t)) = x^{(4)}(t) + e^{-2t} \cdot x^2(t)$ and $f(t) = 2 \cdot e^t + 1$.
  Solving the corresponding equation (4) we obtain the approximation: $x_0 = \frac{1}{120} \cdot t^5 - \frac{331}{60} \cdot t^4 + 2 \cdot e \cdot t^4 + \frac{961}{120} \cdot t^3 - 3 \cdot e \cdot t^3 - \frac{1}{2} \cdot t^3 - t + 2 \cdot e^t - 1$. It follows that the set S_0 consists of the function $\{1, t, t^2, t^3, t^4, t^5, e^t\}$.
  As a consequence we will compute a 0th-order approximate solution of the type:

$$\tilde{x} = c_0 + c_1 \cdot t + c_2 \cdot t^2 + c_3 \cdot t^3 + c_4 \cdot t^4 + c_5 \cdot t^5 + c_6 \cdot e^t.$$

- Imposing the boundary conditions from (2) we obtain:

$$c_0 = 1 - c_6,\ c_1 = 1 - c_6,\ c_2 = \frac{1 - c_6}{2},\ c_3 = c_5 + 8 \cdot c_6 - 3 \cdot e \cdot c_6 + 3 \cdot e - 8,$$
$$c_4 = -2 \cdot c_5 - \frac{11 \cdot c_6}{2} + 2 \cdot e \cdot c_6 + \frac{11}{2} - 2 \cdot e.$$

- Replacing these expressions of c_0, c_1, c_2, c_3, c_4 in the expression of $\tilde{x}$ and then introducing $\tilde{x}$ in the remainder given by (2), (9) we obtain:

$$\begin{aligned} \mathfrak{R}(t, c_5, c_6) = e^{-2t} \cdot \Bigg(& t^4 \left(-2 \cdot c_5 - \frac{11 \cdot c_6}{2} + 2 \cdot e \cdot c_6 + \frac{11}{2} - 2 \cdot e \right) \\ & + t^3 \cdot (c_5 + 8 \cdot c_6 - 3 \cdot e \cdot c_6 + 3 \cdot e - 8) + c_5 \cdot t^5 \\ & + \frac{1}{2} \cdot (1 - c_6) \cdot t^2 + (1 - c_6) \cdot t + c_6 \cdot e^t - c_6 + 1 \Bigg)^2 + 24 \\ & \cdot \left(-2c_5 - \frac{11 \cdot c_6}{2} + 2 \cdot e \cdot c_6 + \frac{11}{2} - 2 \cdot e \right) + 120 \cdot c_5 t + 120 \cdot c_5 + 2 \cdot c_6 \cdot e^t - 2 \cdot e^t - 1. \end{aligned}$$

  Next we compute the functional J of (10) as $J(c_5, c_6) = \int_0^1 \mathfrak{R}^2(t, c_5, c_6) dt$. The expression of the functional is too long to be included here, but relatively easy to compute.
- We compute the minimum of the functional J. The only critical point, determined using Mathematica 9 as the real solution of the system of equations $\{\frac{\partial J}{\partial c_5} = 0, \frac{\partial J}{\partial c_6} = 0\}$, is $c_5 = 0$, $c_6 = 1$.
  It follows that our 0th-order approximation is actually the exact solution, $\tilde{x}_{LSHPM}(t) = e^t$!

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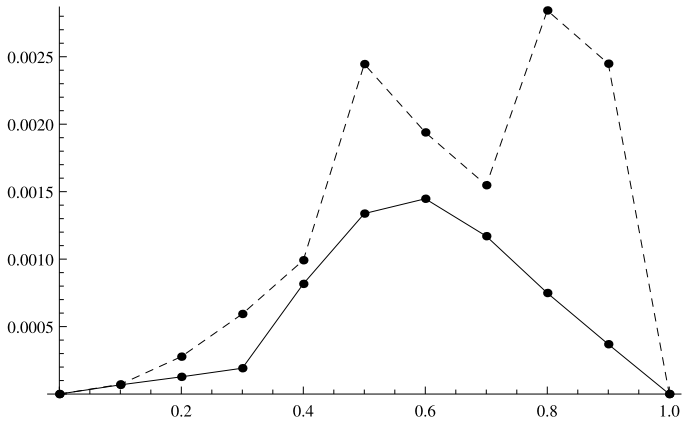


Fig. 1. Comparison of the absolute errors for problem (13). The HPM solution $\tilde{x}_0(t)$ is the dotted line and the LSHPM solution $\tilde{x}_1(t)$ is the solid line.

3.2. Application 2 – nonlinear MHD Jeffry–Hamel problem

Our second application is the nonlinear MHD Jeffry–Hamel problem [9]:

$$\begin{cases} x^{(3)}(t) + 2 \cdot \alpha \cdot Re \cdot x(t) \cdot x^{(1)}(t) + (4 - Ha) \cdot \alpha^2 \cdot x^{(1)}(t) \\ x(0) = 1,\ x^{(1)}(0) = 0,\ x(1) = 0 \end{cases} \tag{13}$$

In [9] approximate solutions of (13) are computed using the Homotopy Perturbation Method.

Choosing in Eq. (1) for the linear operator the expression $\mathfrak{L} = x^{(3)}(t) - x^{(1)}(t)$, for the nonlinear operator the expression $\mathfrak{N} = 2 \cdot \alpha \cdot Re \cdot x(t) \cdot x^{(1)}(t) + (4 - Ha) \cdot \alpha^2 \cdot x^{(1)}(t) + x^{(1)}(t)$ and taking $f(t) = 0$, we obtain by LSHPM for the case $\alpha = 7.5°$, $Re = 50$ and $Ha = 0$ (studied in [9]) the first-order approximate solution:

$$\tilde{x}_1(t) = -151.361 + 26.8645 \cdot e^{-2t} + 74.1501 \cdot e^{-t} + 49.6059 \cdot e^{t} + 1.74009 \cdot e^{2t} + 98.0949 \cdot t \cdot e^{-t} - 23.3018 \cdot t \cdot e^{t}$$

Fig. 1 presents the comparison of the absolute errors corresponding to the higher order approximation $\tilde{x}_0(t)$ obtained in [9] by HPM (dashed line) and to the first order approximation $\tilde{x}_1(t)$ obtained by using LSHPM (solid line). While the difference is not large, we remark that our approximation, even though it is only of the first order, is more accurate than the higher order approximation in [9].

Changing the expressions of $\mathfrak{L}$ and $\mathfrak{N}$ and thus changing the homotopy (3) we may obtain different approximate solutions. Choosing $\mathfrak{L} = x^{(3)}(t)$, $\mathfrak{N} = 2 \cdot \alpha \cdot Re \cdot x(t) \cdot x^{(1)}(t) + (4 - Ha) \cdot \alpha^2 \cdot x^{(1)}(t)$ and taking $f(t) = 0$, we obtain by LSHPM (also for the case $\alpha = 7.5°$, $Re = 50$ and $Ha = 0$) the following approximate solutions:

$$\begin{aligned} \tilde{x}_2(t) = {} & -0.0171482 \cdot t^{18} + 0.120978 \cdot t^{16} - 0.400527 \cdot t^{14} + 0.85334 \cdot t^{12} - 1.37331 \cdot t^{10} \\ & + 1.86358 \cdot t^{8} - 2.26938 \cdot t^{6} + 2.53329 \cdot t^{4} - 2.31083 \cdot t^{2} + 1 \\ \tilde{x}_3(t) = {} & -0.00526539 \cdot t^{22} + 0.0424623 \cdot t^{20} - 0.16167 \cdot t^{18} + 0.39558 \cdot t^{16} - 0.722964 \cdot t^{14} \\ & + 1.09679 \cdot t^{12} - 1.49206 \cdot t^{10} + 1.90009 \cdot t^{8} - 2.27603 \cdot t^{6} + 2.53392 \cdot t^{4} - 2.31085 \cdot t^{2} + 1 \end{aligned}$$

Fig. 2 presents the residuals (Eq. (3)) corresponding to the solutions $\tilde{x}_2(t)$ and $\tilde{x}_3(t)$. We omitted the residual corresponding to $\tilde{x}_1(t)$ since it was rather large. We also omitted the graphical representation of the comparison of the absolute errors corresponding to the solutions $\tilde{x}_2(t)$ and $\tilde{x}_3(t)$ due to the fact that the errors are too small compared to the error of the solution $\tilde{x}_0(t)$ from [9]. The comparison is instead made in the Table 1 below.

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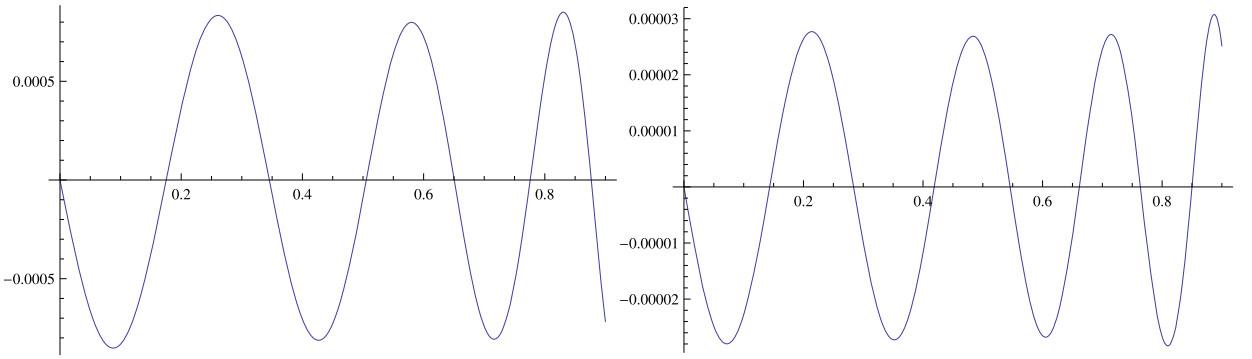


Fig. 2. The residuals $R(t, \tilde{x}_2)$ (left) and $R(t, \tilde{x}_3)$ (right).

Table 1
Comparison of the absolute errors for problem (13).

t	$\tilde{x}_0(t)$ (HPM)	$\tilde{x}_1(t)$ (LSHPM)	$\tilde{x}_2(t)$ (LSHPM)	$\tilde{x}_3(t)$ (LSHPM)
0	0.0	0.0	0.0	0.0
0.1	7.15×10^{-5}	5.04×10^{-5}	7.22×10^{-6}	1.94×10^{-7}
0.2	2.77×10^{-4}	7.18×10^{-5}	1.72×10^{-5}	2.94×10^{-7}
0.3	5.94×10^{-4}	2.21×10^{-4}	1.42×10^{-5}	4.99×10^{-8}
0.4	9.92×10^{-4}	6.96×10^{-4}	3.82×10^{-6}	1.88×10^{-8}
0.5	2.44×10^{-3}	9.76×10^{-4}	2.22×10^{-6}	1.59×10^{-7}
0.6	1.94×10^{-3}	8.78×10^{-4}	7.35×10^{-6}	2.47×10^{-8}
0.7	1.55×10^{-3}	5.61×10^{-4}	5.33×10^{-6}	2.07×10^{-8}
0.8	2.84×10^{-3}	3.09×10^{-4}	1.55×10^{-7}	3.32×10^{-8}
0.9	2.45×10^{-3}	1.93×10^{-4}	1.09×10^{-6}	2.35×10^{-8}
1	0.0	8.31×10^{-9}	0.0	8.31×10^{-9}

4. Conclusions

The new Least Squares Homotopy Perturbation Method introduced in this paper is a straightforward and efficient method to compute approximate solutions for nonlinear differential equations.

Based on the Homotopy Perturbation Method, LSHPM has an accelerated convergence compared to the regular Homotopy Perturbation Method. This is proved by the applications presented, which include boundary value problems on both finite and infinite intervals. Moreover, the comparison with previous results illustrated the accuracy of the method.

Conflict of interest statement

The authors declare that there is no conflict of interests regarding the publication of this paper.

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