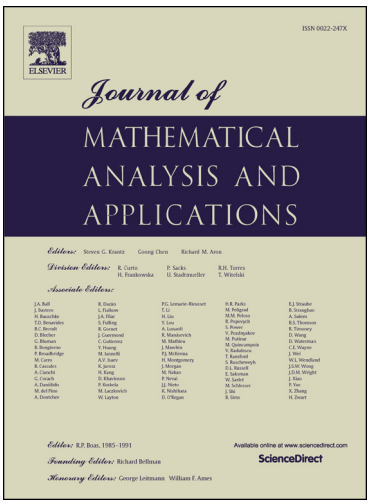
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Fourier Transformation of O(p,q)-Invariant Distributions. Fundamental Solutions of Ultra-Hyperbolic Operators

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Abstract. We present some formulas for the Fourier transform of $O(p,q)$-invariant temperate distributions. The formulas are used to calculate fundamental solutions of homogeneous and non-homogeneous ultra-hyperbolic operators.

1. Introduction and notation

The Fourier transform of a rotational invariant function $f(|x|) \in L^1(\mathbb{R}^n)$ is given by the so-called Poisson-Bochner formula:

$$\mathcal{F}(f(|x|)) = \frac{(2\pi)^{n/2}}{|\xi|^{n/2-1}} \int_0^\infty f(\rho) J_{n/2-1}(\rho|\xi|) \rho^{n/2} \, \mathrm{d}\rho, \tag{1.1}$$

see [37, (VII, 7; 22), p. 259], [40, Thm. 3.10, p. 158]. Formula (1.1) expresses the Fourier transform of $f(|x|)$ as a one-dimensional integral over f multiplied by a Bessel kernel. Of course, formula (1.1) is equivalent to L. Schwartz' expression for the Fourier transform of a uniformly distributed density on a sphere, i.e.,

$$\mathcal{F}(\delta(|x|^2 - s)) = \frac{1}{2\sqrt{s}} \mathcal{F}(\delta(|x| - \sqrt{s})) = \frac{2^{n/2-1} \pi^{n/2} s^{n/4-1/2}}{|\xi|^{n/2-1}} J_{n/2-1}(\sqrt{s}|\xi|) \tag{1.2}$$

for $s > 0$, see [33, p. 127].

For rapidly decreasing $\mathcal{C}^\infty$ functions, i.e., for $f \in \mathcal{S}(\mathbb{R}^1)$, formula (1.1) was generalized from the positive quadratic form $|x|^2$ to indefinite quadratic forms $[x,x]$ by

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R. Strichartz, see [41] and note the erratum. In this article, we derive a different generalization of (1.1) by expressing the Fourier transform $\mathcal{F}(S([x,x]))$ as an integral over $\mathcal{F}S$. This procedure yields formulas for larger spaces of functions and distributions, namely for $S \in \mathcal{O}_M(\mathbb{R}^1)$ (see [25, Prop. 5]) and even for $S \in \mathcal{D}'_{L^1,-n/2}(\mathbb{R}^1)$, see Theorem 2.4 below. In this way, we also arrive at Strichartz' result, which is now reformulated in an analogous way as (1.2) corresponds to (1.1) in the special case of the Euclidean quadratic form $|x|^2$, see Theorem 2.7. We furthermore describe in Corollary 2.6 how the Fourier transforms of $S([x,x])$ are connected by the Hilbert transform or by fractional derivation if the signatures of the quadratic form $[x,x]$ change.

The new tools in comparison with the classical approaches (e.g., deRham, Gelfand-Shilov, Jeanquartier) consist in the use of L. Schwartz' theory of vector-valued distributions and in the replacement of limit procedures by the employment of weighted $\mathcal{D}'_{L^p}$-spaces for the definition of the pull-back $S([x,x])$.

Our method leads to explicit representations of fundamental solutions of the ultrahyperbolic operators $(z-[\partial,\partial])^k$, see Section 3, of Jeanquartier's operator $\partial_0 - [\partial,\partial]$ and of the generalized Sobolev operator $\partial_{y_1} + [\partial,\partial]\partial_{y_2}$, see Section 4. In contrast to [30] and [18], our formulas do not contain approximations or higher-order derivatives. E.g., we straightforwardly derive in Proposition 3.1 fundamental solutions of $[\partial,\partial]$, in contrast to the complicated and partly incorrect formulae (2.22), p. 192, (4.9), p. 202, (5.9), p. 206, in the encyclopedia article [20].

Let us yet explain heuristically our new approach in the Euclidean case, i.e., for $\mathcal{F}(S(|x|^2))$. Upon writing

$$\mathcal{F}(S(|x|^2)) = \mathcal{F}_x\left(\int_0^\infty S(s)\delta(|x|^2-s)\,\mathrm{d}s\right)$$

and using $\delta(|x|^2-s) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\sigma(s-|x|^2)}\mathrm{d}\sigma$, we derive

$$\begin{aligned}\mathcal{F}(S(|x|^2)) &= \frac{1}{2\pi}\int_{\mathbb{R}}\left(\int_0^\infty S(s)\mathrm{e}^{-\mathrm{i}\sigma s}\,\mathrm{d}s\right)\mathcal{F}_x(\mathrm{e}^{\mathrm{i}\sigma|x|^2})\,\mathrm{d}\sigma\\ &= \langle(\mathcal{F}S)(\sigma), K(\sigma,\xi)\rangle.\end{aligned}$$

This formula expresses $\mathcal{F}(S(|x|^2))$ as a one-dimensional integral over $\mathcal{F}S$ multiplied by the kernel $K(\sigma,\xi) = \frac{1}{2\pi}\mathcal{F}_x(\mathrm{e}^{\mathrm{i}\sigma|x|^2})$. It is made precise for distributions $S([x,x])$ depending on general non-degenerate quadratic forms in Theorem 2.4 below.

Let us introduce some notation. In the following, we work in the space $\mathbb{R}^n$, $n \in \mathbb{N}$, and fix integers $p,q \in \mathbb{N}_0$ such that $n = p+q$. We consider the splitting of $\mathbb{R}^n$ into the two Euclidean subspaces $\mathbb{R}^p$ and $\mathbb{R}^q$, i.e., $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$ and collect the corresponding variables in the form

$$x = (x', x'') \in \mathbb{R}^n,\ x' = (x_1,\dots,x_p) \in \mathbb{R}^p,\ x'' = (x_{p+1},\dots,x_n) \in \mathbb{R}^q.$$

On $\mathbb{R}^n$ we use the canonical bilinear form

$$[x,y] = x'y' - x''y'' = x_1y_1 + \cdots + x_py_p - x_{p+1}y_{p+1} - \cdots - x_ny_n.$$

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The group of linear mappings $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ preserving $[x, y]$ is denoted by $O(p, q)$, i.e.,

$$O(p,q) = \{A \in \mathrm{Gl}_n(\mathbb{R});\, \forall x, y \in \mathbb{R}^n : [Ax, Ay] = [x, y]\}.$$

This group is sometimes called "pseudo-orthogonal group", see [41, p. 500].

We employ the standard notation for the distribution spaces $\mathcal{D}'$, $\mathcal{S}'$, the dual spaces of the spaces $\mathcal{D}$, $\mathcal{S}$ of "test functions" and of "rapidly decreasing functions", respectively, see [16], [37]. In order to display the active variable in a distribution, say $x \in \mathbb{R}^n$, we use notation as $T \in \mathcal{D}'(\mathbb{R}^n_x)$. Furthermore, we use the spaces $\mathcal{D}_{L^p}, \mathcal{D}'_{L^p}$, $1 \le p \le \infty$, which were introduced in [37, Ch. VI, § 8, p. 199], and the so-called "weighted" $\mathcal{D}_{L^p}$ and $\mathcal{D}'_{L^p}$-spaces, i.e.,

$$\mathcal{D}_{L^p,\mu} := \mathcal{D}_{L^p}(\mathbb{R}^n) \cdot (1 + |x|^2)^{-\mu/2}, \quad \mathcal{D}'_{L^p,\mu} := \mathcal{D}'_{L^p}(\mathbb{R}^n) \cdot (1 + |x|^2)^{-\mu/2}$$

for $1 \le p \le \infty$, $\mu \in \mathbb{R}$, see [26], [28]. Analogously, we use the abbreviation L^1_μ, $\mu \in \mathbb{R}$, for the Banach space $L^1 \cdot (1 + |x|)^{-\mu}$. For the spaces $\mathcal{O}_M, \mathcal{O}'_C$, see [37, Ch. VII, § 5, p. 243].

The Heaviside function is denoted by Y, and we write $\delta_s \in \mathcal{D}'(\mathbb{R}^1)$, $s \in \mathbb{R}$, for the delta distribution with support in s, which is the derivative of $Y(x - s)$, i.e., $\langle \phi, \delta_s \rangle = \phi(s)$ for $\phi \in \mathcal{D}(\mathbb{R}^1)$. In contrast, δ without any subscript stands for the delta distribution at the origin. For a distribution $T \in \mathcal{D}' = \mathcal{D}'(\mathbb{R}^n)$, we denote by $\check{T}$ its reflection at the origin. For the distributions $t^\lambda_+, t^\lambda_- = (t^\lambda_+)^{\check{}} \in \mathcal{D}'(\mathbb{R}^1_t)$, $\lambda \in \mathbb{C}$, see [13, Ch. I, 3.2, p. 48] or [29, Ex. 1.3.2, p. 32, Ex. 1.4.8, p. 49].

The pull-back $h^*T = T \circ h \in \mathcal{D}'(\Omega)$ of a distribution T in one variable t with respect to a submersive $\mathcal{C}^\infty$ function $h : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ open, is defined as in [11, (7.2.4/5), p. 82] or [29, Def. 1.2.12, p. 19], i.e.,

$$\langle \phi, h^*T \rangle = \Big\langle \frac{\mathrm{d}}{\mathrm{d}t} \Big(\int_\Omega Y(t - h(x))\phi(x)\,\mathrm{d}x \Big), T \Big\rangle, \qquad \phi \in \mathcal{D}(\Omega).$$

We use the Fourier transform $\mathcal{F}$ in the form

$$(\mathcal{F}\phi)(\xi) := \int \mathrm{e}^{-\mathrm{i}\xi x}\phi(x)\,\mathrm{d}x, \qquad \phi \in \mathcal{S}(\mathbb{R}^n),$$

this being extended to $\mathcal{S}'$ by continuity. (Herein and also elsewhere, the Euclidean inner product $(\xi, x) \mapsto \xi x$ is simply expressed by juxtaposition.)

2. Representations of the Fourier transform of $O(p,q)$-invariant distributions

Distributions invariant under $O(p, q)$-transformations can, in the complement of the origin, be represented as pull-backs with the quadratic function $[x, x]$ of one-dimensional distributions. In fact, since the mapping

$$h : \mathbb{R}^n \longrightarrow \mathbb{R} : x \longmapsto [x, x]$$

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is submersive outside the origin and $O(p,q)$ operates transitively on the level sets, we obtain an isomorphism

$$h^* : \mathcal{D}'(\mathbb{R}) \xrightarrow{\sim} \big\{T \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\});\, \forall A \in O(p,q) : T \circ A = T\big\} \tag{2.1}$$
$$S \longmapsto h^*S = S([x,x]).$$

(In the two marginal cases $p = n, q = 0$ and $p = 0, q = n$, the space $\mathcal{D}'(\mathbb{R})$ has to be replaced by $\mathcal{D}'((0,\infty))$ and $\mathcal{D}'((-\infty,0))$, respectively.)

In particular, for fixed $s \in \mathbb{R} \setminus \{0\}$, the pull-back of $\delta_s = \delta(s-t) \in \mathcal{D}'(\mathbb{R}^1_t)$ by h yields the temperate measures $\delta(s-[x,x]) \in \mathcal{S}'(\mathbb{R}^n)$, and each $O(p,q)$-invariant distribution can be represented through the kernels $\delta(s-[x,x])$ by integration with respect to s. In the next lemma, we describe how the "mean-values"

$$M_\phi(s) = \langle \phi, \delta(s-[x,x])\rangle, \qquad \phi \in \mathcal{S}(\mathbb{R}^n_x),\ s \in \mathbb{R}\setminus\{0\},$$

depend on s, in particular for $s \to 0$, compare [42], [43]. Note, incidentally, that the evaluation of $h^*S = S([x,x])$ on $\phi \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ is given by

$$\langle \phi, h^*S\rangle = \langle \phi, S([x,x])\rangle = \langle M_\phi, S\rangle.$$

With an additional assumption on S, we shall then extend $h^*S \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ unambiguously to a distribution in $\mathcal{D}'(\mathbb{R}^n)$ by using a suitable property of M_ϕ, which shall be stated in Lemma 2.2.

Lemma 2.1. *Let $s \in \mathbb{R}$. If $n \geq 3$ or $s \neq 0$, then $\delta(s-[x,x])$ is a well-defined Radon measure, i.e.,*

$$\delta(s-[x,x]) \in \mathcal{K}'(\mathbb{R}^n) = \mathcal{M}(\mathbb{R}^n)$$

where

$$\mathcal{K}(\mathbb{R}^n) = \{\phi : \mathbb{R}^n \longrightarrow \mathbb{C};\ \phi \text{ continuous with compact support}\}$$

endowed with its strict inductive limit topology. For $s \neq 0$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$M_\phi(s) := \langle \phi, \delta(s-[x,x])\rangle = \frac{1}{2}\int_{[x,x]=s} \frac{\phi(x)}{|x|}\,\mathrm{d}\sigma(x) \tag{2.2}$$

where $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$ and $\mathrm{d}\sigma$ denotes the surface measure on the hyper-surface $[x,x] = s$. Furthermore, M_ϕ has the following properties:
(i) $M_\phi \in \mathcal{C}^\infty(\mathbb{R}\setminus\{0\})$; (ii) $M_\phi \in L^1(\mathbb{R})$; (iii) *if* $n \geq 3$, *then* $M_\phi \in \mathcal{C}(\mathbb{R})$;
(iv) *if* $\chi \in \mathcal{D}(\mathbb{R})$ *with* $\chi = 1$ *near* 0, *then* $(1-\chi)\cdot M_\phi \in \mathcal{S}(\mathbb{R})$.

Proof. (a) For $s \neq 0$, the hyper-surface $H : [x,x] = s$ is non-singular and, using a partition of unity and renumbering the coordinates, we can suppose that, in a neighborhood of $\operatorname{supp}\phi$, H is represented by $x_1 = f(\tilde{x}) = \sqrt{s - [\tilde{x},\tilde{x}]} > 0$ where $\tilde{x} = (x_2,\ldots,x_n)$ and $[\tilde{x},\tilde{x}] = x_2^2+\cdots+x_p^2-(x_{p+1}^2+\cdots+x_n^2)$. (We assume here $p \geq 1$. The case $p = 0$ is similar.)

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Then, by the definition of the pull-back,

$$\begin{aligned}\langle\phi,\delta(s-[x,x])\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} Y(t-[x,x])\phi(x)\,\mathrm{d}x\bigg|_{t=s}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{n-1}}\int_{-\infty}^{\sqrt{t-[\tilde{x},\tilde{x}]}}\phi(x)\,\mathrm{d}x_1\mathrm{d}\tilde{x}\bigg|_{t=s}\\ &= \frac{1}{2}\int_{\mathbb{R}^{n-1}}\phi\big(\sqrt{s-[\tilde{x},\tilde{x}]},\tilde{x}\big)\,\frac{\mathrm{d}\tilde{x}}{\sqrt{s-[\tilde{x},\tilde{x}]}}.\end{aligned}\tag{2.3}$$

On the other hand,

$$\begin{aligned}\mathrm{d}\sigma &= \sqrt{1+|\nabla f|^2}\,\mathrm{d}\tilde{x} = \sqrt{1+\frac{|\tilde{x}|^2}{s-[\tilde{x},\tilde{x}]}}\,\mathrm{d}\tilde{x}\\ &= \frac{|x|}{\sqrt{s-[\tilde{x},\tilde{x}]}}\,\mathrm{d}\tilde{x}\quad\text{if } [x,x]=s,\ \text{i.e., } x_1^2=s-[\tilde{x},\tilde{x}].\end{aligned}$$

Hence $\langle\phi,\delta(s-[x,x])\rangle = \frac{1}{2}\int_{[x,x]=s}\frac{\phi(x)}{|x|}\,\mathrm{d}\sigma(x)$.

Note that the limit of $\delta(s-[x,x])$ for $s\to 0$ converges in $\mathcal{M}(\mathbb{R}^n)$ due to the assumption $n\geq 3$. In fact, if $p=0$ or $q=0$, then this limit yields 0, if $p,q\geq 1$, then $H:[x,x]=0$ is non-singular outside the origin and the measure $\delta([x,x]) = \frac{1}{2}\mathrm{d}\sigma(x)/|x|$ is locally integrable at 0.

(b) The assertion in (i) follows by differentiation under the integral sign in formula (2.3), compare [42, Lemma 1, p. 334]. For $s\to 0$, the function M_ϕ is continuous if $n\geq 3$, grows at most as a multiple of $\log|s|$ if $n=2$ and of $|s|^{-1/2}$ if $n=1$, respectively, see [42, Lemma 4, p. 343] or [43, Lemma 4.3, p. 210]. Therefore (ii) and (iii) hold. Finally, formula (2.3) shows that $M_\phi(s)$ and all its derivatives converge rapidly to 0 for $|s|\to\infty$ on account of $\phi\in\mathcal{S}(\mathbb{R}^n)$. □

Lemma 2.2. *The function* $\mathrm{e}^{\mathrm{i}\sigma[x,x]}\in\mathcal{C}^\infty(\mathbb{R}^{n+1}_{\sigma,x})$ *belongs to* $\mathcal{D}_{L^\infty,n/2}(\mathbb{R}^1_\sigma)\,\hat{\otimes}\,\mathcal{S}'(\mathbb{R}^n_x)$, *i.e., the map*

$$\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{D}_{L^\infty,n/2}(\mathbb{R}^1_\sigma):\phi\longmapsto\left(\sigma\mapsto\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\sigma[x,x]}\phi(x)\,\mathrm{d}x\right)$$

is well-defined, linear and continuous. Moreover,

$$\mathcal{F}^{-1}(M_\phi)(\sigma)=\frac{1}{2\pi}\int_{\mathbb{R}^n}\mathrm{e}^{\mathrm{i}\sigma[x,x]}\,\phi(x)\,\mathrm{d}x,\qquad\phi\in\mathcal{S}(\mathbb{R}^n),\ \sigma\in\mathbb{R}.$$

Proof. (a) Let us first assume $n=1$ and $\phi\in\mathcal{S}(\mathbb{R}^1)$. Then

$$g(\sigma)=\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\sigma t^2}\,\phi(t)\,\mathrm{d}t$$

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is continuous and even $\mathcal{C}^\infty$ since

$$g^{(k)}(\sigma) = \mathrm{i}^k \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\sigma t^2}\, t^{2k}\, \phi(t)\, \mathrm{d}t, \quad k \in \mathbb{N}.$$

Furthermore, the method of stationary phase (see [1, Lemma 6.3.3, p. 444], [16, p. 232], or [39, (1.1.16), (1.1.20), pp. 45, 49]) implies that

$$g(\sigma) = \phi(0)\sqrt{\frac{\mathrm{i}\pi}{\sigma}} + o(|\sigma|^{-1/2}) \text{ for } |\sigma| \to \infty.$$

Hence $g \in \mathcal{D}_{L^\infty,1/2}$ and the linear mapping

$$\mathcal{S}(\mathbb{R}^1) \longrightarrow \mathcal{D}_{L^\infty,1/2}(\mathbb{R}^1) : \phi \longmapsto g$$

is continuous by the closed graph theorem.

(b) For $n \geq 2$, let us consider the multi-linear mapping

$$\begin{aligned} A : \mathcal{S}(\mathbb{R}^1)^n &\longrightarrow \mathcal{D}_{L^\infty,1/2}(\mathbb{R}^1)^n \longrightarrow \mathcal{D}_{L^\infty,n/2}(\mathbb{R}^1) \\ (\phi_1, \dots, \phi_n) \longmapsto (g_1, \dots, g_n) \quad &\longmapsto \prod_{j=1}^n g_j(\sigma) = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\sigma[x,x]} \prod_{j=1}^n \phi_j(x_j)\, \mathrm{d}x \end{aligned}$$

where

$$g_j(\sigma) = \int_{-\infty}^{\infty} \mathrm{e}^{\epsilon_j \mathrm{i}\sigma t^2} \phi_j(t)\, \mathrm{d}t, \quad j = 1, \dots, n,\ \epsilon_j = \operatorname{sign}(p + \tfrac{1}{2} - j).$$

By (a), the application A is well-defined and continuous and hence it can be continuously extended to the completion of the tensor product. From Schwartz' kernel theorem (see [44, Thm. 51.6, p. 530]), we obtain

$$\begin{aligned} \tilde{A} : \mathcal{S}(\mathbb{R}^n) \simeq \mathcal{S}(\mathbb{R}^1) \hat{\otimes} \dots \hat{\otimes} \mathcal{S}(\mathbb{R}^1) &\longrightarrow \mathcal{D}_{L^\infty,n/2}(\mathbb{R}^1_\sigma) \\ \phi \longmapsto G(\sigma) &:= \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\sigma[x,x]}\, \phi(x)\, \mathrm{d}x, \end{aligned}$$

i.e., $\mathrm{e}^{\mathrm{i}\sigma[x,x]} \in \mathcal{D}_{L^\infty,n/2}(\mathbb{R}^1_\sigma) \hat{\otimes} \mathcal{S}'(\mathbb{R}^n_x)$, compare [44, Prop. 50.5, p. 522].

(c) It remains to verify the identity $G(\sigma) = 2\pi \mathcal{F}^{-1}(M_\phi)(\sigma)$. From Lemma 2.1, we conclude that $M_\phi \in L^1(\mathbb{R})$ and thus $\mathcal{F}^{-1}(M_\phi)$ is a bounded continuous function given by

$$\begin{aligned} \mathcal{F}^{-1}(M_\phi)(\sigma) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} M_\phi(s)\, \mathrm{e}^{\mathrm{i}\sigma s}\, \mathrm{d}s \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\sigma s} \left\langle \phi(x), \delta(s - [x,x]) \right\rangle \mathrm{d}s \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\sigma s} \int_{[x,x]=s} \phi(x)\, \frac{\mathrm{d}\sigma(x)}{2|x|}\, \mathrm{d}s \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\sigma[x,x]}\, \phi(x)\, \mathrm{d}x. \end{aligned}$$

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(If $0 \notin \operatorname{supp}\phi$, then these equalities are consequences of the definition of the pull-back, and in 0 they hold by approximation.) □

Definition 2.3. For $n \in \mathbb{N}$ and $S \in \mathcal{F}\big(\mathcal{D}'_{L^1,-n/2}(\mathbb{R}^1)\big)$, the pull-back $S([x,x]) \in \mathcal{S}'(\mathbb{R}^n)$ is defined by

$$\langle \phi, S([x,x])\rangle = {}_{\mathcal{D}_{L^\infty,n/2}(\mathbb{R}^1)}\langle \mathcal{F}^{-1}(M_\phi), \mathcal{F}S\rangle_{\mathcal{D}'_{L^1,-n/2}(\mathbb{R}^1)}, \qquad \phi \in \mathcal{S}(\mathbb{R}^n).$$

Note that the restriction $S([x,x])|_{\mathbb{R}^n\setminus\{0\}}$ coincides with the pull-back defined before Lemma 2.1. In fact, for $\phi \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$, we have $M_\phi \in \mathcal{S}(\mathbb{R}^1)$ and hence

$${}_{\mathcal{D}_{L^\infty,n/2}}\langle \mathcal{F}^{-1}(M_\phi), \mathcal{F}S\rangle_{\mathcal{D}'_{L^1,-n/2}} = {}_{\mathcal{S}}\langle \mathcal{F}^{-1}(M_\phi), \mathcal{F}S\rangle_{\mathcal{S}'} = {}_{\mathcal{S}}\langle M_\phi, S\rangle_{\mathcal{S}'} = \langle \phi, S([x,x])\rangle.$$

In the following theorem, we slightly generalize Prop. 5 in [25], where formula (2.4) below was proven under the assumption of $S \in \mathcal{O}_M$. Note that $\mathcal{O}_M = \mathcal{F}\mathcal{O}'_C$ is contained in $\mathcal{F}\mathcal{D}'_{L^1,-n/2}$.

Theorem 2.4. *Let $n \in \mathbb{N}$, $p,q \in \mathbb{N}_0$ with $p+q=n$ and $S \in \mathcal{F}(\mathcal{D}'_{L^1,-n/2}(\mathbb{R}^1))$.*

(a) *The kernel $K(\sigma,\xi) = \frac{1}{2\pi}\,\mathcal{F}_x(\mathrm{e}^{\mathrm{i}\sigma[x,x]})(\xi)$ belongs to $\mathcal{D}_{L^\infty,n/2}(\mathbb{R}^1_\sigma)\,\hat\otimes\,\mathcal{S}'(\mathbb{R}^n_\xi)$. For $\sigma \neq 0$, it is given by the $\mathcal{C}^\infty$ function*

$$K(\sigma,\xi) = \frac{\pi^{n/2-1}}{2|\sigma|^{n/2}}\,\exp\Big(\frac{\mathrm{i}\pi}{4}(p-q)\operatorname{sign}\sigma - \frac{\mathrm{i}[\xi,\xi]}{4\sigma}\Big). \tag{2.4}$$

(b) *For the Fourier transform of the $O(p,q)$-invariant distribution $S([x,x])$, the following identity holds:*

$$\mathcal{F}(S([x,x])) = {}_{\mathcal{D}'_{L^1,-n/2}}\langle (\mathcal{F}S)(\sigma), K(\sigma,\xi)\rangle_{\mathcal{D}_{L^\infty,n/2}\,\hat\otimes\,\mathcal{S}'_\xi}. \tag{2.5}$$

Proof. (a) From $\mathrm{e}^{\mathrm{i}\sigma[x,x]} \in \mathcal{D}_{L^\infty,n/2}\,\hat\otimes\,\mathcal{S}'$ we immediately conclude that the distribution $K(\sigma,\xi) = \frac{1}{2\pi}\,\mathcal{F}_x(\mathrm{e}^{\mathrm{i}\sigma[x,x]})$ belongs to $\mathcal{D}_{L^\infty,n/2}\,\hat\otimes\,\mathcal{S}'$ as well. Furthermore, the formula in (2.4) follows from the classical Fourier transform of the Gauss kernel (see, e.g., [16, p. 207] or [29, Ex. 1.6.14, p. 103]).

(b) For $\phi \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\langle \phi, \mathcal{F}(S([x,x]))\rangle = \langle \mathcal{F}\phi, S([x,x])\rangle = {}_{\mathcal{D}_{L^\infty,n/2}}\langle \mathcal{F}^{-1}(M_{\mathcal{F}\phi}), \mathcal{F}S\rangle_{\mathcal{D}'_{L^1,-n/2}}.$$

On the other hand, by Lemma 2.2, for $\sigma \in \mathbb{R}$,

$$\begin{aligned}\mathcal{F}^{-1}(M_{\mathcal{F}\phi})(\sigma) &= \frac{1}{2\pi}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\sigma[x,x]}\,\mathcal{F}\phi(x)\,\mathrm{d}x \\ &= {}_{\mathcal{S}(\mathbb{R}^n_x)}\langle \mathcal{F}\phi, \tfrac{1}{2\pi}\,\mathrm{e}^{\mathrm{i}\sigma[x,x]}\rangle_{\mathcal{S}'(\mathbb{R}^n_x)} \\ &= \langle \phi, \mathcal{F}(\tfrac{1}{2\pi}\,\mathrm{e}^{\mathrm{i}\sigma[x,x]})\rangle = \langle \phi(\xi), K(\sigma,\xi)\rangle\end{aligned}$$

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and hence

$$\langle \phi, \mathcal{F}(S([x,x])) \rangle = \big\langle \langle \phi(\xi), K(\sigma,\xi) \rangle, (\mathcal{F}S)(\sigma) \big\rangle. \tag{2.6}$$

We shall now use Fubini's Theorem for integrable distributions as given in [34, Cor., p. 136] in order to conclude that

$$\big\langle \langle \phi(\xi), K(\sigma,\xi) \rangle, (\mathcal{F}S)(\sigma) \big\rangle = \big\langle \phi(\xi), {}_{\mathcal{D}_{L^\infty,n/2}(\mathcal{S}'_\xi)} \langle K(\sigma,\xi), (\mathcal{F}S)(\sigma) \rangle_{\mathcal{D}'_{L^1,-n/2}} \big\rangle_{\mathcal{S}'_\xi}. \tag{2.7}$$

(The combination of (2.6) and (2.7) then implies formula (2.5).)

In fact, the application of Fubini's Theorem is legitimate, once we show that the product $K(\sigma,\xi)\phi(\xi)(\mathcal{F}S)(\sigma)$ belongs to $\mathcal{D}'_{L^1}(\mathbb{R}^{n+1}_{\sigma,\xi})$. From the hypocontinuity of the multiplication mapping $\mathcal{S}' \times \mathcal{S} \to \mathcal{O}'_C$ follows $K(\sigma,\xi)\phi(\xi) \in \mathcal{D}_{L^\infty,n/2}(\mathbb{R}^1_\sigma) \hat{\otimes} \mathcal{O}'_C(\mathbb{R}^n_\xi)$ (see [35, Thm. 7.1, p. 31]). The hypocontinuity of the multiplication mapping $\mathcal{D}_{L^\infty,n/2} \times \mathcal{D}'_{L^1,-n/2} \to \mathcal{D}'_{L^1}$ implies in the same way $K(\sigma,\xi)\phi(\xi)(\mathcal{F}S)(\sigma) \in \mathcal{D}'_{L^1}(\mathbb{R}^1_\sigma) \hat{\otimes} \mathcal{O}'_C(\mathbb{R}^n_\xi)$, and the last completed tensor product is a subspace of $\mathcal{D}'_{L^1}(\mathbb{R}^{n+1}_{\sigma,\xi})$, see [34, Prop. 38, p. 135]. This completes the proof. □

Let us add two remarks concerning Theorem 2.4. First, note that $\mathcal{F}(S([x,x]))$ cannot always be written in the form $T([\xi,\xi])$ though formula (2.5) seems to imply that. E.g., if $S = 1$, then $\mathcal{F}(S([x,x])) = (2\pi)^n \delta$ and this is in accordance with formulas (2.4), (2.5) since K is a continuous function of σ with values in $\mathcal{S}'(\mathbb{R}^n_\xi)$ and $K(0,\xi) = \frac{1}{2\pi}\mathcal{F}_x 1 = (2\pi)^{n-1}\delta$ and $\mathcal{F}S = 2\pi\delta$ and hence

$$\langle (\mathcal{F}S)(\sigma), K(\sigma,\xi) \rangle = 2\pi K(0,\xi) = (2\pi)^n \delta(\xi).$$

Second, as we have seen in the beginning, the behavior of S outside 0 does not matter for the possibility of defining the pull-back $S([x,x])$; we just need a certain amount of regularity of S near 0 in order to extend $S([x,x])$ in an unequivocal way from $\mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ to $\mathcal{D}'(\mathbb{R}^n)$. Therefore the assumption $\mathcal{F}S \in \mathcal{D}'_{L^1,-n/2}$ in Def. 2.3 is too restrictive since it affects S globally and not just at 0. E.g., for $S = \delta_1^{(m)} = \delta^{(m)}(s-1), m \in \mathbb{N}_0$, we have $(\mathcal{F}S)(\sigma) = (\mathrm{i}\sigma)^m \mathrm{e}^{-\mathrm{i}\sigma} \notin \mathcal{D}'_{L^1,-n/2}(\mathbb{R}^1)$ if $n \leq 2m+2$, but $\delta_1^{(m)}([x,x]) \in \mathcal{S}'(\mathbb{R}^n_x)$ is clearly well-defined for each n since $0 \notin \operatorname{supp}(\delta_1^{(m)}([x,x]))$.

A localized version of Theorem 2.4 reads as follows: For $S \in \mathcal{D}'(\mathbb{R})$ with the property that

$$\exists \chi \in \mathcal{D}(\mathbb{R}) : \chi = 1 \text{ near } 0 \text{ and } \chi \cdot S \in \mathcal{F}(\mathcal{D}'_{L^1,-n/2}),$$

we can, as before, define $S([x,x]) \in \mathcal{D}'(\mathbb{R}^n)$ by

$$\begin{aligned} \langle \phi, S([x,x]) \rangle &= \langle \phi, (\chi \cdot S)([x,x]) \rangle + \langle \phi, ((1-\chi)\cdot S)([x,x]) \rangle \\ &= \langle \mathcal{F}^{-1}(M_\phi), \mathcal{F}(\chi \cdot S) \rangle + \langle M_\phi \cdot (1-\chi), S \rangle. \end{aligned}$$

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(Note that $M_\phi \cdot (1-\chi) \in \mathcal{D}(\mathbb{R})$. Furthermore, this definition of $S([x,x])$ does not depend on the choice of the particular function χ, and if $S \in \mathcal{S}'(\mathbb{R})$, we obtain $S([x,x]) \in \mathcal{S}'(\mathbb{R}^n)$ as before.)

Then the formula for the Fourier transform $\mathcal{F}(S([x,x]))$ in Theorem 2.4 takes the form

$$\begin{aligned}(2.8)\qquad \mathcal{F}(S([x,x])) = {}& {}_{\mathcal{D}'_{L^1,-n/2}(\mathbb{R}^1_\sigma)}\langle \mathcal{F}(\chi\cdot S)(\sigma), K(\sigma,\xi)\rangle_{\mathcal{D}_{L^\infty,n/2}(\mathbb{R}^1_\sigma)\,\hat\otimes\,\mathcal{S}'(\mathbb{R}^n_\xi)} + \\ & + {}_{\mathcal{S}'(\mathbb{R}^1_\sigma)}\langle (\mathcal{F}S)(\sigma), \mathcal{F}_x\big(\tfrac{1}{2\pi}\,\mathrm{e}^{\mathrm{i}\sigma[x,x]}\cdot(1-\chi)([x,x])\big)\rangle_{\mathcal{S}(\mathbb{R}^1_\sigma)\,\hat\otimes\,\mathcal{S}'(\mathbb{R}^n_\xi)}.\end{aligned}$$

Note, however, that in cases as the one considered above, e.g., $S=\delta_1'$, one would rather calculate $\mathcal{F}(\delta_1'([x,x]))$ by differentiation with respect to a parameter, i.e.,

$$\mathcal{F}(\delta_1'([x,x])) = -\Big(\frac{\mathrm{d}}{\mathrm{d}a}\,\mathcal{F}\big(\delta([x,x]-a)\big)\Big)\Big|_{a=1},$$

instead of using formula (2.8).

Before continuing, let us yet give another representation of $\mathcal{F}(S([x,x]))$ under an additional restriction on S.

Corollary 2.5. *Let $n\in\mathbb{N}$, $p,q\in\mathbb{N}_0$ with $p+q=n$ and $S\in\mathcal{F}(L^1_{-n/2}(\mathbb{R}^1))$. Then $\mathcal{F}(S([x,x])) = T([\xi,\xi])$ with $T=\mathcal{F}U$ and*

$$(2.9)\qquad U = 2^{n-3}\pi^{n/2-1}\exp\Big(\frac{\mathrm{i}\pi}{4}\,(p-q)\operatorname{sign}u\Big)|u|^{n/2-2}\cdot(\mathcal{F}S)(\tfrac{1}{4u}) \in L^1_{-n/2}(\mathbb{R}^1_u).$$

Proof. (a) Clearly, $\mathcal{F}S \in (1+|\sigma|)^{n/2}\cdot L^1 \subset \mathcal{D}'_{L^1,-n/2}$ and hence $S([x,x])$ is well-defined according to Def. 2.3. Furthermore, applying the substitution $\sigma=\frac{1}{4u}$ to $(1+|\sigma|)^{-n/2}\cdot\mathcal{F}S\in L^1(\mathbb{R}^1_\sigma)$ we obtain that

$$u^{-2}(1+|u|^{-1})^{-n/2}(\mathcal{F}S)(\tfrac{1}{4u}) \in L^1(\mathbb{R}^1_u)$$

and hence $(1+|u|)^{-n/2}U\in L^1$ and $U\in L^1_{-n/2}\subset\mathcal{D}'_{L^1,-n/2}$. Therefore, $T([\xi,\xi])$ is also well-defined.

Let us now first additionally assume that $\operatorname{supp}(\mathcal{F}S)\subset\mathbb{R}\setminus\{0\}$. If $\phi\in\mathcal{S}(\mathbb{R}^n)$, then the function $(\mathcal{F}S)(\sigma)K(\sigma,\xi)\phi(\xi)$ is absolutely integrable, and formulas (2.4/5) together with Fubini's theorem imply

$$\begin{aligned}\langle\phi,\mathcal{F}(S([x,x]))\rangle &= \int_{\mathbb{R}^{n+1}}(\mathcal{F}S)(\sigma)K(\sigma,\xi)\phi(\xi)\,\mathrm{d}\sigma\mathrm{d}\xi \\ &= \int_{\mathbb{R}^n}\phi(\xi)\int_{-\infty}^{\infty}(\mathcal{F}S)(\sigma)K(\sigma,\xi)\,\mathrm{d}\sigma\mathrm{d}\xi \\ &= \int_{\mathbb{R}^n}\phi(\xi)\int_{-\infty}^{\infty}U(u)\mathrm{e}^{-\mathrm{i}u[\xi,\xi]}\,\mathrm{d}u\mathrm{d}\xi = \int_{\mathbb{R}^n}\phi(\xi)\cdot(\mathcal{F}U)([\xi,\xi])\,\mathrm{d}\xi.\end{aligned}$$

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(b) For the general case, we note that the two mappings from $\mathcal{F}(L^1_{-n/2}(\mathbb{R}^1))$ to $\mathcal{S}'(\mathbb{R}^n)$ which associate with $S \in \mathcal{F}(L^1_{-n/2}(\mathbb{R}^1))$ the images $\mathcal{F}(S([x,x]))$ and $(\mathcal{F}U)([\xi,\xi])$, respectively, with U being as in (2.9), are both linear and continuous. Since these two maps coincide on a dense subset according to (a), they must do so everywhere. □

As formula (2.9) reveals, U and consequently also T are multiplied by -1 if we increase one of the numbers p,q by two and simultaneously decrease the other one by two. More interestingly, if p,q are simultaneously increased respectively decreased by one (letting thus n unchanged), then U is multiplied by $\pm\mathrm{i}\operatorname{sign} u$ and hence T undergoes a Hilbert transform. Let us define the Hilbert transform $\mathcal{H}$ on the space $\mathcal{F}(L^1_{-n/2}(\mathbb{R}))$ by the following formula:

$$\mathcal{H} : \mathcal{F}(L^1_{-n/2}(\mathbb{R})) \longrightarrow \mathcal{F}(L^1_{-n/2}(\mathbb{R})) : S \longmapsto \mathcal{F}^{-1}(-\mathrm{i}\operatorname{sign}\sigma \cdot \mathcal{F}S) = \mathcal{F}(\mathrm{i}\operatorname{sign}\sigma \cdot \mathcal{F}^{-1}S).$$

Note that $\mathcal{H}S = \frac{1}{\pi}\operatorname{vp}\frac{1}{s} * S$ (since $\mathcal{F}(\operatorname{vp}\frac{1}{s}) = -\mathrm{i}\pi\operatorname{sign}\sigma$) if S and $\operatorname{vp}\frac{1}{s}$ are convolvable, i.e., if $S \in \mathcal{D}'_{L^1,-1}$, and that $\mathcal{H}^2 = -\mathrm{id}$. From formula (2.9) we also obtain that $T_2 = -\mathcal{H}T_1$ if $S_2 = \mathcal{H}S_1$ for $S_j \in \mathcal{F}(L^1_{-n/2}(\mathbb{R}))$ with $\mathcal{F}(S_j([x,x])) = T_j([\xi,\xi])$, $j = 1,2$, and p,q remain fixed.

Furthermore, if we leave q unchanged and increase or decrease p (and thus also n by one), respectively, then U in formula (2.9) is multiplied or divided by $2\sqrt{\mathrm{i}\pi u}$, respectively. With respect to T, this amounts to a fractional derivation or integration, respectively. This recursion formula goes in the Euclidean case $q = 0$ back to [22, (13.2), p. 30], see also [32, (25.14′), Lemma 25.1′, p. 359; Engl. transl. p. 486]. In the Euclidean case where the spaces $\mathbb{R}^n$ and $\mathbb{R}^{n+2}$ are involved, it was re-derived in [15], the general Euclidean case, i.e., the comparison of $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ by a fractional derivative, was taken up in [7]. In the Lorentz case, the corresponding recursion formula was first given in [46, Thm. 4, p. 312].

We shall collect all these formulas in Corollary 2.6 below. In order to compare $\mathcal{F}(S([x,x]))$ for different p,q, let us write now

$$[x,x]_{p,q} = x_1^2 + \cdots + x_p^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2) \tag{2.10}$$

and set $\mathcal{F}(S([x,x]_{p,q})) = T_{p,q}([\xi,\xi]_{p,q})$.

Corollary 2.6. *Let* $n \in \mathbb{N}$, $p,q \in \mathbb{N}_0$ *with* $p+q = n$.

(a) *If* $S \in \mathcal{F}(L^1_{-n/2}(\mathbb{R}))$ *and* $T_{p,q}$ *are as above, then* $T_{p+1,q-1} = \mathcal{H}(T_{p,q})$ *(for* $q \geq 1$*) and* $T_{p-1,q+1} = -\mathcal{H}(T_{p,q})$ *(for* $p \geq 1$*).*

(b) *If* $S_j, T_j \in \mathcal{F}(L^1_{-n/2}(\mathbb{R}))$ *with* $\mathcal{F}(S_j([x,x]_{p,q})) = T_j([\xi,\xi]_{p,q})$, $j = 1,2$, *and* $S_2 = \mathcal{H}S_1$, *then* $T_2 = -\mathcal{H}T_1$.

(c) *If* $S \in \mathcal{F}(L^1_{-n/2}(\mathbb{R}))$, *then* $T_{p+2,q} = -4\pi\frac{\mathrm{d}}{\mathrm{d}t}T_{p,q} = -T_{p,q+2}$.

(d) *If* $S \in \mathcal{F}(L^1_{-(n-1)/2}(\mathbb{R}))$, $p \geq 1$, *and* $T_{p,q}$ *is convolvable with* $t_-^{-1/2}$ (i.e., *if* $\mathrm{e}^{-1/t}t_+^{-1/2}\cdot T_{p,q} \in \mathcal{D}'_{L^1}$), *then* $T_{p-1,q} = \frac{1}{2\pi}T_{p,q} * t_-^{-1/2}$. *Similarly, if* $S \in \mathcal{F}(L^1_{-(n-1)/2}(\mathbb{R}))$,

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$q \geq 1$, and $T_{p,q}$ is convolvable with $t_+^{-1/2}$, then $T_{p,q-1} = \frac{1}{2\pi}\, T_{p,q} * t_+^{-1/2}$.
If $S \in \mathcal{F}(L^1_{-n/2}(\mathbb{R}))$, then $T_{p+1,q} = -T_{p,q} * t_-^{-3/2}$ and $T_{p,q+1} = -T_{p,q} * t_+^{-3/2}$.

(e) *If $S \in \mathcal{F}(L^1_{-n/2}(\mathbb{R}))$ and $\mathcal{F}(S([x,x]_{p,q})) = T_{p,q}([\xi,\xi]_{p,q})$ and $\mathcal{F}(\check{S}([x,x]_{p,q})) = \tilde{T}_{p,q}([\xi,\xi]_{p,q})$, then $\tilde{T}_{p,q} = \check{T}_{q,p}$.*

Proof. (a) If $T_{p,q} = \mathcal{F}U_{p,q}$, then

$$\begin{aligned}\exp\Big(\frac{\mathrm{i}\pi}{4}\big(p+1-(q-1)\big)\operatorname{sign} u\Big) &= \mathrm{i}(\operatorname{sign} u)\cdot\exp\Big(\frac{\mathrm{i}\pi}{4}(p-q)\operatorname{sign} u\Big)\\ &= -\exp\Big(\frac{\mathrm{i}\pi}{4}\big(p-1-(q+1)\big)\operatorname{sign} u\Big)\end{aligned}$$

and the definition of the Hilbert transform imply $U_{p+1,q-1} = \mathrm{i}\operatorname{sign} u \cdot U_{p,q} = -U_{p-1,q+1}$ and $T_{p+1,q-1} = \mathcal{H}(T_{p,q}) = -T_{p-1,q+1}$.

(b) If $S_2 = \mathcal{H}S_1$, then $\mathcal{F}S_2 = -\mathrm{i}\operatorname{sign}\sigma \cdot \mathcal{F}S_1$. If therefore $T_j = \mathcal{F}U_j$, then U_j are given by formula (2.9) and hence $U_2 = -\mathrm{i}\operatorname{sign} u \cdot U_1$ and hence $T_2 = \mathcal{F}U_2 = \mathcal{F}(-\mathrm{i}\operatorname{sign} u \cdot U_1) = -\mathcal{H}T_1$.

(c) As in (a), we set $T_{p,q} = \mathcal{F}U_{p,q}$ and obtain $U_{p+2,q} = -U_{p,q+2} = 4\pi\mathrm{i}u \cdot U_{p,q}$. This immediately implies the asserted formula.

(d) With notation as in (c), we obtain $U_{p-1,q} = (4\pi\mathrm{i}u)^{-1/2} \cdot U_{p,q}$ and hence

$$T_{p-1,q} = \mathcal{F}U_{p-1,q} = T_{p,q} * \mathcal{F}^{-1}\Big(\frac{1}{2\sqrt{-\mathrm{i}\pi u}}\Big) = T_{p,q} * \frac{1}{2\pi}\, t_-^{-1/2}$$

if $p \geq 1$ and $T_{p,q}$ and $t_-^{-1/2}$ are convolvable (and then are also $\mathcal{S}'$-convolvable). Here we use that $\mathcal{F}(t_+^{-1/2}/\sqrt{\pi}) = (\mathrm{i}u)^{-1/2}$, see [29, (1.6.2), p. 87]. The other three identities are shown in an analogous way. For the last two equations in Corollary 2.6 (d), let us observe that $\mathcal{F}(L^1_{-n/2}(\mathbb{R})) \subset \mathcal{D}'_{L^\infty}(\mathbb{R})$ and hence $T_{p,q} \in \mathcal{F}(L^1_{-n/2}(\mathbb{R}))$ is convolvable with $x_\pm^{-3/2} \in \mathcal{D}'_{L^1}(\mathbb{R})$.

(e) If $T_{p,q} = \mathcal{F}(U_{p,q})$ and $\tilde{T}_{p,q} = \mathcal{F}(\tilde{U}_{p,q})$, then (2.9) implies $\tilde{U}_{p,q} = \check{U}_{q,p}$ and hence also $\tilde{T}_{p,q} = \check{T}_{q,p}$. This completes the proof. □

We now return to the simplified notation $[x,y] = x'y' - x''y'' = x_1y_1 + \cdots + x_py_p - x_{p+1}y_{p+1} - \cdots - x_ny_n$ from the beginning. Whereas Theorem 2.4 yields a representation of $\mathcal{F}(S([x,x]))$ in the form "$\int(\mathcal{F}S)(\sigma)\cdot K(\sigma,\xi)\,\mathrm{d}\sigma$" involving the Fourier transform $\mathcal{F}S$ of S, the classical Poisson-Bochner formula corresponds to a representation of the form "$\int S(s)\cdot L(s,\xi)\,\mathrm{d}s$" with a suitable kernel L. Evidently,

$$L(s,\xi) = \mathcal{F}(\delta([x,x]-s)), \qquad s \in \mathbb{R}.$$

If $s \neq 0$, we can, without restriction of generality, assume that $s > 0$ since $\delta_{-s} = (\delta_s)^{\check{}}$ and hence $\mathcal{F}(\delta([x,x]+s))$, $s > 0$, is obtained by interchanging p and q in formula

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(2.11) below and replacing s,t by $-s,-t$, respectively, see Corollary 2.6 (e). We adjourn the case $s=0$ to Proposition 2.8.

Theorem 2.7. *Let $p\in\mathbb{N}$, $q\in\mathbb{N}_0$ with $n=p+q$ and $s>0$ be fixed. Then $\mathcal{F}(\delta([x,x]-s))=T([\xi,\xi])$ where $T\in\mathcal{F}(\mathcal{D}'_{L^1,-n/2})$ is given by*

(2.11)
$$T(t)=\begin{cases}(2\pi)^{n/2}\dfrac{s^{n/4-1/2}}{|t|^{n/4-1/2}}\Big\{\dfrac{1}{2}Y(t)\Big[\cos\Big(\dfrac{\pi q}{2}\Big)J_{n/2-1}\big(\sqrt{st}\big)-\sin\Big(\dfrac{\pi q}{2}\Big)N_{n/2-1}\big(\sqrt{st}\big)\Big]\\ \qquad\qquad+\dfrac{1}{\pi}\,Y(-t)\sin\Big(\dfrac{\pi p}{2}\Big)K_{n/2-1}\big(\sqrt{-st}\big)\Big\} & :p \text{ or } q \text{ odd},\\ 2^{n-2}\pi^{n/2}\Big\{(-1)^{q/2}Y(t)\Big(\dfrac{4t}{s}\Big)^{1/2-n/4}J_{n/2-1}(\sqrt{st})\\ \qquad\qquad-(-1)^{p/2}\displaystyle\sum_{j=0}^{n/2-2}\frac{(-s)^j}{j!\,2^{2j}}\,\delta^{(n/2-2-j)}\Big\} & :p,q \text{ even}.\end{cases}$$

For $n\le 3$, T is a locally integrable function. If $n\ge 4$ and p or q are odd, then T contains singular terms originating from the negative powers of $\pm t$ in the series expansion of $Y(t)t^{1/2-n/4}N_{n/2-1}(\sqrt{st})$ or of $Y(-t)|t|^{1/2-n/4}K_{n/2-1}(\sqrt{-st})$, respectively. In the composition $T([\xi,\xi])$, these negative powers of $\pm t$ have to be replaced by $t_\pm^\mu$, respectively.

Proof. If we set $S=\delta_s=\delta(s-t)\in\mathcal{S}'(\mathbb{R}^1_t)$, then $\mathcal{F}S=\mathrm{e}^{-\mathrm{i}\sigma s}\in L^1_{-3/2}(\mathbb{R}^1_\sigma)$. Hence Corollary 2.5 is applicable for $n\ge 3$. Since we will determine $T=\mathcal{F}U$ by analytic continuation, this procedure will work also in the cases $n=1,2$ where $U\in\mathcal{D}'_{L^1,-n/2}\setminus L^1_{-n/2}$ is given by $U=\lim_{\epsilon\searrow 0}Y(|u|-\epsilon)U$ in $\mathcal{S}'(\mathbb{R}^1_u)$. From formula (2.9), we obtain $\mathcal{F}(\delta([x,x]-s))=T([\xi,\xi])$ with $T=\mathcal{F}U$ and

$$U=2^{n-3}\pi^{n/2-1}\exp\Big(\frac{\mathrm{i}\pi}{4}(p-q)\operatorname{sign}u\Big)|u|^{n/2-2}\cdot\mathrm{e}^{-\mathrm{i}s/4u}.$$

Instead of U, let us consider first the holomorphic distribution-valued functions $V_\lambda^\pm=u_\pm^\lambda\mathrm{e}^{-\mathrm{i}s/(4u)}\in\mathcal{S}'(\mathbb{R}^1_u)$. Note that the mappings $\lambda\mapsto V_\lambda^\pm$ are entire due to the recursion formulas $V_\lambda^\pm=\frac{4}{\mathrm{i}s}[(V_{\lambda+2}^\pm)'\mp(\lambda+2)V_{\lambda+1}^\pm]$. In the strip $-2<\operatorname{Re}\lambda<0$, we obtain

(2.12)
$$(\mathcal{F}V_\lambda^+)(t)=\int_0^\infty\mathrm{e}^{-\mathrm{i}s/(4u)-\mathrm{i}tu}u^\lambda\,\mathrm{d}u=\begin{cases}-\pi\mathrm{e}^{-\lambda\mathrm{i}\pi/2}\big(\frac{s}{4t}\big)^{(\lambda+1)/2}H^{(2)}_{\lambda+1}(\sqrt{st}) & :t>0,\\ 2\mathrm{i}\,\mathrm{e}^{\lambda\mathrm{i}\pi/2}\big(-\frac{s}{4t}\big)^{(\lambda+1)/2}K_{\lambda+1}(\sqrt{-st}) & :t<0.\end{cases}$$

according to [14, formula 3.871].

Using also the complex conjugate of formula (2.12), we infer that

$$2^{n-3}\pi^{n/2-1}\mathcal{F}\big(\mathrm{e}^{\mathrm{i}\pi(p-q)/4}V_\lambda^+ + \mathrm{e}^{-\mathrm{i}\pi(p-q)/4}V_\lambda^-\big)(t)=-2^{n-1}\pi^{n/2}\big(\tfrac{s}{4|t|}\big)^{(\lambda+1)/2}\times$$
$$\times\begin{cases}\frac{1}{2}\sin\big(\frac{p-q-2\lambda}{4}\,\pi\big)N_{\lambda+1}(\sqrt{st})+\frac{1}{2}\cos\big(\frac{p-q-2\lambda}{4}\,\pi\big)J_{\lambda+1}(\sqrt{st}) & :t>0,\\ \frac{1}{\pi}\sin\big(\frac{p-q+2\lambda}{4}\,\pi\big)K_{\lambda+1}(\sqrt{-st}) & :t<0.\end{cases}$$

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In order to continue analytically the last expressions with respect to λ into the value $\lambda = \frac{n}{2} - 2$, we insert the series expansions of the Bessel functions. We then have to evaluate the expressions

$$\sin\Big(\frac{p-q\mp 2\lambda}{4}\,\pi\Big)t_{\pm}^{-\lambda-1+k}, \quad k\in\mathbb{N}_0,$$

at $\lambda = \frac{n}{2} - 2$.

If n is odd, then $t_{\pm}^{-\lambda-1+k}$ are analytic at $\lambda = \frac{n}{2} - 2$ and we directly arrive at the first formula of (2.11). Similarly, if both p and q are odd, then

$$\operatorname*{Pf}_{\lambda=n/2-2} \sin\Big(\frac{p-q\mp 2\lambda}{4}\,\pi\Big)t_{\pm}^{-\lambda-1+k} = -\sin\Big(\frac{p-q\mp n}{4}\,\pi\Big)t_{\pm}^{1-n/2+k}, \quad k\in\mathbb{N}_0,$$

since the derivative of $\sin((p-q\mp 2\lambda)\pi/4)$ vanishes at $\lambda = \frac{n}{2} - 2$. So this case also leads to the first formula in (2.11).

Finally, if both p and q are even, then we use formula 8.433 in [14] to infer that

$$\begin{aligned}
&\lim_{\lambda\to\frac{n}{2}-2} Y(t)\Big(\frac{s}{4t}\Big)^{(\lambda+1)/2}\sin\Big(\frac{p-q-2\lambda}{4}\,\pi\Big)N_{\lambda+1}(\sqrt{st})\\
&\qquad = -\lim_{\lambda\to\frac{n}{2}-2}\frac{\sin(\frac{p-q-2\lambda}{4}\,\pi)}{\sin((\lambda+1)\pi)}\cdot\lim_{\nu\to\frac{n}{2}-1}\sum_{j=0}^{n/2-2}\frac{(-s)^j\,t_+^{j-\nu}}{j!\,2^{2j}\,\Gamma(j+1-\nu)}\\
&\qquad = \frac{1}{2}(-1)^{p/2}\sum_{j=0}^{n/2-2}\frac{(-s)^j}{j!\,2^{2j}}\,\delta^{(n/2-2-j)}.
\end{aligned}$$

(A similar calculation can be found in [20, Ch. IV, (1.12), p. 188].) We obtain the same contribution to the singular part of T from $t<0$ and that leads to the second formula in (2.11). The proof is complete. □

Setting $\epsilon = \sigma = 0$ in [41, Thm. 1, p. 509] and approximating $f(t) = \delta(t-\sqrt{s})$ by functions in $\mathcal{S}(\mathbb{R}^1_t)$ yields the formula in Theorem 2.7 if we use

$$\delta([x,x]-s) = \frac{1}{2\sqrt{s}}\,\delta(\sqrt{[x,x]}-\sqrt{s})Y([x,x]) \text{ for } s>0.$$

If both p,q are even, the erratum in [41] has to be taken into account. The special cases $p=1$ respectively $q=1$ in formula (2.11) coincide with [27, Prop. 1, p. 136].

Note that the classical Poisson-Bochner formula is also a special case of Theorem 2.7. In fact, for $p=n$, $q=0$ and $s=\rho^2$, $\rho>0$, we have

$$\delta([x,x]-s) = \delta(|x|^2-\rho^2) = \frac{1}{2\rho}\,\delta(\rho-|x|)$$

and $\delta^{(k)}(|x|^2)=0$ for $0\le k\le \frac{n}{2}-2$ (see Corollary 4.4 below) and hence

$$\mathcal{F}(\delta(\rho-|x|)) = 2\rho\cdot\mathcal{F}(\delta(|x|^2-s)) = (2\pi\rho)^{n/2}|\xi|^{-n/2+1}J_{n/2-1}(\rho|\xi|),$$

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compare formulas (1.1/2), [37, (VII, 7; 22), p. 259] or [29, (1.6.15), p. 98].

Proposition 2.8. *Let* $p, q \in \mathbb{N}$ *with* $n = p + q \geq 3$. *Then* $\mathcal{F}(\delta([x,x])) = T([\xi,\xi])$ *where* $T \in \mathcal{F}(\mathcal{D}'_{L^1,-n/2})$ *is given by*

$$T = (4\pi)^{n/2-1} \cdot \begin{cases} \Gamma(\frac{n}{2}-1)(-1)^{(p-1)/2} t_-^{1-n/2} & : n, p \text{ odd}, \\ \Gamma(\frac{n}{2}-1)(-1)^{(q-1)/2} t_+^{1-n/2} & : n, q \text{ odd}, \\ (-1)^{p/2+1}\pi\delta^{(n/2-2)} & : p, q \text{ even}, \\ (-1)^{(p+1)/2}(\operatorname{vp}\frac{1}{t})^{(n/2-2)} & : p, q \text{ odd}. \end{cases} \tag{2.13}$$

Proof. We set $S = \delta$ and apply Corollary 2.5. Then formula (2.9) yields

$$U = 2^{n-3}\pi^{n/2-1}\big(\mathrm{e}^{\mathrm{i}\pi(p-q)/4} u_+^{n/2-2} + \mathrm{e}^{\mathrm{i}\pi(q-p)/4} u_-^{n/2-2}\big)$$

and $T = \mathcal{F}U$ results as in (2.13) upon inserting the well-known Fourier transforms of $u_\pm^{n/2-2}$, see [29, Ex. 1.6.7, p. 86]. □

Let us observe that $\mathcal{F}(\delta([x,x]))$ could also be deduced from Theorem 2.7 by inserting $s = 0$. The formula $\mathcal{F}(\delta([x,x])) = T([\xi,\xi])$ can also be derived from [13, p. 365, no. 14] by employing Sokhotskij's formula $\delta(t) = \pi^{-1}\mathrm{Im}((t - \mathrm{i}0)^{-1})$ in $\mathcal{S}'(\mathbb{R}^1_t)$. Also note that the different cases (with respect to p, q, n) in formula (2.13) are connected by the Hilbert transform or by (fractional) differentiation, see Corollary 2.6 (a), (c) and (d).

Let us compile some special cases. If we set $p = 1, q = 2, n = 3$, we obtain

$$\mathcal{F}\big(\delta([x,x])\big) = 2\pi\,\frac{Y(-[\xi,\xi])}{\sqrt{-[\xi,\xi]}} \in L^1_{\mathrm{loc}}(\mathbb{R}^3), \qquad [\xi,\xi] = \xi_1^2 - \xi_2^2 - \xi_3^2,$$

in accordance with [27, Cor. 1, (c), p. 139], where also $\mathcal{F}\big(Y(x_1)\delta([x,x])\big)$ is calculated and hints to the literature are given. Similarly, for $p = 1, q = 3, n = 4$, we obtain

$$\mathcal{F}\big(\delta([x,x])\big) = -4\pi\ \operatorname{vp}\Big(\frac{1}{[\xi,\xi]}\Big) \in \mathcal{D}'(\mathbb{R}^4), \qquad [\xi,\xi] = \xi_1^2 - \xi_2^2 - \xi_3^2 - \xi_4^2,$$

see [27, Cor. 1, (e), p. 140]. Finally, for $p = q = 2, n = 4$, we obtain

$$\mathcal{F}\big(\delta([x,x])\big) = 4\pi^2\,\delta([\xi,\xi]) \in \mathcal{D}'(\mathbb{R}^4), \qquad [\xi,\xi] = \xi_1^2 + \xi_2^2 - \xi_3^2 - \xi_4^2.$$

In particular, this implies that the measure $U(x) = \delta([x,x])$ is a solution of the ultrahyperbolic equation $(\partial_1^2 + \partial_2^2 - \partial_3^2 - \partial_4^2)U = 0$ in $\mathbb{R}^4$. (The last two Fourier transforms are connected by the Hilbert transform, see Corollary 2.6 (a).)

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3. Fundamental solutions of $O(p,q)$-invariant operators

If an operator is invariant under the pseudo-orthogonal group $O(p,q)$, it has the form $P(\partial) = Q([\partial,\partial])$ where $Q(u)$ is a polynomial in one variable, i.e., $Q(u) = c\prod_{j=1}^m (u-z_j)^{r_j}$ with $c \in \mathbb{C}\setminus\{0\}, r_j \in \mathbb{N}$ and $z_j \in \mathbb{C}$ pairwise different. Fundamental solutions of particular cases of such operators have been investigated in different form since 1950 in [2], [8], [9], [10], [13], [17], [19], [20], [30], [42], [43], [46].

We take up the task of constructing fundamental solutions and, more generally, convolution groups of $O(p,q)$-invariant operators since Theorem 2.4 and Corollary 2.5 allow to consider these examples from a new view-point. Let us start with the ultra-hyperbolic operator itself, i.e.,

$$P(\partial) = [\partial,\partial] = \partial_1^2 + \cdots + \partial_p^2 - \partial_{p+1}^2 - \cdots - \partial_n^2.$$

As in Proposition 2.8, we assume here that $p,q \in \mathbb{N}_0$ with $n = p+q \geq 3$.

Generally, if $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\psi(x) = [x,x]\cdot\phi(x)$, then $M_\psi(s) = s \cdot M_\phi(s)$ and $\mathcal{F}^{-1}(M_\psi) = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\sigma}\mathcal{F}^{-1}(M_\phi)$. If therefore $\mathcal{F}S \in \mathcal{D}'_{L^1,-n/2}$, then

$$\begin{aligned}\langle\phi,[x,x]\cdot S([x,x])\rangle &= \langle\psi,S([x,x])\rangle = \langle\mathcal{F}^{-1}(M_\psi),\mathcal{F}S\rangle = \langle -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\sigma}\mathcal{F}^{-1}(M_\phi),\mathcal{F}S\rangle \\ &= \langle\mathcal{F}^{-1}(M_\phi),\mathcal{F}(s\cdot S)\rangle = \langle\phi,(s\cdot S)([x,x])\rangle,\end{aligned}$$

i.e., $[x,x]\cdot S([x,x]) = (sS)([x,x])$ as one would expect.

In particular, if $S = \mathrm{vp}\,\frac{1}{s}$, then $\mathcal{F}S = -\mathrm{i}\pi\,\mathrm{sign}(\sigma) \in \mathcal{D}'_{L^1,-n/2}$, and hence the composition $\mathrm{vp}\,\frac{1}{[x,x]}$ is well-defined and fulfills $[x,x]\cdot\mathrm{vp}\,\frac{1}{[x,x]} = 1$. This implies that

$$E = -\mathcal{F}^{-1}\Big(\mathrm{vp}\,\frac{1}{[\xi,\xi]}\Big) = -(2\pi)^{-n}\mathcal{F}\Big(\mathrm{vp}\,\frac{1}{[\xi,\xi]}\Big) \tag{3.1}$$

is a fundamental solution of $[\partial,\partial]$, i.e., $[\partial,\partial]E = \delta$.

We next employ Corollary 2.5 in order to calculate E.

Proposition 3.1. *Let $p,q \in \mathbb{N}_0$ with $n = p+q \geq 3$. Then the fundamental solution E of the ultra-hyperbolic operator $[\partial,\partial]$ defined in* (3.1) *is given explicitly by $E = T([x,x])$ where $T \in \mathcal{F}(\mathcal{D}'_{L^1,-n/2})$ and*

$$T = \frac{1}{4\pi^{n/2}}\cdot\begin{cases}\Gamma(\frac{n}{2}-1)(-1)^{q/2+1}t_+^{1-n/2} & : n,p \text{ odd,}\\ \Gamma(\frac{n}{2}-1)(-1)^{p/2}t_-^{1-n/2} & : n,q \text{ odd,}\\ (-1)^{p/2+1}(\mathrm{vp}\,\frac{1}{t})^{(n/2-2)} & : p,q \text{ even,}\\ (-1)^{(p-1)/2}\pi\delta^{(n/2-2)} & : p,q \text{ odd.}\end{cases} \tag{3.2}$$

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Proof. We set $S = \operatorname{vp}\frac{1}{s}$ and apply Corollary 2.5. Due to

$$S = \operatorname{vp}\tfrac{1}{s} \Longrightarrow \mathcal{F}S = -\mathrm{i}\pi \operatorname{sign}\sigma \Longrightarrow (\mathcal{F}S)(\tfrac{1}{4u}) = -\mathrm{i}\pi \operatorname{sign} u,$$

formula (2.9) yields $E = T([x,x])$ with $T = \mathcal{F}U$ and

$$U = \frac{\mathrm{i}}{2^3\pi^{n/2}}\big(\mathrm{e}^{\mathrm{i}\pi(p-q)/4}u_+^{n/2-2} - \mathrm{e}^{\mathrm{i}\pi(q-p)/4}u_-^{n/2-2}\big).$$

Upon calculating the Fourier transform of U using [29, Ex. 1.6.7, p. 86] (as in the proof of Proposition 2.8) we obtain formula (3.2). □

Note that the distributions T in (2.13) and in (3.2), respectively, are related by the Hilbert transform according to Corollary 2.6 (b). Furthermore, the different cases (with respect to p, q, n) in formula (3.2) are also connected by the Hilbert transform and by (fractional) differentiation, see Corollary 2.6 (a), (c) and (d). E.g., let us observe that the fundamental solutions

$$E_{4,0} = -\frac{1}{4\pi^2|x|^2},\ E_{3,1} = -\frac{\delta(x_1^2+x_2^2+x_3^2-x_4^2)}{4\pi},\ E_{2,2} = \frac{1}{4\pi^2}\operatorname{vp}\Big(\frac{1}{x_1^2+x_2^2-x_3^2-x_4^2}\Big)$$

of the operators $\Delta_4, \Delta_3 - \partial_4^2, \partial_1^2+\partial_2^2-\partial_3^2-\partial_4^2,$ respectively, are all connected by the Hilbert transform of the corresponding one-dimensional distributions

$$T_{4,0} = -\tfrac{1}{4\pi^2}\ \operatorname{vp}\tfrac{1}{t},\quad T_{3,1} = -\tfrac{\delta}{4\pi},\quad T_{2,2} = \tfrac{1}{4\pi^2}\ \operatorname{vp}\tfrac{1}{t}.$$

Formula (3.2) coincides with [28, (2.7.10/11/12), pp. 80, 81], where the more general case $[\partial,\partial]^l,\ \ l \in \mathbb{N},$ is tackled, see Corollary 3.6 below. Fundamental solutions for the ultra-hyperbolic operator $[\partial,\partial]$ were given for the first time in [30, pp. 340, 358–362], where E is represented as a limit for $\epsilon \searrow 0$ with respect to the regions $|[x,x]| > \epsilon$. In [13, Ch. III, 2.5, pp. 281, 282], representations of the fundamental solutions E are given as limits of powers of $[x,x] \pm \mathrm{i}\epsilon|x|^2,\ \ \epsilon \searrow 0$. Another representation of E is derived in [43, Thm. 6.1, p. 214]. In [8, p. 11], E is represented in the form $t_\pm^\lambda \circ [x,x]$, in [9, pp. 219–221] representations by residues are given. In [10], fundamental solutions of $[\partial,\partial]$ were constructed first for the case of odd p, q and thereafter, by the method of descent, for the other cases.

In [20], the formula (4.8), p. 202, for a fundamental solution E of $[\partial,\partial]$, i.e.,

$$E = \frac{1}{4\Omega_{p,q}} \cdot \frac{t_+^\lambda}{\Gamma(\lambda+1)}\bigg|_{\lambda=-n/2+1} \circ [x,x]$$

is meaningful and coincides with (3.2) above in the case of *odd* p. In contrast, for *even* p, the constant $\Omega_{p,q}$ vanishes and the distribution $F = (t_+^\lambda/\Gamma(\lambda+1))|_{\lambda=-n/2+1} \circ [x,x]$ coincides, up to a factor, with $\mathcal{F}(\delta([x,x]))$, see Prop. 2.8, and hence fulfills the homogeneous equation $[\partial,\partial]F = 0$. Therefore formula (4.8) in [20] is meaningless for even p, and, what is more conspicuous, if p is even, then there do not exist any

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fundamental solutions of $[\partial,\partial]$ of the form $T\circ[x,x]$ with $\operatorname{supp} T\subset[0,\infty)$. Thus the whole procedure in [20], which relies on the analytic continuation of $Y([x,x])[x,x]^\lambda$ with respect to λ, has a flaw in this case.

Let us next consider the operator $P(\partial)=z-[\partial,\partial]$ with $z\in\mathbb{C}\setminus\mathbb{R}$. If the logarithm is defined as usually on the slit plane $\mathbb{C}\setminus(-\infty,0]$, then the application

$$\mathbb{C}\longrightarrow\mathcal{O}_M(\mathbb{R}^n_\xi):\lambda\longmapsto(z+[\xi,\xi])^\lambda=\mathrm{e}^{\lambda\log(z+[\xi,\xi])}=:S_\lambda([\xi,\xi])$$

is well-defined and holomorphic. It is also a homomorphism, i.e.,

$$S_{\lambda+\mu}([\xi,\xi])=S_\lambda([\xi,\xi])\cdot S_\mu([\xi,\xi]).$$

Therefore, the inverse Fourier transform of $S_\lambda([\xi,\xi])$ yields the so-called "convolution group" $E_\lambda=\mathcal{F}^{-1}\big(S_\lambda([\xi,\xi])\big)\in\mathcal{O}'_C(\mathbb{R}^n)$ of the operator $P(\partial)$. Again, the mapping

$$E:\mathbb{C}\longrightarrow\mathcal{O}'_C(\mathbb{R}^n_x):\lambda\longmapsto E_\lambda=\mathcal{F}^{-1}\big(S_\lambda([\xi,\xi])\big)$$

is holomorphic, it fulfills $E_{\lambda+\mu}=E_\lambda*E_\mu$, and E_{-k}, $k\in\mathbb{N}$, is the uniquely determined temperate fundamental solution of $(z-[\partial,\partial])^k$, see also [29, Prop. 2.3.3, p. 137]. In the next proposition, we represent E_λ explicitly by Bessel functions.

Proposition 3.2. *Let $n\in\mathbb{N}$, $p,q\in\mathbb{N}_0$ with $n=p+q$, and $z\in\mathbb{C}\setminus\mathbb{R}$, $\rho=\operatorname{sign}(\operatorname{Im} z)$, and set $E_\lambda=\mathcal{F}^{-1}\big((z+[\xi,\xi])^\lambda\big)$ for $\lambda\in\mathbb{C}$.*

If $\operatorname{Re}\lambda<0$, then $E_\lambda=T_\lambda([x,x])$ where

$$T:\{\lambda\in\mathbb{C};\operatorname{Re}\lambda<0\}\longrightarrow\mathcal{F}(\mathcal{D}'_{L^1,-n/2}(\mathbb{R}^1)):\lambda\longmapsto T_\lambda$$

is holomorphic. Furthermore, T_λ is a continuous function for $\operatorname{Re}\lambda<-\frac{n}{2}$, and outside the origin it is $\mathcal{C}^\infty$ for each λ with $\operatorname{Re}\lambda<0$ and given by

$$T_\lambda(t)=\frac{\mathrm{e}^{-\mathrm{i}\pi q\rho/2}z^{\lambda+n/2}}{2^{n/2-\lambda-1}\pi^{n/2}\Gamma(-\lambda)}\,(zt)^{-\lambda/2-n/4}K_{\lambda+n/2}(\sqrt{zt}),\quad t\neq0. \tag{3.3}$$

Finally, the recursion formula $T_{\lambda+1}=4(\lambda+1)T'_\lambda$ holds for $\operatorname{Re}\lambda<-1$.

[*The power $(zt)^{-\lambda/2-n/4}$ and the square root $\sqrt{zt}$ are defined as usually by taking* $\arg(zt)\in(-\pi,\pi)$.]

Proof. (a) We first assume that $\operatorname{Im} z>0$ and calculate

$$\mathcal{F}S_\lambda=\mathcal{F}((z+s)^\lambda)\in\mathcal{O}'_C(\mathbb{R}^1_\sigma)\subset\mathcal{D}'_{L^1,-n/2}(\mathbb{R}^1_\sigma).$$

The holomorphic mapping

$$\chi:\{\mu\in\mathbb{C};\operatorname{Re}\mu>-1\}\longrightarrow\mathcal{S}'(\mathbb{R}^1_\sigma):\mu\longmapsto\chi_\mu=\frac{\sigma_+^\mu}{\Gamma(\mu+1)}$$

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can analytically be continued to an entire function, and its Fourier transform is given by

$$\mathcal{F}(\chi_\mu) = \lim_{\epsilon \searrow 0} (\mathrm{i}s + \epsilon)^{-\mu-1}, \qquad \mu \in \mathbb{C},$$

see [16, Ex. 7.1.17, p. 167], [29, (1.6.8), p. 88], [13, p. 360, (21)]. By translation, we obtain

$$\mathcal{F}^{-1}\Big(\frac{\sigma_+^\mu}{\Gamma(\mu+1)}\,\mathrm{e}^{\mathrm{i}\sigma z}\Big) = \frac{1}{2\pi}\,\mathrm{e}^{\mathrm{i}\pi(\mu+1)/2}(s+z)^{-\mu-1}$$

and hence

$$\mathcal{F}S_\lambda = 2\pi \mathrm{e}^{\mathrm{i}\pi\lambda/2}\mathrm{e}^{\mathrm{i}\sigma z}\chi_{-\lambda-1}(\sigma).$$

(b) Let us observe that $\mathcal{F}S_\lambda \in L^1_{-n/2}$ if and only if $\operatorname{Re}\lambda < 0$. For such λ, we can apply Corollary 2.5 and obtain

$$E_\lambda = \mathcal{F}^{-1}(S_\lambda([\xi,\xi])) = (2\pi)^{-n}\mathcal{F}(S_\lambda([\xi,\xi])) = T_\lambda([x,x])$$

with $T_\lambda = \mathcal{F}U_\lambda$ and

$$U_\lambda = \frac{2^{2\lambda}\mathrm{e}^{\mathrm{i}\pi(2\lambda+p-q)/4}}{\pi^{n/2}\Gamma(-\lambda)}\,u_+^{n/2+\lambda-1}\mathrm{e}^{\mathrm{i}z/(4u)}.$$

If $\operatorname{Re}\lambda < -\frac{n}{2}$, then $U_\lambda \in L^1$ and hence T_λ is continuous and given by the absolutely convergent integral

$$T_\lambda(t) = \frac{2^{2\lambda}\mathrm{e}^{\mathrm{i}\pi(2\lambda+p-q)/4}}{\pi^{n/2}\Gamma(-\lambda)}\int_0^\infty u^{n/2+\lambda-1}\mathrm{e}^{\mathrm{i}z/(4u)-\mathrm{i}tu}\,\mathrm{d}u. \tag{3.4}$$

We note that formula (3.4) implies $T_{\lambda+1} = 4(\lambda+1)T'_\lambda$ if $\operatorname{Re}\lambda < -\frac{n}{2}-1$. Clearly, this recurrence relation is then preserved for all λ with $\operatorname{Re}\lambda < -1$ by analytic continuation.

Finally, we use formula 8.432.7 in [14] to infer formula (3.3) from (3.4) in the case $\operatorname{Im} z > 0$. For $\operatorname{Im} z < 0$, we use complex conjugation, i.e., $T_{\lambda,\bar z} = \overline{T_{\bar\lambda,z}}$. The proof is complete. □

Let us note that T_λ is still locally integrable and given by the formula in (3.3) if $\operatorname{Re}\lambda < 1 - \frac{n}{2}$. Furthermore, outside the cone $[x,x] = 0$, the distribution E_λ is $\mathcal{C}^\infty$ for each $\lambda \in \mathbb{C}$ and the equation $E_\lambda(x) = T_\lambda([x,x])$ holds if $T_\lambda(t)$ is defined as in (3.3) for $t \neq 0$. In particular, if $\lambda = k \in \mathbb{N}_0$, then $E_k = (z - [\partial,\partial])^k\delta$. Furthermore, we mention that differentiation with respect to z yields the recursion formula $\partial T_{\lambda,z}/\partial z = \lambda T_{\lambda-1,z}$, $\operatorname{Re}\lambda < 0$, which is also immediate from the definition $E_\lambda = \mathcal{F}^{-1}((z+[\xi,\xi])^\lambda)$.

In the Lorentz case, i.e., for $p = 1, q = n-1$, formula (3.3) coincides with formula (4.1) in [27, Prop. 6, p. 146] if we set $z = c^2$, $\operatorname{Re} c > 0$, $\operatorname{Im} c \neq 0$, and replace λ by $-\lambda$.

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We next aim at a representation of the fundamental solutions E_{-k}, $k \in \mathbb{N}$, which is as explicit as possible. For this purpose, let us first observe that, for fixed $z \in \mathbb{C} \setminus \mathbb{R}$, the meromorphic function

$$(zt)^\lambda : \mathbb{C} \setminus (-\mathbb{N}) \longrightarrow \mathcal{S}'(\mathbb{R}) : \lambda \longmapsto (zt)^\lambda := z^\lambda t_+^\lambda + (-z)^\lambda t_-^\lambda$$

(where, as always, $(\pm z)^\lambda = \exp(\lambda \cdot \log(\pm z))$ and the complex logarithm is taken with a cut in $(-\infty, 0]$) is in fact entire, since the residues at $\lambda = -k$, $k \in \mathbb{N}$, vanish:

$$\operatorname*{Res}_{\lambda=-k} (zt)^\lambda = z^{-k} \frac{(-1)^{k-1}}{(k-1)!}\, \delta^{(k-1)} + (-z)^{-k} \frac{1}{(k-1)!}\, \delta^{(k-1)} = 0,$$

cf. [29, (1.4.1), p. 49]. Note, however, that for $\lambda = -k$, $k \in \mathbb{N}$, the following formula holds:

$$(zt)^{-k} = \lim_{\lambda \to -k} (zt)^\lambda = z^{-k} t_+^{-k} + (-z)^{-k} t_-^{-k} + \mathrm{i}\pi \operatorname{sign}(\operatorname{Im} z) z^{-k} \cdot \frac{(-1)^{k-1}}{(k-1)!}\, \delta^{(k-1)}.$$

Furthermore, also $\lambda \mapsto (zt)^\lambda \cdot \log(zt) = \partial (zt)^\lambda / \partial \lambda$ is entire. This implies that the power series representation of $T_\lambda(t)$ makes sense as a distribution in $\mathcal{S}'(\mathbb{R}^1_t)$ for each $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda < 0$.

Corollary 3.3. *Let $n \in \mathbb{N}$, $p, q \in \mathbb{N}_0$ with $n = p+q$, and $z \in \mathbb{C} \setminus \mathbb{R}$, $\rho = \operatorname{sign}(\operatorname{Im} z)$, and $k \in \mathbb{N}$. Then the uniquely determined temperate fundamental solution E_{-k} of the operator $(z - [\partial, \partial])^k$ is given by $E_{-k} = T_{-k}([x,x])$ where*

$$T_{-k}(t) = \frac{\mathrm{e}^{-\mathrm{i}\pi q\rho/2} z^{n/2-k}}{2^{n/2+k-1} \pi^{n/2} (k-1)!} (zt)^{k/2-n/4} K_{k-n/2}(\sqrt{zt}). \tag{3.5}$$

If $k > \frac{n}{2}$, E_{-k} and T_{-k} are continuous functions, if $k > \frac{n}{2} - 1$, E_{-k} and T_{-k} are locally integrable, and if $1 \le k \le \frac{n}{2} - 1$, $T_{-k} \in \mathcal{F}(L^1_{-n/2})$ has to be interpreted as a distribution in $\mathcal{S}'(\mathbb{R}^1_t)$ as explained above or by the recursion formula

$$T_{-k} = (-4)^m \frac{(k+m-1)!}{(k-1)!} \left(T_{-k-m}\right)^{(m)}, \qquad m \in \mathbb{N}. \tag{3.6}$$

More explicitly still, we have

(3.7)

$$T_{-k} = \begin{cases} \dfrac{\mathrm{e}^{-\mathrm{i}\pi q\rho/2} z^{n/2-k} \mathrm{e}^{-\sqrt{zt}}}{2^{(n-1)/2+k} \pi^{(n-1)/2} (k-1)!} \displaystyle\sum_{j=0}^{l} \frac{(l+j)!}{j!(l-j)!\, 2^j} (zt)^{(k-j)/2-(n+1)/4} : \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \text{ odd}, \ l = [|k - \tfrac{n}{2}|], \\ \dfrac{\mathrm{e}^{-\mathrm{i}\pi q\rho/2} z^{n/2-k}}{2^{n/2+k} \pi^{n/2} (k-1)!} \Bigg\{ \displaystyle\sum_{j=0}^{l-1} \frac{(-1)^j (l-j-1)!}{j!\, 2^{2j-l}} (zt)^{j+(k-n/2)Y(n/2-k)} + \\ + \displaystyle\sum_{j=0}^{\infty} \frac{(-1)^{l+1} (zt)^{j+(k-n/2)Y(k-n/2)}}{j!(l+j)!\, 2^{2j+l}} \left(\log(\tfrac{zt}{4}) - \psi(j+1) - \psi(l+j+1) \right) \Bigg\} : \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n \text{ even}, \ l = |k - \tfrac{n}{2}|. \end{cases}$$

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Proof. We put $\lambda = -k$ into formula (3.3) and observe that $(zt)^{-\lambda/2-n/4}K_{\lambda+n/2}(\sqrt{tz}) \in \mathcal{S}'(\mathbb{R}^1_t)$ analytically depends on λ if it is interpreted with the help of the series expansion for K_ν as explained before Corollary 3.3. Therefore, formula (3.3) in Proposition 3.2 implies equation (3.5). Inserting the explicit formulas for K_ν in [14, 8.446 and 8.468] yields the series expansions in (3.7). Finally, formula (3.6) is a consequence of the recursion formula in Proposition 3.2. The proof is complete. □

Let us write down formulas for the temperate fundamental solution $E_{-1} = T_{-1}([x,x])$ of $P(\partial) = z - \partial_1^2 + \partial_2^2 + \cdots + \partial_n^2$ in the cases $n = 1, \dots, 4$. If $n = 1$, then $T_{-1}(t) = \mathrm{e}^{-\sqrt{zt}}/(2\sqrt{z})$ is still continuous, if $n = 2, 3$, the distribution T_{-1} is a locally integrable function and (3.5) and (3.7) furnish

$$T_{-1}(t) = \begin{cases} \dfrac{K_0\big(\sqrt{zt}\big)}{2\pi\mathrm{i}\rho} & : n = 2, \\ \dfrac{-\sqrt{z}\,\mathrm{e}^{-\sqrt{zt}}}{4\pi\sqrt{zt}} & : n = 3. \end{cases} \tag{3.8}$$

If $n = 4$, then (3.5) yields

$$T_{-1} = \frac{\mathrm{i}\rho z}{4\pi^2}\,(zt)^{-1/2}K_1(\sqrt{zt}),$$

which has to be interpreted according to what has been said before Corollary 3.3. From formula (3.7), we infer that T_{-1} has the singular term

$$\frac{\mathrm{i}\rho z}{4\pi^2}\,(zt)^{-1} = \frac{\mathrm{i}\rho z}{4\pi^2}\,\big[z^{-1}\,\mathrm{vp}(\tfrac{1}{t}) + \mathrm{i}\pi\rho z^{-1}\delta\big] = \frac{\mathrm{i}\rho}{4\pi^2}\,\mathrm{vp}(\tfrac{1}{t}) - \frac{1}{4\pi}\,\delta.$$

Therefore, we obtain

$$T_{-1} = \frac{\mathrm{i}\rho}{4\pi^2}\,\mathrm{vp}\Big(\frac{zK_1(\sqrt{zt})}{\sqrt{zt}}\Big) - \frac{1}{4\pi}\,\delta(t). \tag{3.9}$$

Formulas (3.8/9) are in complete agreement with [27, Cor. 2, p. 148] if we set $z = c^2$ and note that there the operator $-P(\partial)$ is considered.

From formula (3.7), we can also infer the singular part of T_{-k} if $1 \le k \le \frac{n}{2} - 1$. Let us first assume that n is even. Then $l = \frac{n}{2} - k$ and $T_{-k} = S_{-k} + f_{-k}$ with $f_{-k} \in L^1_{\mathrm{loc}}(\mathbb{R}^1_t)$ and

$$S_{-k} = \frac{\mathrm{e}^{-\mathrm{i}\pi q\rho/2}(-1)^{n/2-k-1}}{2^{2k}\pi^{n/2}(k-1)!} \sum_{j=0}^{n/2-k-1} \frac{z^j}{j!\,2^{2j}}\,(\mathrm{vp}\,\tfrac{1}{t} + \mathrm{i}\pi\rho\delta)^{(n/2-k-1-j)}. \tag{3.10}$$

In contrast, if n is odd, we start with the locally integrable function

$$T_{-(n-1)/2} = \frac{\mathrm{e}^{-\mathrm{i}\pi q\rho/2}\sqrt{z}\,\mathrm{e}^{-\sqrt{zt}}}{2^{n-1}\pi^{(n-1)/2}\,\frac{n-3}{2}!\sqrt{zt}}$$

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and use the recursion formula (3.6):

$$T_{-k} = T_{-(n-1)/2+(n-1)/2-k} = (-4)^{(n-1)/2-k} \cdot \frac{\frac{n-3}{2}!}{(k-1)!} \cdot T^{((n-1)/2-k)}_{-(n-1)/2}$$
$$= \frac{\mathrm{e}^{-\mathrm{i}\pi q\rho/2}\sqrt{z}\,(-1)^{(n-1)/2-k}}{2^{2k}\pi^{(n-1)/2}(k-1)!}\left(\frac{\mathrm{e}^{-\sqrt{zt}}}{\sqrt{zt}}\right)^{((n-1)/2-k)}.$$

Therefore,

$$S_{-k} = \frac{\mathrm{e}^{-\mathrm{i}\pi q\rho/2}\sqrt{z}\,(-1)^{(n-1)/2-k}}{2^{2k}\pi^{(n-1)/2}(k-1)!} \sum_{j=0}^{(n-3)/2-k} \frac{1}{(2j)!}\big((zt)^{j-1/2}\big)^{((n-1)/2-k)}$$

(3.11)

$$= \frac{\mathrm{e}^{-\mathrm{i}\pi q\rho/2}(-1)^{(n-1)/2-k}}{2^{2k}\pi^{(n-1)/2}(k-1)!} \sum_{j=0}^{(n-3)/2-k} \frac{z^j}{j!\,2^{2j}}\left(t_+^{-1/2} + \mathrm{i}\rho t_-^{-1/2}\right)^{((n-1)/2-k-j)}.$$

We next consider the generalized *Klein-Gordon operator* $m^2 - [\partial,\partial]$ for $m > 0$.

Proposition 3.4. *Let* $n \in \mathbb{N}$, $p, q \in \mathbb{N}_0$ *with* $n = p + q$, *and* $m > 0$ *and set*

$$E_\lambda^\pm := \lim_{\epsilon \searrow 0} \mathcal{F}^{-1}\big((m^2 \pm \mathrm{i}\epsilon + [\xi,\xi])^\lambda\big).$$

Then $E_\lambda^\pm$ *is well-defined, the mappings* $\lambda \mapsto E_\lambda^\pm$ *are entire, and if* $\operatorname{Re}\lambda < 0$, *then* $E_\lambda^\pm = T_\lambda^\pm([x,x])$ *where*

$$T^\pm : \{\lambda \in \mathbb{C};\, \operatorname{Re}\lambda < 0\} \longrightarrow \mathcal{F}(\mathcal{D}'_{L^1,-n/2}(\mathbb{R}^1)) : \lambda \longmapsto T_\lambda^\pm$$

is holomorphic. Furthermore, $T_\lambda^\pm$ *is a continuous function for* $\operatorname{Re}\lambda < -\frac{n}{2}$, *and outside the origin it is* $\mathcal{C}^\infty$ *for each* λ *with* $\operatorname{Re}\lambda < 0$ *and given by*

$$T_\lambda^\pm(t) = \frac{\mathrm{e}^{\mp \mathrm{i}\pi q/2} m^{\lambda+n/2}|t|^{-\lambda/2-n/4}}{2^{n/2-\lambda-1}\pi^{n/2}\Gamma(-\lambda)}\Big[Y(t)K_{-\lambda-n/2}\big(m\sqrt{t}\big)$$
$$\pm \frac{\mathrm{i}\pi}{2}\,Y(-t)J_{-\lambda-n/2}\big(m\sqrt{|t|}\big) - \frac{\pi}{2}\,Y(-t)N_{-\lambda-n/2}\big(m\sqrt{|t|}\big)\Big], \quad t \neq 0. \tag{3.12}$$

Finally, the recursion formula $T_{\lambda+1}^\pm = 4(\lambda+1)(T_\lambda^\pm)'$ *holds for* $\operatorname{Re}\lambda < -1$.

Proof. Since

$$h : \mathbb{R}^n \longrightarrow \mathbb{R} : \xi \longmapsto m^2 + [\xi,\xi]$$

is submersive on $h^{-1}(0)$ and the two mappings

$$\mathbb{C} \longrightarrow \mathcal{S}'(\mathbb{R}^1_t) : \lambda \longmapsto (t \pm \mathrm{i}0)^\lambda := \lim_{\epsilon \searrow 0}(t \pm \mathrm{i}\epsilon)^\lambda$$

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are entire, we conclude that also the maps

$$E^{\pm} : \mathbb{C} \longrightarrow \mathcal{S}'(\mathbb{R}^n) : \lambda \longmapsto E^{\pm}_{\lambda} = \lim_{\epsilon \searrow 0} \mathcal{F}^{-1}\big(h^*((t \pm \mathrm{i}\epsilon)^{\lambda})\big) = \mathcal{F}^{-1}\big(h^*((t \pm \mathrm{i}0)^{\lambda})\big)$$

are well-defined and entire.

From the proof of Propositon 3.2, we infer that $E^{\pm}_{\lambda} = T^{\pm}_{\lambda}([x,x])$ if $\operatorname{Re}\lambda > 0$, and that $T^{\pm}_{\lambda} = \mathcal{F}U^{\pm}_{\lambda}$ with

$$U^{\pm}_{\lambda} = \frac{2^{2\lambda}\mathrm{e}^{\pm \mathrm{i}\pi(2\lambda+p-q)/4}}{\pi^{n/2}\Gamma(-\lambda)}\, u_{\pm}^{n/2+\lambda-1}\mathrm{e}^{\pm \mathrm{i}m^2/(4u)} \in \mathcal{D}'_{L^1,-n/2}(\mathbb{R}^1_u).$$

(Note that $\mu \mapsto V^{\pm}_{\mu} := u^{\mu}_{\pm}\mathrm{e}^{\mathrm{i}\alpha/u} \in \mathcal{S}'(\mathbb{R}^1_u)$ are entire distribution-valued functions for fixed $\alpha \in \mathbb{R}\setminus\{0\}$, see the proof of Thm. 2.7.)

Finally, formula (3.12) for $T^{\pm}_{\lambda}(t)$, $t \neq 0$, follows from formula (3.3) upon using

$$K_{\nu}(\pm \mathrm{i}\alpha) = \mp\frac{\pi \mathrm{i}}{2}\,\mathrm{e}^{\mp\nu\pi\mathrm{i}/2}\cdot\big(J_{\nu}(\alpha) \mp \mathrm{i}N_{\nu}(\alpha)\big), \qquad \nu \in \mathbb{C},\ \alpha > 0,$$

see [14, 8.407]. The proof is complete. □

Before continuing, let us specialize formula (3.12) to two important cases. First, if $q = 0$, then $E_{\lambda} = E^{+}_{\lambda} = E^{-}_{\lambda}$ yields the Bessel kernels

$$E_{\lambda} = \mathcal{F}^{-1}\big((m^2 + |\xi|^2)^{\lambda}\big) = \frac{2^{\lambda+1-n/2}}{\pi^{n/2}\Gamma(-\lambda)}\left(\frac{m}{|x|}\right)^{\lambda+n/2} K_{\lambda+n/2}(m|x|),$$

cf. [37, (VII, 7; 23), p. 260], [3, p. 292], [29, Ex. 1.6.11, p. 98]. We mention that the "composition law" $E_{\lambda} * E_{\mu} = E_{\lambda+\mu}$ is still valid in this case, comp. [37, (VI, 8; 5), p. 204], [48, p. 135], in contrast to the general case of arbitrary q where $E^{+}_{\lambda}, E^{+}_{\mu}$ are in general not convolvable. The special case of $q \in 1 + 4\mathbb{N}_0$ is treated in [45].

Let us next consider the case $q = 1$. We remark that in this case, and more generally for odd q, the sum $E^{+}_{\lambda} + E^{-}_{\lambda}$ vanishes for $[x,x] > 0$. This is by no means accidental. In fact, the operator $m^2 - \Delta_{n-1} + \partial_n^2$ is hyperbolic in the x_n-direction, and hence the distributions

$$F^{\pm}_{\lambda} := \lim_{\epsilon \searrow 0} \mathcal{F}^{-1}\big((m^2 + |\xi'|^2 - (\xi_n \mp \mathrm{i}\epsilon)^2)^{\lambda}\big), \qquad \xi' = (\xi_1, \ldots, \xi_{n-1}),$$

have their support in the forward and in the backward light-cone, respectively, i.e.,

$$\operatorname{supp} F^{\pm}_{\lambda} \subset \{x \in \mathbb{R}^n;\ \pm x_n \geq |x'|\}.$$

Since $E^{+}_{\lambda} + E^{-}_{\lambda} = F^{+}_{\lambda} + F^{-}_{\lambda}$, we conclude that $E^{+}_{\lambda} + E^{-}_{\lambda}$ must vanish if $[x,x] = |x'|^2 - x_n^2$ is positive.

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For $\operatorname{Re}\lambda < 1 - \frac{n}{2}$ and $q = 1$, we obtain from formula (3.12), setting $x' = (x_1, \dots, x_{n-1})$,

$$\frac{1}{2}(E_\lambda^+ + E_\lambda^-) = \frac{m^{\lambda+n/2} Y(x_n^2 - |x'|^2)(x_n^2 - |x'|^2)^{-\lambda/2-n/4}}{2^{n/2-\lambda}\pi^{n/2-1}\Gamma(-\lambda)}\, J_{-\lambda-n/2}\big(m\sqrt{x_n^2 - |x'|^2}\big),$$

which coincides with $\frac{1}{2}(F_\lambda^+ + F_\lambda^-)$ given in [29, (2.3.14), p. 145]. This formula can be traced back to [37, (VI, 5; 30), p. 179], where [31, pp. 89–90] is cited as primary source. We also refer to [24, op. 60, p. 166], where the number $n+1$ of the variables should be replaced by n.

In the next corollary, we describe the fundamental solutions $E_{-k} := \frac{1}{2}(E_{-k}^+ + E_{-k}^-)$ of the operators $(m^2 - [\partial,\partial])^k$, $k \in \mathbb{N}$. The special case of n even, p, q odd and $k \le \frac{n}{2} - 1$ can be found already in [45, (II, 12), p. 234]. Let us also remark that the case $k = 1$ is equivalent to Theorem 2.7 via the Hilbert transformation, see Corollary 2.6 (b).

Corollary 3.5. *Let $n \in \mathbb{N}$, $p, q \in \mathbb{N}_0$ with $n = p + q$, and $m > 0$, $k \in \mathbb{N}$. If $E_\lambda^\pm$ are as in* Proposition 3.4, *then $E_{-k} := \frac{1}{2}(E_{-k}^+ + E_{-k}^-)$ is a temperate fundamental solution of the operator $(m^2 - [\partial,\partial])^k$ and $E_{-k} = T_{-k}([x,x])$ where $T_{-k} \in \mathcal{F}(\mathcal{D}'_{L^1,-n/2})$ and*

$$\begin{aligned}(3.13)\quad T_{-k}(t) = {} & \frac{m^{n/2-k}|t|^{k/2-n/4}}{2^{n/2+k-1}\pi^{n/2}(k-1)!}\Big[Y(t)\cos(\tfrac{\pi q}{2})K_{k-n/2}\big(m\sqrt{t}\big) \\ & + \frac{\pi}{2}\,Y(-t)\sin(\tfrac{\pi q}{2})J_{k-n/2}\big(m\sqrt{|t|}\big) - \frac{\pi}{2}\,Y(-t)\cos(\tfrac{\pi q}{2})N_{k-n/2}\big(m\sqrt{|t|}\big)\Big], \quad t \neq 0.\end{aligned}$$

If $k > \frac{n}{2}$, E_{-k} and T_{-k} are continuous functions, if $k > \frac{n}{2} - 1$, E_{-k} and T_{-k} are locally integrable, and if $1 \le k \le \frac{n}{2} - 1$, we have

$$(3.14)\quad T_{-k} = \begin{cases} \dfrac{(-1)^{q/2} m^{n/2-k}|t|^{k/2-n/4}}{2^{n/2+k-1}\pi^{n/2}(k-1)!}\Big[Y(t)K_{k-n/2}\big(m\sqrt{t}\big) \\ \qquad + (-1)^{(n+1)/2+k}\dfrac{\pi}{2}\,Y(-t)J_{n/2-k}\big(m\sqrt{|t|}\big)\Big] & : n, p \text{ odd}, \\ \dfrac{(-1)^{(q-1)/2} m^{n/2-k}|t|^{k/2-n/4}}{2^{n/2+k}\pi^{n/2-1}(k-1)!}\,Y(-t)J_{k-n/2}\big(m\sqrt{|t|}\big) & : n, q \text{ odd}, \\ \dfrac{(-1)^{q/2} m^{n/2-k}|t|^{k/2-n/4}}{2^{n/2+k-1}\pi^{n/2}(k-1)!}\Big[Y(t)K_{k-n/2}\big(m\sqrt{t}\big) \\ \qquad - \dfrac{\pi}{2}\,Y(-t)N_{k-n/2}\big(m\sqrt{|t|}\big)\Big] & : p, q \text{ even}, \\ \dfrac{(-1)^{(p+1)/2+k}}{2^{2k}\pi^{n/2-1}(k-1)!}\Big[Y(-t)\Big(\dfrac{4|t|}{m^2}\Big)^{k/2-n/4}J_{n/2-k}\big(m\sqrt{|t|}\big) \\ \qquad - \displaystyle\sum_{j=0}^{n/2-k-1}\frac{m^{2j}}{j!\,2^{2j}}\,\delta^{(n/2-k-1-j)}\Big] & : p, q \text{ odd}. \end{cases}$$

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[*In formula* (3.14) *negative powers of* t *or* $|t|$ *in the series expansions have to replaced by the distributions* t_+^μ *or* t_-^μ, *respectively.*]

Proof. If we put $\lambda = -k$ into formula (3.12) and set $T_{-k} = \frac{1}{2}(T_{-k}^+ + T_{-k}^-)$, we immediately obtain formula (3.13). Let us now assume $1 \le k \le \frac{n}{2} - 1$. Then the singular part of T_{-k} is just the real part of S_{-k} in (3.10/11) with $z = m^2$. We have to distinguish four cases.

If n and q are odd, then $T_{-k}(t) = 0$ for $t > 0$ and $|t|^{k/2-n/4} J_{k-n/2}(m\sqrt{|t|})$ accounts for the singular part of T_{-k} given by formula (3.11).

If n and p are odd, then T_{-k} is given by a multiple of the bounded function $|t|^{k/2-n/4} J_{n/2-k}(m\sqrt{|t|})$ in the region $t < 0$, due to

$$N_{k-n/2}(m\sqrt{|t|}) = (-1)^{(n-1)/2-k} J_{n/2-k}(m\sqrt{|t|}), \qquad 1 \le k \le \frac{n}{2},\ n \text{ odd},$$

and $t^{k/2-n/4} K_{k-n/2}(m\sqrt{|t|})$ accounts for the singular part of T_{-k} given by formula (3.11).

If n is even and p, q are odd, then $\cos(\frac{\pi q}{2}) = 0$ and the regular part of T_{-k} is given by formula (3.13) since $|t|^{k/2-n/4} J_{n/2-k}(m\sqrt{|t|})$ is a bounded function.

Eventually, if n, p, q are even, then the singular part of T_{-k} is represented by the functions $Y(t) K_{k-n/2}(m\sqrt{t})$ and $-\frac{\pi}{2} Y(-t) N_{k-n/2}\big(m\sqrt{|t|}\big)$ and, as formula (3.10) shows, no additional delta terms appear. The proof is complete. □

For illustration, let us enumerate some special cases. If we set $x' = (x_1, \dots, x_{n-1})$ and consider the Klein-Gordon operator

$$P(\partial) = \partial_n^2 - \Delta_{n-1} + m^2 = \partial_n^2 - \partial_1^2 - \cdots - \partial_{n-1}^2 + m^2, \quad m > 0,$$

in low dimensions, then Corollary 3.5 furnishes (with $q = 1, k = 1$) the fundamental solutions

$$E_{-1} = \begin{cases} \frac{1}{4}\, Y(|x_2| - |x_1|)\, J_0\big(m\sqrt{x_2^2 - x_1^2}\big) & : n = 2, \\ \frac{Y(|x_3| - |x'|)\cos(m\sqrt{x_3^2 - |x'|^2})}{4\pi\sqrt{x_3^2 - |x'|^2}} & : n = 3, \\ \frac{1}{4\pi}\,\delta(x_4^2 - |x'|^2) - \frac{mY(|x_4| - |x'|)J_1(m\sqrt{x_4^2 - |x'|^2})}{8\pi\sqrt{x_4^2 - |x'|^2}} & : n = 4, \\ \frac{1}{8\pi^2}\Big[-t_+^{-3/2} \circ (x_5^2 - |x'|^2) - \frac{Y(|x_5| - |x'|)m\sin(m\sqrt{x_5^2 - |x'|^2})}{x_5^2 - |x'|^2} & \\ \qquad + \frac{Y(|x_5| - |x'|)\big[1 - \cos(m\sqrt{x_5^2 - |x'|^2})\big]}{(x_5^2 - |x'|^2)^{3/2}}\Big] & : n = 5. \end{cases}$$

The physically important forward and backward fundamental solutions, i.e., $\pm 2Y(x_n - |x'|)E_{-1}$, are well-known and can be found in many places, for $n = 4$, e.g., see [12, p. 274], [36, 29.4, p. 186, and 31.5, p. 200], [21, (IV.1), p. 67], [17, (4.13), p. 955], [2, p. 58], [47, III, § 10.12], [29, Ex. 1.6.18, p. 107].

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For a non-trivial example in dimension six, let us consider the operator $P(\partial) = m^2 - \partial_1^2 - \partial_2^2 - \partial_3^2 + \partial_4^2 + \partial_5^2 + \partial_6^2$. Then Corollary 3.5 yields the fundamental solution

$$E_{-1} = \tfrac{1}{4\pi^2}\big[\delta'(|x'|^2 - |x''|^2) + \tfrac{m^2}{4}\,\delta(|x'|^2 - |x''|^2) - Y(|x''| - |x'|)\tfrac{m^2 J_2(m\sqrt{|x''|^2-|x'|^2})}{4(|x''|^2-|x'|^2)}\big],$$

with $x' = (x_1, x_2, x_3)$, $x'' = (x_4, x_5, x_6)$, comp. [10, p. 288] for the case $m = 0$. Note that the fundamental solution $-\frac{1}{4\pi^2}\,\delta'(|x'|^2 - |x''|^2)$ of $\Delta_3' - \Delta_3''$ results also directly from the fundamental solution $-\frac{1}{4\pi^3|x|^4}$ of Δ_6 by the Hilbert transform, see Corollary 2.6 (a).

Eventually, let us come back to the operator $[\partial, \partial]$ itself and give formulas for fundamental solutions of the iterated ultra-hyperbolic operator $[\partial, \partial]^k$ by analyzing the limit of $(-1)^k E_{-k}$ in Corollary 3.5 when $m \to 0$. This generalizes Proposition 3.1, where the case $k = 1$ was considered, and the result is in accordance with [28, pp. 80, 81], see also the literature cited therein.

Corollary 3.6. *Let $n \in \mathbb{N}$, $p, q \in \mathbb{N}_0$ with $n = p + q$, and $k \in \mathbb{N}$. If T is as in formula* (3.15) *below, then $E = T([x, x])$ is an O(p,q)-invariant fundamental solution of $[\partial, \partial]^k$.*

$$(3.15)\quad T = \begin{cases} \dfrac{(-1)^{q/2+k}\,\Gamma(\frac{n}{2}-k)}{2^{2k}(k-1)!\,\pi^{n/2}}\,t_+^{k-n/2} & : n, p \text{ odd}, \\ \dfrac{(-1)^{p/2}\,\Gamma(\frac{n}{2}-k)}{2^{2k}(k-1)!\,\pi^{n/2}}\,t_-^{k-n/2} & : n, q \text{ odd}, \\ \dfrac{(-1)^{p/2+1}}{2^{2k}(k-1)!\,\pi^{n/2}}\,(\mathrm{vp}\,\frac{1}{t})^{(n/2-k-1)} & : n, p \text{ even}, 1 \le k \le \frac{n}{2}-1, \\ \dfrac{(-1)^{p/2+1}}{2^{2k}(k-1)!(k-\frac{n}{2})!\,\pi^{n/2}}\,t^{k-n/2}\log|t| & : n, p \text{ even}, k > \frac{n}{2}-1, \\ \dfrac{(-1)^{(p-1)/2}}{2^{2k}(k-1)!\,\pi^{n/2-1}}\,\delta^{(n/2-k-1)} & : n \text{ even}, p \text{ odd}, 1 \le k \le \frac{n}{2}-1, \\ \dfrac{(-1)^{(q-1)/2+k}}{2^{2k}(k-1)!(k-\frac{n}{2})!\,\pi^{n/2-1}}\,t_-^{k-n/2} & : n \text{ even}, p \text{ odd}, k > \frac{n}{2}-1. \end{cases}$$

Proof. If $1 \le k \le \frac{n}{2} - 1$, then the limit of the regular part of T_{-k} in (3.14) vanishes for $m \to 0$. Hence if T_{-k}, S_{-k} are as in (3.14) and in (3.10/11), respectively, then $E = T([x, x])$ with

$$T = (-1)^k \lim_{m\to 0} T_{-k} = (-1)^k \lim_{z\to 0} \mathrm{Re}\, S_{-k},$$

and this yields readily the results for the considered range of the powers k.

Let us turn now to the case $k > \frac{n}{2} - 1$. If q is odd and T_{-k} is as in (3.13), then the limit $T = (-1)^k \lim_{m\to 0} T_{-k}$ exists since

$$\lim_{m\to 0} m^{n/2-k}|t|^{k/2-n/4}Y(-t)J_{k-n/2}(m\sqrt{|t|}) = \frac{t_-^{k-n/2}}{2^{k-n/2}\Gamma(k-\frac{n}{2}+1)} \in L^1_{\mathrm{loc}}(\mathbb{R}).$$

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For even n, this yields the last formula in (3.15), for odd n, we use the complement formula for the gamma function and obtain the same formula as in the case $k \leq \frac{n}{2} - 1$.

For n, p odd and $k > \frac{n}{2} - 1$, we use the quadratic form

$$-[x,x] = x_{p+1}^2 + \cdots + x_n^2 - x_1^2 - \cdots - x_p^2$$

in which p and q are swapped. Then we obtain the first formula in (3.15) from the second one.

Finally, let us consider the case of n, p, q even and $k \geq \frac{n}{2}$. Note that $\lim_{m\to 0} T_{-k}$ diverges for T_{-k} from (3.13). In fact, we can write $T_{-k} = D_{-k} + C_{-k}$ where the divergent part has the form

$$D_{-k} = \frac{(-1)^{q/2}}{\pi^{n/2}(k-1)!}\Big[\sum_{j=0}^{k-n/2-1} \frac{(k-\frac{n}{2}-1-j)!}{m^{2k-n-2j}2^{n+2j}j!}\,(-t)^j - \frac{(-t)^{k-n/2}}{2^{2k}(k-\frac{n}{2})!}\,\log(m^2\alpha_k)\Big]$$

with $\alpha_k > 0$ and $\lim_{m\to 0} C_{-k}$ exists. Since

$$[\partial,\partial][x,x]^j = 2j(n+2j-2)[x,x]^{j-1},$$

we conclude that

$$\lim_{m\to 0}([\partial,\partial] - m^2)^{k-n/2+1} D_{-k}([x,x]) = 0 \text{ in } \mathcal{S}'(\mathbb{R}^n).$$

Hence we obtain a fundamental solution E of $[\partial,\partial]^k$ in the form $E = T([x,x])$ with $T = \lim_{m\to 0}(-1)^k C_{-k}$. This yields the fourth formula in (3.15) and concludes the proof. □

Finally, let us investigate general $O(p,q)$-invariant operators. For simplicity, we assume that $[\partial,\partial]$ itself does not appear as a factor.

Corollary 3.7. *Let $z_j \in \mathbb{C}\setminus\{0\}$ be pairwise different and $r_j \in \mathbb{N}_0$ for $j = 1,\dots,m$, and define*

$$P(\partial) = \prod_{j=1}^m ([\partial,\partial] - z_j)^{r_j+1} \quad \text{and} \quad c_j = \prod_{\substack{l=1\\ l\neq j}}^m (z_j - z_l)^{-r_l-1}.$$

For $z \in \mathbb{C}\setminus\mathbb{R}$ and $k \in \mathbb{N}$, let $E_{k,z} = (-1)^k T_{-k}([x,x])$ be the unique temperate fundamental solution of $([\partial,\partial]-z)^k$ described in Corollary 3.3*, for $z \in \mathbb{R}\setminus\{0\}$, set $E_{k,z} := \lim_{\epsilon\searrow 0} \frac{1}{2}(E_{k,z+\mathrm{i}\epsilon} + E_{k,z-\mathrm{i}\epsilon})$, see* Corollary 3.5*.*

Then we obtain an $O(p,q)$-invariant fundamental solution E of $P(\partial)$ by the formula

$$E = \sum_{j=1}^m \sum_{i=0}^{r_j} E_{i+1,z_j}\,\frac{1}{(r_j-i)!}\Big(\frac{\partial}{\partial z_j}\Big)^{r_j-i} c_j. \tag{3.16}$$

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Proof. Note that $E_{i+1,z_j} = \frac{1}{i!}\,\partial_{z_j}^i E_{1,z_j}$. Formula (3.16) then follows immediately from [29, Prop. 1.4.4, p. 47], which covers the case of real z_j, but is easily extendable to the complex case. □

The fundamental solution given in Corollary 3.7 can be found, in a less explicit manner, in [42, pp. 355–360].

4. $O(p,q)$-invariant operators depending on further variables

If $Q(u,v)$ is a polynomial in two variables, then we call

$$P(D) = Q(\partial_0, [\partial,\partial]) \quad \text{with } D = (\partial_0, \partial_1, \dots, \partial_n),\ \partial = (\partial_1, \dots, \partial_n),$$

a *time-dependent $O(p,q)$-invariant operator.* The simplest operator of this kind is the generalized Schrödinger operator $\partial_0 - \mathrm{i}[\partial,\partial]$. More generally, in [16, Thm. 3.3.5, p. 82], [29, Ex. 1.4.13, p. 60], fundamental solutions of $\partial_0 - \partial A \partial^T$ are constructed under the conditions that $A = A^T \in \mathbb{C}^{n\times n}$, $\operatorname{Re} A$ is positive semidefinite and $\det A \neq 0$. Starting from the classical fundamental solution of the heat equation one obtains by analytic continuation with respect to the matrices A the following formula for a fundamental solution E:

$$E = \frac{Y(x_0)}{(4\pi x_0)^{n/2}\sqrt{\det A}}\ \exp\Big(-\frac{xA^{-1}x^T}{4x_0}\Big) \in \mathcal{C}([0,\infty), \mathcal{S}'(\mathbb{R}^n_x)) \subset \mathcal{S}'(\mathbb{R}^{n+1}_{x_0,x}),$$

where $x = (x_1, \dots, x_n)$, see [16, (3.3.5), p. 82], [29, (1.4.14), p. 61].

In contrast, the "parabolic-hyperbolic operator" $P(D) = \partial_0 - [\partial,\partial]$ considered in [18] cannot be subsumed under the form $\partial_0 - \partial A \partial^T$ above (except for $q = 0$) since $[x,x]$ has signature (p,q) and hence is not positive semidefinite. Since the function $f(\xi_0,\xi) = (\mathrm{i}\xi_0 + [\xi,\xi])^{-1}$ is locally integrable on $\mathbb{R}^{n+1}$, we obtain a fundamental solution E of $P(D)$ in the form

$$E = \mathcal{F}^{-1}\Big(\frac{1}{\mathrm{i}\xi_0 + [\xi,\xi]}\Big) \in \mathcal{S}'(\mathbb{R}^{n+1}), \tag{4.1}$$

and E is uniquely determined if we assume additionally that E is quasihomogeneous of the type $(2,1,\dots,1)$, i.e., if

$$\exists \lambda \in \mathbb{C} : \forall c > 0 : E(c^2 x_0, cx) = c^\lambda E,$$

see [29, Def. 2.1.1, p. 33], [18, Thm. 1, p. 304]. In fact, in our case $\lambda = -n$ holds.

In the next proposition, we aim at representing explicitly E, which is given as an inverse Fourier transform in (4.1).

Proposition 4.1. *Let $n \in \mathbb{N}$, $p,q \in \mathbb{N}_0$ with $n = p+q$ and set $\partial = (\partial_1,\dots,\partial_n)$, $x = (x_1,\dots,x_n)$, $D = (\partial_0,\partial_1,\dots,\partial_n)$ and*

$$P(D) = \partial_0 - [\partial,\partial] = \partial_0 - \partial_1^2 - \cdots - \partial_p^2 + \partial_{p+1}^2 + \cdots + \partial_n^2.$$

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Then the uniquely determined quasihomogeneous fundamental solution E *of* $P(D)$ *belongs to* $\mathcal{C}^\infty(\mathbb{R}^1_{x_0}\setminus\{0\},\mathcal{S}'(\mathbb{R}^n_x))$. *Furthermore, the two limits* $\lim_{x_0\nearrow 0}E(x_0,x)$ *and* $\lim_{x_0\searrow 0}E(x_0,x)$ *exist in* $\mathcal{S}'(\mathbb{R}^n_x)$. *For each* $x_0\neq 0$, *we have* $E(x_0,x)=T_{x_0}([x,x])$ *with* $T_{x_0}\in\mathcal{F}(L^1_{-n/2}(\mathbb{R}^1))$ *given by*

(4.2)

$$T_{x_0}(t)=\begin{cases}\dfrac{(-1)^{q/2}}{(4\pi x_0)^{(n+1)/2}}\Big[\sqrt{4\pi x_0-\mathrm{i}0}\,\mathrm{e}^{-t/(4x_0)}\Big(Y(x_0)\\ -Y(-t)\,\mathrm{Erf}\Big(\sqrt{\mathrm{i}0-\frac{t}{4x_0}}\Big)\Big)+\displaystyle\sum_{j=0}^{(n-3)/2}(-4x_0)^{j+1}(t_-^{-1/2})^{(j)}(t)\Big] & : n,p\ \textit{odd},\\ \dfrac{(-1)^{(q+1)/2}}{(4\pi x_0)^{(n+1)/2}}\Big[\sqrt{-4\pi x_0-\mathrm{i}0}\,\mathrm{e}^{-t/(4x_0)}\Big(-Y(-x_0)\\ +Y(t)\,\mathrm{Erf}\Big(\sqrt{\mathrm{i}0-\frac{t}{4x_0}}\Big)\Big)+\displaystyle\sum_{j=0}^{(n-3)/2}(-4x_0)^{j+1}(t_+^{-1/2})^{(j)}(t)\Big] & : n,q\ \textit{odd},\\ \dfrac{(-1)^{q/2}}{(4\pi x_0)^{n/2}}\Big[\mathrm{sign}(t)Y(tx_0)\mathrm{e}^{-t/(4x_0)}+\displaystyle\sum_{j=0}^{n/2-2}(-4x_0)^{j+1}\delta^{(j)}(t)\Big] & : p,q\ \textit{even},\\ \dfrac{(-1)^{(q+1)/2}}{(4\pi x_0)^{n/2}\pi}\Big[\mathrm{e}^{-t/(4x_0)}\,\mathrm{Ei}\Big(\dfrac{t}{4x_0}\Big)+\displaystyle\sum_{j=0}^{n/2-2}(-4x_0)^{j+1}(\mathrm{vp}\,\tfrac{1}{t})^{(j)}(t)\Big] & : p,q\ \textit{odd}.\end{cases}$$

[*Here the "exponential integral" is given by* $\mathrm{Ei}(u)=Y(u)*\mathrm{vp}(\mathrm{e}^u/u)=\pi\mathrm{e}^u\mathcal{H}(Y(u)\mathrm{e}^{-u})$, i.e., $\mathrm{Ei}(u)=\int_{-\infty}^u\mathrm{e}^s s^{-1}\mathrm{d}s$ *for* $u<0$ *and*

$$\mathrm{Ei}(u)=\oint_{-\infty}^u\frac{\mathrm{e}^s}{s}\,\mathrm{d}s=\lim_{\epsilon\searrow 0}\Big(\int_{-\infty}^{-\epsilon}\frac{\mathrm{e}^s}{s}\,\mathrm{d}s+\int_\epsilon^u\frac{\mathrm{e}^s}{s}\,\mathrm{d}s\Big)\quad\textit{for } u>0.$$

Furthermore, $\sqrt{4\pi x_0-\mathrm{i}0}=\lim_{\epsilon\searrow 0}\sqrt{4\pi x_0-\mathrm{i}\epsilon}$ *etc. and the error function* Erf *denotes the entire function*

$$\mathrm{Erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z\mathrm{e}^{-s^2}\,\mathrm{d}s,\qquad z\in\mathbb{C}.]$$

Proof. (a) If we denote by $\mathcal{F}_{\xi_0},\mathcal{F}_\xi$ the partial Fourier transforms with respect to ξ_0 and ξ, respectively, then

$$\begin{aligned}E&=(2\pi)^{-n}\mathcal{F}_\xi\Big(\mathcal{F}^{-1}_{\xi_0}\Big(\frac{1}{\mathrm{i}\xi_0+[\xi,\xi]}\Big)\Big)\\&=(2\pi)^{-n}\mathcal{F}_\xi\big(\mathrm{sign}([\xi,\xi])Y(x_0[\xi,\xi])\mathrm{e}^{-x_0[\xi,\xi]}\big)\end{aligned}$$

This shows that E continuously depends on x_0 for $x_0\neq 0$ and that

$$\lim_{x_0\searrow 0}E(x_0,x)=(2\pi)^{-n}\mathcal{F}_\xi\big(Y([\xi,\xi])\big)=\delta+\lim_{x_0\nearrow 0}E(x_0,x).$$

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In the following let us fix $x_0 \in \mathbb{R} \setminus \{0\}$ and calculate $E(x_0, x) \in \mathcal{S}'(\mathbb{R}^n_x)$. Then $E(x_0,x) = (2\pi)^{-n}\mathcal{F}(S([\xi,\xi]))$ with $S(s) = \operatorname{sign}(s)Y(x_0 s)\mathrm{e}^{-x_0 s}$. Evidently, $(\mathcal{F}S)(\sigma) = (x_0 + \mathrm{i}\sigma)^{-1} \in L^1_{-n/2}(\mathbb{R}^1_\sigma)$ and hence Corollary 2.5 can be applied and yields $E(x_0,x) = T_{x_0}([x,x])$ with $T_{x_0} = \mathcal{F}U_{x_0}$ and

$$U_{x_0} = \frac{1}{2}\pi^{-n/2-1}\exp\Big(\frac{\mathrm{i}\pi}{4}(p-q)\operatorname{sign} u\Big)|u|^{n/2-1}\frac{\operatorname{sign} u}{\mathrm{i}+4x_0u} \in L^1_{-n/2}.$$

(b) We shall calculate now $E(x_0,x)$ in the case of even p,q,n and deduce the other cases therefrom by employing Corollary 2.6. For even p,q,n, we have

$$\begin{aligned}\exp\Big(\frac{\mathrm{i}\pi}{4}(p-q)\operatorname{sign} u\Big)|u|^{n/2-1} &= (\mathrm{i}\operatorname{sign} u)^{(p-q)/2}\cdot(\operatorname{sign} u)^{n/2-1}\cdot u^{n/2-1}\\ &= (-1)^{p/2-1}\mathrm{i}\cdot(\operatorname{sign} u)\cdot(-\mathrm{i}u)^{n/2-1}\end{aligned}$$

and hence $T_{x_0} = \mathcal{F}U_{x_0} = (\mathcal{F}V_{x_0})^{(n/2-1)}$, where

$$V_{x_0} = \frac{(-1)^{p/2-1}\mathrm{i}}{2\pi^{n/2+1}}\cdot\frac{1}{\mathrm{i}+4x_0u}.$$

$\mathcal{F}V_{x_0}$ is a classical Fourier transform:

$$\begin{aligned}\mathcal{F}V_{x_0}(t) &= \frac{(-1)^{p/2-1}\mathrm{i}}{2\pi^{n/2+1}}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}tu}}{\mathrm{i}+4x_0u}\,\mathrm{d}u\\ &= \frac{(-1)^{p/2-1}}{4\pi^{n/2}x_0}\operatorname{sign}(t)Y(tx_0)\mathrm{e}^{-t/(4x_0)}\end{aligned}$$

and differentiation eventually yields

$$T_{x_0}(t) = (\mathcal{F}V_{x_0})^{(n/2-1)} = \frac{(-1)^{q/2}}{(4\pi x_0)^{n/2}}\Big[\operatorname{sign}(t)Y(tx_0)\mathrm{e}^{-t/(4x_0)} + \sum_{j=0}^{n/2-2}(-4x_0)^{j+1}\delta^{(j)}(t)\Big]$$

in accordance with the third equation in formula (4.2).

(c) If n is even and p,q are odd, we use the Hilbert transform. Denoting the dependence of T_{x_0} on p,q now by superscripts, Corollary 2.6 (a) implies (e.g., for $q \geq 1$)

$$\begin{aligned}T^{p,q}_{x_0} &= -\mathcal{H}(T^{p+1,q-1}_{x_0})\\ &= \frac{(-1)^{(q+1)/2}}{(4\pi x_0)^{n/2}}\Big[\mathcal{H}\big(\operatorname{sign}(t)Y(tx_0)\mathrm{e}^{-t/(4x_0)}\big) + \frac{1}{\pi}\sum_{j=0}^{n/2-2}(-4x_0)^{j+1}(\operatorname{vp}\tfrac{1}{t})^{(j)}(t)\Big].\end{aligned}$$

Since

$$\mathcal{H}\big(\operatorname{sign}(t)Y(tx_0)\mathrm{e}^{-t/(4x_0)}\big) = \frac{1}{\pi}\,\mathrm{e}^{-t/(4x_0)}\,\operatorname{Ei}\Big(\frac{t}{4x_0}\Big),$$

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which follows immediately from the definition of Ei, see also [6, 15.2, (37), p. 251], we obtain the last formula in (4.2).

If n is odd and p is even, say, we use Corollary 2.6 (d) to infer that

$$\begin{aligned} T^{p,q}_{x_0} &= T^{p,q+1}_{x_0} * t_+^{-1/2} \\ &= \frac{(-1)^{(q+1)/2}}{(4\pi x_0)^{(n+1)/2}} \Big[\big(\operatorname{sign}(t) Y(tx_0) \mathrm{e}^{-t/(4x_0)}\big) * t_+^{-1/2} + \sum_{j=0}^{(n-3)/2} (-4x_0)^{j+1} (t_+^{-1/2})^{(j)}(t) \Big]. \end{aligned}$$

For $a > 0$, we obtain from the definition of Erf that

$$Y(t)\mathrm{e}^{-at} * t_+^{-1/2} = -\mathrm{i} Y(t) \sqrt{\frac{\pi}{a}}\, \mathrm{e}^{-at} \operatorname{Erf}(\mathrm{i}\sqrt{at})$$

and

$$Y(-t)\mathrm{e}^{at} * t_+^{-1/2} = \sqrt{\frac{\pi}{a}}\, \mathrm{e}^{at} \big(1 - Y(t) \operatorname{Erf}(\sqrt{at})\big)$$

and hence

$$\begin{aligned} &\big(\operatorname{sign}(t) Y(tx_0) \mathrm{e}^{-t/(4x_0)}\big) * t_+^{-1/2} \\ &\qquad = \sqrt{-4\pi x_0 - \mathrm{i}0}\, \mathrm{e}^{-t/(4x_0)} \big[-Y(-x_0) + Y(t) \operatorname{Erf}(\sqrt{\mathrm{i}0 - t/(4x_0)}) \big]. \end{aligned}$$

This yields the second formula in (4.2). The result for n, p odd is derived similarly from $T^{p,q}_{x_0} = T^{p+1,q}_{x_0} * t_-^{-1/2}$ or by using Corollary 2.6 (e). This completes the proof. □

The fundamental solution E in Proposition 4.1 was represented in [18, Thm. 3, p. 318], though in a less explicit form, i.e., $E = [\partial, \partial]^{\bar n} F$, $\bar n = [\frac{n-2}{2}]$, $F \in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$. Hence, in Jeanquartier's representation, the singular terms of E are not visible and must be calculated by distributional differentiation. Furthermore, the locally integrable function F is defined by series expansions only.

Let us observe that the leading singular term in formula (4.2) results from the formula for $\mathcal{F}(\delta([x,x]))$ given in Proposition 2.8, (2.13). In fact, since the function

$$\mathbb{R} \setminus \{0\} \longrightarrow \mathcal{S}'(\mathbb{R}^1_t) : \xi_0 \longmapsto (\mathrm{i}\xi_0 + t)^{-1} = g_{\xi_0}(t)$$

has, by Sokhotskij's formula, a discontinuity of $-2\pi\mathrm{i}\delta(t)$ at $\xi_0 = 0$, we conclude that $E = \mathcal{F}^{-1}_{\xi_0} \mathcal{F}^{-1}_{\xi} (g_{\xi_0}([\xi,\xi]))$ fulfills

$$\lim_{x_0 \to \infty} x_0 E = \lim_{x_0 \to \infty} \mathrm{i} \mathcal{F}^{-1}_{\xi_0} \mathcal{F}^{-1}_{\xi} \Big(\frac{\partial g_{\xi_0}}{\partial \xi_0}([\xi,\xi]) \Big) = \mathcal{F}^{-1}_{\xi} (\delta([\xi,\xi])).$$

E.g., in the case p, q even, Proposition 2.8 then yields

$$\lim_{x_0 \to \infty} x_0 T_{x_0}(t) = \frac{(-1)^{p/2+1}}{4\pi^{n/2}}\, \delta^{(n/2-2)}(t)$$

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in accordance with the third formula in (4.2). We also mention that E could be inferred from Corollary 3.3 by Fourier transformation. In fact, if $S(\xi_0) := T_{-1}|_{z=\mathrm{i}\xi_0}$ with T_{-k} as in (3.5), then T_{x_0} in Proposition 4.1 fulfills $T_{x_0} = \mathcal{F}^{-1}_{\xi_0}(S(\xi_0))$.

Let us generalize now the procedure we used in Proposition 4.1 and study operators containing derivatives with respect to m additional variables $y_1, \ldots, y_m$ (instead of the one additional variable x_0 in Proposition 4.1).

Proposition 4.2. *Let $m, n \in \mathbb{N}$, $p, q \in \mathbb{N}_0$ with $n = p + q$ and set $[x, x] = x_1^2 + \cdots + x_p^2 - (x_{p+1}^2 + \cdots + x_n^2)$ as before and*

$$\partial_{x_i} = \partial/\partial x_i,\ \partial_x = (\partial_{x_1}, \ldots, \partial_{x_n}),\ \partial_{y_j} = \partial/\partial y_j,\ \partial_y = (\partial_{y_1}, \ldots, \partial_{y_m}) \text{ and } D = (\partial_x, \partial_y).$$

For $Q \in \mathbb{C}[s, y_1, \ldots, y_m]$, we consider the linear partial differential operator $P(D) = Q([\partial_x, \partial_x], \partial_{y_1}, \ldots, \partial_{y_m})$.

(a) *If $S \in \mathcal{F}(\mathcal{D}'_{L^1,-n/2}(\mathbb{R}^1)) \hat{\otimes} \mathcal{D}'(\mathbb{R}^m)$ fulfills $Q(-s, \partial_y)S(s, y) = 1(s) \otimes \delta(y)$ in $\mathcal{D}'(\mathbb{R}^{m+1}_{s,y})$, then*

$$E(x, y) = (2\pi)^{-n} \mathcal{F}_\xi\big(S([\xi, \xi], y)\big) \in \mathcal{S}'(\mathbb{R}^n_x) \hat{\otimes} \mathcal{D}'(\mathbb{R}^m_y)$$

is a fundamental solution of $P(D)$.

(b) *In particular, if additionally $S \in \mathcal{F}(L^1_{-n/2}) \hat{\otimes} \mathcal{D}'(\mathbb{R}^m)$, then $E(x, y) = T([x, x], y)$ with $T(t, y) = \mathcal{F}_u(U(u, y)) \in \mathcal{F}(L^1_{-n/2}) \hat{\otimes} \mathcal{D}'(\mathbb{R}^m)$ and*

$$U = \frac{1}{2^3 \pi^{n/2+1}} \exp\Big(\frac{\mathrm{i}\pi}{4}(p - q) \operatorname{sign} u\Big) |u|^{n/2-2} \cdot (\mathcal{F}S)(\tfrac{1}{4u}, y). \tag{4.3}$$

Proof. (a) If we use, as before, the pull-back

$$f^* : \mathcal{F}(\mathcal{D}'_{L^1,-n/2}(\mathbb{R}^1)) \longrightarrow \mathcal{S}'(\mathbb{R}^n) : S \longmapsto S([x, x])$$

with respect to the function $f(x) = [x, x]$, see Def. 2.3, we obtain a linear continuous mapping

$$f^* \otimes \mathrm{id} : \mathcal{F}(\mathcal{D}'_{L^1,-n/2}(\mathbb{R}^1)) \hat{\otimes} \mathcal{D}'(\mathbb{R}^n_y) \longrightarrow \mathcal{S}'(\mathbb{R}^n) \hat{\otimes} \mathcal{D}'(\mathbb{R}^n_y).$$

This yields $Q(-[\xi, \xi], \partial_y)S([\xi, \xi], y) = 1(\xi) \otimes \delta(y)$ and, by partial Fourier transform with respect to ξ,

$$P(D)E = P(D)\mathcal{F}^{-1}_\xi(S([\xi, \xi], y)) = \mathcal{F}^{-1}_\xi(Q(-[\xi, \xi], \partial_y)S([\xi, \xi], y)) = \delta.$$

(b) Formula (4.3) is a consequence of formula (2.9) in Corollary 2.5. □

We apply Proposition 4.2 in order to generalize Sobolev's operator $\partial^3/\partial x \partial y \partial z - \frac{1}{4}\partial/\partial t$, see [38], [25, Prop. 6, p. 158], [29, Ex. 2.3.8, p. 148, Ex. 2.6.4, p. 172]. If we replace the operator $\partial^2/\partial x \partial y$ by $[\partial_x, \partial_x]$, we arrive at the generalization $P(D) =$

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$\partial_{y_1} + [\partial_x, \partial_x]\partial_{y_2}$, which belongs to the class of operators considered in Proposition 4.2. Let us observe that $P(D)$ is quasihyperbolic in the y_1-direction (see [29, 2.4.13, p. 162]) and hence possesses exactly one temperate fundamental solution E with support in the half-space $\{(x,y) \in \mathbb{R}^{n+2};\, y_1 \geq 0\}$. We represent E by Bessel functions in the next proposition.

Proposition 4.3. *Let $n \in \mathbb{N}$, $p, q \in \mathbb{N}_0$ with $n = p + q$ and set $[\partial_x, \partial_x]$ as above. The operator*

$$P(D) = P(\partial_{x_1}, \dots, \partial_{x_n}, \partial_{y_1}, \partial_{y_2}) = \partial_{y_1} + [\partial_x, \partial_x]\partial_{y_2}$$

has one and only one temperate fundamental solution $E \in \mathcal{S}'(\mathbb{R}^{n+2})$ with support in the half-space $\{(x,y) \in \mathbb{R}^{n+2};\, y_1 \geq 0\}$. Furthermore, E depends $\mathcal{C}^\infty$ on y_1 for $y_1 \geq 0$, i.e., E is represented through the $\mathcal{C}^\infty$ mapping

$$E : [0,\infty) \longrightarrow \mathcal{S}'(\mathbb{R}^{n+1}_{x,y_2}) : y_1 \longmapsto E_{y_1}$$

by

$$\langle \phi, E\rangle = \int_0^\infty \langle \phi(x, y_1, y_2), E_{y_1}(x, y_2)\rangle \, \mathrm{d}y_1, \qquad \phi \in \mathcal{S}(\mathbb{R}^{n+2}).$$

In particular, $E_0 = \lim_{y_1 \searrow 0} E_{y_1} = \delta(x) \otimes \delta(y_2)$.

For fixed $y_1 > 0$, we have $E_{y_1}(x, y_2) = T_{y_1}([x,x], y_2)$ where $T_{y_1} \in \mathcal{F}(L^1_{-n/2}(\mathbb{R}^1)) \hat{\otimes} \mathcal{S}'(\mathbb{R}^1_{y_2})$. If p or q are odd and $y_1 > 0$, then

$$\begin{aligned} T_{y_1}(t) = (2\pi)^{-n/2} y_1^{-n/4-1/2} \Big|\frac{y_2}{t}\Big|^{n/4-1/2} \times \\ \times \Big\{ \tfrac{1}{2}\Big[Y(-t)Y(y_2)\cos\Big(\frac{\pi p}{2}\Big) + Y(t)Y(-y_2)\cos\Big(\frac{\pi q}{2}\Big)\Big] J_{n/2-1}\Big(\sqrt{-\tfrac{y_2 t}{y_1}}\Big) \\ - \tfrac{1}{2}\Big[Y(-t)Y(y_2)\sin\Big(\frac{\pi p}{2}\Big) + Y(t)Y(-y_2)\sin\Big(\frac{\pi q}{2}\Big)\Big] N_{n/2-1}\Big(\sqrt{-\tfrac{y_2 t}{y_1}}\Big) \\ + \tfrac{1}{\pi}\Big[Y(t)Y(y_2)\sin\Big(\frac{\pi q}{2}\Big) + Y(-t)Y(-y_2)\sin\Big(\frac{\pi p}{2}\Big)\Big] K_{n/2-1}\Big(\sqrt{\tfrac{y_2 t}{y_1}}\Big)\Big\}. \end{aligned} \tag{4.4}$$

[*In formula* (4.4)*, negative powers of t or $|t|$ in the series expansions have to replaced by the distributions t_+^μ or t_-^μ, respectively.*]

If instead p and q are even and $y_1 > 0$, then

$$\begin{aligned} T_{y_1}(t) = \frac{1}{4\pi^{n/2} y_1}\Big\{ \Big[Y(-t)Y(y_2)(-1)^{p/2} + Y(t)Y(-y_2)(-1)^{q/2}\Big] \times \\ \times \Big(-\frac{4y_1 t}{y_2}\Big)^{1/2-n/4} J_{n/2-1}\Big(\sqrt{-\frac{y_2 t}{y_1}}\Big) - (-1)^{p/2} \sum_{j=0}^{n/2-2} \frac{1}{j!}\Big(\frac{y_2}{4y_1}\Big)^j \delta^{(n/2-2-j)}(t)\Big\}. \end{aligned} \tag{4.5}$$

Proof. The uniqueness of E is a consequence of [29, Prop. 2.4.13, p. 162]. Setting $Q(s, y_1, y_2) = y_1 + s y_2$ we obtain that $Q(-s, \partial_y) S(s,y) = 1(s) \otimes \delta(y)$ holds for $S =$

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$Y(y_1)\delta(y_2+sy_1)$. In particular, S depends $\mathcal{C}^\infty$ on y_1 for $y_1\geq 0$ and $\lim_{y_1\searrow 0} S = 1(s)\otimes\delta(y_2)$. Hence also E depends $\mathcal{C}^\infty$ on y_1 for $y_1\geq 0$ and $\lim_{y_1\searrow 0} E_{y_1} = \delta(x)\otimes\delta(y_2)$.

We observe that $S\in\mathcal{F}(L^1(\mathbb{R}^1))\,\hat{\otimes}\,\mathcal{S}'(\mathbb{R}^2_y)$, i.e., the hypothesis of Proposition 4.2 is fulfilled. In fact, $(\mathcal{F}_s S)(\sigma) = (y_1)_+^{-1}\mathrm{e}^{\mathrm{i}y_2\sigma/y_1}$ and $\mathcal{F}_s S\in L^1(\mathbb{R}^1)\,\hat{\otimes}\,\mathcal{S}'(\mathbb{R}^2_y)$ can be shown by considering the partial Fourier transform $\mathcal{F}_{y_2}\mathcal{F}_s S = 2\pi Y(y_1)\delta(y_1\eta_2-\sigma)$. Although we could therefore employ Proposition 4.2 in order to evaluate $E = T([x,x],y)$ with $T=\mathcal{F}_u U$ and U as in formula (4.3), it is easier to recur to formula (2.11) in Theorem 2.7. For $y_1>0$, this yields the following:

$$\begin{aligned} E_{y_1} &= (2\pi)^{-n}\mathcal{F}_\xi(S([\xi,\xi],y)) = (2\pi)^{-n}\mathcal{F}_\xi(\delta(y_2+[\xi,\xi]y_1)) \\ &= (2\pi)^{-n}y_1^{-1}\mathcal{F}_\xi(\delta([\xi,\xi]-s)) = T_{y_1}([x,x]), \end{aligned}$$

where $s=-y_2/y_1$ and $T_{y_1} = (2\pi)^{-n}y_1^{-1}\tilde{T}$ with $\tilde{T}$ given by Theorem 2.7. If $y_2<0$, then $s=-y_2/y_1>0$ and hence $\tilde{T}$ is given directly by formula (2.11). If instead $y_2>0$, we use Corollary 2.6 (e), i.e., we set $s=y_2/y_1$, we swap p and q and we replace t by $-t$. This yields formulas (4.4) and (4.5) and completes the proof. □

Let us illustrate Proposition 4.3 by collecting some special cases. In particular, we derive the fundamental solution of Sobolev's operator $\partial_t-\partial^3_{xyz}$, which was first found in [38], see also [25, Prop. 6, p. 158], [29, Ex. 2.3.8, p. 148, Ex. 2.6.4, p. 172].

Corollary 4.4. *Let n,p,q be as in* Proposition 4.3 *and E be the uniquely determined temperate fundamental solution of $P(D)=\partial_{y_1}+[\partial_x,\partial_x]\partial_{y_2}$ with support in the half-space $\{(x,y)\in\mathbb{R}^{n+2};\, y_1\geq 0\}$.*

(a) *If $p=0$ and $q=n$, i.e., $P(D)=\partial_{y_1}-\Delta_n\partial_{y_2}$, then E_{y_1}, $y_1>0$, is given by*

$$E_{y_1}(x,y_2) = \frac{y_1^{-n/4-1/2}Y(y_2)}{2(2\pi)^{n/2}}\Big(\frac{\sqrt{y_2}}{|x|}\Big)^{n/2-1}J_{n/2-1}\Big(\sqrt{\frac{y_2}{y_1}}|x|\Big)\in L^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_{x,y_2}). \tag{4.6}$$

(b) *If $p=q=1$, $n=2$, i.e., $P(D)=\partial_{y_1}+(\partial^2_{x_1}-\partial^2_{x_2})\partial_{y_2}$, then*

$$\begin{aligned} E = &-\frac{Y(y_1)}{4\pi y_1}Y(-y_2(x_1^2-x_2^2))N_0\Big(\sqrt{-\frac{y_2(x_1^2-x_2^2)}{y_1}}\Big) \\ &+\frac{Y(y_1)}{2\pi^2 y_1}Y(y_2(x_1^2-x_2^2))K_0\Big(\sqrt{\frac{y_2(x_1^2-x_2^2)}{y_1}}\Big)\in L^1_{\mathrm{loc}}(\mathbb{R}^4_{x,y}). \end{aligned} \tag{4.7}$$

After rotation in the variables x_1,x_2 and a change in notation, this yields the fundamental solution F of the original "Sobolev operator" $\partial_t-\partial^3_{xyz}$:

$$F = \frac{Y(t)}{2\pi^2 t}\Big[2Y(-xyz)K_0\big(2\sqrt{-xyz/t}\big)-\pi Y(xyz)N_0\big(2\sqrt{xyz/t}\big)\Big].$$

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(c) *If* $p = q = 2$, $n = 4$, *i.e.,* $P(D) = \partial_{y_1} + (\partial_{x_1}^2 + \partial_{x_2}^2 - \partial_{x_3}^2 - \partial_{x_4}^2)\partial_{y_2}$, *then* E_{y_1}, $y_1 > 0$, *is given by*

(4.8)
$$E_{y_1}(x, y_2) = \frac{1}{4\pi^2 y_1}\Big\{\delta([x,x]) - \tfrac{1}{2}Y(-y_2[x,x])\sqrt{-y_2/(y_1[x,x])}\, J_1(\sqrt{-y_2[x,x]/y_1})\Big\}.$$

(d) *If* $p = 3$, $q = 1$, $n = 4$, *i.e.,* $P(D) = \partial_{y_1} + (\Delta_3 - \partial_{x_4}^2)\partial_{y_2}$, *then* E_{y_1}, $y_1 > 0$, *is given by* $E_{y_1}(x, y_2) = T_y([x,x])$ *with*

(4.9)
$$T_y(t) = \frac{\operatorname{sign} y_2}{8\pi^3 y_1}\operatorname{vp}\Big\{2Y(ty_2)\sqrt{\frac{y_2}{y_1 t}}K_1(\sqrt{y_2 t/y_1}) + \pi Y(-ty_2)\sqrt{-\frac{y_2}{y_1 t}}N_1(\sqrt{-y_2 t/y_1})\Big\},$$

where $\operatorname{vp} U(t) = \lim_{\epsilon\searrow 0} Y(|t| - \epsilon)U(t)$.

Proof. (a) If $q = n$ is odd, then (4.6) follows directly from (4.4). Moreover, formula (4.6) remains correct also in case $q = n$ is even since the additional delta terms in formula (4.5) vanish in this case. In fact, if $0 \le k \le \frac{n}{2} - 2$ and $p = 0$, i.e., $[x,x] = -|x|^2$, then $\delta^{(k)}([x,x]) = (-1)^k\delta^{(k)}(|x|^2) = 0$, because

$$\begin{aligned}\langle \phi, \delta^{(k)}(|x|^2)\rangle = \langle M_\phi, \delta^{(k)}\rangle &= \lim_{\epsilon\searrow 0}\Big(-\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big)^k\langle M_\phi, \delta_\epsilon\rangle = \lim_{\epsilon\searrow 0}\Big(-\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big)^k\frac{1}{2}\int_{|x|=\sqrt{\epsilon}}\frac{\phi(x)}{\sqrt{\epsilon}}\,\mathrm{d}\sigma(x)\\ &= \lim_{\epsilon\searrow 0}\Big(-\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big)^k\frac{1}{2}\int_{|\omega|=1}\phi(\sqrt{\epsilon}\,\omega)\epsilon^{n/2-1}\,\mathrm{d}\sigma(\omega) = 0\end{aligned}$$

holds for $\phi \in \mathcal{D}(\mathbb{R}^n)$.

(b, c) Formulas (4.7) and (4.8) follow directly from Proposition 4.3.

(d) Formula (4.9) follows from formula (4.4) if we take into account that

$$\frac{Y(s)}{\sqrt{s}}\,K_1(\sqrt{s}) + \frac{\pi Y(-s)}{2\sqrt{-s}}\,N_1(\sqrt{-s}) - \operatorname{vp}\frac{1}{s} \in L^1_{\mathrm{loc}}(\mathbb{R}^1_s).$$

The proof is complete. □

As an application of Corollary 4.4, let us deduce the only temperate fundamental solution E of the operator $\partial_1^2\partial_2 - \partial_3 - 1$. In [5, Rem. 6.25, p. 29], it is shown that E (i.e., the fundamental solution "with the best global properties") is not regular (i.e., is not one with the best possible local regularity). Note that E is not calculated in [5]. Explicitly, we have $E = -\mathrm{e}^{-x_3}\cdot F$, where F is the fundamental solution of $\partial_3 - \partial_1\partial_2^2$ given in (4.6), and thus

$$E = -\frac{Y(x_2)Y(x_3)\mathrm{e}^{-x_3}}{2\pi\sqrt{x_2 x_3}}\,\cos\Big(\sqrt{\frac{x_2}{x_3}}\,x_1\Big).$$

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