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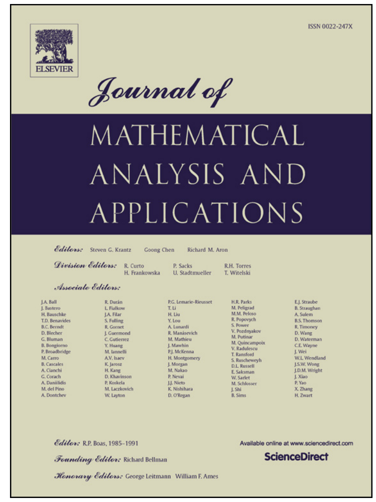

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On the Problem of Geodesic Mappings and Deformations of Generalized Riemannian Spaces

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Abstract

This paper is devoted to a study of geodesic mappings and infinitesimal geodesic deformations of generalized Riemannian spaces. While a geodesic mapping between two generalized Riemannian spaces any geodesic line of one space sends to a geodesic line of the other space, under an infinitesimal geodesic deformation any geodesic line is mapped to a curve approximating a geodesic with a given precision. Basic equations of the theory of geodesic mappings in the case of generalized Riemannian spaces are obtained in this paper. A new generalization of the famous Levi Civita's equation is found. Necessary and sufficient conditions for a nontrivial infinitesimal geodesic deformation are given. It is proven that a generalized Riemannian space admits nontrivial infinitesimal geodesic deformations if and only if it admits nontrivial geodesic mappings. At last it is shown that generalized equidistant spaces of primary type admit nontrivial geodesic deformations.

Keywords: Geodesic mapping, Infinitesimal deformation, Infinitesimal geodesic deformation, Generalized Riemannian space, Generalized equidistant space.
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1. Introduction

The problem of geodesic mappings of Riemannian spaces is related to the Levi Civita's investigations [1] and stems from his study of a dynamics equation. Basics of the theory of geodesic mappings of spaces with symmetric affine connection can be found in monographs of the authors J. Mikeš, V. Kiosak and A. Vanžurová [2], and also N. Sinyukov [3]. A lot of papers is dedicated to the theory of geodesic mappings of nonsymmetric affine connection spaces, specially of generalized Riemannian spaces (see [4, 5, 6, 7, 8, 9, 10]). Note that the nonsymmetric affine connection is related to Einstein's Unified Field Theory

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(UFT), which unites the gravitation theory and the theory of electromagnetism, while the symmetric affine connection is related to General Theory of Relativity (GTR) and gravitation theory.

Infinitesimal deformations of different spaces are very important from the physical point of view because they estimate stability of different magnitudes. Spaces with symmetric affine connection under infinitesimal deformations were thoroughly studied in [2, 11, 12, 13, 14, 15, 16]. Some papers [17, 18, 19, 20] are dedicated to the theory of infinitesimal deformations of nonsymmetric affine connection spaces, specially to deformations of generalized Riemannian spaces.

By setting various special conditions, we get different kinds of infinitesimal deformations. In this paper we are interested in such a kind of infinitesimal deformations called infinitesimal geodesic deformations under which any geodesic is mapped to a curve approximating a geodesic with a given precision. This approach in the studying of geodesics is appropriate for different applications, for example, for simulating real physical situations when evolution of gravity fields (electromagnetic fields, mechanical systems etc.) is considered [2].

This paper is organized as follows: In Section 2, some used notations and preliminaries are given. In Section 3, a generalization of the famous Levi Civita's equation is obtained in the case of generalized Riemannian spaces. Necessary and sufficient conditions for nontrivial geodesic mapping of a generalized Riemannian space $\mathbb{GR}_N$ are found. In Section 4, necessary and sufficient conditions for an infinitesimal geodesic deformation of the space $\mathbb{GR}_N$ are given. It is proven that a generalized Riemannian space $\mathbb{GR}_N$ admits nontrivial infinitesimal geodesic deformations if and only if $\mathbb{GR}_N$ admits nontrivial geodesic mappings. At last it is shown that generalized equidistant spaces of primary type admit nontrivial geodesic deformations.

2. Notation and preliminaries

Generalized Riemannian spaces. We are giving some basic facts about generalized Riemannian spaces according to [7, 21].

A generalized Riemannian space $\mathbb{GR}_N$ is a differentiable N-dimensional manifold endowed with a non-symmetric metric tensor $g_{ij}(x^1, ..., x^N)$, where x^i are local coordinates. Generally we have

$$g_{ij}(x) \neq g_{ji}(x). \tag{2.1}$$

If the metric tensor is symmetric, then we get *the Riemannian space* $\mathbb{R}_N$.

Because of the non-symmetry it is defined symmetric and antisymmetric part of the metric tensor g_{ij} as

$$g_{\underline{ij}} = \frac{1}{2}\big(g_{ij} + g_{ji}\big) \quad \text{and} \quad g_{\underset{\vee}{ij}} = \frac{1}{2}\big(g_{ij} - g_{ji}\big). \tag{2.2}$$

For the lowering and raising of indices in $\mathbb{GR}_N$ one uses the tensors $g_{\underline{ij}}$ respectively $g^{\underline{ij}}$, where

$$\|g^{\underline{ij}}\| = \|g_{\underline{ij}}\|^{-1} \quad (\det \|g_{\underline{ij}}\| \neq 0).$$

First and second kind Christoffel symbols of the space $\mathbb{GR}_N$ are:

$$\Gamma_{i.jk} = \frac{1}{2}\big(g_{ji,k} - g_{jk,i} + g_{ik,j}\big) \quad \text{and} \quad \Gamma^i_{jk} = g^{\underline{i\alpha}}\Gamma_{\alpha.jk}, \tag{2.3}$$

where comma denotes partial derivation $\partial/\partial x^k$. Generally, we have $\Gamma^i_{jk} \neq \Gamma^i_{kj}$. The symbols Γ^i_{jk} present *connection coefficients* of the space $\mathbb{GR}_N$. For Γ^i_{jk} the next equations are valid

$$\Gamma^i_{ik} = \Gamma^i_{ki} = \frac{\partial}{\partial x^k} \ln \sqrt{|\underline{g}|}, \quad \underline{g} = \det \|g_{\underline{ij}}\|, \tag{2.4}$$

$$\Gamma^i_{\underset{\vee}{ik}} = 0. \tag{2.5}$$

Using the non-symmetry of the connection coefficients, it is possible to define four kinds of covariant differentiation of a tensor. Thus, for a tensor a^i_j we have:

$$\begin{aligned} a^i_{j\underset{1}{|}k} &= a^i_{j,k} + \Gamma^i_{\alpha k}a^\alpha_j - \Gamma^\alpha_{jk}a^i_\alpha, & a^i_{j\underset{2}{|}k} &= a^i_{j,k} + \Gamma^i_{k\alpha}a^\alpha_j - \Gamma^\alpha_{kj}a^i_\alpha, \\ a^i_{j\underset{3}{|}k} &= a^i_{j,k} + \Gamma^i_{\alpha k}a^\alpha_j - \Gamma^\alpha_{kj}a^i_\alpha, & a^i_{j\underset{4}{|}k} &= a^i_{j,k} + \Gamma^i_{k\alpha}a^\alpha_j - \Gamma^\alpha_{jk}a^i_\alpha. \end{aligned} \tag{2.6}$$

A Riemannian space $\mathbb{R}_N$ endowed with a symmetric part of the connection $\Gamma^i_{\underline{jk}} = \frac{1}{2}\big(\Gamma^i_{jk} + \Gamma^i_{kj}\big)$ is *the associated space* to the space $\mathbb{GR}_N$. An antisymmetric part of the connection, a magnitude $\Gamma^i_{\underset{\vee}{jk}} = \frac{1}{2}\big(\Gamma^i_{jk} - \Gamma^i_{kj}\big)$ is a tensor called *the torsion tensor*.

Geodesic mappings. Basic facts related to the geodesic mappings of generalized Riemannian spaces are given in the sequel according to [5, 7].

A *geodesic mapping* of a generalized Riemannian space $\mathbb{GR}_N$ onto $\mathbb{G}\overline{\mathbb{R}}_N$ is a diffeomorphism $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ under which the geodesics of the space $\mathbb{GR}_N$ correspond to the geodesics of the space $\mathbb{G}\overline{\mathbb{R}}_N$.

Let the spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ be considered in the common system of local coordinates $x^1, x^2, ..., x^N$, with respect to the mapping f. Then the connection coefficients of these spaces in the corresponding points $M(x)$ and $\overline{M}(x)$, can be connected with the next relation:

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + P^i_{jk}. \tag{2.7}$$

A magnitude P^i_{jk} is a tensor called *deformation tensor* of the connection Γ of $\mathbb{GR}_N$ according to the mapping $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$.

A necessary and sufficient condition for the mapping f to be a geodesic one is that the deformation tensor P^i_{jk} has the next form

$$P^i_{jk} = \delta^i_j\psi_k + \delta^i_k\psi_j + \xi^i_{jk}, \tag{2.8}$$

where ψ_i is a covariant vector and ξ^i_{jk} is an antisymmetric tensor. With respect to (2.7), the equation (2.8) becomes

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j\psi_k + \delta^i_k\psi_j + \xi^i_{jk}. \tag{2.9}$$

Obviously,

$$\psi_i = \frac{1}{1+N} P^p_{\underline{ip}} = \frac{1}{1+N}(\overline{\Gamma}^p_{\underline{ip}} - \Gamma^p_{\underline{ip}}), \quad \xi^i_{jk} = P^i_{\underset{\vee}{jk}} = \overline{\Gamma}^i_{\underset{\vee}{jk}} - \Gamma^i_{\underset{\vee}{jk}} \tag{2.10}$$

A vector ψ_i has also the next form

$$\psi_i = \frac{1}{N+1}\frac{\partial}{\partial x^i} \ln\sqrt{\left|\frac{\overline{g}}{\underline{g}}\right|}, \tag{2.11}$$

where $\underline{g} = \det\|g_{\underline{ij}}\|$, $\overline{g} = \det\|\overline{g}_{\underline{ij}}\|$. As the magnitude $|\overline{g}/\underline{g}|$ is an invariant, we conclude that the vector ψ_i is a gradient. Also, for an antisymmetric tensor ξ^i_{jk}, it is valid the next relation:

$$\xi^p_{pk} = \xi^p_{kp} = 0. \tag{2.12}$$

In [5] the next theorem was proved.

Theorem 2.1. *a) A mapping $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ is geodesic if and only if the 2nd kind Cristoffel symbols of these spaces satisfy* (2.9).
b)If the mapping f is geodesic, then the equations

$$\overline{g}_{ij\underset{1}{|}k} - \overline{g}_{ij\underset{\vee\, 1}{||}k} = 2\psi_k\,\overline{g}_{ij} + \psi_i\,\overline{g}_{kj} + \psi_j\,\overline{g}_{ik} + \xi^p_{ik}\overline{g}_{pj} + \xi^p_{jk}\overline{g}_{ip}, \tag{2.13}$$

$$\overline{g}_{ij\underset{2}{|}k} - \overline{g}_{ij\underset{\vee\, 2}{||}k} = 2\psi_k\,\overline{g}_{ij} + \psi_i\,\overline{g}_{kj} + \psi_j\,\overline{g}_{ik} + \xi^p_{ki}\overline{g}_{pj} + \xi^p_{kj}\overline{g}_{ip}, \tag{2.14}$$

are satisfied, where $(|)$ *and* $(||)$ *denote covariant derivatives of the corresponding kind in the spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$, respectively. Conversely, if one of these equations is satisfied, then this mapping is geodesic, and the other is satisfied too.* ■

Infinitesimal deformations. We are going to define basic terms about infinitesimal deformations following the research of J. Mikeš et al. in [2] about Riemannian spaces and generalizing their considerations to the case of generalized Riemannian spaces.

Let $\mathbb{GR}_M$ be a generalized Riemannian space with a metric tensor $a_{\alpha\beta}$ and local coordinates $y^1, ..., y^M$. The equations

$$y^\alpha = y^\alpha(x^1, ..., x^N), \quad rank\|y^\alpha_{,i}\| = N < M \tag{2.15}$$

determine *the subspace* $\mathbb{GR}_N$ of the space $\mathbb{GR}_M$ ($\mathbb{GR}_N \subset \mathbb{GR}_M$) with *induced metric* g_{ij}. The components of metric tensors $a_{\alpha\beta}$ and g_{ij} are related with the next relation according to [6]

$$g_{ij} = a_{\alpha\beta} y^\alpha_{,i} y^\beta_{,j}. \tag{2.16}$$

Greek indices $\alpha, \beta, ...$ take values $1, ..., M$ and refer to the space $\mathbb{GR}_M$ but Latin indices $i, j, ...$ take values $1, ..., N$ and refer to the subspace $\mathbb{GR}_N$.

Let $z^\alpha(x^1, ..., x^N)$ be a restriction of a vector field defined on a generalized Riemannian space $\mathbb{GR}_M$ onto the subspace $\mathbb{GR}_N$.

Definition 2.1. *The equations*

$$\tilde{y}^\alpha = y^\alpha(x^i) + \epsilon z^\alpha(x^i), \tag{2.17}$$

where ϵ is an infinitesimal, define a family of generalized Riemannian subspaces $\mathbb{G}\tilde{\mathbb{R}}_N$ of the space $\mathbb{GR}_M$ which is said to be the infinitesimal deformation (of the first order) of the space $\mathbb{GR}_N$. The field $z^\alpha(x^1, ..., x^N)$ is **the infinitesimal deformation field** *or* **the displacement field**.

Definition 2.2. *Let $A = A(x^1, ..., x^N)$ be a geometric object from the generalized Riemannian space $\mathbb{GR}_N$ (a scalar function, a vector, a tensor, a connection,...). An object $\tilde{A} = \tilde{A}(x^1, ..., x^N, \epsilon)$ from the space $\mathbb{G}\tilde{\mathbb{R}}_N$ is* **the deformed object** *A, with respect to the infinitesimal deformation* (2.17)*, if*

$$\tilde{A}(y^\alpha) = A(\tilde{y}^\alpha). \tag{2.18}$$

Definition 2.3. *Let $A = A(x^1, ..., x^N)$ be a geometric object from the space $\mathbb{GR}_N$ and let $\tilde{A} = \tilde{A}(x^1, ..., x^N, \epsilon)$ be an infinitesimally deformed object A, with respect to the deformation* (2.17). *Also, let the equation*

$$\Delta A = \tilde{A}(x^i, \epsilon) - A(x^i) = \epsilon \delta A + \epsilon^2 \delta^2 A + ... + \epsilon^n \delta^n A + \dots \tag{2.19}$$

be a valid one, then the coefficients $\delta A, \delta^2 A, \dots, \delta^n A, \dots$ are **first, second,** *etc.* **variation** *of the magnitude A under this infinitesimal deformation.*

Summands which have the infinitesimal ϵ of a degree higher than one are going to be omitted in this research. We will omit mention of the order of variation as well as infinitesimal deformation supposing it is the first variation of infinitesimal deformation of the first order in the sequel.

For a variation of a geometric object A from the space $\mathbb{GR}_N$ the next properties hold [20] as same as it is valid in the case of a Euclidean space $\mathbb{E}_N$:

- A variation of addition of geometric objects of the same kind is equal to the sum of variations of these objects.;
- For multiplication and composition (multiplication with contraction) of geometric objects, Leibnitz role holds.

Definition 2.4. **An infinitesimal geodesic deformation** *is an infinitesimal deformation $\widetilde{\mathbb{GR}}_N$ of a generalized Riemannian space $\mathbb{GR}_N$ if it preserves the geodesic curves of the space $\mathbb{GR}_N$.*

3. Necessary and sufficient conditions for a geodesic mapping $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$

Theorem 2.1 gives necessary and sufficient conditions for a geodesic mapping of two generalized Riemannian spaces. Here we give their equivalent, which presents simpler expression of the conditions (2.13) and (2.14).

Theorem 3.1. *A mapping f of a generalized Riemannian space $\mathbb{GR}_N$ onto a generalized Riemannian space $\mathbb{G}\overline{\mathbb{R}}_N$ is geodesic if and only if in the common system of local coordinates with respect to the mapping f, basic metric tensor of the space $\mathbb{G}\overline{\mathbb{R}}_N$ satisfies the next relation*

$$\overline{g}_{\underline{ij}\underset{1}{|}k} = 2\psi_k\, \overline{g}_{\underline{ij}} + \psi_i\, \overline{g}_{\underline{kj}} + \psi_j\, \overline{g}_{\underline{ik}} + \xi^p_{ik}\overline{g}_{\underline{pj}} + \xi^p_{jk}\overline{g}_{\underline{ip}}, \tag{3.1}$$

where $(\underset{1}{|})$ denotes covariant derivative of the first kind in the space $\mathbb{GR}_N$.

Proof. $(\Rightarrow:)$ Let us suppose that the mapping $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ is geodesic. According to Theorem 2.1, a the next relation is a valid one

$$\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \psi_j\delta^i_k + \psi_k\delta^i_j + \xi^i_{jk}. \tag{3.2}$$

From the definition of the covariant derivative (2.6) and from the equation (3.2) we obtain

$$\begin{aligned}
\overline{g}_{\underset{\vee}{ij}\underset{1}{|}k} - \overline{g}_{\underset{\vee}{ij}\underset{1}{||}k} &= \overline{g}_{\underset{\vee}{ij},k} - \Gamma^p_{ik}\overline{g}_{\underset{\vee}{pj}} - \Gamma^p_{jk}\overline{g}_{\underset{\vee}{ip}} - \overline{g}_{\underset{\vee}{ij},k} + \overline{\Gamma}^p_{ik}\overline{g}_{\underset{\vee}{pj}} + \overline{\Gamma}^p_{jk}\overline{g}_{\underset{\vee}{ip}} \\
&= (\overline{\Gamma}^p_{ik} - \Gamma^p_{ik})\overline{g}_{\underset{\vee}{pj}} + (\overline{\Gamma}^p_{jk} - \Gamma^p_{jk})\overline{g}_{\underset{\vee}{ip}} \\
&= (\delta^p_i\psi_k + \delta^p_k\psi_i + \xi^p_{ik})\overline{g}_{\underset{\vee}{pj}} + (\delta^p_j\psi_k + \delta^p_k\psi_j + \xi^p_{jk})\overline{g}_{\underset{\vee}{ip}} \\
&= 2\psi_k\, \overline{g}_{\underset{\vee}{ij}} + \psi_i\, \overline{g}_{\underset{\vee}{kj}} + \psi_j\, \overline{g}_{\underset{\vee}{ik}} + \xi^p_{ik}\overline{g}_{\underset{\vee}{pj}} + \xi^p_{jk}\overline{g}_{\underset{\vee}{ip}}.
\end{aligned} \tag{3.3}$$

If we exchange $\overline{g}_{\underset{\vee}{ij}\underset{1}{|}k}$ from (3.3) into (2.13), we obtain the equation (3.1).

$(\Leftarrow:)$ Let us suppose that the equation (3.1) holds. It is known that in the space $\mathbb{GR}_N$ the next equations are valid $g_{\underline{ij}\underset{\theta}{|}k} = 0$, $\theta = 1, \ldots, 4$ (see [7, 21]), so using the definition of the covariant derivative we have

$$\overline{g}_{\underline{ij}\underset{1}{|}k} - \overbrace{\overline{g}_{\underline{ij}\underset{1}{||}k}}^{0} = (\overline{\Gamma}^p_{ik} - \Gamma^p_{ik})\overline{g}_{\underline{pj}} + (\overline{\Gamma}^p_{jk} - \Gamma^p_{jk})\overline{g}_{\underline{ip}}. \tag{3.4}$$

On the other hand, from (3.1) we have

$$\begin{aligned}
\overline{g}_{\underline{ij}\underset{1}{|}k} &= 2\psi_k\, \overline{g}_{\underline{ij}} + \psi_i\, \overline{g}_{\underline{kj}} + \psi_j\, \overline{g}_{\underline{ik}} + \xi^p_{ik}\overline{g}_{\underline{pj}} + \xi^p_{jk}\overline{g}_{\underline{ip}} \\
&= (\psi_i\delta^p_k + \psi_k\delta^p_i + \xi^p_{ik})\overline{g}_{\underline{pj}} + (\psi_j\delta^p_k + \psi_k\delta^p_j + \xi^p_{jk})\overline{g}_{\underline{ip}}.
\end{aligned} \tag{3.5}$$

By comparing the equations (3.4) and (3.5) we conclude the relation (2.9) is a valid one, i.e. the mapping is geodesic. ■

It is easy to prove the next theorem.

Theorem 3.2. *A mapping f of a generalized Riemannian space $\mathbb{GR}_N$ onto a generalized Riemannian space $\mathbb{G}\overline{\mathbb{R}}_N$ is geodesic if and only if in the common system of local coordinates with respect to the mapping f, basic metric tensor of the space $\mathbb{G}\overline{\mathbb{R}}_N$ satisfies the next relation*

$$\overline{g}_{\underline{ij}\underset{2}{|}k} = 2\psi_k\,\overline{g}_{\underline{ij}} + \psi_i\,\overline{g}_{\underline{kj}} + \psi_j\,\overline{g}_{\underline{ik}} + \xi^p_{ki}\overline{g}_{\underline{pj}} + \xi^p_{kj}\overline{g}_{\underline{ip}}, \tag{3.6}$$

where $(\underset{2}{|})$ denotes covariant derivative of the second kind in the space $\mathbb{GR}_N$. ■

In connection with the previous exposure we give a basic equation of the theory of geodesic mappings of Riemannian spaces which was proved by N. S. Sinyukov [22].

Theorem 3.3. *A Riemannian space $\mathbb{R}_N$ admits a nontrivial geodesic mapping if and only if there exists a nonsingular symmetric tensor a_{ij} satisfying*

$$a_{ij;k} = \lambda_i g_{jk} + \lambda_j g_{ik}, \tag{3.7}$$

for a gradient $\lambda_i \neq 0$. ■

In the sequel we are giving a generalization of that theorem in the case of generalized Riemannian spaces.

Theorem 3.4. *A generalized Riemannian space $\mathbb{GR}_N$ admits a nontrivial geodesic mapping, if and only if there exists a nonsingular symmetric tensor a_{ij} satisfying*

$$a_{ij\underset{1}{|}k} = \lambda_i g_{\underline{jk}} + \lambda_j g_{\underline{ik}} + \mu^p_{ik} g_{\underline{pj}} + \mu^p_{jk} g_{\underline{pi}}, \tag{3.8}$$

for a gradient $\lambda_i \neq 0$ and an antisymmetric tensor μ^i_{jk} so that $\mu^i_{ij} = \mu^i_{ji} = 0$.

Proof. Let us suppose that $\mathbb{GR}_N$ admits a nontrivial geodesic mapping. Then the equality (3.1) holds, where ψ_k is a gradient vector and ξ^i_{jk} is an antisymmetric tensor given in (2.10).

As the vector ψ_k is a gradient, i. e. $\psi_k = \partial\psi/\partial x^k$, we can introduce the next quantity

$$\tilde{g}_{ij} = e^{-2\psi}\overline{g}_{ij}. \tag{3.9}$$

After covariant derivative of the first kind of symmetric part in the previous equation in $\mathbb{GR}_N$, i. e.

$$\tilde{g}_{\underline{ij}} = e^{-2\psi}\overline{g}_{\underline{ij}}, \tag{3.10}$$

having in mind the condition (3.1), we have

$$\tilde{g}_{\underline{ij}\underset{1}{|}k} = \psi_i\tilde{g}_{\underline{kj}} + \psi_j\tilde{g}_{\underline{ik}} + \xi^p_{ik}\tilde{g}_{\underline{pj}} + \xi^p_{jk}\tilde{g}_{\underline{ip}}. \tag{3.11}$$

As the tensor $\bar{g}_{\underline{ij}}$ is nonsingular, i. e. $\det\|\bar{g}_{\underline{ij}}\| \neq 0$, we conclude from the equation (3.10) that $\tilde{g}_{\underline{ij}}$ is also nonsingular. Let us denote an element of the matrix, which is the inverse of the matrix $\|\tilde{g}_{\underline{ij}}\|$, with $\tilde{g}^{\underline{kl}}$. Then it will be valid

$$\tilde{g}_{\underline{i\alpha}}\,\tilde{g}^{\underline{\alpha j}} = \delta_i^j. \tag{3.12}$$

Let us covariant differentiate this equation in $\mathbb{GR}_N$. We obtain

$$\tilde{g}_{\underline{i\alpha}\underset{1}{|}k}\,\tilde{g}^{\underline{\alpha j}} + \tilde{g}_{\underline{i\alpha}}\,\tilde{g}^{\underline{\alpha j}}{}_{\underset{1}{|}k} = 0.$$

If we multiply this equation with $\tilde{g}^{\underline{i\beta}}$ and use (3.12), then we obtain

$$\tilde{g}^{\underline{ij}}{}_{\underset{1}{|}k} = -\tilde{g}_{\underline{\alpha\beta}\underset{1}{|}k}\tilde{g}^{\underline{\alpha i}}\tilde{g}^{\underline{\beta j}}. \tag{3.13}$$

Comparing with (3.11), we obtain

$$\tilde{g}^{\underline{ij}}{}_{\underset{1}{|}k} = -\psi_\alpha\tilde{g}^{\underline{\alpha i}}\delta_k^j - \psi_\beta\tilde{g}^{\underline{\beta j}}\delta_k^i - \xi_{\alpha k}^j\tilde{g}^{\underline{\alpha i}} - \xi_{\beta k}^i\tilde{g}^{\underline{\beta j}}. \tag{3.14}$$

Let us introduce the next notation:

$$\lambda^i = -\psi_\alpha\tilde{g}^{\underline{\alpha i}}, \quad \mu_k^{pi} = -\xi_{\alpha k}^p\tilde{g}^{\underline{\alpha i}}, \tag{3.15}$$

we obtain

$$\tilde{g}^{\underline{ij}}{}_{\underset{1}{|}k} = \lambda^i\delta_k^j + \lambda^j\delta_k^i + \mu_k^{ji} + \mu_k^{ij}. \tag{3.16}$$

By lowering of indices i and j in $\mathbb{GR}_N$ in the equation (3.16), we obtain

$$\tilde{g}^{\underline{ij}}{}_{\underset{1}{|}k}g_{\underline{i\alpha}}g_{\underline{j\beta}} = \lambda^i\delta_k^j g_{\underline{i\alpha}}g_{\underline{j\beta}} + \lambda^j\delta_k^i g_{\underline{i\alpha}}g_{\underline{j\beta}} + \mu_k^{ji}g_{\underline{i\alpha}}g_{\underline{j\beta}} + \mu_k^{ij}g_{\underline{i\alpha}}g_{\underline{j\beta}}. \tag{3.17}$$

Let us use the next notation

$$a_{\alpha\beta} = \tilde{g}^{\underline{ij}}g_{\underline{i\alpha}}g_{\underline{j\beta}} \quad \lambda_i = g_{\underline{i\alpha}}\lambda^\alpha, \quad \mu_{jk}^i = g_{\underline{j\alpha}}\mu_k^{i\alpha}, \tag{3.18}$$

then we obtain

$$a_{\alpha\beta\underset{1}{|}k} = \lambda_\alpha g_{\underline{k\beta}} + \lambda_\beta g_{\underline{k\alpha}} + \mu_{\alpha k}^p g_{\underline{p\beta}} + \mu_{\beta k}^p g_{\underline{p\alpha}}. \tag{3.19}$$

Obviously, a_{ij} is a nonsingular symmetric tensor, λ_i is a covariant vector, μ_{jk}^i is a tensor.

From (3.10), (3.15) and (3.18) we conclude:

$$a_{\underline{ij}} = e^{2\psi}\overline{g}^{\underline{\alpha\beta}}g_{\underline{\alpha i}}g_{\underline{\beta j}}, \quad \lambda_i = -e^{2\psi}\psi_\alpha\overline{g}^{\underline{\alpha\beta}}g_{\underline{i\beta}}, \quad \mu_{jk}^i = -\xi_{\beta k}^i e^{2\psi}\overline{g}^{\underline{\alpha\beta}}g_{\underline{j\alpha}}. \tag{3.20}$$

Let us multiply (3.19) with $g^{\underline{\alpha\beta}}$, we obtain

$$(a_{\alpha\beta}g^{\underline{\alpha\beta}})_{\underset{1}{|}k} = 2(\lambda_k + \mu_{pk}^p) = 2\eta_k. \tag{3.21}$$

We conclude that η_k is a gradient, so it can be presented as the derivative of the function

$$\eta = \frac{1}{2} a_{\alpha\beta} g^{\underline{\alpha\beta}}. \tag{3.22}$$

Form here it is $a_{\alpha\beta} = (2\eta/N) g_{\underline{\alpha\beta}}$, which comparing with the first equation in (3.18) gives $\tilde{g}^{\underline{ij}} = (2\lambda/N) g^{\underline{ij}}$. Using this equation and the expression for the tensor μ^i_{jk} from (3.15) and (3.18), we obtain $\mu^i_{jk} = -(2\eta/N)\xi^i_{jk}$. It is clear that the tensor μ^i_{jk} is antisymmetric, and also $\mu^p_{pk} = -(2\eta/N)\xi^p_{pk} = 0$. Therefore, $\eta_k = \lambda_k$, so λ_k is a gradient, and from (3.20) we conclude that $\lambda_i \neq 0$ if and only if $\psi_i \neq 0$. In this way, if the space $\mathbb{GR}_N$ admits a nontrivial geodesic mapping, then there exists a nonsingular symmetric tensor a_{ij}, which satisfies the equality (3.19) for a gradient $\lambda_i \neq 0$ and an antisymmetric tensor μ^i_{jk}, $\mu^i_{ik} = \mu^i_{ki} = 0$.

Let us prove conversely statement. By the raising of indices α and β in $\mathbb{GR}_N$ in the equation(3.19), we conclude in accordance with (3.18) that the tensor

$$\tilde{g}^{\underline{ij}} = a_{\alpha\beta} g^{\underline{\alpha i}} g^{\underline{\beta j}} \tag{3.23}$$

satisfies the equality (3.16), for $\lambda^i = \lambda_\alpha g^{\underline{\alpha i}}$, $\mu^{ij}_k = \mu^i_{\alpha k} g^{\underline{\alpha j}}$, where the tensor $\tilde{g}^{\underline{ij}}$ is symmetric and nonsingular. But than the element $\tilde{g}_{\underline{ij}}$ of the inverse matrix satisfies (3.11) for $\psi_i = -\lambda^\alpha \tilde{g}_{\underline{\alpha i}}$, $\xi^p_{jk} = -\mu^{p\alpha}_k \tilde{g}_{\underline{\alpha j}}$. In the similar way as in the first part of the proof we obtain that ξ^i_{jk} is an antisymmetric tensor which satisfies $\xi^p_{pk} = \xi^p_{kp} = 0$.

Further, the tensor $\tilde{g}_{ij}$ can be observed as a metric tensor of a generalized Riemannian space $\mathbb{G}\tilde{\mathbb{R}}_{\mathbb{N}}$. For the second kind Christoffel symbols $\tilde{\Gamma}^i_{jk}$ of that space we have, according to (2.4),

$$\tilde{\Gamma}^\alpha_{\alpha k} = \tilde{\Gamma}^\alpha_{k\alpha} = \frac{\partial}{\partial x^k} \ln\sqrt{|\underline{\tilde{g}}|}, \quad \underline{\tilde{g}} = \det \|\tilde{g}_{\underline{ij}}\|. \tag{3.24}$$

According to (2.3) we conclude

$$\tilde{\Gamma}^\alpha_{\alpha k} = \frac{1}{2} \tilde{g}^{\underline{\alpha\beta}} \tilde{g}_{\alpha\beta,k}. \tag{3.25}$$

As it is $\tilde{g}^{\underline{\alpha\beta}} \tilde{g}_{\alpha\beta,k} = \tilde{g}^{\underline{\alpha\beta}} \tilde{g}_{\beta\alpha,k}$, we have $\tilde{g}^{\underline{\alpha\beta}} \tilde{g}_{\underset{\vee}{\alpha\beta},k} = 0$, so the previous equation reduces to

$$\tilde{\Gamma}^\alpha_{\alpha k} = \frac{1}{2} \tilde{g}^{\underline{\alpha\beta}} \tilde{g}_{\underline{\alpha\beta},k}. \tag{3.26}$$

Let us use (3.11) and the definition of the covariant derivative of the first kind. We have

$$\begin{aligned} \tilde{\Gamma}^\alpha_{\alpha k} &= \frac{1}{2} \tilde{g}^{\underline{\alpha\beta}} \big(\Gamma^p_{\alpha k} \tilde{g}_{\underline{p\beta}} + \Gamma^p_{\beta k} \tilde{g}_{\underline{p\alpha}} + \psi_\alpha \tilde{g}_{\underline{k\beta}} + \psi_\beta \tilde{g}_{\underline{k\alpha}} + \xi^p_{\alpha k} \tilde{g}_{\underline{p\beta}} + \xi^p_{\beta k} \tilde{g}_{\underline{p\alpha}} \big) \\ &= \frac{1}{2} \big(\Gamma^p_{\alpha k} \delta^\alpha_p + \Gamma^p_{\beta k} \delta^\beta_p + \psi_\alpha \delta^\alpha_k + \psi_\beta \delta^\beta_k + \xi^p_{\alpha k} \delta^\alpha_p + \xi^p_{\beta k} \delta^\beta_p \big) \\ &= \Gamma^\alpha_{\alpha k} + \psi_k, \end{aligned} \tag{3.27}$$

after using $\xi^p_{pk} = 0$. Therefore,

$$\psi_k = \tilde{\Gamma}^\alpha_{\alpha k} - \Gamma^\alpha_{\alpha k} = \frac{\partial}{\partial x^k} \ln \sqrt{|\underline{\tilde{g}}|} - \frac{\partial}{\partial x^k} \ln \sqrt{|\underline{g}|} = \frac{\partial}{\partial x^k} \ln \sqrt{\left|\frac{\underline{\tilde{g}}}{\underline{g}}\right|}, \tag{3.28}$$

which means that ψ_k is a gradient, i. e. $\psi_k = \frac{\partial \psi}{\partial x^k}$. But then for the tensor

$$\overline{g}_{ij} = e^{2\psi} \tilde{g}_{ij} \tag{3.29}$$

from (3.11) obviously the condition (3.1) is valid. Therefore, the theorem is proved. ■

Analogously we prove the next theorem:

Theorem 3.5. *A generalized Riemannian space $\mathbb{GR}_N$ admits a nontrivial geodesic mapping, if and only if there exists a nonsingular symmetric tensor a_{ij} satisfying*

$$a_{ij\underset{2}{|}k} = \lambda_i g_{\underline{jk}} + \lambda_j g_{\underline{ik}} + \mu^p_{ki} g_{\underline{pj}} + \mu^p_{kj} g_{\underline{pi}}, \tag{3.30}$$

for a gradient $\lambda_i \neq 0$ and an antisymmetric tensor μ^i_{jk} so that $\mu^i_{ij} = \mu^i_{ji} = 0$. ■

4. Necessary and sufficient conditions for an infinitesimal geodesic deformation of the space $\mathbb{GR}_N$

In his paper [11] M. L. Gavril'chenko gave a necessary and sufficient condition that a Riemannian space admits an infinitesimal geodesic deformation, i. e. he proved following theorem:

Theorem 4.1. *A Riemannian space $\mathbb{R}_N$ admits infinitesimal geodesic deformations if and only if on $\mathbb{R}_N$ there exists a symmetric tensor h_{ij} so that the condition*

$$h_{ij;k} = 2\psi_k g_{ij} + \psi_i g_{jk} + \psi_j g_{ik}, \tag{4.1}$$

is valid for a gradient vector ψ_i. ■

In the sequel we are giving a generalization of that theorem which is valid in the case of generalized Riemannian spaces.

Theorem 4.2. *A generalized Riemannian space $\mathbb{GR}_N$ admits infinitesimal geodesic deformations if and only if on $\mathbb{GR}_N$ there exists a symmetric tensor h_{ij} so that the condition*

$$h_{ij\underset{1}{|}k} = 2\psi_k g_{\underline{ij}} + \psi_i g_{\underline{kj}} + \psi_j g_{\underline{ik}} + \xi^p_{ik} g_{\underline{pj}} + \xi^p_{jk} g_{\underline{ip}}, \tag{4.2}$$

is valid for a gradient vector ψ_i and an antisymmetric tensor ξ^i_{jk} so that $\xi^i_{ik} = \xi^i_{ki} = 0$.

Proof. Let be given an infinitesimal geodesic deformation of the space $\mathbb{GR}_N$ with the equation

$$\tilde{y}^\alpha = y^\alpha(x^i) + \epsilon z^\alpha(x^i), \tag{4.3}$$

where ϵ is a small real parameter. The equations (4.3) define a deformed space $\mathbb{G}\tilde{\mathbb{R}}_{\mathbb{N}}$. Obviously, the spaces $\mathbb{GR}_N$ and $\mathbb{G}\tilde{\mathbb{R}}_{\mathbb{N}}$, in the common system of local coordinates (x^i), admit a geodesic mapping of one to another, so the equality (3.1) holds true:

$$\tilde{g}_{\underline{ij}\underset{1}{|}k} = 2\psi_k\, \tilde{g}_{\underline{ij}} + \psi_i\, \tilde{g}_{\underline{kj}} + \psi_j\, \tilde{g}_{\underline{ik}} + \xi^p_{ik}\tilde{g}_{\underline{pj}} + \xi^p_{jk}\tilde{g}_{\underline{ip}}, \tag{4.4}$$

where ξ^i_{jk} is an antisymmetric part of the deformation tensor, and ψ_i is a gradient vector defined with the function

$$\psi_i = \frac{1}{N+1}\frac{\partial}{\partial x^i} \ln \sqrt{\Big|\frac{\underline{\tilde{g}}}{\underline{g}}\Big|}, \quad \underline{g} = \det \|g_{\underline{ij}}\|, \quad \underline{\tilde{g}} = \det\|\tilde{g}_{\underline{ij}}\|. \tag{4.5}$$

Further,

$$\begin{aligned} 2(N+1)\psi_i &= \frac{\partial}{\partial x^i}\ln\Big|\frac{\underline{\tilde{g}}}{\underline{g}}\Big| = \frac{\partial}{\partial x^i}\ln\Big|1 + \frac{\delta \underline{g}}{\underline{g}}\epsilon\Big| = \frac{\partial}{\partial x^i}\ln\Big(1+\frac{\delta \underline{g}}{\underline{g}}\epsilon\Big) \\ &= \frac{\partial}{\partial x^i}\Big(\frac{\epsilon\delta \underline{g}}{\underline{g}} + \dots\Big) = \epsilon\frac{\partial}{\partial x^i}\Big(\frac{\delta \underline{g}}{\underline{g}}\Big) + \dots \end{aligned}$$

because it is valid $\underline{\tilde{g}} = \underline{g} + \epsilon\delta\underline{g}$, so we can ψ_i in (4.4) exchange with $\epsilon\psi_i$. Also, it is

$$\xi^i_{jk} = \tilde{\Gamma}^i_{\underset{\vee}{jk}} - \Gamma^i_{\underset{\vee}{jk}} = \Gamma^i_{\underset{\vee}{jk}} + \epsilon\,\delta\Gamma^i_{\underset{\vee}{jk}} - \Gamma^i_{\underset{\vee}{jk}} = \epsilon\,\delta\Gamma^i_{\underset{\vee}{jk}},$$

so we can exchange ξ^i_{jk} with $\epsilon\xi^i_{jk}$. Therefore,

$$\begin{aligned} \tilde{g}_{\underline{ij}\underset{1}{|}k} = &\,\epsilon[2\psi_k(g_{\underline{ij}} + \epsilon\,\delta g_{\underline{ij}}) + \psi_i(g_{\underline{kj}} + \epsilon\,\delta g_{\underline{kj}}) + \psi_j(g_{\underline{ik}} + \epsilon\,\delta g_{\underline{ik}}) \\ &+ \xi^p_{ik}(g_{\underline{pj}} + \epsilon\,\delta g_{\underline{pj}}) + \xi^p_{jk}(g_{\underline{ip}} + \epsilon\,\delta g_{\underline{ip}})] \\ &= \epsilon(2\psi_k g_{\underline{ij}} + \psi_i g_{\underline{kj}} + \psi_j g_{\underline{ik}} + \xi^p_{ik}g_{\underline{pj}} + \xi^p_{jk}g_{\underline{ip}}) + \epsilon^2\dots \end{aligned}$$

As it is

$$\tilde{g}_{\underline{ij}\underset{1}{|}k} = \overbrace{g_{\underline{ij}\underset{1}{|}k}}^{0} + \epsilon\,\delta g_{\underline{ij}\underset{1}{|}k} = \epsilon\,\delta g_{\underline{ij}\underset{1}{|}k}, \tag{4.6}$$

we conclude that

$$\delta g_{\underline{ij}\underset{1}{|}k} = 2\psi_k g_{\underline{ij}} + \psi_i g_{\underline{kj}} + \psi_j g_{\underline{ik}} + \xi^p_{ik}g_{\underline{pj}} + \xi^p_{jk}g_{\underline{ip}}. \tag{4.7}$$

Let us denote the symmetric tensor $\delta g_{\underline{ij}}$ with h_{ij}. Then we have (4.2).

Let us prove the opposite part of the theorem. Namely, let the equation (4.2) be a valid one. Let us observe a deformation $\tilde{g}_{\underline{ij}} = g_{\underline{ij}} + \epsilon\,\delta g_{\underline{ij}}$ such that $\delta g_{\underline{ij}} = h_{ij}$.

As it is $\epsilon\tilde{g}_{\underline{ij}} = \epsilon g_{\underline{ij}} + \epsilon^2\,\delta g_{\underline{ij}}$, we can exchange $\epsilon g_{\underline{ij}}$ with $\epsilon\tilde{g}_{\underline{ij}}$, so we have

$$\overbrace{g_{\underline{ij}\underset{1}{|}k}}^{0} + \epsilon\,\delta g_{\underline{ij}\underset{1}{|}k} = \epsilon(2\psi_k g_{\underline{ij}} + \psi_i g_{\underline{kj}} + \psi_j g_{\underline{ik}} + \xi^p_{ik} g_{\underline{pj}} + \xi^p_{jk} g_{\underline{ip}}), \tag{4.8}$$

i. e.

$$\tilde{g}_{\underline{ij}\underset{1}{|}k} = 2\psi_k\tilde{g}_{\underline{ij}} + \psi_i\tilde{g}_{\underline{kj}} + \psi_j\tilde{g}_{\underline{ik}} + \xi^p_{ik}\tilde{g}_{\underline{pj}} + \xi^p_{jk}\tilde{g}_{\underline{ip}}, \tag{4.9}$$

therefore, the deformation is a geodesic one. ■

Also, the next theorem can be analogously proved.

Theorem 4.3. *A generalized Riemannian space $\mathbb{GR}_N$ admits infinitesimal geodesic deformations if and only if on $\mathbb{GR}_N$ there exists a symmetric tensor h_{ij} so that the condition*

$$h_{ij\underset{2}{|}k} = 2\psi_k g_{\underline{ij}} + \psi_i g_{\underline{kj}} + \psi_j g_{\underline{ik}} + \xi^p_{ki} g_{\underline{pj}} + \xi^p_{kj} g_{\underline{ip}}, \tag{4.10}$$

is valid for a gradient vector ψ_i and an antisymmetric tensor ξ^i_{jk} so that $\xi^i_{ik} = \xi^i_{ki} = 0$. ■

Now we can prove the theorem which gives necessary and sufficient conditions that the space $\mathbb{GR}_N$ admits a nontrivial geodesic deformation.

Theorem 4.4. *A generalized Riemannian space $\mathbb{GR}_N$ admits nontrivial infinitesimal geodesic deformations if and only if $\mathbb{GR}_N$ admits nontrivial geodesic mappings.*

proof. ($\Rightarrow$:) Let the space $\mathbb{GR}_N$ admits a nontrivial geodesic deformation. Then the condition (4.2), where $\psi_k \neq 0$, can be written as

$$(h_{ij} - 2\psi g_{\underline{ij}})_{\underset{1}{|}k} = \psi_i g_{\underline{kj}} + \psi_j g_{\underline{ik}} + \xi^p_{ik} g_{\underline{pj}} + \xi^p_{jk} g_{\underline{ip}}, \tag{4.11}$$

i. e. in $\mathbb{GR}_N$ there exists a tensor $a_{ij} = h_{ij} - 2\psi g_{\underline{ij}}$ which satisfies the equation (3.8) for $\lambda_i = \psi_i$ and $\mu^i_{jk} = \xi^i_{jk}$. Therefore, $\mathbb{GR}_N$ admits nontrivial geodesic mapping.

($\Leftarrow$:) On the contrary, a tensor $h_{ij} = a_{ij} + 2\lambda g_{\underline{ij}}$, where a_{ij} is the solution of the equation (3.8), satisfies the condition (4.2), so $\mathbb{GR}_N$ admits nontrivial geodesic deformation according to Theorem 4.2. ■

The corresponding theorem for Riemannian spaces of the first class was proved in 1971 in the paper [16]. Later it was proved for all Riemannian spaces. Also, according to [2, 3], the next Riemannian spaces do not admit nontrivial geodesic deformations: symmetric spaces, recurrent spaces, double symmetric spaces, double recurrent spaces, m-recurrent spaces and semisymmetric spaces

D_n^m of nonconstant curvature. On the contrary, for spaces of constant curvature and for equidistant spaces geodesic deformations exists.

In the case of generalized Riemannian spaces, there are defined so-called generalized equidistant spaces. Namely, according to [4, 8], a generalized Riemannian space GR_N with a nonsymmetric metric tensor g_{ij} is called *generalized equidistant space*, if there exists a non-vanishing one-form φ in GR_N, $\varphi_i \neq 0$, satisfying

$$\varphi_{i;j} = \rho g_{\underline{ij}}, \tag{4.12}$$

where $(;)$ denotes covariant derivative with respect to the symmetric part of the connection of the space GR_N. For $\rho \neq 0$ generalized equidistant spaces belong to the primary type, and for $\rho \equiv 0$ to the particular. The equation (4.12) is equivalent to the next equations:

$$\varphi_{i\underset{1}{|}j} = \rho g_{\underline{ij}} - \Gamma^p_{\underset{\vee}{ij}}\varphi_p, \quad \varphi_{i\underset{2}{|}j} = \rho g_{\underline{ij}} - \Gamma^p_{\underset{\vee}{ji}}\varphi_p \tag{4.13}$$

where $(|)$ denotes covariant derivative of the corresponding kind in the space GR_N. It was proved in [4, 8] that generalized equidistant spaces of the primary type admit nontrivial geodesic mappings. As the direct corollary of Theorem 4.4 we have the next corollary.

Corollary 4.1. *Each generalized equidistant space of primary type admits nontrivial geodesic deformations.*

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