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Pointwise selection theorems for metric space valued bivariate functions

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A B S T R A C T

We introduce a pseudometric TV on the set M^X of all functions mapping a rectangle X on the plane $\mathbb{R}^2$ into a metric space M, called the total joint variation. We prove that if two sequences $\{f_j\}$ and $\{g_j\}$ of functions from M^X are such that $\{f_j\}$ is pointwise precompact on X, $\{g_j\}$ is pointwise convergent on X with the limit $g \in M^X$, and the limit superior of $\mathrm{TV}(f_j, g_j)$ as $j \to \infty$ is finite, then a subsequence of $\{f_j\}$ converges pointwise on X to a function $f \in M^X$ such that $\mathrm{TV}(f, g)$ is finite. One more pointwise selection theorem is given in terms of total ε-variations ($\varepsilon > 0$), which are approximations of the total variation as $\varepsilon \to 0$.

1. Main results

Pointwise selection principles are existence theorems guaranteeing the existence of a pointwise convergent subsequence of a given sequence of functions. The historically first example is the classical Helly's Theorem [25], [32, Section VIII.4]: *a uniformly bounded sequence of real monotone functions on a closed interval* $[a, b]$ *in* $\mathbb{R}$ *contains a pointwise convergent subsequence whose limit is a bounded monotone function on* $[a, b]$. As a corollary, the monotonicity of functions may be replaced by the *uniform boundedness* of their *Jordan's variations*. A far reaching consequence of the latter result is (Theorem C below and) the existence of selections of bounded (generalized) variation of univariate multifunctions of bounded (generalized) variation whose values are compact subsets of a metric space [10].

The purpose of this paper is to provide pointwise selection theorems for functions of several variables valued in an arbitrary metric space. In order to present the results in a simple and principal form and avoid (unnecessary) technicalities, we consider the case of bivariate functions on a closed rectangle.

We begin with reviewing definitions and facts needed for our results.

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Given two points $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$, we write $x < y$ (or $x \le y$) provided $x_1 < y_1$ and $x_2 < y_2$ (or $x_1 \le y_1$ and $x_2 \le y_2$, respectively), and we denote by $I_x^y = \{z \in \mathbb{R}^2 : x \le z \le y\} = [x_1, y_1] \times [x_2, y_2]$ the rectangle in $\mathbb{R}^2$ with the end-points x and y. In what follows, points $a, b \in \mathbb{R}^2$, $a < b$, are fixed, and the domain of bivariate functions is the rectangle I_a^b.

Recall that a function $\nu : I_a^b \to \mathbb{R}$ is said to be *totally monotone* if, for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in I_a^b$ with $x \le y$, we have

$$\nu(y_1, a_2) - \nu(x_1, a_2) \ge 0, \qquad \nu(a_1, y_2) - \nu(a_1, x_2) \ge 0, \text{ and}$$

$$\nu(x_1, x_2) - \nu(y_1, x_2) - \nu(x_1, y_2) + \nu(y_1, y_2) \ge 0.$$

Totally monotone functions are well-studied [1,4,6,23,26–30] (they are called *positively monotonely monotone* in [26, III.4.3]). We recall the following two results for totally monotone functions, also needed below.

Theorem A *([26, III.5.4], [34]). The points of discontinuity of a totally monotone function on I_a^b lie on at most a countable collection of lines parallel to the coordinate axes in $\mathbb{R}^2$.*

Theorem B *(Helly's selection principle from [7], [26, III.6.5]). A uniformly bounded sequence of totally monotone functions on I_a^b contains a subsequence, which converges pointwise on I_a^b to a bounded totally monotone function.*

There are a number of extensions of Theorem B for multivariate functions of bounded variation in various senses: [26,27,30,31] for real valued functions, and [5,19–22] for *metric semigroup* valued functions (see below).

Of main interest in this paper are *metric space* valued functions on I_a^b. Our approach to the pointwise selection theorems for (sequences of) such functions is based on two notions of pseudometrics, the *joint increment* and *joint mixed difference*, to be defined as follows.

Let X be a nonempty set (in the sequel, X is a closed interval $I = [a, b]$ in $\mathbb{R}$, or the rectangle I_a^b in $\mathbb{R}^2$), (M, d) be a metric space with metric d, and M^X be the set of all functions $f : X \to M$ mapping X into M. Given $f \in M^X$ and $u \in M$, we set $f_u(x) = d(u, f(x))$ for all $x \in X$ (so that f_u maps X into $[0, \infty)$) and note that

$$d(f(x), g(y)) = \max_{u \in M} |f_u(x) - g_u(y)| \text{ for all } f, g \in M^X \text{ and } x, y \in X. \tag{1.1}$$

In particular, setting $(f - g)_u(x) = f_u(x) - g_u(x)$ for $u \in M$ and $x \in X$, we find

$$d(f(x), g(x)) = \max_{u \in M} |(f - g)_u(x)|. \tag{1.2}$$

Although the 'subtraction' $f - g$ is given by $(u, x) \mapsto (f - g)_u(x)$ and maps $M \times X$ into $\mathbb{R}$, passing to $h = f - g$ and $h_u(x) = f_u(x) - g_u(x)$, for the sake of brevity, will be a convenient tool in some proofs below.

The *joint increment* of two functions $f, g \in M^X$ on the two-point set $\{x, y\} \subset X$ is (the increment of $f - g$, i.e.) the quantity introduced in [15, Chapter 5] and [16, Section 2] by

$$\begin{aligned} |(f, g)(x, y)| &= \sup_{u \in M} |(f - g)_u(x) - (f - g)_u(y)| \\ &= \sup_{u \in M} \bigl| d(u, f(x)) - d(u, f(y)) - d(u, g(x)) + d(u, g(y)) \bigr|. \end{aligned} \tag{1.3}$$

Now suppose $X = I = [a, b]$ is a closed interval in $\mathbb{R}$ ($a < b$). By a *partition* of I we mean a finite collection of points $\{t_i\}_{i=0}^m \subset I$ for some $m \in \mathbb{N}$ such that $a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b$, which is written as $\{t_i\}_0^m \prec I$.

The *joint variation* of two functions $f, g \in M^I$ is the quantity

$$V_a^b(f,g) = \sup\left\{\sum_{i=1}^m |(f,g)(t_{i-1},t_i)| : m \in \mathbb{N} \text{ and } \{t_i\}_0^m \prec I = [a,b]\right\} \tag{1.4}$$

valued in $[0,\infty]$, where $|(f,g)(x,y)|$ is the joint increment (1.3) with $x = t_{i-1}$ and $y = t_i$ (cf. [15, Section 6.4] and [16, Section 4]).

The following pointwise selection theorem for M-valued functions on $I = [a,b]$ was established recently in [16, Theorem 1 and the beginning of Section 6] in terms of joint variation:

Theorem C. *Suppose $\{f_j\}$, $\{g_j\} \subset M^I$ are two sequences of functions such that:* (a) *$\{f_j\}$ is pointwise precompact on I;* (b) *$\{g_j\}$ is pointwise convergent on I to a function $g \in M^I$; and* (c) $\limsup_{j\to\infty} V_a^b(f_j,g_j) < \infty$.

Then, there is a subsequence of $\{f_j\}$, which converges pointwise on I to a function $f \in M^I$ such that $V_a^b(f,g) < \infty$.[1]

Now, let $I_a^b = [a_1,b_1]\times[a_2,b_2]$ be the rectangle in $\mathbb{R}^2$. The (Vitali-type) *joint mixed difference* $|(f,g)(I_x^y)|_2$ of two functions $f, g \in M^{I_a^b}$ on a subrectangle $I_x^y = I_{x_1,x_2}^{y_1,y_2} \subset I_a^b$ with $x \le y$ is defined by

$$|(f,g)(I_x^y)|_2 = \sup_{u\in M}\big|(f-g)_u(x_1,x_2) - (f-g)_u(y_1,x_2) - (f-g)_u(x_1,y_2) + (f-g)_u(y_1,y_2)\big|. \tag{1.5}$$

If $\{t_i\}_0^m \prec [a_1,b_1]$ with $m \in \mathbb{N}$ and $\{s_k\}_0^n \prec [a_2,b_2]$ with $n \in \mathbb{N}$, we say that the collection of subrectangles of I_a^b, given by

$$I_{i,k} = [t_{i-1},t_i]\times[s_{k-1},s_k] = I_{t_{i-1},s_{k-1}}^{t_i,\,s_k}, \quad i = 1,\dots,m,\ k = 1,\dots,n, \tag{1.6}$$

forms (or is) a *partition* of I_a^b (in symbols, $\{I_{i,k}\}_{1,1}^{m,n} \prec I_a^b$).

The (Vitali-type) *joint double variation* of two functions $f, g \in M^{I_a^b}$ is defined by

$$V_2(f,g,I_a^b) = \sup\left\{\sum_{i=1}^m\sum_{k=1}^n |(f,g)(I_{i,k})|_2 : m,n \in \mathbb{N} \text{ and } \{I_{i,k}\}_{1,1}^{m,n} \prec I_a^b\right\}. \tag{1.7}$$

The *total joint variation* (of Vitali–Hardy–Krause-type) of functions f and g as above is given by (means of (1.4) and (1.7))[2]

$$\mathrm{TV}(f,g,I_a^b) = V_{a_1}^{b_1}\big(f(\cdot,a_2),g(\cdot,a_2)\big) + V_{a_2}^{b_2}\big(f(a_1,\cdot),g(a_1,\cdot)\big) + V_2(f,g,I_a^b). \tag{1.8}$$

Our first main result is the following *pointwise selection theorem.*

Theorem 1. *Let $\{f_j\}$, $\{g_j\} \subset M^{I_a^b}$ be two sequences of functions such that*

(a) *$\{f_j\}$ is pointwise precompact on I_a^b;*

(b) *$\{g_j\}$ is pointwise convergent on I_a^b to a function $g \in M^{I_a^b}$;*

(c) $C \equiv \limsup_{j\to\infty} \mathrm{TV}(f_j,g_j,I_a^b) < \infty$.

Then, there is a subsequence of $\{f_j\}$, which converges pointwise on I_a^b to a function $f \in M^{I_a^b}$ such that $\mathrm{TV}(f,g,I_a^b) \le C$.

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[1] A sequence $\{f_j\} \equiv \{f_j\}_{j=1}^\infty \subset M^X$ *converges pointwise* (or *everywhere*) on X to a function $f \in M^X$ if $\lim_{j\to\infty} d(f_j(x), f(x)) = 0$ for all $x \in X$, and $\{f_j\} \subset M^X$ is *pointwise precompact on* X if the closure in M of the set $\{f_j(x) : j \in \mathbb{N}\}$ is compact for all $x \in X$.

[2] Given $x = (x_1,x_2) \in I_a^b$ and $f \in M^{I_a^b}$, the univariate functions $f(\cdot,x_2) : [a_1,b_1] \to M$ and $f(x_1,\cdot) : [a_2,b_2] \to M$ are defined in the usual manner: $f(\cdot,x_2)(t) = f(t,x_2)$ for all $t \in [a_1,b_1]$, and $f(x_1,\cdot)(s) = f(x_1,s)$ for all $s \in [a_2,b_2]$.

The novelty of this theorem is threefold as compared to the references following Theorem B. First, assumption (b) refers to an arbitrary pointwise convergent sequence $\{g_j\}$ whereas, usually, $\{g_j\}$ consists of a single constant function $c : I_a^b \to M$. Second, assumption (c) is more general than condition $\sup_{j\in\mathbb{N}} \mathrm{TV}(f_j, c, I_a^b) < \infty$ adopted in the literature. Third, (M, d) is an arbitrary metric space in Theorem 1 instead of a metric semigroup from the references. Recall that a triple $(M, d, +)$ is a *metric semigroup* [10, Section 4] if (M, d) is a metric space, $(M, +)$ is an Abelian semigroup with the operation of addition $+$, and $d(u, v) = d(u + w, v + w)$ for all $u, v, w \in M$. In this case, the joint increment (1.3) may be replaced by [10,17,18]

$$|(f, g)(x, y)| = d(f(x) + g(y), f(y) + g(x)), \tag{1.9}$$

and the joint mixed difference (1.5)—by [5,9,11,12,19–22]

$$\begin{aligned} |(f, g)(I_x^y)|_2 = d\big(&f(x_1, x_2) + g(y_1, x_2) + g(x_1, y_2) + f(y_1, y_2),\\ &g(x_1, x_2) + f(y_1, x_2) + f(x_1, y_2) + g(y_1, y_2)\big). \end{aligned} \tag{1.10}$$

Furthermore, if $(M, \|\cdot\|)$ is a *normed linear space* (over $\mathbb{R}$ or $\mathbb{C}$), we may set

$$|(f, g)(x, y)| = \|(f - g)(x) - (f - g)(y)\| = \|[f(x) + g(y)] - [f(y) + g(x)]\|, \tag{1.11}$$

$$|(f, g)(I_x^y)|_2 = \|(f - g)(x_1, x_2) - (f - g)(y_1, x_2) - (f - g)(x_1, y_2) + (f - g)(y_1, y_2)\|. \tag{1.12}$$

Since quantities (1.3), (1.9), and (1.11) (as well as (1.5), (1.10), and (1.12), respectively) have the same properties needed for the selection theorem, the method of proof of Theorem 1 applies to the three mentioned cases, and so, Theorem 1 contains the results from the references above as particular cases.

The second main result, Theorem 2 below, is based on the notion of *total ε-variation* ($\varepsilon > 0$) to be defined as follows.

The quantity $\mathrm{TV}(f, c, I_a^b)$ from (1.8) is independent of a constant function $c : I_a^b \to M$; it is called the *total variation* of $f \in M^{I_a^b}$ and denoted by $\mathrm{TV}(f, I_a^b)$. This notion was employed in [8], [26, III.6.3], [27] (for real bivariate functions), [13,14,28,30] (for real multivariate functions), and [5,9,11,12,19–21,33] (for metric semigroup valued multivariate functions). The set of all functions of *bounded total variation* on I_a^b is denoted by

$$\mathrm{BV}(I_a^b; M) = \{f \in M^{I_a^b} : \mathrm{TV}(f, I_a^b) < \infty\}.\text{[3]} \tag{1.13}$$

We equip the set M^X (in particular when $X = I_a^b$) with the (extended valued) *uniform metric* given, as usual, by

$$d_\infty(f, g) \equiv d_{\infty,X}(f, g) = \sup_{x\in X} d(f(x), g(x)) \quad \text{for all } f, g \in M^X. \tag{1.14}$$

Given $\varepsilon > 0$, the *total ε-variation* of $f \in M^{I_a^b}$ is the quantity

$$\mathrm{TV}_\varepsilon(f, I_a^b) = \inf\big\{\mathrm{TV}(g, I_a^b) : g \in \mathrm{BV}(I_a^b; M) \text{ and } d_\infty(f, g) \le \varepsilon\big\} \tag{1.15}$$

with the convention that $\inf \varnothing = \infty$. In the special case when $(M, \|\cdot\|)$ is a normed linear space, we set $\|f\|_\infty = \sup_{x\in I_a^b} \|f(x)\|$ and $d_\infty(f, g) = \|f - g\|_\infty$.

[3] This notation should not be confused with a similar notation for multivariate functions of bounded variation in the distributional approach [2,35]; cf. also [3, Section 3.12].

For univariate functions $f : I \to M$ with $I = [a, b] \subset \mathbb{R}$, a similar notion of *ε-variation* was introduced in [24, Definition 3.2] for $M = \mathbb{R}^N$ and considered in [16, equality (4.1)] for an arbitrary metric space (M, d). It is to be noted that this notion characterizes *regulated* functions on $I = [a, b]$.

The following theorem is a *pointwise selection principle* in terms of *total ε-variations*:

Theorem 2. *Let $(M, \|\cdot\|)$ be a finite-dimensional normed linear space and $\{f_j\} \subset M^{I_a^b}$ be a sequence of functions such that*

(a) $A \equiv \sup_{j\in\mathbb{N}} \|f_j(a)\| < \infty$, *and*

(b) $\nu_\varepsilon \equiv \limsup_{j\to\infty} \mathrm{TV}_\varepsilon(f_j, I_a^b) < \infty$ *for all* $\varepsilon > 0$.

Then, there is a subsequence of $\{f_j\}$, which converges in M pointwise on I_a^b to a bounded function $f \in M^{I_a^b}$ such that $\mathrm{TV}_\varepsilon(f, I_a^b) \le \nu_\varepsilon$ for all $\varepsilon > 0$.

The paper is organized as follows. In Section 2, we present properties of the joint increment and joint variation for metric space valued functions of one variable. In Section 3, we study the joint mixed difference, joint double variation and total joint variation for metric space valued functions of two real variables: Lemmas 2–4 are the main ingredients in the proof of Theorem 1 given in Section 4. Properties of the total ε-variation and the proof of Theorem 2 are presented in Section 5.

2. The joint increment and joint variation

2.1. Bounded functions

For a set X, a metric space (M, d), and $f \in M^X$, the quantity

$$|f(X)| \equiv |f(X)|_d = \sup_{x,y\in X} d(f(x), f(y)) \in [0, \infty]$$

is known as the *diameter of the image* $f(X) = \{f(x) : x \in X\} \subset M$, or the *oscillation of f on X*. By the triangle inequality for d and (1.14),

$$d_\infty(f, g) \le |f(X)| + d(f(y), g(y)) + |g(X)| \quad \text{for all } y \in X \tag{2.1}$$

and

$$|f(X)| \le |g(X)| + 2d_\infty(f, g) \quad \text{for all } f, g \in M^X. \tag{2.2}$$

Given $g \in M^X$, we denote by $\mathrm{B}_g(X; M) = \{f \in M^X : d_\infty(f, g) < \infty\}$ the set of all *g-bounded functions* on X. The pair $(\mathrm{B}_g(X; M), d_\infty)$ is a metric space, which is complete provided (M, d) is complete. If $c : X \to M$ is a constant function, functions from $\mathrm{B}(X; M) \equiv \mathrm{B}_c(X; M)$ are said to be (simply) *bounded* on X. By (2.1) and (2.2), $\mathrm{B}(X; M) = \{f \in M^X : |f(X)| < \infty\}$.

2.2. The joint increment

For any $f, g \in M^X$ and $x, y \in X$, the joint increment $|(f, g)(x, y)|$ from (1.3) is well-defined, since, by virtue of (1.1) and (1.2),

$$|(f, g)(x, y)| \le d(f(x), f(y)) + d(g(x), g(y)), \tag{2.3}$$

$$|(f, g)(x, y)| \le d(f(x), g(x)) + d(f(y), g(y)); \tag{2.4}$$

and $|(f,c)(x,y)| = d(f(x), f(y))$ is independent of a constant function $c \in M^X$. If $F(u) = |(f-g)_u(x)-(f-g)_u(y)|$ is the quantity under the supremum sign in (1.3), then $F : M \to [0,\infty)$ and $|F(u)-F(v)| \le 4d(u,v)$ for all $u,v \in M$. Hence, the space M in (1.3) may be replaced by any of its dense subsets.

Given $x,y \in X$, the function $(f,g) \mapsto |(f,g)(x,y)|$ is a *pseudometric* on M^X, i.e., $|(f,f)(x,y)| = 0$, $|(f,g)(x,y)| = |(g,f)(x,y)|$, and $|(f,g)(x,y)| \le |(f,h)(x,y)| + |(h,g)(x,y)|$ for all $f,g,h \in M^X$; moreover,

$$d(f(x), f(y)) \le d(g(x), g(y)) + |(f,g)(x,y)|,$$
$$d(f(x), g(x)) \le d(f(y), g(y)) + |(f,g)(x,y)| \tag{2.5}$$

(cf. [15, Section 5.1], [16, Section 3]).

If $M = \mathbb{R}$ with the usual metric $d(u,v) = |u-v|$, then

$$\begin{aligned}|(f,g)(x,y)| &= \max_{u\in\mathbb{R}} F(u) = \max\big\{F(f(x)), F(f(y)), F(g(x)), F(g(y))\big\}\\ &= \min\big\{|f(x)-f(y)| + |g(x)-g(y)|, |f(x)-g(x)| + |f(y)-g(y)|\big\}\end{aligned}$$

for all $f,g : X \to \mathbb{R}$ and $x,y \in X$, and so, $|(f-g)(x)-(f-g)(y)| \le |(f,g)(x,y)|$ (this inequality may be *strict*: put $f(x) = 0$, $g(y) = 2$, and $f(y) = g(x) = 1$).

However, in general both inequalities (2.3) and (2.4) may be strict. In fact, let X be a set with at least two elements $x,y \in X$, M be a set with at least four elements $u_1,u_2,u_3,u_4 \in M$, and d be the discrete metric on M (i.e., $d(u,v) = 0$ if $u = v$, and $d(u,v) = 1$ if $u \ne v$). Suppose $f,g \in M^X$ are such that $f(x) = u_1$, $f(y) = u_2$, $g(x) = u_3$, and $g(y) = u_4$. Since $F(u_i) = 1$ for $i = 1,2,3,4$, and $F(u) = 0$ if $u \in M$ and $u \ne u_i$ for $i = 1,2,3,4$, we find $|(f,g)(x,y)| = \max_{u\in M} F(u) = 1$. On the other hand, the quantities on the right in (2.3) and (2.4) are equal to 2.

2.3. The joint variation of univariate functions

Here we assume that $X = I = [a,b] \subset \mathbb{R}$ and (M,d) is a metric space. It is clear from (1.4) that $|(f,g)(s,t)| \le V_a^b(f,g)$ for all $s,t \in I$ and, by (2.5),

$$d_\infty(f,g) \le d_{\mathrm{BV}}(f,g) \equiv d(f(a), g(a)) + V_a^b(f,g) \ \text{ for all } f,g \in M^I. \tag{2.6}$$

Since $(f,g) \mapsto |(f,g)(s,t)|$ is a pseudometric on M^I, the function V_a^b is a pseudometric on M^I, possibly taking the value ∞. An element from the set $\mathrm{BV}_g(I;M) = \{f \in M^I : V_a^b(f,g) < \infty\}$ is said to be a *function of g-bounded variation* on I. By (2.6), $\mathrm{BV}_g(I;M) \subset \mathrm{B}_g(I;M)$. If $c \in M^I$ is constant, functions from $\mathrm{BV}(I;M) \equiv \mathrm{BV}_c(I;M)$ are the usual functions *of bounded variation*, and the quantity

$$V_a^b(f) \equiv V_a^b(f,c) = \sup\bigg\{\sum_{i=1}^m d(f(t_{i-1}), f(t_i)) : m \in \mathbb{N} \text{ and } \{t_i\}_0^m \prec [a,b]\bigg\} < \infty$$

is the usual *variation* (in the sense of C. Jordan) of $f \in \mathrm{BV}(I;M)$. The triangle inequality for V_a^b also gives, for all $f,g \in M^I$,

$$V_a^b(f) \le V_a^b(g) + V_a^b(f,g) \quad \text{and} \quad V_a^b(f,g) \le V_a^b(f) + V_a^b(g).$$

Two more properties of V_a^b are worth mentioning, namely, the *additivity*:

$$V_a^t(f,g) + V_t^b(f,g) = V_a^b(f,g) \quad \text{for all } f,g \in M^I \text{ and } a \le t \le b \tag{2.7}$$

(see [15, Lemma 6.4.1]); and the (sequential) *lower semicontinuity*: if f, g, f_j, $g_j \in M^I$ ($j \in \mathbb{N}$), $f_j \to f$ and $g_j \to g$ pointwise on $I = [a,b]$ as $j \to \infty$, then

$$V_a^b(f,g) \le \liminf_{j\to\infty} V_a^b(f_j,g_j) \tag{2.8}$$

(see [15, Lemma 6.1.6(a) for $\varphi = \mathrm{id}$]). Property (2.8) and (2.6) imply that if (M,d) is a complete metric space, then $(\mathrm{BV}_g(I;M), d_{\mathrm{BV}})$ is also a complete metric space for all $g \in M^I$.

3. The total joint variation of bivariate functions

In this section, $X = I_a^b$ is the rectangle in $\mathbb{R}^2$ and (M,d) is a metric space.

3.1. The space $\mathrm{BV}_g(I_a^b;M)$

For any $f,g \in M^{I_a^b}$ and $x,y \in I_a^b$ with $x \le y$, the joint mixed difference $|(f,g)(I_x^y)|_2$ from (1.5) is well-defined: in fact, (1.2) implies (for instance)

$$\begin{aligned}|(f,g)(I_x^y)|_2 \le\, & d(f(x_1,x_2),g(x_1,x_2)) + d(f(y_1,x_2),g(y_1,x_2)) \\ & + d(f(x_1,y_2),g(x_1,y_2)) + d(f(y_1,y_2),g(y_1,y_2)).\end{aligned} \tag{3.1}$$

The function $(f,g) \mapsto |(f,g)(I_x^y)|_2$ is a pseudometric on $M^{I_a^b}$ (for $x \le y$), and so, by (1.7) and (1.8), functions $(f,g) \mapsto V_2(f,g,I_a^b)$ and $(f,g) \mapsto \mathrm{TV}(f,g,I_a^b)$ are also pseudometrics on $M^{I_a^b}$, possibly taking the value ∞.

The (Hardy-type) space of all bivariate functions on the rectangle I_a^b with values in M *of g-bounded variation* is defined by

$$\mathrm{BV}_g(I_a^b;M) = \{f \in M^{I_a^b} : \mathrm{TV}(f,g,I_a^b) < \infty\}, \quad g \in M^{I_a^b},$$

and we employ notation (1.13) for a constant function $g = c \in M^{I_a^b}$.

3.2. The additivity of joint double variation V_2

Lemma 1. *Given $f,g \in M^{I_a^b}$ and a partition $\{I_{i,k}\}_{1,1}^{m,n} \prec I_a^b$ with $m,n \in \mathbb{N}$, we have:*

$$V_2(f,g,I_a^b) = \sum_{i=1}^m \sum_{k=1}^n V_2(f,g,I_{i,k}). \tag{3.2}$$

Proof. We divide the proof into three steps for clarity.

1. First, let us show that, for any $x,y \in I_a^b$ with $x \le y$,

$$|(f,g)(I_x^y)|_2 \le |(f,g)(I_{x_1,x_2}^{t,\,y_2})|_2 + |(f,g)(I_{t,\,x_2}^{y_1,y_2})|_2 \quad \text{if } x_1 \le t \le y_1, \tag{3.3}$$

$$|(f,g)(I_x^y)|_2 \le |(f,g)(I_{x_1,x_2}^{y_1,\,s})|_2 + |(f,g)(I_{x_1,\,s}^{y_1,y_2})|_2 \quad \text{if } x_2 \le s \le y_2. \tag{3.4}$$

We prove only (3.3) (inequality (3.4) is established similarly). Let $u \in M$. Setting $h = f - g$ and taking into account definition (1.5), we get

$$|h_u(x_1,x_2) - h_u(y_1,x_2) - h_u(x_1,y_2) + h_u(y_1,y_2)|$$

$$
\begin{aligned}
&\le |h_u(x_1,x_2)-h_u(t,x_2)-h_u(x_1,y_2)+h_u(t,y_2)| \\
&\quad + |h_u(t,x_2)-h_u(y_1,x_2)-h_u(t,y_2)+h_u(y_1,y_2)| \\
&\le |(f,g)(I_{x_1,x_2}^{t,\,y_2})|_2 + |(f,g)(I_{t,\,x_2}^{y_1,y_2})|_2,
\end{aligned}
$$

and it remains to take the supremum over all $u \in M$.

As corollaries of (3.3) and (3.4), we have the following two assertions:

(a) if $\{t_i\}_0^m, \{t_i'\}_0^{m'} \prec [x_1,y_1]$ with $m,m' \in \mathbb{N}$ and $\{t_i'\}_0^{m'} \subset \{t_i\}_0^m$, then

$$
\sum_{i=1}^{m'} |(f,g)(I_{t_{i-1}',x_2}^{t_i',\,y_2})|_2 \le \sum_{i=1}^{m} |(f,g)(I_{t_{i-1},x_2}^{t_i,\,y_2})|_2;
$$

(b) if $\{s_k\}_0^n, \{s_k'\}_0^{n'} \prec [x_2,y_2]$ with $n,n' \in \mathbb{N}$ and $\{s_k'\}_0^{n'} \subset \{s_k\}_0^n$, then

$$
\sum_{k=1}^{n'} |(f,g)(I_{x_1,s_{k-1}'}^{y_1,\,s_k'})|_2 \le \sum_{k=1}^{n} |(f,g)(I_{x_1,s_{k-1}}^{y_1,\,s_k})|_2. \tag{3.5}
$$

2. In this step we show that, for any $x,y \in I_a^b$ with $x \le y$, we have

$$
V_2(f,g,I_{x_1,x_2}^{y_1,y_2}) = V_2(f,g,I_{x_1,x_2}^{t,\,y_2}) + V_2(f,g,I_{t,\,x_2}^{y_1,y_2}) \quad \text{if } x_1 \le t \le y_1, \tag{3.6}
$$

$$
V_2(f,g,I_{x_1,x_2}^{y_1,y_2}) = V_2(f,g,I_{x_1,x_2}^{y_1,\,s}) + V_2(f,g,I_{x_1,\,s}^{y_1,y_2}) \quad \text{if } x_2 \le s \le y_2. \tag{3.7}
$$

Again it suffices to establish only equality (3.6). We may assume that all the quantities in this equality are finite.

($\ge$) Let $\varepsilon > 0$ be arbitrary. By definition (1.7) of $V_2(f,g,I_{x_1,x_2}^{t,\,y_2})$, there are partitions $\{t_i^1\}_0^{m_1} \prec [x_1,t]$ and $\{s_k^1\}_0^{n_1} \prec [x_2,y_2]$ with $m_1,n_1 \in \mathbb{N}$ such that if $I_{i,k}^1 = [t_{i-1}^1,t_i^1] \times [s_{k-1}^1,s_k^1]$, $i = 1,\dots,m_1$, $k = 1,\dots,n_1$, then

$$
\sum_{i=1}^{m_1}\sum_{k=1}^{n_1} |(f,g)(I_{i,k}^1)|_2 \ge V_2(f,g,I_{x_1,x_2}^{t,\,y_2}) - \varepsilon. \tag{3.8}
$$

Similarly, there are partitions $\{t_i^2\}_0^{m_2} \prec [t,y_1]$ and $\{s_k^2\}_0^{n_2} \prec [x_2,y_2]$ with $m_2,n_2 \in \mathbb{N}$ such that if $I_{i,k}^2 = [t_{i-1}^2,t_i^2] \times [s_{k-1}^2,s_k^2]$, $i = 1,\dots,m_2$, $k = 1,\dots,n_2$, then

$$
\sum_{i=1}^{m_2}\sum_{k=1}^{n_2} |(f,g)(I_{i,k}^2)|_2 \ge V_2(f,g,I_{t,\,x_2}^{y_1,y_2}) - \varepsilon. \tag{3.9}
$$

The union $\{t_i^1\}_0^{m_1} \cup \{t_i^2\}_0^{m_2}$ is a partition of $[x_1,y_1]$ (note that $x_1 = t_0^1 < t_1^1 < \dots < t_{m_1}^1 = t = t_0^2 < t_1^2 < \dots < t_{m_2}^2 = y_1$), which we denote by $\{t_i\}_0^m$. Here $m = m_1 + m_2$, because $t_i = t_i^1$ if $i = 0,1,\dots,m_1$, and $t_i = t_{i-m_1}^2$ if $i = m_1+1,\dots,m_1+m_2 = m$. Furthermore, $\{s_k^1\}_0^{n_1} \cup \{s_k^2\}_0^{n_2}$ is a partition of $[x_2,y_2]$, which we denote by $\{s_k\}_0^n$. Note that $\max\{n_1,n_2\} \le n < n_1 + n_2$. The collection of rectangles $I_{i,k} = [t_{i-1},t_i] \times [s_{k-1},s_k]$, $i = 1,\dots,m$, $k = 1,\dots,n$, is a partition of I_a^b. Since $\{s_k^1\}_0^{n_1} \subset \{s_k\}_0^n$ and $\{s_k^2\}_0^{n_2} \subset \{s_k\}_0^n$, inequality (3.5) implies

$$
\sum_{k=1}^{n_1} |(f,g)(I_{i,k}^1)|_2 \le \sum_{k=1}^{n} |(f,g)(I_{i,k})|_2 \quad \text{if } i = 1,\dots,m_1,
$$

$$
\sum_{k=1}^{n_2} |(f,g)(I_{i-m_1,k}^2)|_2 \le \sum_{k=1}^{n} |(f,g)(I_{i,k})|_2 \quad \text{if } i = m_1+1,\dots,m.
$$

Now it follows from (3.8) and (3.9) that

$$\begin{aligned}
V_2(f,g,I_{x_1,x_2}^{y_1,y_2}) &\geq \sum_{i=1}^{m}\sum_{k=1}^{n}|(f,g)(I_{i,k})|_2 \\
&= \sum_{i=1}^{m_1}\sum_{k=1}^{n}|(f,g)(I_{i,k})|_2 + \sum_{i=m_1+1}^{m_1+m_2}\sum_{k=1}^{n}|(f,g)(I_{i,k})|_2 \\
&\geq \sum_{i=1}^{m_1}\sum_{k=1}^{n_1}|(f,g)(I_{i,k}^1)|_2 + \sum_{i=m_1+1}^{m_1+m_2}\sum_{k=1}^{n_2}|(f,g)(I_{i-m_1,k}^2)|_2 \\
&= \sum_{i=1}^{m_1}\sum_{k=1}^{n_1}|(f,g)(I_{i,k}^1)|_2 + \sum_{i=1}^{m_2}\sum_{k=1}^{n_2}|(f,g)(I_{i,k}^2)|_2 \\
&\geq V_2(f,g,I_{x_1,x_2}^{t,\,y_2}) - \varepsilon + V_2(f,g,I_{t,\,x_2}^{y_1,y_2}) - \varepsilon.
\end{aligned}$$

Due to the arbitrariness of $\varepsilon > 0$, this establishes the inequality $\geq$ in (3.6).

($\leq$) By the definition of $V_2(f,g,I_{x_1,x_2}^{y_1,y_2})$, given $\varepsilon > 0$, there are partitions $\{t_i\}_0^m \prec [x_1,y_1]$ and $\{s_k\}_0^n \prec [x_2,y_2]$ with $m,n \in \mathbb{N}$ such that, making use of notation (1.6), we find

$$V_2(f,g,I_{x_1,x_2}^{y_1,y_2}) \leq \sum_{i=1}^{m}\sum_{k=1}^{n}|(f,g)(I_{i,k})|_2 + \varepsilon. \tag{3.10}$$

With no loss of generality we may assume that $t \in \{t_i\}_0^m$: in fact, if, on the contrary, $t_{i_*-1} < t < t_{i_*}$ for some $i_* \in \{1,\dots,m\}$, then, by virtue of (3.3), we get, for all $k = 1,\dots,n$,

$$|(f,g)(I_{i_*,k})|_2 = |(f,g)(I_{t_{i_*-1},s_{k-1}}^{t_{i_*},\,s_k})|_2 \leq |(f,g)(I_{t_{i_*-1},s_{k-1}}^{t,\,s_k})|_2 + |(f,g)(I_{t,\,s_{k-1}}^{t_{i_*},\,s_k})|_2,$$

so that the double sum on the right in (3.10) does not decrease when the point t is added to $\{t_i\}_0^m$. So, since $t \in \{t_i\}_0^m$, $t = t_{i_0}$ for some $i_0 \in \{0,1,\dots,m\}$, whence (3.10) implies

$$\begin{aligned}
V_2(f,g,I_{x_1,x_2}^{y_1,y_2}) &\leq \sum_{i=1}^{i_0}\sum_{k=1}^{n}|(f,g)(I_{i,k})|_2 + \sum_{i=i_0+1}^{m}\sum_{k=1}^{n}|(f,g)(I_{i,k})|_2 + \varepsilon \\
&\leq V_2(f,g,I_{x_1,x_2}^{t,\,y_2}) + V_2(f,g,I_{t,\,x_2}^{y_1,y_2}) + \varepsilon.
\end{aligned}$$

It remains to take into account the arbitrariness of $\varepsilon > 0$.

3. To prove (3.2), let (1.6) be a partition of I_a^b. Applying successively (3.6), we get

$$\begin{aligned}
V_2(f,g,I_a^b) &= V_2(f,g,I_{a_1,a_2}^{t_1,b_2}) + V_2(f,g,I_{t_1,a_2}^{b_1,b_2}) \\
&= V_2(f,g,I_{t_0,a_2}^{t_1,b_2}) + V_2(f,g,I_{t_1,a_2}^{t_2,b_2}) + V_2(f,g,I_{t_2,a_2}^{b_1,b_2}) \\
&= \cdots = \sum_{i=1}^{m} V_2(f,g,I_{t_{i-1},a_2}^{t_i,\,b_2}).
\end{aligned}$$

Similarly, applying successively (3.7), for each $i = 1,\dots,m$, we find

$$V_2(f,g,I_{t_{i-1},a_2}^{t_i,\,b_2}) = \sum_{k=1}^{n} V_2(f,g,I_{t_{i-1},s_{k-1}}^{t_i,\,s_k}) = \sum_{k=1}^{n} V_2(f,g,I_{i,k}),$$

and the desired equality (3.2) readily follows. $\square$

3.3. Estimates with the total joint variation

Lemma 2. *Given $f, g \in M^{I_a^b}$ and $x, y \in I_a^b$ with $x \le y$, we have:*

$$|(f,g)(x,y)| \le \mathrm{TV}(f,g,I_x^y), \tag{3.11}$$

$$\mathrm{TV}(f,g,I_a^x) + \mathrm{TV}(f,g,I_x^y) \le \mathrm{TV}(f,g,I_a^y). \tag{3.12}$$

Proof. 1. Let us prove (3.11). Setting $h = f - g$ and taking into account (1.3), (1.5), (1.4), and (1.7), we get, for all $u \in M$,

$$\begin{aligned}
|h_u(x) - h_u(y)| &= |h_u(x_1,x_2) - h_u(y_1,y_2)| \\
&\le |h_u(x_1,x_2) - h_u(y_1,x_2)| + |h_u(x_1,x_2) - h_u(x_1,y_2)| \\
&\quad + |-h_u(x_1,x_2) + h_u(y_1,x_2) + h_u(x_1,y_2) - h_u(y_1,y_2)| \\
&\le |(f(\cdot,x_2),g(\cdot,x_2))(x_1,y_1)| + |(f(x_1,\cdot),g(x_1,\cdot))(x_2,y_2)| + |(f,g)(I_x^y)|_2 \\
&\le V_{x_1}^{y_1}(f(\cdot,x_2),g(\cdot,x_2)) + V_{x_2}^{y_2}(f(x_1,\cdot),g(x_1,\cdot)) + V_2(f,g,I_x^y) \\
&= \mathrm{TV}(f,g,I_x^y).
\end{aligned} \tag{3.13}$$

It suffices to note that, by (1.3), $|(f,g)(x,y)| = \sup_{u\in M} |h_u(x) - h_u(y)|$.

2. Before we prove (3.12), we establish the following two inequalities:

$$V_{x_1}^{y_1}(f(\cdot,x_2),g(\cdot,x_2)) \le V_{x_1}^{y_1}(f(\cdot,a_2),g(\cdot,a_2)) + V_2(f,g,I_{x_1,a_2}^{y_1,x_2}), \tag{3.14}$$

$$V_{x_2}^{y_2}(f(x_1,\cdot),g(x_1,\cdot)) \le V_{x_2}^{y_2}(f(a_1,\cdot),g(a_1,\cdot)) + V_2(f,g,I_{a_1,x_2}^{x_1,y_2}). \tag{3.15}$$

We prove only (3.14) (a similar proof applies to (3.15)). Given $u \in M$ and $x_1 \le s \le t \le y_1$, setting $h = f - g$, we find

$$\begin{aligned}
|h_u(s,x_2) - h_u(t,x_2)| &\le |h_u(s,a_2) - h_u(t,a_2)| \\
&\quad + |-h_u(s,a_2) + h_u(t,a_2) + h_u(s,x_2) - h_u(t,x_2)| \\
&\le |(f(\cdot,a_2),g(\cdot,a_2))(s,t)| + |(f,g)(I_{s,a_2}^{t,x_2})|_2.
\end{aligned}$$

Since $|(f(\cdot,x_2),g(\cdot,x_2))(s,t)| = \sup_{u\in M} |h_u(s,x_2) - h_u(t,x_2)|$, we get

$$|(f(\cdot,x_2),g(\cdot,x_2))(s,t)| \le |(f(\cdot,a_2),g(\cdot,a_2))(s,t)| + |(f,g)(I_{s,a_2}^{t,x_2})|_2.$$

Let $\{t_i\}_0^m \prec [x_1,y_1]$ with $m \in \mathbb{N}$ be a partition of $[x_1,y_1]$. Setting $s = t_{i-1}$ and $t = t_i$, summing over $i = 1,\dots,m$, and noting that the collection of rectangles $I_{t_{i-1},a_2}^{t_i,x_2} = [t_{i-1},t_i] \times [a_2,x_2]$, $i = 1,\dots,m$, is a partition of $I_{x_1,a_2}^{y_1,x_2} = [x_1,y_1] \times [a_2,x_2]$, we have

$$\begin{aligned}
\sum_{i=1}^m |(f(\cdot,x_2),g(\cdot,x_2))(t_{i-1},t_i)| &\le \sum_{i=1}^m |(f(\cdot,a_2),g(\cdot,a_2))(t_{i-1},t_i)| + \sum_{i=1}^m |(f,g)(I_{t_{i-1},a_2}^{t_i,x_2})|_2 \\
&\le V_{x_1}^{y_1}(f(\cdot,a_2),g(\cdot,a_2)) + V_2(f,g,I_{x_1,a_2}^{y_1,x_2}).
\end{aligned}$$

Taking the supremum over all $\{t_i\}_0^m \prec [x_1,y_1]$, we obtain (3.14).

3. In order to prove (3.12), we note that $\{a_1, x_1, y_1\} \prec [a_1, y_1]$ and $\{a_2, x_2, y_2\} \prec [a_2, y_2]$, and so, the four rectangles $I_{a_1,a_2}^{x_1,x_2} = I_a^x$, $I_{x_1,a_2}^{y_1,x_2}$, $I_{a_1,x_2}^{x_1,y_2}$, and $I_{x_1,x_2}^{y_1,y_2} = I_x^y$ form a partition of $I_a^y = I_{a_1,a_2}^{y_1,y_2}$ (draw the picture on the plane). By the additivity properties (2.7) and (3.2), we have:

$$\begin{aligned} V_{a_1}^{x_1}(f(\cdot,a_2),g(\cdot,a_2)) + V_{x_1}^{y_1}(f(\cdot,a_2),g(\cdot,a_2)) &= V_{a_1}^{y_1}(f(\cdot,a_2),g(\cdot,a_2)), \\ V_{a_2}^{x_2}(f(a_1,\cdot),g(a_1,\cdot)) + V_{x_2}^{y_2}(f(a_1,\cdot),g(a_1,\cdot)) &= V_{a_2}^{y_2}(f(a_1,\cdot),g(a_1,\cdot)), \\ V_2(f,g,I_a^y) = V_2(f,g,I_a^x) + V_2(f,g,I_{x_1,a_2}^{y_1,x_2}) &+ V_2(f,g,I_{a_1,x_2}^{x_1,y_2}) + V_2(f,g,I_x^y). \end{aligned} \tag{3.16}$$

Since

$$\mathrm{TV}(f,g,I_a^x) = V_{a_1}^{x_1}(f(\cdot,a_2),g(\cdot,a_2)) + V_{a_2}^{x_2}(f(a_1,\cdot),g(a_1,\cdot)) + V_2(f,g,I_a^x),$$

and by virtue of the line preceding (3.13), (3.14), and (3.15),

$$\begin{aligned} \mathrm{TV}(f,g,I_x^y) \le{}& V_{x_1}^{y_1}(f(\cdot,a_2),g(\cdot,a_2)) + V_2(f,g,I_{x_1,a_2}^{y_1,x_2}) \\ &+ V_{x_2}^{y_2}(f(a_1,\cdot),g(a_1,\cdot)) + V_2(f,g,I_{a_1,x_2}^{x_1,y_2}) + V_2(f,g,I_x^y), \end{aligned}$$

it follows from (3.16) that

$$\begin{aligned} \mathrm{TV}(f,g,I_a^x) + \mathrm{TV}(f,g,I_x^y) & \\ \le{}& V_{a_1}^{y_1}(f(\cdot,a_2),g(\cdot,a_2)) + V_{a_2}^{y_2}(f(a_1,\cdot),g(a_1,\cdot)) \\ &+ V_2(f,g,I_a^x) + V_2(f,g,I_{x_1,a_2}^{y_1,x_2}) + V_2(f,g,I_{a_1,x_2}^{x_1,y_2}) + V_2(f,g,I_x^y) \\ ={}& \mathrm{TV}(f,g,I_a^y), \end{aligned}$$

which completes the proof of inequality (3.12) and Lemma 2. □

As a corollary of (3.11) and (2.5), we get a counterpart of inequality (2.6) for bivariate functions:

$$d_\infty(f,g) \le d_{\mathrm{BV}}(f,g) \equiv d(f(a),g(a)) + \mathrm{TV}(f,g,I_a^b) \text{ for all } f,g \in M^{I_a^b},$$

which implies

$$\mathrm{BV}_g(I_a^b;M) \subset \mathrm{B}_g(I_a^b;M) \quad \text{for all } g \in M^{I_a^b}. \tag{3.17}$$

The pair $(\mathrm{BV}_g(I_a^b;M), d_{\mathrm{BV}})$ is a metric space, which (taking into account Lemma 4 below) is complete provided (M,d) is complete (this can be shown along the same lines as Lemma 3 from [11]).

Lemma 3. *Given $f,g \in M^{I_a^b}$ with $\mathrm{TV}(f,g,I_a^b) < \infty$, the function $\nu : I_a^b \to \mathbb{R}$, defined by $\nu(x) = \mathrm{TV}(f,g,I_a^x)$ for all $x \in I_a^b$, is totally monotone.*

Proof. Suppose $x, y \in I_a^b$ are such that $x \le y$. Since

$$\nu(t,a_2) = \mathrm{TV}(f,g,I_{a_1,a_2}^{t,a_2}) = V_{a_1}^t(f(\cdot,a_2),g(\cdot,a_2)) \quad \text{for } a_1 \le t \le b_1,$$

we find from (2.7) that $\nu(y_1,a_2) - \nu(x_1,a_2) = V_{x_1}^{y_1}(f(\cdot,a_2),g(\cdot,a_2)) \ge 0$. Similarly, $\nu(a_1,y_2) - \nu(a_1,x_2) = V_{x_2}^{y_2}(f(a_1,\cdot),g(a_1,\cdot)) \ge 0$. Furthermore, by the definition of the total joint variation (1.8),

$$\nu(x_1,x_2)=V_{a_1}^{x_1}(f(\cdot,a_2),g(\cdot,a_2))+V_{a_2}^{x_2}(f(a_1,\cdot),g(a_1,\cdot))+V_2(f,g,I_a^x),$$

$$\nu(y_1,x_2)=V_{a_1}^{y_1}(f(\cdot,a_2),g(\cdot,a_2))+V_{a_2}^{x_2}(f(a_1,\cdot),g(a_1,\cdot))+V_2(f,g,I_{a_1,a_2}^{y_1,x_2}),$$

$$\nu(x_1,y_2)=V_{a_1}^{x_1}(f(\cdot,a_2),g(\cdot,a_2))+V_{a_2}^{y_2}(f(a_1,\cdot),g(a_1,\cdot))+V_2(f,g,I_{a_1,a_2}^{x_1,y_2}),$$

$$\nu(y_1,y_2)=V_{a_1}^{y_1}(f(\cdot,a_2),g(\cdot,a_2))+V_{a_2}^{y_2}(f(a_1,\cdot),g(a_1,\cdot))+V_2(f,g,I_a^y),$$

and so, the additivity property (3.2) of V_2 and equality (3.16) imply

$$\begin{aligned}
&\nu(x_1,x_2)-\nu(y_1,x_2)-\nu(x_1,y_2)+\nu(y_1,y_2)\\
&\qquad=V_2(f,g,I_a^x)-\big[V_2(f,g,I_a^x)+V_2(f,g,I_{x_1,a_2}^{y_1,x_2})\big]\\
&\qquad\quad-\big[V_2(f,g,I_a^x)+V_2(f,g,I_{a_1,x_2}^{x_1,y_2})\big]\\
&\qquad\quad+\big[V_2(f,g,I_a^x)+V_2(f,g,I_{x_1,a_2}^{y_1,x_2})+V_2(f,g,I_{a_1,x_2}^{x_1,y_2})+V_2(f,g,I_x^y)\big]\\
&\qquad=V_2(f,g,I_x^y)\ge 0,
\end{aligned}$$

which was to be proved. □

3.4. The sequential lower semicontinuity of V_2 and TV

Lemma 4. *If $\{f_j\},\{g_j\}\subset M^{I_a^b}$ and $f,g\in M^{I_a^b}$ are such that $f_j\to f$ and $g_j\to g$ pointwise on I_a^b as $j\to\infty$, then*

$$V_2(f,g,I_a^b)\le\liminf_{j\to\infty}V_2(f_j,g_j,I_a^b), \tag{3.18}$$

$$\mathrm{TV}(f,g,I_a^b)\le\liminf_{j\to\infty}\mathrm{TV}(f_j,g_j,I_a^b). \tag{3.19}$$

Proof. Let us prove (3.18). First, we note that

$$\lim_{j\to\infty}|(f_j,g_j)(I_x^y)|_2=|(f,g)(I_x^y)|_2\quad\text{for all } x,y\in I_a^b \text{ with } x\le y. \tag{3.20}$$

In fact, the triangle inequality for function $(f,g)\mapsto|(f,g)(I_x^y)|_2$ implies

$$\big||(f_j,g_j)(I_x^y)|_2-|(f,g)(I_x^y)|_2\big|\le|(f_j,f)(I_x^y)|_2+|(g,g_j)(I_x^y)|_2.$$

By virtue of inequality (3.1),

$$\begin{aligned}|(f_j,f)(I_x^y)|_2\le{}& d(f_j(x_1,x_2),f(x_1,x_2))+d(f_j(y_1,x_2),f(y_1,x_2))\\
&+d(f_j(x_1,y_2),f(x_1,y_2))+d(f_j(y_1,y_2),f(y_1,y_2)),\end{aligned}$$

and a similar estimate holds for $|(g,g_j)(I_x^y)|_2=|(g_j,g)(I_x^y)|_2$. It remains to take into account the pointwise convergence of f_j to f and g_j to g.

By definition (1.7), given $\{I_{i,k}\}_{1,1}^{m,n}\prec I_a^b$ with $m,n\in\mathbb{N}$, we have

$$\sum_{i=1}^m\sum_{k=1}^n|(f_j,g_j)(I_{i,k})|_2\le V_2(f_j,g_j,I_a^b)\quad\text{for all } j\in\mathbb{N}.$$

Passing to the limit inferior as $j \to \infty$, we find, by virtue of (3.20),

$$\sum_{i=1}^{m}\sum_{k=1}^{n}|(f,g)(I_{i,k})|_2 \le \liminf_{j\to\infty} V_2(f_j, g_j, I_a^b).$$

Now, (3.18) follows by taking the supremum over all partitions of I_a^b.

Inequality (3.19) is a consequence of (1.8), (2.8), (3.18), and the fact that, given a finite collection of sequences in $[0,\infty]$, the sum of their limits inferior does not exceed the limit inferior of their sum. □

4. Proof of Theorem 1

Proof of Theorem 1. 1. By extracting appropriate subsequences of $\{f_j\}$ and $\{g_j\}$, again denoted by $\{f_j\}$ and $\{g_j\}$, respectively, we may assume that $\mathrm{TV}(f_j, g_j, I_a^b)$ is finite for all $j \in \mathbb{N}$ and $\lim_{j\to\infty} \mathrm{TV}(f_j, g_j, I_a^b) = C$. It follows that the sequence $\{\mathrm{TV}(f_j, g_j, I_a^b)\}_{j=1}^{\infty}$ is bounded in $[0,\infty)$.

Given $j \in \mathbb{N}$ and $x \in I_a^b$, we set $\nu_j(x) = \mathrm{TV}(f_j, g_j, I_a^x)$. Since

$$\nu_j(x) \le \mathrm{TV}(f_j, g_j, I_a^b) \quad \text{for all } j \in \mathbb{N} \text{ and } x \in I_a^b,$$

the sequence $\{\nu_j\}_{j=1}^{\infty}$ of functions $\nu_j : I_a^b \to [0,\infty)$ is uniformly bounded. By Lemma 3, each function ν_j is totally monotone, and so, Theorem B implies the existence of subsequences of $\{f_j\}$ and $\{g_j\}$, again denoted by $\{f_j\}$ and $\{g_j\}$, respectively, and a totally monotone function $\nu : I_a^b \to [0,\infty)$ such that

$$\lim_{j\to\infty} \nu_j(x) = \nu(x) \quad \text{for all } \ x \in I_a^b. \tag{4.1}$$

By Theorem A, the points of discontinuity of ν lie on at most a countable set of lines parallel to the coordinate axes in $\mathbb{R}^2$.

2. Let Q_1 be the subset of $[a_1, b_1]$ consisting of all rational points from $[a_1, b_1]$, points a_1 and b_1, and those points $t \in [a_1, b_1]$, for which the line segment $\{t\} \times [a_2, b_2]$ contains points of discontinuity of ν. Similarly, denote by Q_2 the subset of $[a_2, b_2]$ consisting of all rational points from $[a_2, b_2]$, points a_2 and b_2, and those points $s \in [a_2, b_2]$, for which the line segment $[a_1, b_1] \times \{s\}$ contains points of discontinuity of ν. Since the sets Q_1 and Q_2 are countable, we may assume that $Q_1 = \{t_i\}_{i=1}^{\infty}$ and $Q_2 = \{s_k\}_{k=1}^{\infty}$.

Now, we apply Theorem C and the Cantor diagonal procedure. The sequence of univariate functions $\{f_j(t_1, \cdot)\}_{j=1}^{\infty} \subset M^{[a_2,b_2]}$ has the following properties. By inequality (3.15) and definition (1.8), we get

$$\begin{aligned} V_{a_2}^{b_2}(f_j(t_1,\cdot), g_j(t_1,\cdot)) &\le V_{a_2}^{b_2}(f_j(a_1,\cdot), g_j(a_1,\cdot)) + V_2(f_j, g_j, I_{a_1,a_2}^{t_1,b_2}) \\ &\le \mathrm{TV}(f_j, g_j, I_a^b) \quad \text{for all } \ j \in \mathbb{N}, \end{aligned}$$

and so,

$$\limsup_{j\to\infty} V_{a_2}^{b_2}(f_j(t_1,\cdot), g_j(t_1,\cdot)) \le \lim_{j\to\infty} \mathrm{TV}(f_j, g_j, I_a^b) = C < \infty.$$

Furthermore, assumption (a) in Theorem 1 implies that the set $\{f_j(t_1, s) : j \in \mathbb{N}\}$ is precompact in M for all $s \in [a_2, b_2]$, and assumption (b) implies that the sequence $\{g_j(t_1,\cdot)\}_{j=1}^{\infty}$ is pointwise convergent on $[a_2, b_2]$ to the function $g(t_1, \cdot) \in M^{[a_2,b_2]}$. By Theorem C, there is an increasing sequence $\{J_1(j)\}_{j=1}^{\infty} \subset \mathbb{N}$ (i.e., a subsequence of $\{j\}_{j=1}^{\infty}$) such that the sequence of functions $\{f_{J_1(j)}(t_1,\cdot)\}_{j=1}^{\infty} \subset M^{[a_2,b_2]}$ converges in M pointwise on $[a_2, b_2]$ to a function denoted by $f(t_1, \cdot) \in M^{[a_2,b_2]}$.

Inductively, if $k \in \mathbb{N}$, $k \ge 2$, and a subsequence $\{J_{k-1}(j)\}_{j=1}^{\infty}$ of $\{j\}_{j=1}^{\infty}$ is already chosen, we consider the sequence $\{f_{J_{k-1}(j)}(t_k,\cdot)\}_{j=1}^{\infty} \subset M^{[a_2,b_2]}$, which has the following properties. Inequality (3.15) and definition (1.8) imply

$$\begin{aligned}V_{a_2}^{b_2}\big(f_{J_{k-1}(j)}(t_k,\cdot),g_{J_{k-1}(j)}(t_k,\cdot)\big)&\le V_{a_2}^{b_2}\big(f_{J_{k-1}(j)}(a_1,\cdot),g_{J_{k-1}(j)}(a_1,\cdot)\big)\\&\quad+V_2\big(f_{J_{k-1}(j)},g_{J_{k-1}(j)},I_{a_1,a_2}^{t_k,b_2}\big)\\&\le \mathrm{TV}\big(f_{J_{k-1}(j)},g_{J_{k-1}(j)},I_a^b\big)\quad\text{for all } j\in\mathbb{N},\end{aligned}$$

and so,

$$\begin{aligned}\limsup_{j\to\infty}V_{a_2}^{b_2}\big(f_{J_{k-1}(j)}(t_k,\cdot),g_{J_{k-1}(j)}(t_k,\cdot)\big)&\le\limsup_{j\to\infty}\mathrm{TV}(f_{J_{k-1}(j)},g_{J_{k-1}(j)},I_a^b)\\&\le\lim_{j\to\infty}\mathrm{TV}(f_j,g_j,I_a^b)=C<\infty.\end{aligned}$$

Furthermore, the set $\{f_{J_{k-1}(j)}(t_k,s):j\in\mathbb{N}\}$ is precompact in M for all $s\in[a_2,b_2]$, and the sequence $\{g_{J_{k-1}(j)}(t_k,\cdot)\}_{j=1}^{\infty}$ is pointwise convergent on $[a_2,b_2]$ to the function $g(t_k,\cdot)\in M^{[a_2,b_2]}$. Applying Theorem C, we find a subsequence $\{J_k(j)\}_{j=1}^{\infty}$ of $\{J_{k-1}(j)\}_{j=1}^{\infty}$ such that the sequence of functions $\{f_{J_k(j)}(t_k,\cdot)\}_{j=1}^{\infty}$ converges in M pointwise on $[a_2,b_2]$ to a function denoted by $f(t_k,\cdot)\in M^{[a_2,b_2]}$.

Since, for all $k\in\mathbb{N}$, the sequence $\{J_j(j)\}_{j=k}^{\infty}$ is a subsequence of $\{J_k(j)\}_{j=1}^{\infty}$, denoting the diagonal sequences $\{f_{J_j(j)}\}_{j=1}^{\infty}$ and $\{g_{J_j(j)}\}_{j=1}^{\infty}$ again by $\{f_j\}$ and $\{g_j\}$, respectively, we find that $\{f_j\}$ converges in M pointwise on the set $Q_1\times[a_2,b_2]$ to the function $f:Q_1\times[a_2,b_2]\to M$.

Similarly, starting from the just defined sequences $\{f_j\}$ and $\{g_j\}$, applying inequality (3.14) (in place of (3.15)) and the 'diagonal arguments' similar to the above, we extract a new subsequence $\{f_{J_j(j)}\}_{j=1}^{\infty}$ of $\{f_j\}$, which converges in M pointwise on the set $[a_1,b_1]\times Q_2$ to the function $f:[a_1,b_1]\times Q_2\to M$.

Thus, with no loss of generality we may assume that the sequence $\{f_j\}$ (the corresponding g-sequence being denoted by $\{g_j\}$) converges in M pointwise on the set $Q=(Q_1\times[a_2,b_2])\cup([a_1,b_1]\times Q_2)$ to the function $f:Q\to M$.

3. Let us prove now that the sequence $\{f_j(x)\}_{j=1}^{\infty}$ converges in M at each point $x\in I_a^b\setminus Q$. Let $\varepsilon>0$ be arbitrary. By the density of Q in I_a^b and the continuity of ν at x, there exists $y\in Q$ such that $x<y$ and $|\nu(y)-\nu(x)|\le\varepsilon$. From (4.1), there is $N_0=N_0(\varepsilon)\in\mathbb{N}$ such that $|\nu_j(x)-\nu(x)|\le\varepsilon$ and $|\nu_j(y)-\nu(y)|\le\varepsilon$ for all $j\ge N_0$. By (3.11) and (3.12), if $j\ge N_0$, we have

$$\begin{aligned}|(f_j,g_j)(x,y)|&\le\mathrm{TV}(f_j,g_j,I_x^y)\\&\le\mathrm{TV}(f_j,g_j,I_a^y)-\mathrm{TV}(f_j,g_j,I_a^x)=\nu_j(y)-\nu_j(x)\\&\le|\nu_j(y)-\nu(y)|+|\nu(y)-\nu(x)|+|\nu(x)-\nu_j(x)|\le3\varepsilon.\end{aligned}$$

Being convergent, the sequences $\{f_j(y)\}_{j=1}^{\infty}$, $\{g_j(x)\}_{j=1}^{\infty}$, and $\{g_j(y)\}_{j=1}^{\infty}$ are Cauchy in M, and so, there is $N_1=N_1(\varepsilon,x,y)\in\mathbb{N}$ such that, for all $j,k\ge N_1$,

$$d(f_j(y),f_k(y))\le\varepsilon,\quad d(g_j(x),g_k(x))\le\varepsilon,\ \text{ and }\ d(g_j(y),g_k(y))\le\varepsilon.$$

By (2.4), we get

$$|(g_j,g_k)(x,y)|\le d(g_j(x),g_k(x))+d(g_j(y),g_k(y))\le2\varepsilon,\qquad j,k\ge N_1.$$

Applying (2.5) and the triangle inequality for $(f,g)\mapsto|(f,g)(x,y)|$, we find

$$\begin{aligned}d(f_j(x),f_k(x))&\le d(f_j(y),f_k(y))+|(f_j,f_k)(x,y)|\\&\le\varepsilon+|(f_j,g_j)(x,y)|+|(g_j,g_k)(x,y)|+|(g_k,f_k)(x,y)|\\&\le\varepsilon+3\varepsilon+2\varepsilon+3\varepsilon=9\varepsilon\quad\text{for all } j,k\ge\max\{N_0,N_1\}.\end{aligned}$$

Thus, $\{f_j(x)\}_{j=1}^{\infty}$ is a Cauchy sequence in M, which together with assumption (a) establishes its convergence in M to an element denoted by $f(x) \in M$.

4. At the end of step 2 and in step 3 we have shown that the function $f : I_a^b = Q \cup (I_a^b \setminus Q) \to M$ is the pointwise limit on I_a^b of a subsequence $\{f_{j_k}\}_{k=1}^{\infty}$ of the original sequence $\{f_j\}_{j=1}^{\infty}$. Since $g_{j_k} \to g$ pointwise on I_a^b as $k \to \infty$, we conclude from (3.19) that

$$\mathrm{TV}(f, g, I_a^b) \le \liminf_{k\to\infty} \mathrm{TV}(f_{j_k}, g_{j_k}, I_a^b) \le \limsup_{j\to\infty} \mathrm{TV}(f_j, g_j, I_a^b) = C < \infty,$$

and so, $f \in \mathrm{BV}_g(I_a^b; M)$. This completes the proof of Theorem 1. $\square$

Clearly, Theorem C follows from Theorem 1: it suffices to consider functions of two variables depending on one fixed variable.

5. The total ε-variation and proof of Theorem 2

In order to prove Theorem 2, we need two lemmas. Note that, by (1.15), the function $\varepsilon \mapsto \mathrm{TV}_\varepsilon(f, I_a^b)$, which maps $(0, \infty)$ into $[0, \infty]$, is *nonincreasing.*

Lemma 5. *Given $f \in M^{I_a^b}$, where (M, d) is a metric space, we have*:

(a) $\lim_{\varepsilon\to+0} \mathrm{TV}_\varepsilon(f, I_a^b) = \mathrm{TV}(f, I_a^b)$ *in* $[0, \infty]$;

(b) $|f(I_a^b)| \le \mathrm{TV}_\varepsilon(f, I_a^b) + 2\varepsilon$ *for all* $\varepsilon > 0$;

(c) *if* $f \in \mathrm{B}(I_a^b; M)$ *and* $\varepsilon \ge |f(I_a^b)|$, *then* $\mathrm{TV}_\varepsilon(f, I_a^b) = 0$.

Proof. (a) First, suppose $\mathrm{TV}(f, I_a^b) < \infty$, i.e., $f \in \mathrm{BV}(I_a^b; M)$. Definition (1.15) implies $\mathrm{TV}_\varepsilon(f, I_a^b) \le \mathrm{TV}(f, I_a^b)$ for all $\varepsilon > 0$, and so,

$$C \equiv \lim_{\varepsilon\to+0} \mathrm{TV}_\varepsilon(f, I_a^b) \le \mathrm{TV}(f, I_a^b). \tag{5.1}$$

To prove the reverse inequality, we apply the definition of C: given $\eta > 0$, there is $\delta = \delta(\eta) > 0$ such that $\mathrm{TV}_\varepsilon(f, I_a^b) < C + \eta$ for all $\varepsilon \in (0, \delta)$. Let $\{\varepsilon_k\}_{k=1}^{\infty} \subset (0, \delta)$ be such that $\varepsilon_k \to 0$ as $k \to \infty$. Then, for each $k \in \mathbb{N}$, by the definition of $\mathrm{TV}_{\varepsilon_k}(f, I_a^b)$, there is $g_k \in \mathrm{BV}(I_a^b; M)$ such that $d_\infty(f, g_k) \le \varepsilon_k$ and $\mathrm{TV}(g_k, I_a^b) \le C + \eta$. Since $\varepsilon_k \to 0$, g_k converges uniformly (hence, pointwise) on I_a^b to f as $k \to \infty$, and so, inequality (3.19) implies

$$\mathrm{TV}(f, I_a^b) \le \liminf_{k\to\infty} \mathrm{TV}(g_k, I_a^b) \le C + \eta \quad \text{for all } \eta > 0.$$

Thus, $\mathrm{TV}(f, I_a^b) \le C$.

Now, if $\mathrm{TV}(f, I_a^b) = \infty$, we claim that the quantity C from (5.1) is infinite as well; in fact, the arguments following (5.1) show that $\mathrm{TV}(f, I_a^b) \le C < \infty$, which is a contradiction.

(b) Given $\varepsilon > 0$, we may assume that $\mathrm{TV}_\varepsilon(f, I_a^b) < \infty$. By definition (1.15), for each $\eta > 0$ there is $g = g_\eta \in \mathrm{BV}(I_a^b; M)$ such that $d_\infty(f, g) \le \varepsilon$ and $\mathrm{TV}(g, I_a^b) \le \mathrm{TV}_\varepsilon(f, I_a^b) + \eta$. Making use of (2.2) with $X = I_a^b$ and (3.11) (when function f in (3.11) is constant), we get

$$|f(I_a^b)| \le |g(I_a^b)| + 2d_\infty(f, g) \le \mathrm{TV}(g, I_a^b) + 2\varepsilon \le \mathrm{TV}_\varepsilon(f, I_a^b) + \eta + 2\varepsilon,$$

and the desired inequality follows due to the arbitrariness of $\eta > 0$.

(c) Setting $g(x) = f(a)$ for all $x \in I_a^b$, we find

$$d_\infty(f, g) = \sup_{x \in I_a^b} d(f(x), f(a)) \leq |f(I_a^b)| \leq \varepsilon,$$

and so, by (1.15), $0 \leq \mathrm{TV}_\varepsilon(f, I_a^b) \leq \mathrm{TV}(g, I_a^b) = 0$. $\square$

In the case when $(M, \|\cdot\|)$ is a normed linear space (over $\mathbb{R}$ or $\mathbb{C}$), the joint increment $|(f,g)(x,y)|$ and joint mixed difference $|(f,g)(I_x^y)|_2$ for functions $f \in M^{I_a^b}$ may be understood either (i) as in (1.3) and (1.5) via the induced metric $d(u,v) = \|u - v\|$ on M, or (ii) as in (1.11) and (1.12) directly via the norm $\|\cdot\|$. In both cases (i) and (ii), definitions (1.8) of $\mathrm{TV}(f, g, I_a^b)$ and (1.15) of $\mathrm{TV}_\varepsilon(f, I_a^b)$ remain unchanged. In the proofs of Lemma 6 and Theorem 2 below, Theorem 1 is applied in the particular case when $g_j = c$ for all $j \in \mathbb{N}$ and some (no matter which) constant function $c \in M^{I_a^b}$. Recall also that $\|f\|_\infty = \sup_{x \in I_a^b} \|f(x)\|$ and $d_\infty(f, g) = \|f - g\|_\infty$.

Lemma 6. *If $(M, \|\cdot\|)$ is a finite-dimensional normed linear space, $f \in M^{I_a^b}$, $\{f_j\} \subset M^{I_a^b}$, and $f_j \to f$ pointwise on I_a^b as $j \to \infty$, then*

$$\mathrm{TV}_\varepsilon(f, I_a^b) \leq \liminf_{j \to \infty} \mathrm{TV}_\varepsilon(f_j, I_a^b) \quad \textit{for all } \varepsilon > 0. \tag{5.2}$$

Proof. Given $\varepsilon > 0$, we may assume (passing to a subsequence of $\{f_j\}$ if necessary) that the right-hand side in (5.2) is $C_\varepsilon \equiv \lim_{j \to \infty} \mathrm{TV}_\varepsilon(f_j, I_a^b) < \infty$. Then, for every $\eta > C_\varepsilon$, there is $j_0 = j_0(\varepsilon, \eta) \in \mathbb{N}$ such that $\eta > \mathrm{TV}_\varepsilon(f_j, I_a^b)$ for all $j \geq j_0$, and so, definition (1.15) implies the existence of $g_j \in \mathrm{BV}(I_a^b; M)$ such that

$$\|f_j - g_j\|_\infty = d_\infty(f_j, g_j) \leq \varepsilon \quad \text{and} \quad \mathrm{TV}(g_j, I_a^b) \leq \eta. \tag{5.3}$$

Since $f_j \to f$ pointwise on I_a^b as $j \to \infty$, the sequence $\{f_j\}$ is pointwise bounded on I_a^b, i.e., for each $x \in I_a^b$ there is a constant $A(x) > 0$ such that $\|f_j(x)\| \leq A(x)$ for all $j \in \mathbb{N}$. This implies, for all $x \in I_a^b$ and $j \geq j_0$,

$$\|g_j(x)\| \leq \|g_j(x) - f_j(x)\| + \|f_j(x)\| \leq \|g_j - f_j\|_\infty + A(x) \leq \varepsilon + A(x),$$

and so, $\{g_j\}_{j=j_0}^\infty$ is a pointwise bounded sequence on I_a^b. Since M is finite-dimensional, the sequence $\{g_j\}_{j=j_0}^\infty$ is pointwise precompact on I_a^b. Furthermore, by (5.3), $\{g_j\}_{j=j_0}^\infty$ has uniformly bounded total variations on I_a^b. By Theorem 1, there are a subsequence $\{g_{j_k}\}_{k=1}^\infty$ of $\{g_j\}_{j=j_0}^\infty$ and a function $g \in \mathrm{BV}(I_a^b; M)$ such that $g_{j_k} \to g$ pointwise on I_a^b as $k \to \infty$. Noting that $f_{j_k} \to f$ pointwise on I_a^b as $k \to \infty$, from (5.3) we get

$$\|f - g\|_\infty \leq \liminf_{k \to \infty} \|f_{j_k} - g_{j_k}\|_\infty \leq \varepsilon.$$

By virtue of (1.15), (3.19), and (5.3), it follows that

$$\mathrm{TV}_\varepsilon(f, I_a^b) \leq \mathrm{TV}(g, I_a^b) \leq \liminf_{k \to \infty} \mathrm{TV}(g_{j_k}, I_a^b) \leq \eta \quad \text{for all } \eta > C_\varepsilon,$$

and so, $\mathrm{TV}_\varepsilon(f, I_a^b) \leq C_\varepsilon$, which was to be proved. $\square$

For functions of one variable $f : [a, b] \to \mathbb{R}^N$, a counterpart of Lemma 6 was established in [24, Proposition 3.6].

Proof of Theorem 2. 1. Assumption (b) implies that, given $\varepsilon > 0$, there are $j_0(\varepsilon) \in \mathbb{N}$ and a constant $K(\varepsilon) > 0$ such that $\mathrm{TV}_\varepsilon(f_j, I_a^b) < K(\varepsilon)$ for all $j \geq j_0(\varepsilon)$.

Let $\{\varepsilon_k\}_{k=1}^\infty \subset (0,\infty)$ be such that $\varepsilon_k \to 0$ as $k \to \infty$. Then, for each $k \in \mathbb{N}$, we have $\mathrm{TV}_{\varepsilon_k}(f_j, I_a^b) < K(\varepsilon_k)$ for all $j \ge j_0(\varepsilon_k)$, and so, by definition (1.15), for each $j \ge j_0(\varepsilon_k)$ there is $g_j^{(k)} \in \mathrm{BV}(I_a^b; M)$ such that

$$\big\|f_j - g_j^{(k)}\big\|_\infty = d_\infty\big(f_j, g_j^{(k)}\big) \le \varepsilon_k \quad \text{and} \quad \mathrm{TV}\big(g_j^{(k)}, I_a^b\big) \le K(\varepsilon_k). \tag{5.4}$$

By assumption (a), given $k \in \mathbb{N}$, $j \ge j_0(\varepsilon_k)$, and $x \in I_a^b$, we have

$$\begin{aligned}\|g_j^{(k)}(x)\| &\le \|g_j^{(k)}(x) - f_j(x)\| + \|f_j(x) - f_j(a)\| + \|f_j(a)\| \\ &\le \|g_j^{(k)} - f_j\|_\infty + |f_j(I_a^x)| + A,\end{aligned}$$

where the term in the middle is estimated from Lemma 5(b):

$$|f_j(I_a^x)| \le |f_j(I_a^b)| \le \mathrm{TV}_{\varepsilon_k}(f_j, I_a^b) + 2\varepsilon_k \le K(\varepsilon_k) + 2\varepsilon_k.$$

In this way we have shown that

$$\sup_{j \ge j_0(\varepsilon_k)} \big\|g_j^{(k)}(x)\big\| \le 3\varepsilon_k + K(\varepsilon_k) + A \quad \text{for all } k \in \mathbb{N} \text{ and } x \in I_a^b, \tag{5.5}$$

and, by the second inequality in (5.4),

$$\sup_{j \ge j_0(\varepsilon_k)} \mathrm{TV}\big(g_j^{(k)}, I_a^b\big) \le K(\varepsilon_k) \quad \text{for all } \ k \in \mathbb{N}. \tag{5.6}$$

2. Applying the Cantor diagonal procedure, let us show that, for each $k \in \mathbb{N}$, there are a subsequence of $\{g_j^{(k)}\}_{j=j_0(\varepsilon_k)}^\infty$, denoted by $\{g_j^{(k)}\}_{j=1}^\infty$, and a function $g^{(k)} \in \mathrm{BV}(I_a^b; M)$ such that

$$\lim_{j\to\infty} d\big(g_j^{(k)}(x), g^{(k)}(x)\big) \equiv \lim_{j\to\infty} \big\|g_j^{(k)}(x) - g^{(k)}(x)\big\| = 0 \quad \text{for all } x \in I_a^b. \tag{5.7}$$

Putting $k = 1$ in (5.5) and (5.6), we find that the sequence of functions $\{g_j^{(1)}\}_{j=j_0(\varepsilon_1)}^\infty$ has uniformly bounded total variations (bounded by $K(\varepsilon_1)$) and is uniformly bounded on I_a^b (by constant $3\varepsilon_1 + K(\varepsilon_1) + A$), and so, since M is finite-dimensional, the sequence is pointwise precompact on I_a^b. By Theorem 1, there are a subsequence $\{J_1(j)\}_{j=1}^\infty$ of $\{j\}_{j=j_0(\varepsilon_1)}^\infty$ and a function $g^{(1)} \in \mathrm{BV}(I_a^b; M)$ such that $g_{J_1(j)}^{(1)}(x) \to g^{(1)}(x)$ in M as $j \to \infty$ for all $x \in I_a^b$. Choose the least number $j_1 \in \mathbb{N}$ such that $J_1(j_1) \ge j_0(\varepsilon_2)$. Inductively, if $k \in \mathbb{N}$, $k \ge 2$, a subsequence $\{J_{k-1}(j)\}_{j=1}^\infty$ of $\{j\}_{j=j_0(\varepsilon_1)}^\infty$ and a number $j_{k-1} \in \mathbb{N}$ such that $J_{k-1}(j_{k-1}) \ge j_0(\varepsilon_k)$ are already chosen, we get the sequence of functions $\{g_{J_{k-1}(j)}^{(k)}\}_{j=j_{k-1}}^\infty \subset \mathrm{BV}(I_a^b; M)$, which, by virtue of (5.5) and (5.6), satisfies the following two conditions:

$$\sup_{j \ge j_{k-1}} \big\|g_{J_{k-1}(j)}^{(k)}\big\|_\infty \le 3\varepsilon_k + K(\varepsilon_k) + A \quad \text{and} \quad \sup_{j \ge j_{k-1}} \mathrm{TV}\big(g_{J_{k-1}(j)}^{(k)}, I_a^b\big) \le K(\varepsilon_k).$$

By Theorem 1, there are a subsequence $\{J_k(j)\}_{j=1}^\infty$ of $\{J_{k-1}(j)\}_{j=j_{k-1}}^\infty$ and a function $g^{(k)} \in \mathrm{BV}(I_a^b; M)$ such that $g_{J_k(j)}^{(k)}(x) \to g^{(k)}(x)$ in M as $j \to \infty$ for all $x \in I_a^b$. Noting that, for each $k \in \mathbb{N}$, $\{J_j(j)\}_{j=k}^\infty$ is a subsequence of $\{J_k(j)\}_{j=j_{k-1}}^\infty \subset \{J_k(j)\}_{j=1}^\infty$, we conclude that the diagonal sequence $\{g_{J_j(j)}^{(k)}\}_{j=1}^\infty$, which was denoted by $\{g_j^{(k)}\}_{j=1}^\infty$ at the beginning of step 2, satisfies (5.7).

The corresponding (diagonal) subsequence $\{f_{J_j(j)}\}_{j=1}^\infty$ of $\{f_j\}_{j=1}^\infty$ is again denoted by $\{f_j\}_{j=1}^\infty$.

3. By virtue of (3.17), $\{g^{(k)}\}_{k=1}^\infty \subset \mathrm{BV}(I_a^b; M) \subset \mathrm{B}(I_a^b; M)$. In this step, we show that $\{g^{(k)}\}_{k=1}^\infty$ is a Cauchy sequence in $\mathrm{B}(I_a^b; M)$ with respect to the uniform metric $d_\infty(f,g) = \|f - g\|_\infty$. To do this, we make use of an idea from [24, p. 49], which has been applied for univariate functions.

Let $\eta > 0$ be arbitrary. Since $\varepsilon_k \to 0$ as $k \to \infty$, there is $k_0 = k_0(\eta) \in \mathbb{N}$ such that $\varepsilon_k \le \eta$ for all $k \ge k_0$. Now, let $k_1, k_2 \in \mathbb{N}$ be arbitrary such that $k_1 \ge k_0$ and $k_2 \ge k_0$. By (5.7), for each $x \in I_a^b$, there is $j^1 \in \mathbb{N}$, depending on x, η, k_1, and k_2, such that if $j \ge j^1$, we have

$$d\big(g_j^{(k_1)}(x), g^{(k_1)}(x)\big) \le \eta \quad \text{and} \quad d\big(g_j^{(k_2)}(x), g^{(k_2)}(x)\big) \le \eta.$$

Thus, if $j \ge j^1$, it follows from (5.4) and the triangle inequality for d that

$$\begin{aligned} d\big(g^{(k_1)}(x), g^{(k_2)}(x)\big) &\le d\big(g^{(k_1)}(x), g_j^{(k_1)}(x)\big) + d\big(g_j^{(k_1)}(x), f_j(x)\big) \\ &\qquad + d\big(f_j(x), g_j^{(k_2)}(x)\big) + d\big(g_j^{(k_2)}(x), g^{(k_2)}(x)\big) \\ &\le \eta + \varepsilon_{k_1} + \varepsilon_{k_2} + \eta \le 4\eta. \end{aligned}$$

Since $x \in I_a^b$ is arbitrary, we get $d_\infty\big(g^{(k_1)}, g^{(k_2)}\big) \le 4\eta$ for all $k_1, k_2 \ge k_0$.

4. Being finite-dimensional, (M, d) is complete, and so, the space $\mathrm{B}(I_a^b; M)$ with the uniform metric d_∞ is also complete. By step 3, there is $g \in \mathrm{B}(I_a^b; M)$ such that $\{g^{(k)}\}_{k=1}^\infty$ converges uniformly on I_a^b to g (i.e., $d_\infty(g^{(k)}, g) \to 0$) as $k \to \infty$. Now, we show that $f_j \to g$ pointwise on I_a^b as $j \to \infty$.

Let $x \in I_a^b$ and $\eta > 0$ be arbitrary. Choose and fix $k = k(\eta) \in \mathbb{N}$ such that $\varepsilon_k \le \eta$ and $d_\infty(g^{(k)}, g) \le \eta$. By (5.7), there is $j^2 \in \mathbb{N}$, depending on x, η, and k, such that $d(g_j^{(k)}(x), g^{(k)}(x)) \le \eta$ for all $j \ge j^2$, and so, (5.4) implies

$$\begin{aligned} d\big(f_j(x), g(x)\big) &\le d\big(f_j(x), g_j^{(k)}(x)\big) + d\big(g_j^{(k)}(x), g^{(k)}(x)\big) + d\big(g^{(k)}(x), g(x)\big) \\ &\le \varepsilon_k + d\big(g_j^{(k)}(x), g^{(k)}(x)\big) + d_\infty\big(g^{(k)}, g\big) \\ &\le \eta + \eta + \eta = 3\eta \quad \text{for all } \ j \ge j^2. \end{aligned}$$

Thus, we have shown that a suitable subsequence $\{f_{j_k}\}_{k=1}^\infty$ of the original sequence $\{f_j\}_{j=1}^\infty$ converges pointwise on I_a^b to the function $g \in \mathrm{B}(I_a^b; M)$. Applying Lemma 6 and setting $f = g$, we conclude that

$$\mathrm{TV}_\varepsilon(f, I_a^b) \le \liminf_{k\to\infty} \mathrm{TV}_\varepsilon(f_{j_k}, I_a^b) \le \limsup_{j\to\infty} \mathrm{TV}_\varepsilon(f_j, I_a^b) = \nu_\varepsilon < \infty$$

for all $\varepsilon > 0$. This completes the proof of Theorem 2. □

A simple consequence of Theorem 2 is as follows. Assume that assumption (b) in Theorem 2 is replaced by condition $\lim_{j\to\infty} |f_j(I_a^b)| = 0$. Then a subsequence of $\{f_j\}$ converges pointwise on I_a^b to a constant function. In fact, given $\varepsilon > 0$, there is $j_0 = j_0(\varepsilon) \in \mathbb{N}$ such that $|f_j(I_a^b)| \le \varepsilon$ for all $j \ge j_0$, and so, Lemma 5(c) implies $\mathrm{TV}_\varepsilon(f_j, I_a^b) = 0$ for all $j \ge j_0$. This yields

$$\nu_\varepsilon = \limsup_{j\to\infty} \mathrm{TV}_\varepsilon(f_j, I_a^b) \le \sup_{j\ge j_0} \mathrm{TV}_\varepsilon(f_j, I_a^b) = 0 \quad \text{for all } \varepsilon > 0.$$

By Theorem 2, a subsequence of $\{f_j\}$ converges pointwise on I_a^b to a function $f \in M^{I_a^b}$ such that $\mathrm{TV}_\varepsilon(f, I_a^b) = 0$ for all $\varepsilon > 0$. Now, Lemma 5(b) gives $|f(I_a^b)| = 0$, i.e., f is a constant function on I_a^b.

In order to compare Theorems 1 and 2, we suppose that, in Theorem 1, $g_j = c$ is a constant function for all $j \in \mathbb{N}$. Let $u_j, v_j, u \in M$, $u_j \ne v_j$ $(j \in \mathbb{N})$, be such that $u_j \to u$ and $v_j \to u$ in M as $j \to \infty$, and define the sequence $\{f_j\} \subset M^{I_a^b}$ of Dirichlet-type functions by

$$f_j(x_1, x_2) = \begin{cases} u_j & \text{if } x_1 \in [a_1, b_1] \text{ and } x_2 \in [a_2, b_2] \text{ are rational,} \\ v_j & \text{otherwise.} \end{cases}$$

Clearly, $|f_j(I_a^b)| = d(u_j, v_j) \to 0$ as $j \to \infty$, and so, as it was shown above, Theorem 2 can be applied to the sequence $\{f_j\}$. On the other hand, we already have $V_{a_1}^{b_1}(f_j(\cdot, a_2)) = \infty$ (e.g., if a_2 is rational), which implies $\mathrm{TV}(f_j, I_a^b) = \infty$ for all $j \in \mathbb{N}$, and so, Theorem 1 is inapplicable.

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