



Revisiting the Hahn–Banach theorem and nonlinear infinite programming



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ABSTRACT

The aim of this paper is to state a sharp version of the König supremum theorem, an equivalent reformulation of the Hahn–Banach theorem. We apply it to derive statements of the Lagrange multipliers, Karush–Kuhn–Tucker and Fritz John types, for nonlinear infinite programs. We also show that a weak concept of convexity coming from minimax theory, inf-sup-convexity, is the adequate one for this kind of results.

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1. Introduction

Without a doubt, the Hahn–Banach theorem is not only a central result in functional and convex analysis, but also provides endless applications in many other fields, even outside of mathematics. One of its powerful equivalent reformulations is the so-called *Mazur–Orlicz* theorem (see [20, Théorème 2.41], [22, Theorem, p. 365] and [15, Satz, p. 482], [27, Theorem 28], and its generalizations [15, Satz, p. 482 and Zusatz, p. 483], [17, Theorem 1.1], [28, Theorem 2.9], [19, Theorem 2], [8, Theorem 12], [3, Theorem 3.1], [31, Theorem 3.5] and [29, Theorem 3.5 and Theorem 6.1]), which allows one to find a linear functional dominated by another sublinear functional, and states in addition a control of the infimum of both functionals on a given convex set. Such a control is not trivial, and generates numerous applications in functional analysis, minimax theory, variational analysis, monotone multifunctions theory or optimization. For instance, one can check [28,19,7,3]. Along these same lines one can consider the *König supremum theorem* ([16, Erweiterter Maximumssatz, p. 501]), which is an extension result in a space of

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bounded functions, although it is again equivalent to the Hahn–Banach theorem (the details can be found in [Proposition 2.3](#)).

The main result in this paper, [Theorem 3.1](#), establishes as a consequence of the Mazur–Orlicz theorem, a generalization of the König supremum theorem in terms of a not very restrictive kind of convexity (*infsup-convexity*, see [Definition 2.4](#) below). This notion of convexity arises in minimax theory: see [\[30, Definition 2.11\]](#), [\[14, p. 653\]](#) and [\[24, Definition 2.1\]](#). In this setting, infsup-convexity is the adequate type of convexity to state some general characterizations of the minimax inequality ([\[26, Corollary 3.12\]](#)). Finally, from such a generalization we deduce several theorems for nonlinear infinite programs – Lagrange multipliers, Karush–Kuhn–Tucker, Fritz John – extending those in the finite case in [\[25,21\]](#).

2. Preliminaries

Let us begin by evoking the Mazur–Orlicz theorem. Recall that a real-valued functional on a real linear space is *sublinear* if it is subadditive and positively homogeneous.

Theorem 2.1 (*Mazur–Orlicz*). *Suppose that E is a real vector space, C is a nonempty and convex subset of E , and that $S : E \rightarrow \mathbb{R}$ is a sublinear functional. Then, there exists a linear functional $L : E \rightarrow \mathbb{R}$ such that*

$$x \in E \Rightarrow L(x) \leq S(x)$$

and

$$\inf_{x \in C} L(x) = \inf_{x \in C} S(x).$$

Infinite values are allowed in the equality.

Let us also mention the König supremum theorem, which we will generalize in [Section 3](#). To this end, if Λ is a nonempty set, Δ_Λ stands for the subset of the topological dual $\ell^\infty(\Lambda)^*$ of the real Banach space $\ell^\infty(\Lambda)$ (usual sup-norm) of all the bounded real-valued functions defined on Λ

$$\Delta_\Lambda := \{\Phi \in \ell^\infty(\Lambda)^* : \Phi \leq \sup_\Lambda\},$$

that is, $\Phi \in \ell^\infty(\Lambda)^*$ belonging to Δ_Λ means that

$$\varphi \in \ell^\infty(\Lambda) \Rightarrow \Phi(\varphi) \leq \sup_{\lambda \in \Lambda} \varphi(\lambda).$$

It is very easy to prove that $\Phi \in \Delta_\Lambda$ if, and only if, Φ is positive, i.e., $\varphi \geq 0$ on Λ implies $\Phi(\varphi) \geq 0$, and $\Phi(\mathbf{1}) = 1$, where $\mathbf{1} \in \ell^\infty(\Lambda)$ is the constant function 1. In particular, when Λ is a nonempty finite set, $\ell^\infty(\Lambda)$ is of the form \mathbb{R}^N for some $N \in \mathbb{N}$, and Δ_Λ is the probability simplex

$$\Delta_N := \{(t_1, \dots, t_N) \in \mathbb{R}^N : t_1, \dots, t_N \geq 0 \text{ and } \sum_{j=1}^N t_j = 1\}.$$

As we will check in the proof of [Proposition 2.3](#), the elements in Δ_Λ act as extension functionals.

Specifically, the König supremum theorem reads as follows:

Theorem 2.2 (*König*). *Let E be a real linear space, Λ be a nonempty set, $L : E \rightarrow \mathbb{R}$ be a linear functional and assume that, for each $\lambda \in \Lambda$, $S_\lambda : E \rightarrow \mathbb{R}$ is a sublinear functional in such a way that*

$$x \in E \Rightarrow (S_\lambda(x))_{\lambda \in \Lambda} \in \ell^\infty(\Lambda)$$

and

$$x \in E \Rightarrow L(x) \leq \sup_{\lambda \in \Lambda} S_\lambda(x).$$

Then, there exists $\Phi \in \Delta_\Lambda$ satisfying

$$x \in E \Rightarrow L(x) \leq \Phi((S_\lambda(x))_{\lambda \in \Lambda}).$$

Even in the finite case, this result implies a wide variety of applications: see [16,17,21].

Before stating a generalization of the preceding result, which will turn out to be sharp in terms of the functions under consideration, let us observe that not only is it a consequence of the Hahn–Banach theorem (see the proof of König in [16, *Erweiterter Maximumssatz*, p. 501] from a variant of the Mazur–Orlicz theorem), but also an equivalent reformulation. Now we exactly prove that the validity of [Theorem 2.2](#) implies (and therefore is equivalent to) that of the norm preserving extension version of the Hahn–Banach theorem.

Proposition 2.3. *The König supremum theorem implies the Hahn–Banach theorem.*

Proof. Suppose that E is a real normed space, F is a vector subspace of E , and that $y_0^* : F \rightarrow \mathbb{R}$ is a continuous and linear functional. We are going to find a continuous and linear functional $x_0^* : E \rightarrow \mathbb{R}$ with

$$x_0^*|_F = y_0^*$$

and

$$\|x_0^*\| = \|y_0^*\|$$

(usual dual norms). Assume that $\|y_0^*\| = 1$; then it suffices to consider the linear space F , the linear functional y_0^* , the index set

$$\Lambda := \{x^* \in E^* : \|x^*\| \leq 1\},$$

and, for all $(x^*, y) \in \Lambda \times F$,

$$S_{x^*}(y) := x^*(y),$$

which clearly satisfy the assumptions in König’s supremum theorem. Then, there exists $\Phi \in \Delta_\Lambda$ such that

$$y \in F \Rightarrow y_0^*(y) \leq \Phi((x^*(y))_{x^* \in \Lambda}). \quad (2.1)$$

Now we can construct the required $x_0^* \in E^*$. Let $\rho : E \rightarrow \ell^\infty(\Lambda)$ be the linear operator assigning to each $x \in E$ the function $\rho(x) : \Lambda \rightarrow \mathbb{R}$ given for all $x^* \in \Lambda$ by

$$\rho(x)(x^*) := x^*(x).$$

It is clear that ρ is well defined and moreover is a linear isomorphism from E into $\ell^\infty(\Lambda)$. Then we can define x_0^* at each $x \in E$ as

$$x_0^*(x) := \Phi(\rho(x)).$$

This functional is obviously linear and in addition is continuous, since given $x \in E$,

$$\begin{aligned} x_0^*(x) &= \Phi(\rho(x)) \\ &\leq \sup_{x^* \in \Lambda} \rho(x)(x^*), \quad (\text{since } \Phi \in \Delta_\Lambda) \\ &\leq \|x\|, \end{aligned}$$

and thus, in particular,

$$\|x_0^*\| \leq 1.$$

In view of this inequality, it only remains to show that x_0^* extends y_0^* to E . But that is true, because for $y \in F$ it follows from (2.1) that

$$\begin{aligned} y_0^*(y) &\leq \Phi((x^*(y))_{x^* \in \Lambda}) \\ &= \Phi(\rho(y)) \\ &= x_0^*(y). \quad \square \end{aligned}$$

In the next section we will establish a sharp version of the König supremum theorem in terms of the following weak notion of convexity, useful in minimax theory as mentioned in the Introduction:

Definition 2.4. Let X and Λ be nonempty sets and for each $\lambda \in \Lambda$ let $f_\lambda : X \rightarrow \mathbb{R}$ be a function. The family $(f_\lambda)_{\lambda \in \Lambda}$ is said to be *infsup-convex* on X provided that

$$\left. \begin{array}{l} m \geq 1, \mathbf{t} \in \Delta_m \\ x_1, \dots, x_m \in X \end{array} \right\} \Rightarrow \inf_{x \in X} \sup_{\lambda \in \Lambda} f_\lambda(x) \leq \sup_{\lambda \in \Lambda} \sum_{j=1}^m t_j f_\lambda(x_j).$$

Infsup-convexity not only properly extends the notion of convexity of a family functions, but also that of convexlikeness for a family of functions, due to K. Fan ([4, p. 42]).

3. A generalized version of König's supremum theorem

Now we focus on stating a general König supremum theorem, by replacing the vector space with a set, and the linear functional and the family of sublinear functionals with a suitable infsup-convex family of functions, so no linear structure is required. Furthermore, we prove that such a result is sharp.

Theorem 3.1. Let X and Λ be nonempty sets, $f : X \rightarrow \mathbb{R}$ be a function and $(f_\lambda)_{\lambda \in \Lambda}$ be a family of real valued functions defined on X such that the family $(f_\lambda - f)_{\lambda \in \Lambda}$ is infsup-convex on X . Assume in addition that

$$x \in X \Rightarrow (f_\lambda(x))_{\lambda \in \Lambda} \in \ell^\infty(\Lambda)$$

and

$$x \in X \Rightarrow f(x) \leq \sup_{\lambda \in \Lambda} f_\lambda(x).$$

Then there exists $\Phi \in \Delta_\Lambda$ such that

$$x \in X \Rightarrow f(x) \leq \Phi((f_\lambda)_{\lambda \in \Lambda}).$$

Proof. Apply the Mazur–Orlicz theorem, [Theorem 2.1](#), to the real vector space $\ell^\infty(\Lambda)$, its nonempty convex subset

$$C := \text{conv}\{(f_\lambda(x) - f(x))_{\lambda \in \Lambda} : x \in X\},$$

and the sublinear functional $S : \ell^\infty(\Lambda) \longrightarrow \mathbb{R}$ given for each $\varphi \in \ell^\infty(\Lambda)$ by

$$S(\varphi) := \sup_{\lambda \in \Lambda} \varphi(\lambda).$$

Then, there exists $\Phi \in \Delta_\Lambda$ such that

$$\inf_{\varphi \in C} \Phi(\varphi) = \inf_{\varphi \in C} S(\varphi).$$

To conclude, let us observe, on the one hand, that

$$\inf_{\varphi \in C} \Phi(\varphi) = \inf_{x \in X} \Phi((f_\lambda(x) - f(x))_{\lambda \in \Lambda}),$$

and on the other hand, that the infsup-convexity of the family $(f_\lambda - f)_{\lambda \in \Lambda}$ on X and the assumption $f \leq S((f_\lambda)_{\lambda \in \Lambda})$ yield

$$\begin{aligned} \inf_{\varphi \in C} S(\varphi) &= \inf_{\substack{m \geq 1, t \in \Delta_m \\ x_1, \dots, x_m \in X}} \sup_{\lambda \in \Lambda} \sum_{j=1}^m t_j (f_\lambda(x_j) - f(x_j)) \\ &\geq \inf_{x \in X} \sup_{\lambda \in \Lambda} (f_\lambda(x) - f(x)) \\ &\geq 0. \end{aligned}$$

Therefore, for some $\Phi \in \Delta_\Lambda$,

$$0 \leq \inf_{x \in X} \Phi((f_\lambda(x) - f(x))_{\lambda \in \Lambda}),$$

and taking into account that Φ is linear and $\Phi(\mathbf{1}) = 1$, because $\Phi \in \Delta_\Lambda$, then

$$x \in X \Rightarrow f(x) \leq \Phi((f_\lambda)_{\lambda \in \Lambda}),$$

as announced. \square

The equivalence of [Theorem 3.1](#) and the Hahn–Banach theorem follows from that of the Hahn–Banach theorem and the Mazur–Orlicz theorem and from [Proposition 2.3](#).

Since infsup-convexity is a concept invariant by adding a constant, then [Theorem 3.1](#) can be equivalently reformulated as follows: assume that $\alpha \in \mathbb{R}$, X and Λ are nonempty sets, $f : X \longrightarrow \mathbb{R}$ is a function, and that $(f_\lambda)_{\lambda \in \Lambda}$ is a family of real valued functions defined on X such that the family $(f_\lambda - f)_{\lambda \in \Lambda}$ is infsup-convex on X and for all $x \in X$, $(f_\lambda(x))_{\lambda \in \Lambda} \in \ell^\infty(\Lambda)$. If in addition

$$x \in X \Rightarrow f(x) + \alpha \leq \sup_{\lambda \in \Lambda} f_\lambda(x),$$

then, there exists $\Phi \in \Delta_\Lambda$ such that

$$x \in X \Rightarrow f(x) + \alpha \leq \Phi((f_\lambda)_{\lambda \in \Lambda}).$$

Surprisingly, the converse is also true, as we state in the following result, which is the above-mentioned sharpness of [Theorem 3.1](#):

Theorem 3.2. Suppose that X and Λ are nonempty sets, $f : X \rightarrow \mathbb{R}$ is a function, and that $(f_\lambda)_{\lambda \in \Lambda}$ is a family of real valued functions defined on X such that, for each $x \in X$, $(f_\lambda(x))_{\lambda \in \Lambda} \in \ell^\infty(\Lambda)$. Then, the family $(f_\lambda - f)_{\lambda \in \Lambda}$ is infsup-convex on X if, and only if, for all $\alpha \in \mathbb{R}$ satisfying

$$x \in X \Rightarrow f(x) + \alpha \leq \sup_{\lambda \in \Lambda} f_\lambda(x),$$

there exists $\Phi \in \Delta_\Lambda$ such that

$$x \in X \Rightarrow f(x) + \alpha \leq \Phi((f_\lambda(x))_{\lambda \in \Lambda}).$$

Proof. According to the preceding argument, we focus on proving the sufficiency. Hence, let $m \geq 1$, $\mathbf{t} \in \Delta_m$ and $x_1, \dots, x_m \in X$. Let $\alpha := \inf_{x \in X} \sup_{\lambda \in \Lambda} (f_\lambda(x) - f(x))$, which can be assumed finite without any loss of generality. Since for all $x \in X$,

$$\begin{aligned} f(x) + \alpha &= f(x) + \inf_{x \in X} \sup_{\lambda \in \Lambda} (f_\lambda(x) - f(x)) \\ &\leq \sup_{\lambda \in \Lambda} f_\lambda(x), \end{aligned}$$

in view of our assumption, we arrive at

$$x \in X \Rightarrow f(x) + \alpha \leq \Phi((f_\lambda(x))_{\lambda \in \Lambda})$$

for some $\Phi \in \Delta_\Lambda$, and therefore,

$$\begin{aligned} \alpha &\leq \inf_{x \in X} \Phi((f_\lambda(x) - f(x))_{\lambda \in \Lambda}) \quad (\text{since } \Phi(\mathbf{1}) = 1) \\ &\leq \min_{j=1, \dots, m} \Phi((f_\lambda(x_j) - f(x_j))_{\lambda \in \Lambda}) \\ &\leq \sum_{j=1}^m t_j \Phi((f_\lambda(x_j) - f(x_j))_{\lambda \in \Lambda}) \\ &= \Phi \left(\sum_{j=1}^m t_j (f_\lambda(x_j) - f(x_j))_{\lambda \in \Lambda} \right) \\ &\leq \sup_{\lambda \in \Lambda} \sum_{j=1}^m t_j (f_\lambda(x_j) - f(x_j)) \quad (\text{because } \Phi \leq \sup_\Lambda). \end{aligned}$$

The arbitrariness of $m \geq 1$, $\mathbf{t} \in \Delta_m$ and $x_1, \dots, x_m \in X$ yields the announced infsup-convexity. \square

Let us point out that [Theorem 3.1](#) and [Theorem 3.2](#) were proven for Λ finite in [\[21, Theorem 2.3\]](#) and [\[21, Theorem 2.4\]](#), respectively.

4. Consequences in infinite programming

Assume that X and Λ are nonempty sets, $f : X \rightarrow \mathbb{R}$, $(f_\lambda)_{\lambda \in \Lambda}$ is a family of real valued functions on X satisfying

$$x \in X \Rightarrow (f_\lambda(x))_{\lambda \in \Lambda} \in \ell^\infty(\Lambda),$$

and that the set

$$X_0 := \left\{ x \in X : \sup_{\lambda \in \Lambda} f_\lambda(x) \leq 0 \right\}$$

is nonempty. Let us consider the nonlinear infinite program

$$\inf_{x \in X_0} f(x). \quad (4.1)$$

If we denote by $\ell^\infty(\Lambda)_+^*$ the cone of the positive functionals in $\ell^\infty(\Lambda)^*$, then the associated *Lagrangian* $\mathbf{L} : X \times \ell^\infty(\Lambda)_+^* \rightarrow \mathbb{R}$ is defined at each $(x, \Phi) \in X \times \ell^\infty(\Lambda)_+^*$ as

$$\mathbf{L}(x, \Phi) := f(x) + \Phi((f_\lambda(x))_{\lambda \in \Lambda}).$$

In addition, $(x^0, \Phi_0) \in X \times \ell^\infty(\Lambda)_+^*$ is said to be a *saddle point* of \mathbf{L} provided that

$$(x, \Phi) \in X \times \ell^\infty(\Lambda)_+^* \Rightarrow \mathbf{L}(x^0, \Phi) \leq \mathbf{L}(x^0, \Phi_0) \leq \mathbf{L}(x, \Phi_0).$$

In such a case, Φ_0 is a *Lagrange multiplier* for \mathbf{L} . It is an elementary fact that $x^0 \in X$ is an optimal solution of the nonlinear problem (4.1) provided there exists $\Phi_0 \in \ell^\infty(\Lambda)_+^*$ such that (x^0, Φ_0) is a saddle point for the Lagrangian. Now we go the other way, by proving that the infsup-convexity of a certain family of functions is exactly the assumption required for deriving – under a natural Slater condition – the existence of a Lagrange multiplier $\Phi_0 \in \ell^\infty(\Lambda)_+^*$ from that of an optimal solution $x^0 \in X$. It is a Lagrange multiplier type result: see the classical theorem [33,23] and its extensions in convexlike and quasiconvex contexts [1,9,10,32,5].

Before it, an easy technical result:

Lemma 4.1. *Let X and Λ be nonempty sets, $(f_\lambda)_{\lambda \in \Lambda}$ be a family of real valued functions defined on X such that*

$$x \in X \Rightarrow (f_\lambda(x))_{\lambda \in \Lambda} \in \ell^\infty(\Lambda),$$

and suppose that the set $X_0 := \{x \in X : \sup_{\lambda \in \Lambda} f_\lambda(x) \leq 0\}$ is nonempty. If $x^0 \in X$ is a solution of the nonlinear program (4.1), then

$$\inf_{x \in X} \max \left\{ \sup_{\lambda \in \Lambda} f_\lambda(x), f(x) - f(x^0) \right\} = 0.$$

Proof. We are assuming that

$$f(x^0) = \inf_{x \in X_0} f(x),$$

hence

$$0 \leq \inf_{x \in X_0} \max \left\{ \sup_{\lambda \in \Lambda} f_\lambda(x), f(x) - f(x^0) \right\}.$$

But for all $x \in X \setminus X_0$ there exists $\lambda \in \Lambda$ with $0 < f_\lambda(x)$, so

$$0 \leq \inf_{x \in X \setminus X_0} \max \left\{ \sup_{\lambda \in \Lambda} f_\lambda(x), f(x) - f(x^0) \right\}.$$

According to these two previous inequalities, we arrive at

$$0 \leq \inf_{x \in X} \max \left\{ \sup_{\lambda \in \Lambda} f_\lambda(x), f(x) - f(x^0) \right\}$$

and, since

$$0 = \max \left\{ \sup_{\lambda \in \Lambda} f_{\lambda}(x^0), f(x^0) - f(x^0) \right\},$$

we have concluded the proof. \square

Now we are in a position to state the aforementioned relationship between optimal solutions and saddle points (or Lagrange multipliers). This is our main statement on nonlinear infinite programming.

Theorem 4.2. *Suppose that X and Λ are nonempty sets, $f : X \rightarrow \mathbb{R}$ and that $(f_{\lambda})_{\lambda \in \Lambda}$ is a family of real valued functions defined on X such that*

$$x \in X \Rightarrow (f_{\lambda}(x))_{\lambda \in \Lambda} \in \ell^{\infty}(\Lambda),$$

and the feasible set $X_0 := \{x \in X : \sup_{\lambda \in \Lambda} f_{\lambda}(x) \leq 0\}$ is nonempty. Let us also assume that $x^0 \in X$ is an optimal solution of the nonlinear problem (4.1) and that the following Slater condition is fulfilled:

$$\text{there exists } x^1 \in X \text{ such that } \sup_{\lambda \in \Lambda} f_{\lambda}(x^1) < 0.$$

Then there exists $\Phi_0 \in \ell^{\infty}(\Lambda)_+^$ such that (x^0, Φ_0) is a saddle point for \mathbf{L} if, and only if, the family $(f_{\lambda})_{\lambda \in \Lambda} \cup (f - f(x^0))$ is infsup-convex on X .*

Proof. We first assume that for some $\Phi_0 \in \ell^{\infty}(\Lambda)_+^*$, (x^0, Φ_0) is a saddle point of \mathbf{L} . Then $f(x^0) = \mathbf{L}(x^0, \Phi_0)$ and

$$x \in X \Rightarrow \mathbf{L}(x^0, \Phi_0) \leq \mathbf{L}(x, \Phi_0),$$

i.e.,

$$0 \leq \inf_{x \in X} (\Phi_0((f_{\lambda}(x))_{\lambda \in \Lambda}) + f(x) - f(x^0)). \quad (4.2)$$

Let $\Lambda_0 := \Lambda \cup \{\mu\}$, where $\mu \notin \Lambda$, and define $\Psi \in \ell^{\infty}(\Lambda_0)^* = \ell^{\infty}(\Lambda)^* \times \mathbb{R}$ as

$$\Psi := \frac{1}{1 + \Phi_0(\mathbf{1})}(\Phi_0, 1),$$

which clearly belongs to Δ_{Λ_0} and, thanks to (4.2) satisfies

$$0 \leq \inf_{x \in X} \Psi((f_{\lambda}(x))_{\lambda \in \Lambda} \cup (f(x) - f(x^0))).$$

But, since x^0 is an optimal solution for (4.1), in view of Lemma 4.1,

$$0 = \inf_{x \in X} \max \left\{ \sup_{\lambda \in \Lambda} f_{\lambda}(x), f(x) - f(x^0) \right\}.$$

Therefore, we have that

$$\inf_{x \in X} \max \left\{ \sup_{\lambda \in \Lambda} f_{\lambda}(x), f(x) - f(x^0) \right\} \leq \inf_{x \in X} \Psi((f_{\lambda}(x))_{\lambda \in \Lambda} \cup (f(x) - f(x^0)))$$

and this inequality and the fact that $\Psi \in \Delta_{\Lambda_0}$ easily imply, as in the last part of the proof of [Theorem 3.2](#), the infsup-convexity of the family $(f_\lambda)_{\lambda \in \Lambda} \cup (f - f(x^0))$ on X .

And conversely, let us suppose that $x^0 \in X$ is an optimal solution of the nonlinear program under consideration and that the Slater condition is satisfied. The first assumption and [Lemma 4.1](#) yield

$$\inf_{x \in X} \max \left\{ \sup_{\lambda \in \Lambda} f_\lambda(x), f(x) - f(x^0) \right\} = 0.$$

Then, [Theorem 3.1](#), when applied with the function $f : X \rightarrow \mathbb{R}$, assigning to each $x \in X$ the value

$$f(x) := 0,$$

and the infsup-convexity on X of the family $(f_\lambda)_{\lambda \in \Lambda} \cup (f - f(x^0))$, provides us with a positive and linear functional $\Phi : \ell^\infty(\Lambda) \rightarrow \mathbb{R}$ and $\rho \geq 0$ with

$$\Phi(\mathbf{1}) + \rho = 1$$

and

$$x \in X \Rightarrow 0 \leq \Phi((f_\lambda(x))_{\lambda \in \Lambda}) + \rho(f(x) - f(x^0)). \quad (4.3)$$

Let us notice that $\rho > 0$, because otherwise $\Phi \in \Delta_\Lambda$ and we would arrive at

$$\begin{aligned} 0 &\leq \inf_{x \in X} \Phi((f_\lambda(x))_{\lambda \in \Lambda}) \\ &\leq \Phi((f_\lambda(x^1))_{\lambda \in \Lambda}) \\ &\leq \sup_{\lambda \in \Lambda} f_\lambda(x^1) \\ &< 0, \end{aligned}$$

a contradiction. So $\rho > 0$ and we take $\Phi_0 := \Phi/\rho \in \ell^\infty(\Lambda)_+^*$. Then, according to [\(4.3\)](#) we have that

$$x \in X \Rightarrow f(x^0) \leq f(x) + \Phi_0((f_\lambda(x))_{\lambda \in \Lambda}).$$

Then, given $x \in X$,

$$\begin{aligned} \mathbf{L}(x^0, \Phi_0) &= f(x^0) + \Phi_0((f_\lambda(x^0))_{\lambda \in \Lambda}) \\ &\leq f(x^0) \\ &\leq f(x) + \Phi_0((f_\lambda(x))_{\lambda \in \Lambda}) \\ &= \mathbf{L}(x, \Phi_0). \end{aligned}$$

Finally, given $\Upsilon \in \ell^\infty(\Lambda)_+^*$ it follows that

$$\Upsilon((f_\lambda(x^0))_{\lambda \in \Lambda}) \leq 0,$$

so

$$\mathbf{L}(x^0, \Upsilon) \leq \mathbf{L}(x^0, \Phi_0)$$

and we have completed the proof. \square

We would hope that the Slater condition in [Theorem 4.2](#) could be replaced with this weaker one:

$$\text{there exists } x^1 \in X \text{ such that } \lambda \in \Lambda \Rightarrow f_\lambda(x^1) < 0, \quad (4.4)$$

but it is false in general:

Example 4.3. Let $X := \mathbb{R}$, $\Lambda := \mathbb{N}$, $f : X \rightarrow \mathbb{R}$ be the function assigning to each $x \in X$

$$f(x) := x,$$

and for all $n \in \mathbb{N}$, let $f_n : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined at each $x \in \mathbb{R}$ by

$$f_n(x) = -\frac{x^3}{n}.$$

Then, the feasible set is

$$X_0 = \mathbb{R}_+.$$

The Slater condition fails, for if there exists $x^1 \in \mathbb{R}$ with

$$\sup_{n \in \mathbb{N}} f_n(x^1) < 0,$$

then, in particular, $x^1 \in X_0 = \mathbb{R}_+$, but for such an x^1 ,

$$\sup_{n \in \mathbb{N}} f_n(x^1) = 0,$$

which contradicts the Slater condition. However, clearly any $x^1 \in X_0 \setminus \{0\}$ fulfills (4.4). To conclude, let us show that, despite the fact that the nonlinear program

$$\inf_{x \in X_0} f(x)$$

admits the optimal solution $x^0 = 0$, there exists no corresponding Lagrange multiplier $\Phi_0 \in \ell^\infty(\mathbb{N})_+^*$ for the associated Lagrangian \mathbf{L} . To this end, in order to check the hypotheses in [Theorem 4.2](#) (obviously except the Slater condition), we prove the unique non trivial fact that the family $(f_n)_{n \in \mathbb{N}} \cup f$ is infsup-convex on \mathbb{R} ($f(x^0) = 0$), that is,

$$\inf_{x \in \mathbb{R}} \left(\sup_{n \in \mathbb{N}} f_n(x) \vee f(x) \right) \leq \sup_{n \in \mathbb{N}} \sum_{j=1}^m t_j f_n(x_j) \vee \sum_{j=1}^m t_j f(x_j),$$

whenever $m \geq 1$, $\mathbf{t} \in \Delta_m$ and $x_1, \dots, x_m \in \mathbb{R}$. By [Lemma 4.1](#) we know for the left-hand side that

$$\inf_{x \in \mathbb{R}} \left(\sup_{n \in \mathbb{N}} f_n(x) \vee f(x) \right) = 0,$$

so we have to show that

$$0 \leq \sup_{n \in \mathbb{N}} \sum_{j=1}^m t_j f_n(x_j) \vee \sum_{j=1}^m t_j f(x_j),$$

i.e.,

$$0 \leq \sup_{n \in \mathbb{N}} \left(\sum_{j=1}^m t_j x_j^3 \right) \left(-\frac{1}{n} \right) \vee \left(\sum_{j=1}^m t_j x_j \right).$$

But this inequality is clearly satisfied, because if

$$0 \leq \sum_{j=1}^m t_j x_j^3$$

then

$$\sup_{n \in \mathbb{N}} \left(\sum_{j=1}^m t_j x_j^3 \right) \left(-\frac{1}{n} \right) = 0$$

and the inequality holds, while if

$$\sum_{j=1}^m t_j x_j^3 < 0$$

then

$$0 \leq \sup_{n \in \mathbb{N}} \left(\sum_{j=1}^m t_j x_j^3 \right) \left(-\frac{1}{n} \right),$$

which implies the validity of the inequality and thus the above mentioned infsup-convexity.

We finish by arguing by contradiction, so let us assume that the nonlinear problem under consideration admits a Lagrange multiplier $\Phi_0 \in \ell^\infty(\mathbb{N})_+^*$. Since, in particular, $\Phi_0 \in \ell^\infty(\mathbb{N})^*$, making use of the Dixmier decomposition of $\ell^\infty(\mathbb{N})^*$, there exist $y_0 \in \ell^1(\mathbb{N})$ and $\varphi_0 \in c_0(\mathbb{N})^\perp$ such that

$$\Phi_0 = y_0 + \varphi_0,$$

where $c_0(\mathbb{N})$ is the closed linear subspace of $\ell^\infty(\mathbb{N})$ of those null sequences and $c_0(\mathbb{N})^\perp$ is its annihilator.

Observe that for all $n \in \mathbb{N}$, $y_0(n) \geq 0$, because $\Phi_0 \in \ell^\infty(\mathbb{N})_+^*$ and $e_n = (0, \dots, 0, \overbrace{1}^n, 0, \dots, 0, \dots) \in \ell^\infty(\mathbb{N})_+$, so

$$\begin{aligned} 0 &\leq \Phi_0(e_n) \\ &= y_0(e_n) + \varphi_0(e_n) \\ &= y_0(n) \quad (\varphi_0 \in c_0(\mathbb{N})^\perp, e_n \in c_0(\mathbb{N})). \end{aligned}$$

Then, taking into account that we are assuming that $(0, \Phi_0)$ is a saddle point for the Lagrangian, in particular there holds for all $x \in \mathbb{R}$

$$f(x^0) + \Phi_0((f_n(x^0))_{n \in \mathbb{N}}) \leq f(x) + \Phi_0((f_n(x))_{n \in \mathbb{N}}).$$

But, for each $x \in \mathbb{R}$ $(f_n(x))_{n \in \mathbb{N}} \in c_0(\mathbb{N})$, so this inequality is nothing more than

$$x \in \mathbb{R} \Rightarrow 0 \leq x - x^3 \sum_{n=1}^{\infty} \frac{y_0(n)}{n} \quad (4.5)$$

which is absurd, because if $\sum_{n=1}^{\infty} \frac{y_0(n)}{n} = 0$, then it suffices to take $x < 0$ in (4.5) to arrive at a contradiction, while if $\sum_{n=1}^{\infty} \frac{y_0(n)}{n} > 0$, a large enough $x > 0$ yields

$$x - x^3 \sum_{n=1}^{\infty} \frac{y_0(n)}{n} < 0,$$

once again against the inequality (4.5). \square

We finish by deriving from Theorem 4.2 and Theorem 3.1 some Karush–Kuhn–Tucker and Fritz John results (see [13,12,18,2,11,6]), respectively, for which the sharp concept of convexity turns out to be once again infsup-convexity. The first of them is just an equivalent reformulation of Theorem 4.2, according to the easy-to-prove fact that the Karush–Kuhn–Tucker conditions below are equivalent to the existence of a Lagrange multiplier:

Theorem 4.4. Assume that X and Λ are nonempty sets, $x^0 \in X$, $f : X \rightarrow \mathbb{R}$ and that $(f_\lambda)_{\lambda \in \Lambda}$ is a family of real valued functions defined on X in such a way that the feasible set $X_0 := \{x \in X : \sup_{\lambda \in \Lambda} f_\lambda(x) \leq 0\}$ is nonempty and that

$$x \in X \Rightarrow (f_\lambda(x))_{\lambda \in \Lambda} \in \ell^\infty(\Lambda).$$

If in addition the family $(f_\lambda)_{\lambda \in \Lambda} \cup (f - f(x^0))$ is infsup-convex on X and the Slater condition

$$\text{there exists } x^1 \in X \text{ such that } \sup_{\lambda \in \Lambda} f_\lambda(x^1) < 0$$

is valid, then x^0 is an optimal solution for the nonlinear problem (4.1) if, and only if, $x^0 \in X_0$ and there exists $\Phi_0 \in \ell^\infty(\Lambda)_+^*$ such that

$$f + \Phi_0((f_\lambda(\cdot))_{\lambda \in \Lambda}) \text{ attains its infimum on } X \text{ at } x^0$$

and

$$\Phi_0((f_\lambda(x^0))_{\lambda \in \Lambda}) = 0.$$

The Fritz John result needs some additional work:

Theorem 4.5. Let X and Λ be nonempty sets, $f : X \rightarrow \mathbb{R}$ and for all $\lambda \in \Lambda$, $f_\lambda : X \rightarrow \mathbb{R}$ such that the feasible set $X_0 := \{x \in X : \sup_{\lambda \in \Lambda} f_\lambda(x) \leq 0\}$ is nonempty, x^0 is an optimal solution of the infinite program (4.1), and

$$x \in X \Rightarrow (f_\lambda(x))_{\lambda \in \Lambda} \in \ell^\infty(\Lambda).$$

Then, there exist $\rho \geq 0$ and $\Phi_0 \in \ell^\infty(\Lambda)_+^*$ with $\rho + \Phi_0(\mathbf{1}) = 1$ satisfying the following Fritz John conditions

$$\rho f + \Phi_0((f_\lambda(\cdot))_{\lambda \in \Lambda}) \text{ attains its infimum on } X \text{ at } x^0 \quad (4.6)$$

and

$$\Phi_0((f_\lambda(x^0))_{\lambda \in \Lambda}) = 0 \quad (4.7)$$

if, and only if, the family $(f_\lambda)_{\lambda \in \Lambda} \cup (f - f(x^0))$ is infsup-convex on X .

Proof. Suppose that the family $(f_\lambda)_{\lambda \in \Lambda} \cup (f - f(x^0))$ is infsup-convex on X . Apply Lemma 4.1 to arrive at

$$\inf_{x \in X} \max \left\{ \sup_{\lambda \in \Lambda} f_\lambda(x), f(x) - f(x^0) \right\} = 0.$$

Then, Theorem 3.1 provides us with a $\Phi \in \Delta_{\Lambda_0}$ (same notation as in the proof of Theorem 4.2) such that

$$0 \leq \Phi((f_\lambda)_{\lambda \in \Lambda} \cup (f - f(x^0))),$$

i.e., for some $\rho \geq 0$ and $\Phi_0 \in \ell^\infty(\Lambda)_+^*$ we have that

$$\begin{aligned} \Phi_0(\mathbf{1}) &\geq 0, \\ \Phi_0(\mathbf{1}) + \rho &= 1 \end{aligned}$$

and

$$x \in X \Rightarrow \rho f(x^0) \leq \rho f(x) + \Phi_0((f_\lambda(x))_{\lambda \in \Lambda}). \quad (4.8)$$

Condition (4.7) follows from both the fact that $x^0 \in X_0$ and $\Phi_0 \in \ell^\infty(\Lambda)_+^*$, and inequality (4.8) for $x = x^0$. And conditions (4.7) and (4.8) clearly imply (4.6).

And conversely, if $\rho \geq 0$ and $\Phi_0 \in \ell^\infty(\Lambda)_+^*$ with $\rho + \Phi_0(\mathbf{1}) = 1$ fulfill conditions (4.6) and (4.7), then

$$x \in X \Rightarrow 0 \leq \rho(f(x) - f(x^0)) + \Phi_0((f_\lambda(x))_{\lambda \in \Lambda}).$$

In particular, $\Psi := (\Phi_0, \rho) \in \Delta_{\Lambda_0}$ (notation in the proof of Theorem 4.2) satisfies, according to Lemma 4.1, that

$$\inf_{x \in X} \max \left\{ \sup_{\lambda \in \Lambda} f_\lambda(x), f(x) - f(x^0) \right\} \leq \inf_{x \in X} \Psi((f_\lambda(x))_{\lambda \in \Lambda} \cup (f - f(x^0))),$$

which, as mentioned in the proof of Theorem 4.2, implies the infsup-convexity of the family $(f_\lambda)_{\lambda \in \Lambda} \cup (f - f(x^0))$ on X . \square

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