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Invariant measures for multivalued semigroups

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A B S T R A C T

In this work we extend the concept of an invariant measure for a multivalued semigroup and, when it has a global attractor, we give different, but equivalent, definitions for such a measure. As a consequence we can apply the Birkhoff Ergodic Theorem to conclude that time averages converge almost everywhere to the spatial average.

1. Introduction

There have been various attempt to define invariant measures for multivalued applications. The most precocious introduced by Vershik in [35], followed by a definition from Aubin, Frankowska and Lasota in [8] and, because of the increasing attention paid to multivalued dynamical systems during the 80's and 90's, some other definitions have been proposed, as we can see in [31] and [3–6]. In [5] and [32], authors prove the equivalence, under appropriated conditions, of the main definitions in the discrete dynamical system context, with some applications to ordinary differential equations in [3–6].

The study of invariant measures for well posed partial differential equations is done in Wang, Luckaszewicz, Robinson and Real, and Checkroun's works, respectively in [38], [29,30] and [17] for autonomous dissipative problems. Invariant measures for non-autonomous well posed problems have been studied by Luckaszewicz and Robinson in [28].

We can define an *invariant measure for a semigroup* $\{S(t)\}_{t\in\mathbb{R}^+}$ on a complete metric space X, as a Borel probability measure μ satisfying

$$\mu(A) = \mu(S(t)^{-1}(A)) \quad \forall\, t \geq 0 \text{ and for any Borel subset } A \subset X.$$

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Equivalently we can say that μ is invariant for $S(\cdot)$ if

$$\int_X \varphi(x)\mu(dx) = \int_X \varphi(S(t)x)\mu(dx)$$

for any $t \geq 0$ and φ continuous and bounded on X, [17,28].

In [30] and [28] the authors define a family of Borel probability measures $\{\mu_t\}$ as invariant for a non-autonomous system $\{U(t,\tau)\}_{t\geq\tau}$ if

$$\mu_t(B) = \mu_\tau\left(U(t,\tau)^{-1}(B)\right) = \left(U(t,\tau)_*(\nu_\tau)\right)(B), \quad t \geq \tau \text{ (in [30])},$$

or equivalently,

$$\int_{A(t)} \varphi(x)\mu_t(dx) = \int_{A(\tau)} \varphi(U(t,\tau)x)\mu_\tau(dx) \text{ for each } t \geq \tau \text{ and each } \varphi \in \mathcal{C}(X), \text{ (in [28])}.$$

For multivalued infinite dimensional dynamical systems, the main references related with invariant measures are connected to Navier–Stokes problems and the concept of statistic solutions, introduced by Foias in [20], following Hopf and Prodi previous ideas respectively found in [23] and [33]. Later Vishik and Fursikov introduced in [37] a new concept of statistical solutions, defined on a trajectory space determined by an evolution problem. More recently, Foias, Rosa and Temam, focusing on Navier–Stokes problems, improved this concept, defining a measure on a trajectory space, the Vishik–Fursikov measure, whose projection on the phase space at each time t generates a projected statistic solution, called Vishik–Fursikov statistic solution, which recover the former concept proposed by Prodi and Foias (see [22] and references therein). The particular case when the Vishik–Fursikov statistic solutions are stationary coincides, as we are going to prove, with an invariant measure for the evolution system associated with the above mentioned trajectory space. It is worth to mention the works [12,14] and [13] where we can find the first ideas of an abstract theory on statistic solutions for more general evolution problems. The equivalence between invariant measures and statistic solutions are pointed in [16] and [25] for a well posed version of the Navier Stokes problem.

In this work, we extend to multivalued evolution problems the main definitions of invariant measures and, under suitable conditions, we prove the equivalence of such definitions. As a relevant consequence, we can apply the Birkhoff Ergodic Theorem to multivalued dissipative evolution problems in order to conclude the convergence of time averages almost everywhere with respect to an invariant measure.

2. Basic definitions, notations and terminologies

For a nonempty metric space X, we use the notations $\mathbb{B}(X)$ and $\mathbb{P}(X)$ to indicate the Borel σ-algebra on X and the set of Borel probability measures on X respectively. $\mathbb{P}(X)$ is embedded with the weak topology, the coarsest which makes continuous the function $\mu \mapsto \int_X f(x)\mu(dx)$ for each $f \in C(X,\mathbb{R})$. The convergence of a sequence $\{\mu_n\}$ in $\mathbb{P}(X)$ is given by

$$\mu_n \to \mu \Leftrightarrow \int_X f(x)\mu_n(dx) \to \int_X f(x)\mu(dx)$$

for each $f \in C(X,\mathbb{R})$. Since X is a compact metric space, $\mathbb{P}(X)$ is a compact metrizable space. On $\mathbb{P}(X)$ we consider the Hutchinson metric given by

$$d(\mu_1,\mu_2) = \sup\left\{\int_X f(x)\mu_1(dx) - \int_X f(x)\mu_2(dx)\right\}, \tag{2.1}$$

where the sup is taken over all real-valued functions f with Lipschitz constant less than or equal to 1. It is well known that this metric agrees with the weak topology on $\mathbb{P}(X)$ (Proposition 1, [1]).

Given a probability measure $\mu \in \mathbb{P}(X)$, we denote by $|\mu|$ the *support* of μ, that means, $|\mu| \subset X$ is a closed subset satisfying

(1) $\mu(|\mu|^C) = 0$;
(2) if $G \subset X$ is open and $G \cap |\mu| \neq \emptyset$, then $\mu(G \cap |\mu|) > 0$.

Remark 2.1. Some authors define support as $|\mu| = (\cup\{V; V \text{ is open and } \mu(V) = 0\})^C$. Since we are supposing that X is a metric space, then a probability measure μ on X is *tight* (Theorem 12.5, [2]) and so, in this case, both definitions coincide and μ has a unique well defined support (see Theorem 12.14 and its proof in [2]).

In this work, we are going to use the same notation μ to indicate the Lebesgue extension of a Borel measure μ. As the Lebesgue completion $\Sigma_\mu(X)$ of $\mathbb{B}(X)$ contains all subsets of any Borel null set, we assume that μ is complete. For an introduction to Lebesgue extensions of measures, see [10], Section 1.5.

Let X_1 and X_2 be nonempty metric spaces and F a set-valued map from X_1 to X_2 which we identify with its graph, the relation $F = Graph(F) \subset X_1 \times X_2$. We say that F is a *closed relation* (or a *closed set-valued map*) if $F \subset X_1 \times X_2$ is closed. Analogously we say that F is a *measurable relation* (or a *measurable set-valued map*) if $F \subset X_1 \times X_2$ is measurable. Given $x \in X_1$, $F(x) = \{y \in X_2; (x, y) \in F\}$, for $A \subset X_1$ the *image* of A is the set $F(A) = \bigcup\{F(x); x \in A\}$, and $\mathrm{Dom}(F) = \{x \in X_1; F(x) \neq \emptyset\}$. The *inverse image* of $B \subset X_2$ is given by $F^{-1}(B) = \{x \in X_1; F(x) \cap B \neq \emptyset\}$. We say that F is *closed-valued* (*compact-valued, convex-valued,* or *bounded-valued*) if $F(x)$ is closed (respectively compact, convex, or bounded) for each $x \in X_1$.

Remark 2.2. Observe that in [7] authors call F measurable if the inverse image of each open set is measurable. Our definition of a measurable relation agrees with [1] instead, as it is done in [32]. Some authors define measurability of a multivalued map asking the inverse image of each closed set to be a Borel set. This is the case for example in [5]. Nevertheless, in the framework we adopt through this text (we are considering complete σ-finite measures on complete separable metric spaces) all those concepts are equivalents, see a Remark on p. 307 and Theorem 8.1.4 in [7]. See also Lemma 6.4.2 (i), [11].

A set-valued map $F \subset X_1 \times X_2$ is called *upper semicontinuous at* $x \in \mathrm{Dom}(F)$ if for any neighborhood U of $F(x)$ there exists $\eta > 0$ such that for any x' satisfying $d(x, x') \leq \eta$, $F(x') \subset U$. F is called *upper semicontinuous* if it is upper semicontinuous at any $x \in \mathrm{Dom}(F)$. Observe that if F is upper semicontinuous with closed domain and closed values, then F is closed. The inverse is true if X_2 is compact. We refer the reader to [7] for elementary facts about set-valued maps.

2.1. Invariant measures for closed relations

Let $F \subset X \times X$ be a closed relation on X. If $F(x)$ is a single valued set for each $x \in X$, then F is a function and in this case we use the notation f rather than F. We say that $\mu \in \mathbb{P}(X)$ is an invariant probability measure for a function $f : X \to X$ if

$$\mu(A) = \mu(f^{-1}(A)), \text{ for each Borel subset } A \text{ of } X. \tag{2.2}$$

It is common to use the notation $f_*(\mu)$ to indicate the measure defined by the right side in the above equality, that means, $f_*(\mu)(A) = \mu(f^{-1}(A))$, for $A \in \mathbb{B}(X)$, which is usually referred as *push-forward* of μ. The notation f_* stands for the application

$$\mu \mapsto f_*(\mu) \text{ from } X \text{ to } \mathbb{P}(X). \tag{2.3}$$

When F is set-valued, the application $A \mapsto \mu(F^{-1}(A))$ does not define a measure and the concept of an invariant measure has to be reformulated. There are several ways to do that, and below we present some equivalent different definitions of an invariant measure for a closed relation F which agree, in the present framework, with the classic definition for single valued maps.

We suppose in this section that X is a compact, complete separable metric space.[1]

Definition 2.1. ([5,8,32]) We say that $\mu \in \mathbb{P}(X)$ is an invariant measure for a closed relation F if

$$\mu(A) \leq \mu(F^{-1}(A)), \text{ for each Borel subset } A \text{ of } X. \tag{2.4}$$

Remark 2.3. Recall that F is a closed, and so, a Borel measurable subset of $X \times X$, but we do not have necessarily that $F^{-1}(A) \in \mathbb{B}(X)$ for each Borel subset A of X. However, given $A \in \mathbb{B}(X)$, $F \cap X \times A \in \mathbb{B}(X \times X)$, and then, from Theorem 6.7.3, [11], $\pi_1(F \cap X \times A) = F^{-1}(A)$ is a Souslin (or analytic) set, and so, measurable. It is worth to note that in [5] and the subsequent work [6] this definition is slightly different, namely: $\mu(A) \leq \mu(F^{-1}(A))$, for each *closed* subset A of X. This is probably to avoid the use of the Lebesgue extension of μ, since according to the approach given by the author, the inverse images of closed sets are measurable, as we recall above.

If $F = f$ is a function then (2.4) agrees with (2.2) since

$$\mu(A) \leq \mu(f^{-1}(A)) \text{ and } \mu(A^C) \leq \mu(f^{-1}(A^C)) = \mu((f^{-1}(A))^C) \quad \Rightarrow \quad \mu(A) = \mu(f^{-1}(A)).$$

Definition 2.2. ([6]) We say that μ is invariant for a closed relation F on X if

$$\int_X \varphi(x)\mu(dx) \leq \int_X \overline{\varphi}(F(x))\mu(dx) \text{ and each } \varphi \in \mathcal{C}_b(X), \tag{2.5}$$

where, given a subset $D \subset X$, $\overline{\varphi}(D) = \sup\{\varphi(x), x \in D\}$ and $\mathcal{C}_b(X)$ stands for the set of bounded continuous real-valued functions defined on X.

It follows from Theorem 13.46 and Theorem 15.1 in [2] that $\mu = f_*(\mu)$ for a measurable function $f : X \to X$ if and only if

$$\int_X \varphi(x)\mu(dx) = \int_X \varphi(x) f_*(\mu)(dx) = \int_X \varphi(f(x))\mu(dx), \text{ for } \varphi \in \mathcal{C}_b(X).$$

Therefore, if $F = f$ is a single valued map, Definition 2.2 agrees with the classical concept as well once the inequality in (2.5) is true for both, φ and $-\varphi \in \mathcal{C}_b(X)$.

Definition 2.3. ([3,5,31]) For a closed relation F on X, we say that μ is invariant for F if there exists a Markov kernel $k : X \to \mathbb{P}(X)$ that means, a map k associating each point $x \in X$ with a probability

[1] Therefore X is a Polish and a Souslin space as well. See Definition 6.1.10 and Definition 6.6.1, [11].

measure k_x on X, in such way that $x \mapsto k_x(A)$ is a measurable real function for each $A \in \mathbb{B}(X)$, satisfying additionally

$$|k_x| \subset F(x) \text{ for all } x \in |\mu| \text{ and } \mu = k_*(\mu), \tag{2.6}$$

where

$$k_*(\mu)(A) = \int_X k_x(A)\mu(dx) \text{ for each } A \text{ Borel subset of } X. \tag{2.7}$$

This definition also agrees with the usual concept of invariant measure since, if $F = f$ is a function, the only Markov kernel k satisfying (2.6) is such that $x \mapsto \delta_{f(x)}$, where $\delta_{f(x)}$ is the Dirac measure concentrated at $f(x)$, for any $x \in |\mu|$. Therefore, in this case, k_* as defined in (2.7) coincides with f_* as defined in (2.3). In fact, for $A \in \mathbb{B}(X)$,

$$k_*(\mu)(A) = \int_X \delta_{f(x)}(A)\mu(dx) = \int_{f^{-1}(A)} \mu(dx) = \mu(f^{-1}(A)) = f_*(\mu)(A).$$

Definition 2.4. ([32,36]) For a closed relation $F \subset X \times X$, we can say that μ is an invariant measure if there exists $\mu_{12} \in \mathbb{P}(X \times X)$ such that

$$|\mu_{12}| \subset F \quad \text{ and } \quad \pi_{1*}(\mu_{12}) = \mu = \pi_{2*}(\mu_{12}).$$

If $F = f$ is a function and $graph_f : X \to X \times X$ is given by $x \mapsto (x, f(x))$ then consider $\mu_{12} = graph_{f*}(\mu)$, which is the only possible measure on $X \times X$ supported by $\{(x, f(x)); x \in X\}$ such that $\pi_{1*}(\mu_{12}) = \mu$. In this case one can easily prove that $\mu = \pi_{2*}(\mu_{12})$ if and only if μ satisfies (2.2).

Now we consider the set $X^{\mathbb{Z}}$ of bi-infinite sequences $\xi = (\xi_i)_{i \in \mathbb{Z}}$ in X, endowed with the product topology, which is compact since X is assumed compact. In $X^{\mathbb{Z}}$ we define

$$d(\xi, \eta) = \sup\{\min(d(\xi_i, \eta_i), 1/|i|); i \in \mathbb{Z}\},$$

where $\min(d, 1/0) = d$ by convention. It can be proved that $d(\cdot, \cdot)$ is a metric in $X^{\mathbb{Z}}$ yielding the product topology and, for $\varepsilon > 0$,

$$d(\xi, \eta) \le \varepsilon \quad \Leftrightarrow \quad d(\xi_i, \eta_i) \le \varepsilon \quad \text{ for all } i \text{ such that } |i| \le 1/\varepsilon.$$

In particular, the projection $\pi_0 : X^{\mathbb{Z}} \to X$ is Lipschitz with constant equal 1.

Let $s : X^{\mathbb{Z}} \to X^{\mathbb{Z}}$ be the shift homeomorphism defined by

$$(s(\xi))_i = \xi_{i+1}, \quad i \in \mathbb{Z}.$$

We denote by X_F the sample path space for F, that means, $X_F = \{\xi \in X^{\mathbb{Z}}; \xi_{i+1} \in F(\xi), i \in \mathbb{Z}\}$ and use the same notation s for the shift homeomorphism restricted to X_F.

Definition 2.5. ([5,32]) We call $\mu \in \mathbb{P}(X)$ an invariant measure for F if there exists $\nu \in \mathbb{P}(X^{\mathbb{Z}})$ such that

$$|\nu| \subset X_F, \quad s_*(\nu) = \nu, \quad \text{ and } \quad \pi_{0*}(\nu) = \mu,$$

where $\pi_0(\xi) = \xi_0$ for $\xi \in X^{\mathbb{Z}}$.

If $F = f$ is a bijection on X, $\pi_0 : X_f \to X$ induces an invertible conjugacy of s and f, that means, $f \circ \pi_0 = \pi_0 \circ s$, and then a measure ν is invariant for s if and only if the *push-forward* $\pi_{0*}(\nu)$ is invariant for f.

Remark 2.4. If $\nu \in \mathbb{P}(X^{\mathbb{Z}})$ is invariant for s, then $\pi_{k*}(\nu) = \pi_{0*}(\nu)$ for all $k \in \mathbb{Z}$.

For a closed relation F on a compact space X all those definitions are in fact equivalent, which is proved in [6] and [31]. Our first purpose in this work is to consider analogous definitions of invariant measures for multivalued semigroups on compact spaces and to guarantee that, under some additional conditions, they are still equivalent in that context.

3. Invariant measures for multivalued semigroups

Recall that an *invariant measure for a semigroup* $\{S(t)\}_{t\in\mathbb{R}^+}$ on a complete metric space X is a measure $\mu \in \mathbb{P}(X)$ satisfying

$$\mu(A) = \mu(S(t)^{-1}(A)) \quad \forall \quad t \geq 0 \text{ and for any } A \in \mathbb{B}(X).$$

Equivalently, we can say that $\mu \in \mathbb{P}(X)$ is invariant for $S(\cdot)$ if

$$\int_X \varphi(x)\mu(dx) = \int_X \varphi(S(t)x)\mu(dx)$$

for each $t \geq 0$ and $\varphi \in \mathcal{C}_b(X)$, [17,28]. In what follows we present some different (but equivalent) ways to define an invariant measure for a multivalued system which agree with the above definition for the single-valued case.

Let us firstly suppose that X is a complete separable metric space. We are going to consider a multivalued semigroup $\{V(t)\}_{t\in\mathbb{R}^+}$ on X, that means, a family of set-valued operators $V(t) : X \to \mathcal{P}(X)$ such that for each $t \geq 0$ and $E \subset X$

(1) $V(0) = Id$;
(2) $V(t_1 + t_2)E \subset V(t_1)V(t_2)E$ for $t_1, t_2 \in \mathbb{R}^+$.

Because we want to consider curves linking points on the images of $V(\cdot)$, **we consider in this text multivalued semigroups which are generated by semiflows**, as we describe below.

Definition 3.1. A *generalized semiflow* $\mathcal{G}$ on X is a family of maps $\varphi : [0, +\infty) \to X$ satisfying

(H1) For each $z \in X$ there exists at least one $\varphi \in \mathcal{G}$ with $\varphi(0) = z$;
(H2) If $\varphi \in \mathcal{G}$ and $\tau > 0$, $\varphi^\tau \in \mathcal{G}$, where $\varphi^\tau(\cdot) := \varphi(\cdot + \tau)$;
(H3) If $\varphi, \psi \in \mathcal{G}$ and $\psi(0) = \varphi(t)$ for some $t > 0$, then the map θ given by

$$\theta(\tau) = \begin{cases} \varphi(\tau) & \text{for } \tau \in [0, t], \\ \psi(\tau - t) & \text{for } \tau \in (t, +\infty) \end{cases}$$

belongs to $\mathcal{G}$;
(H4) If $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{G}$ and $\varphi_n(0) \to z$, there exists $\varphi \in \mathcal{G}$ and a subsequence $\{\varphi_{n_j}\} \subset \{\varphi_n\}$ such that $\varphi(0) = z$ and $\varphi_{n_j}(t) \to \varphi(t)$ for each $t \geq 0$.

Definition 3.2. A *multivalued semigroup determined by* $\mathcal{G}$ is a family of set-valued operators $\{V(t)\}_{t\in\mathbb{R}^+}$ on X given by

$$V(t)E := \{\varphi(t); \varphi \in \mathcal{G}, \varphi(0) \in E\}$$

for $E \subset X$. $V(t)x = V(t)\{x\}$, for $x \in X$.

When considering multivalued semigroups determined by a generalized semiflow, thanks to **(H3)** we have the equality

$$V(t_1 + t_2)E = V(t_1)V(t_2)E, \text{ for } t_1, t_2 \in \mathbb{R}^+ \text{ and } E \subset X,$$

instead of having just an inclusion. Furthermore, we also get from **(H4)** that each $V(t)$ is compact-valued, closed and upper semicontinuous, [15].

Therefore, since each $V(t)$ is a closed subset of $X \times X$, we can define the following.

Definition 3.3. We say that $\mu \in \mathbb{P}(X)$ is an invariant measure for a multivalued semigroup $\{V(t)\}_{t\in\mathbb{R}^+}$ on X if

$$\mu(A) \leq \mu(V(t)^{-1}(A)) \text{ for each } t \in [0, \infty)$$

and each $A \in \mathbb{B}(X)$.

We suppose in addition that the semigroup $\{V(t)\}_{t\in\mathbb{R}^+}$ possesses a global attractor $\mathcal{A}$, a compact set which is the minimal closed subset of X that attracts all bounded subsets of X in the following sense, for any bounded subset $B \subset X$, and $\varepsilon > 0$, there exists $T > 0$ such that $t > T$ implies $V(t)B \subset \mathcal{O}_\varepsilon(\mathcal{A})$, where $\mathcal{O}_\varepsilon(\mathcal{A})$ is the ε neighborhood of $\mathcal{A}$. Under our hypotheses $\mathcal{A}$ is invariant, that means, $V(t)\mathcal{A} = \mathcal{A}$ for any $t \geq 0$, (in fact, the maximal compact invariant subset of X). We observe that the invariance of the attractor $\mathcal{A}$ is guaranteed in this case by Hypotheses **(H3)** and **(H4)**, see [15] and [34] for detailed studies on attractors for multivalued semigroups determined by generalized semiflows.

We emphasize that, as it occurs with semigroups, if there exists an invariant measure μ for $\{V(t)\}_{t\in\mathbb{R}^+}$, $|\mu| \subset \mathcal{A}$. The proof follows the ideas in [17], Lemma 4.7 (on the single-valued case) and, in order to adapt it for a multivalued context, we appeal to arguments used in the proof of Poincare's Recurrence Theorem for set-valued dynamical systems, [8], Theorem 2.2.

Proposition 3.1. *Let X be a complete metric space and $\{V(t)\}_{t\in\mathbb{R}^+}$ a multivalued semigroup on X determined by a generalized semiflow $\mathcal{G}$ which has a global attractor $\mathcal{A}$. If μ is an invariant measure for $\{V(t)\}_{t\in\mathbb{R}^+}$, then its support $|\mu|$ is contained in $\mathcal{A}$.*

Proof. Let $\delta > 0$ and $\mathcal{A}_\delta := \{y \in X; dist(y, \mathcal{A}) < \delta\}$. We claim that $\mu(\mathcal{A}_\delta) = 1$ for any $\delta > 0$. In fact, for $R > 0$, $x \in X$ and $A = B_R(x) \cap \mathcal{A}_\delta^C$, we consider the sets

$$A_N = \cup_{n\geq N} V(1)^{-n}(A) \quad \text{ and } \quad A_\infty = \cap_{N\geq 0} A_N.$$

We clearly have that:

(1) $A \subset A_0$;
(2) $A_N \subset A_{N-1}$ for $N \in \mathbb{N}$ and, in particular, each $A_N \subset A_0$ for $N \in \mathbb{N}$;
(3) $A_N = V(1)^{-N} A_0$.

Since μ is invariant for $V(\cdot)$,

$$\mu(A_0) \leq \mu(V(1)^{-N}(A_0)) = \mu(A_N) \leq \mu(A_0).$$

Therefore

$$\mu(A_\infty) = \mu(A_0).$$

Thus we have

$$\mu(A \cap A_\infty) = \mu(A \cap A_0) = \mu(A).$$

However, as $\mathcal{A}$ attracts bounded sets, there exists $T > 0$ such that $V(t)A \subset \mathcal{A}_\delta$ for $t \geq T$, and so we can conclude that $V(1)^N(A) = V(N)A \subset \mathcal{A}_\delta$ if $N > T$, then $A \cap A_N = \emptyset$ if $N > T$. Therefore $\mu(A) = 0$. As R was arbitrarily chosen, $\mu(\mathcal{A}_\delta^C) = 0$. The result follows observing that $\mathcal{A} = \cap_{\delta>0}\mathcal{A}_\delta$. $\square$

The above theorem and the invariance of the attractor enable us to limit our study to a compact space, considering the multivalued semigroup $V(\cdot)$ restricted to the attractor $\mathcal{A}$. **Therefore, from now on, we are going to suppose X a compact, complete and separable metric space.**

Below we give some different definitions for an invariant measure for a multivalued semigroup and then we prove that, under appropriate hypotheses, they are all equivalent.

Definition 3.4. We can say that $\mu \in \mathbb{P}(X)$ is invariant for $V(\cdot)$ if

$$\int_X \varphi(x)\mu(dx) \geq \int_X \overline{\varphi}(V(t)x)\mu(dx) \text{ for each } t \geq 0 \text{ and each } \varphi \in \mathcal{C}_b(X),$$

where, given a subset $D \subset X$, $\overline{\varphi}(D) = \sup\{\varphi(x), x \in D\}$.[2]

Definition 3.5. Suppose that there exists $\mu \in \mathbb{P}(X)$ and a family of Markov kernels $\{k^t\}_{t\in\mathbb{R}^+}$, $k^t : X \to \mathbb{P}(X)$, such that

$$\text{for all } t \geq 0 \quad |k_x^t| \subset V(t)(x) \text{ for all } x \in |\mu| \text{ and } \mu = k_*^t(\mu).$$

Then we call μ invariant for $V(\cdot)$.

Definition 3.6. A measure μ can also be called invariant for $V(\cdot)$ if there exists a family of measures $\{\mu_{12}^t\}_{t\in\mathbb{R}^+} \subset \mathbb{P}(X \times X)$ such that

$$\text{for each } t \geq 0, \quad |\mu_{12}^t| \subset V(t) \subset X \times X \quad \text{and} \quad \pi_{1*}(\mu_{12}^t) = \mu = \pi_{2*}(\mu_{12}^t).$$

Finally, we can define an invariant measure for $V(\cdot)$ as a projection of a measure defined on the space of complete trajectories of $\{V(t)\}_{t\in\mathbb{R}^+}$. For this purpose we are going to suppose that $\mathcal{G}$ is a continuous generalized semiflow, that means, each $\varphi \in \mathcal{G}$ is a continuous function from $[0, +\infty)$ to X.

A function $\xi : \mathbb{R} \to X$ is said to be a **complete trajectory** of $\{V(t)\}_{t\in\mathbb{R}^+}$ if $\xi(t + s) \in V(t)\xi(s)$ for each $t \geq 0$ and $s \in \mathbb{R}$. Let $\mathcal{K}$ be the set of all complete trajectories of $V(\cdot)$ on X, which is nonempty, since we are supposing the existence of a global attractor $\mathcal{A}$ for $V(\cdot)$, and it is known that, given any point $x \in \mathcal{A}$, there

[2] As $V(t)$ is measurable and φ is continuous, $\overline{\varphi}$ is measurable, see [6], Notation 6.1.

is a complete trajectory ξ of $V(\cdot)$ such that $\xi(0) = x$ and $\xi(t) \in \mathcal{A}$ for any $t \in \mathbb{R}$. In fact, $\mathcal{A}$ is precisely the union of all bounded complete trajectory of $V(\cdot)$, see [34]. As we are considering $\{V(t)\}_{t\in\mathbb{R}}$ to be a multivalued semigroup determined by a continuous generalized semiflow, then $\mathcal{K} \subset \mathcal{C}_{loc}(\mathbb{R}, X)$, the set of continuous bounded maps from $\mathbb{R}$ to X with the topology of uniform convergence on compacts. It follows from Theorem 2.3, [9], that $\mathcal{K}$ is a compact subset of $\mathcal{C}_{loc}(\mathbb{R}, X)$.

Definition 3.7. We define in $\mathcal{K}$ the translation group $\{T(\tau)\}_{\tau\in\mathbb{R}}$, given by

$$T(\tau)\xi = \xi^\tau, \xi^\tau(\cdot) = \xi(\cdot + \tau), \tau \in \mathbb{R}.$$

Definition 3.8. Suppose that there exists a measure $\nu \in \mathbb{P}(\mathcal{C}_{loc})$, where $\mathcal{C}_{loc}$ stands for $\mathcal{C}_{loc}(\mathbb{R}, X)$, such that $|\nu| \subset \mathcal{K}$, ν is an invariant measure for the translation group $\{T(\tau)\}_{\tau\in\mathbb{R}}$, and $\pi_{0*}(\nu) = \mu$. That means, for any $\tau \in \mathbb{R}$,

$$\nu = T(\tau)_*(\nu)$$

and, for any $A \in \mathbb{B}(X)$,

$$\mu(A) = \nu(\{\xi \in \mathcal{K}; \xi(0) \in A\}).$$

Then we say that μ is invariant for $\{V(t)\}_{t\in\mathbb{R}^+}$.

Remark 3.1. A measure ν defined on a curve space and carried by the space $\mathcal{K}$ of complete trajectories of a flow is a special case of the trajectory statistical solutions defined in [12], which derive from the *(space-time) Vishik–Fursikov measures* that appear on Navier–Stokes problems [22]. In the particular situation when ν is invariant under the translation semigroup, (as in the above definition), ν is called an *invariant (space-time) Vishik–Fursikov measure*. It is known that each Vishik–Fursikov measure can be projected to a family of statistical solutions on the phase space X and, in this case, the statistical solution is named *Vishik–Fursikov statistical solution*, and a Vishik–Fursikov statistical solution which does not vary with the time is called a *stationary Vishik–Fursikov statistical solution*, a particular case of stationary statistical solutions. We are proving here that an invariant measure on the trajectory space $\mathcal{K}$ (an "invariant trajectory measure") can be projected to an invariant measure for the flow on the phase space X (a "stationary statistical solution") and, reciprocally, an invariant measure on the space phase X can be lifted to an invariant measure on the trajectory space $\mathcal{K}$. However, we are doing that under topological conditions on X and $\mathcal{K}$ that are stronger of those asked on related spaces in [12]. We refer the reader to [22], Remarks 4, 5 and 6, where it can be also found the most relevant related references for the different notions of statistical solutions mentioned above. See also [14] for further references and an abstract approach for trajectory statistical solutions and statistical solutions.

3.1. On the equivalence of the above definitions

Theorem 3.1. *Let X be a compact, complete and separable metric space, and $\{V(t)\}_{t\in\mathbb{R}}$ a multivalued semigroup on X determined by a continuous generalized semiflow for which the set $\mathcal{K}$ of all bounded complete trajectories is nonempty.*[3] *Then Definitions 3.3, 3.4, 3.5, 3.6 and 3.8 are all equivalents.*

Proof. **Definition 3.3** $\Rightarrow$ **Definition 3.6:** It follows exactly as the same proof of (1) $\Rightarrow$ (3) in Theorem 3.1, [32].

[3] Here instead of supposing the existence of a global attractor we assume that X is compact and $\mathcal{K}$ is a nonempty set.

Definition 3.6 ⇒ Definition 3.5: Once X is a complete metric separable space, both X and $X \times X$ are Souslin spaces, (see Definition 6.6.1, p. 19, [11]). Then, for each $t \geq 0$, it follows from Proposition 10.4.12, p. 363, [11], that there exists a Markov kernel $\tilde{k}^t : X \to \mathbb{P}(X \times X)$ such that,

(1) $|\tilde{k}^t_x| \subset \{x\} \times X$;
(2) for each $B \in \mathbb{B}(X \times X)$ and each $E \in \mathbb{B}(X)$,

$$\mu^t_{12}(B \cap \pi_1^{-1}(E)) = \int_E \tilde{k}^t_x(B)(\pi_{1*}\mu^t_{12})(dx).$$

Observe that, from Corollary 6.6.7, p. 22, [11] every Borel subset of a Souslin space is a Souslin space. Therefore, the class of Souslin sets contains the class of Borel sets, and thus the application $x \mapsto \tilde{k}^t_x(B)$ is Borel measurable for each $B \in \mathbb{B}(X)$. For the notion of class of sets see [10], Definition 1.2.1, p. 3.

We are going to prove that there exists a subset $A \in \mathbb{B}(X)$, such that $\pi_{1*}\mu^t_{12}(A^C) = 0$, and $|\tilde{k}^t_x| \subset \{x\} \times V(t)x$ for each $x \in A$. In fact, otherwise, there should exist $A_1 \in \mathbb{B}(X)$ with $\pi_{1*}\mu^t_{12}(A_1) > 0$, and such that for $x \in A_1$, $\tilde{k}^t_x((\{x\} \times V(t)x)^C) = \tilde{k}^t_x(X \times (V(t)x)^C \cup X \setminus \{x\} \times V(t)x) = \tilde{k}^t_x(\{x\} \times (V(t)x)^C) > 0$. (See [2], p. 441 and use the fact that $(V(t)x)^C$ is open.) From item(1) above, $\tilde{k}^t_x(\{x\} \times (V(t)x)^C) = \tilde{k}^t_x(V(t)^C)$ and so,

$$\begin{aligned}\mu^t_{12}(V(t)^C) &= \mu^t_{12}(V(t)^C \cap \pi_1^{-1}(X)) = \textstyle\int_X \tilde{k}^t_x(V(t)^C)(\pi_{1*}\mu^t_{12})(dx) \\ &\geq \textstyle\int_{A_1} \tilde{k}^t_x(V(t)^C)(\pi_{1*}\mu^t_{12})(dx) > 0,\end{aligned}$$

which is a contradiction. Now we define $k^t_x \in \mathbb{P}(X)$, for $x \in A$, as $k^t_x(E) = \pi_{2*}\tilde{k}^t_x(E) = \tilde{k}^t_x(X \times E)$ for $E \in \mathbb{B}(X)$. For $x \in A^C$, we define k^t_x as follows. Let $f : X \to X$ be a measurable selection of $V(t)$ (for a fixed t). The existence of such f is assured by, for example, Lemma 1.1, p. 1204, [32]. Then for $x \in A^C$ we define $k^t_x = \delta_{f(x)}$. Therefore $|k^t_x| \subset V(t)x$ for each $x \in X$, and

$$\mu(E) = \pi_{2*}\mu^t_{12}(E) = \mu^t_{12}(X \times E) = \int_X \tilde{k}^t_x(X \times E)(\pi_{1*}\mu^t_{12})(dx) = \int_X k^t_x(E)\mu(dx).$$

We observe that the definition of k^t_x for $x \in A^C$ does not have any effect on the above integral, since $\pi_{1*}\mu^t_{12}(A^C) = 0$.

Definition 3.5 ⇒ Definition 3.4: (see [6], Proposition 6.1) It follows from Proposition 2.1, p. 442, [6], that

$$\int_X \varphi(x)\mu(dx) = \int_X \left(\int_X \varphi(w)k^t_x(dw) \right) \mu(dx)$$

for any bounded and continuous function $\varphi : X \to \mathbb{R}$. Since $|k^t_x| \subset V(t)x$ for each $t \geq 0$ and $x \in X$, we have that

$$\begin{aligned}&\textstyle\int_X \left(\int_X \varphi(w)k^t_x(dw)\right)\mu(dx) = \int_X \left(\int_{V(t)(x)} \varphi(w)k^t_x(dw)\right)\mu(dx) \leq \\ &\textstyle\int_X \left(\int_{V(t)(x)} \overline{\varphi}(V(t)x)k^t_x(dw)\right)\mu(dx) = \int_X \overline{\varphi}(V(t)x)k^t_x(V(t)(x))\mu(dx) = \\ &\textstyle\int_X \overline{\varphi}(V(t)x)\mu(dx).\end{aligned}$$

Definition 3.4 ⇒ Definition 3.3: From Lemma 6.1, [6], it follows that the inequality in Definition (3.4) remains true for $\varphi = \mathbb{1}_A$ for any Borel subset $A \subset X$, then

$$\mu(A) = \int_X \mathbb{1}_A(x)\mu(dx) \leq \int_X \overline{\mathbb{1}}_A(V(t)x)\mu(dx) = \int_{V(t)^{-1}A} \mu(dx) = \mu(V(t)^{-1}A).$$

Definition 3.3 ⇔ Definition 3.8

Firstly let us observe that, according with Theorem 3.2, [32], Definition 3.3 is equivalent to the existence, for each $t \geq 0$, of a measure ν_t in $X^{\mathbb{Z}}$ whose support is included inside $X_{V(t)}$, the set of bi-infinite sequences $\omega = (\omega_i)_{i\in\mathbb{Z}}$ satisfying $\omega_{i+1} \in V(t)\omega_i$, for $i \in \mathbb{Z}$, and such that $\nu_t = s_*\nu_t$ and $\mu = \pi_{0*}\nu_t$.[4] The equivalence follows from Theorem 2.5, [19].[5] □

Besides, it is worth to mention that in the proof of Theorem 2.5 in [19] it is assumed that $\mathcal{K}_{[0,T]} := \{\xi|_{[0,T]}; \xi \in \mathcal{K}\}$ is compact on uniform topology for each $T > 0$, which we are not supposing in this text. Nevertheless, this assumption is used only to conclude that the map

$$\begin{array}{rcl} K_T : V(t) & \to & \mathcal{P}(\mathcal{K}_{[0,T]}) \\ (x_0, x_t) & \mapsto & \{\xi_{[0,T]}; \xi \in \mathcal{K} \text{ and } \xi(0) = x_0, \xi(t) = x_t\} \end{array}$$

is measurable, which can be concluded just observing that K_T is closed. The measurability follows from Lemma 6.4.2 (i), [11] and Theorem 8.1.4, [7].

The existence of at least one invariant measure for $\{V(t)\}_{t\in\mathbb{R}^+}$ is assured by the following.

Theorem 3.2. *Let $\mathcal{I}_V$ be the set of all Borel probability measures $\mu \in \mathbb{P}(X)$ which are invariant for $\{V(t)\}_{t\in\mathbb{R}^+}$. Then $\mathcal{I}_V$ is a nonempty, compact and convex set.*

This result follows from Theorem 2.5 in [19]. See also Theorem 3.2, [32].

4. On the convergence of the time averages

The first important consequence of the above equivalences is the viability of applying the Birkhoff Ergodic Theorem to multivalued dissipative evolution problems to conclude the convergence of time averages almost everywhere with respect to an invariant measure.

Theorem 4.1. *Let X be a complete separable metric space and $\{V(t)\}_{t\in\mathbb{R}^+}$ a multivalued semigroup determined by a continuous generalized semiflow $\mathcal{G}$ on X which possesses a global attractor $\mathcal{A}$, and let μ be an invariant measure for $\{V(t)\}_{t\in\mathbb{R}^+}$. Let $\varphi \in L^1(X,\mu)$. For μ-almost every $u_0 \in X$, there exists at least one $\xi \in \mathcal{G}$ satisfying $\xi(0) = u_0$ such that the limit*

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau \varphi(\xi(t))dt \tag{4.1}$$

exists.

---

[4] Here s stands for the shift homeomorphism restricted to $X_{V(t)}$.

[5] We observe that the concatenation of solutions (property **H2** in Definition 3.1) is an essential tool in the proof of Theorem 2.5 in [19], despite not being explicitly requested there.

Proof. We know from Proposition 3.1 that the support $|\mu|$ of μ is inside the attractor $\mathcal{A}$ then, if we consider $\overline{\mu} \in \mathbb{P}(\mathcal{A})$ given by $\overline{\mu}(B) = \mu(B)$ for $B \in \mathbb{B}(\mathcal{A})$, clearly $\overline{\mu}$ is an invariant measure for the multivalued semigroup $\{V(t)\}_{t\in\mathbb{R}^+}$ restricted to the attractor $\mathcal{A}$ (which is invariant). Therefore, from Theorem 3.1 and Definition 3.8 there is a measure $\nu \in \mathbb{P}(\mathcal{C}_{\text{loc}})$ whose support $|\nu|$ is inside $\mathcal{K}$ (which is nonempty), invariant for the translation semigroup $\{T(t)\}_{t\in\mathbb{R}^+}$ and such that $\overline{\mu} = \pi_{0*}(\nu)$.[6]

Thus, from the Ergodic Birkhoff Theorem, given any $\psi \in L^1(\mathcal{C}_{\text{loc}}, \nu)$, there is a set $\mathcal{E}_\psi \in \mathbb{B}(\mathcal{C}_{\text{loc}})$, with $\nu(\mathcal{E}_\psi) = 0$ such that for $\xi \in \mathcal{C}_{\text{loc}} \setminus \mathcal{E}_\psi$, the limit

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau \psi(T(t)\xi)dt$$

exists, (see [26], p. 10, and [18], Theorem VIII.7.5). The result follows as in the proof of Theorem 1, [22]. In fact, given $\varphi \in L^1(\mathcal{A}, \overline{\mu})$, consider $\psi = \varphi \circ \pi_0$. Thanks to Theorem 13.46, [2], $\psi \in L^1(\mathcal{C}_{\text{loc}}, \nu)$. Let $E = \mathcal{A} \setminus \pi_0(\mathcal{C}_{\text{loc}} \setminus \mathcal{E}_\psi)$. Note that $\pi_0^{-1}(E) = \pi_0^{-1}(\mathcal{A}) \setminus \pi_0^{-1}(\pi_0(\mathcal{C}_{\text{loc}} \setminus \mathcal{E}_\psi)) \subset \mathcal{E}_\psi$ and then,

$$\overline{\mu}(E) = \nu(\pi_0^{-1}(E)) \leq \nu(\mathcal{E}_\psi) = 0.$$ [7]

Our claim follows by observing that if $u_0 \in \mathcal{A} \setminus E$ and $\xi \in \mathcal{K}$ is such that $\xi(0) = u_0$, then $\xi \in \mathcal{C}_{\text{loc}} \setminus \mathcal{E}_\psi$, since

$$\frac{1}{\tau} \int_0^\tau \psi(T(t)\xi)dt = \frac{1}{\tau} \int_0^\tau \varphi(\xi(t))dt,$$

and it is enough to note that $\mu(E \cup \mathcal{A}^C) = \overline{\mu}(E)$, and $\varphi \in L^1(X, \mu)$ implies $\varphi|_{\mathcal{A}} \in L^1(\mathcal{A}, \overline{\mu})$. $\square$

To carry on the proof of the next theorem we are going to use generalized Banach limits, as defined below.

Definition 4.1. Let $\mathcal{B}_+$ the set of bounded real-valued functions on $\mathbb{R}^+$ endowed with sup norm. A generalized Banach limit, denoted by $\text{LIM}_{t\to\infty}$, is any linear functional on $\mathcal{B}_+$ such that

(1) $\text{LIM}_{t\to\infty} g(t) \geq 0$ for $g \in \mathcal{B}_+$ with $g(s) \geq 0$ for $s \in \mathbb{R}^+$;
(2) $\text{LIM}_{t\to\infty} g(t) = \lim_{t\to\infty} g(t)$ if the usual limit $\lim_{t\to\infty} g(t)$ exists.

For the existence of a generalized Banach limit and its properties see [17,21,27,29].

The next theorems partially extend to the multivalued context some results proved on [17] and [29].

Theorem 4.2. *Let X be a complete separable metric space and $\{V(t)\}_{t\in\mathbb{R}^+}$ a multivalued semigroup determined by a continuous generalized semiflow $\mathcal{G}$ on X which possesses a global attractor $\mathcal{A}$. Given a generalized Banach limit $LIM_{t\to\infty}$ and $\xi \in \mathcal{G}$, there exists a unique $\mu \in \mathbb{P}(X)$, μ invariant for $\{V(t)\}_{t\in\mathbb{R}^+}$, such that*

$$LIM_{t\to\infty} \frac{1}{t} \int_0^t \varphi(\xi(\tau))d\tau = \int_X \varphi(x)\mu(dx), \text{ for any } \varphi \in \mathcal{C}(X).$$

[6] Precisely, $\mathcal{C}_{loc}$ stands here for $\mathcal{C}_{loc}(\mathbb{R}, \mathcal{A})$. Note also that $\mathcal{A} = \cup_{t\in\mathbb{R}}\{\xi(t); \xi \in \mathcal{K}\}$, so $\mathcal{K}$ is in fact the same set introduced before Definition 3.8. In other words, it does not really matter if we define $\mathcal{K}$ as the set of bounded complete trajectories on X or the set of bounded complete trajectories on $\mathcal{A}$.

[7] Despite not necessarily being Borel sets, E and $\pi_0^{-1}(E)$ are both measurable, since their complements are Souslin sets. See Definition 1.5.1, Theorem 1.5.6, (ii), [10]. See also Definition 6.6.1 and Theorem 6.7.3, [11].

We observe additionally that the relation $\overline{\mu} = \pi_{0*}(\nu)$ remains true for the Lebesgue extensions of $\overline{\mu}$ and ν. This is a simple consequence of Corollary 1.5.8, [10].

Proof. Consider the set $\mathcal{G}$, endowed with the topology of uniform convergence on compacts of $(0,\infty)$. Given $\xi \in \mathcal{G}$, we can define a positive linear functional on $\mathcal{C}(\mathcal{G})$ by setting

$$\mathcal{F}_\xi(\varphi) = \mathrm{LIM}_{t\to\infty}\frac{1}{t}\int_0^t \varphi(T(\tau)\xi)d\tau, \text{ for } \varphi \in \mathcal{C}(\mathcal{G}).$$

The set $\mathcal{G}$ can be regarded as a closed subset of $\mathcal{C}_{loc}(\mathbb{R}^+, X)$ (the set of continuous functions from $[0,+\infty)$ to X endowed with the topology of uniform convergence on compacts). In fact, suppose that $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathcal{G}$ converges to ξ uniformly on compacts of $[0,+\infty)$. Then from Definition 3.1, **(H4)**, we have that $\xi \in \mathcal{G}$. From Theorem 11.5, [24], we know that, in this space, the uniform convergence on compacta coincides with the compact open topology, then there is a metric for this topology such that $\mathcal{C}_{loc}(\mathbb{R}^+, X)$ is complete. We also have that $\mathcal{C}_{loc}(\mathbb{R}^+, X)$ is separable, since X is supposed to be. Therefore, $\mathcal{G}$ is a complete separable metric space.

Consider on $\mathcal{G}$ the translation semigroup $\{T(t)\}_{t\in\mathbb{R}^+}$ which is clearly continuous in the following sense: $T(t)\xi$ is continuous at any $t \geq 0$ for fixed $\xi \in \mathcal{G}$ and it is continuous at any $\xi \in \mathcal{G}$ for any fixed $t \in [0,+\infty)$. The set $\mathcal{U} := \pi_+\mathcal{K} := \{\xi|_{[0,\infty)}; \xi \in \mathcal{K}\}$ is a global compact attractor for $\{T(\cdot)\}$. Actually, let $\mathcal{D}$ be a bounded subset of $\mathcal{G}$ and suppose that for some $\varepsilon > 0$, given a sequence $T_k \to \infty$, it is possible to find a sequence $\{\xi_k\} \subset \mathcal{D}$ such that $T(t_k)\xi_k \notin \mathcal{O}_\varepsilon(\mathcal{U})$, with $t_k > T_k$.

Now chose T_k a sequence such that $t \geq T_k$ implies $V(t)D \subset \mathcal{O}_{\varepsilon_k}(\mathcal{A})$ for some sequence $\varepsilon_k \to 0$, with $D := \pi_0(\mathcal{D})$. Then there exists $a \in \mathcal{A}$ such that $\xi_k(t_k) \to a$. From Definition 3.1, **(H4)** and Theorem 2.2, [9] we know that there exists $\xi_a \in \mathcal{G}$, $\xi_a(0) = a$, such that $\xi_k^{t_k} \to \xi_a$ uniformly on compacts of $(0,+\infty)$. From Theorem 15, [34], $\xi_a \in \mathcal{U}$. It remains to prove that $\xi_k^{t_k} \to \xi_a$ on compacts of $[0,+\infty)$ and, in order to do that, it is enough to prove that $\tau_k \to 0$ implies $\xi_k^{t_k}(\tau_k) \to a$. Suppose this is not true. Then there exists a subsequence $\{\xi_{k_n}^{t_{k_n}}\} \subset \{\xi_k^{t_k}\}$ such that $d(\xi_{k_n}^{t_{k_n}}(\tau_{k_n}), a) \geq \varepsilon_0$ for some $\varepsilon_0 > 0$. We can also suppose that $d(\xi_{k_n}^{t_{k_n}}(0), a) < \varepsilon_0$ therefore, for each $n \in \mathbb{N}$, there exists $s_{k_n} \in [0, \tau_{k_n}]$ such that $d(\xi_{k_n}^{t_{k_n}}(s_{k_n}), a) = \varepsilon_0$. From the compactness of $\mathcal{A}$ and the choice of $\{t_k\}$ we know that there is some $a_1 \in \mathcal{A}$ such that $\xi_{k_n}^{t_{k_n}}(s_{k_n}) = \xi_{k_n}(t_{k_n} + s_{k_n}) \to a_1$ with $d(a_1, a) = \varepsilon_0$. Thus there exists ξ_{a_1}, with $\xi_{a_1}(0) = a_1$ such that $\xi_{k_n}^{t_{k_n}}(s_{k_n} + t) \to \xi_{a_1}(t)$ for $t > 0$. Since $\xi_{k_n}^{t_{k_n}}(s_{k_n} + t) \to \xi_a(t)$ for $t > 0$ we have that $\xi_a(t) = \xi_{a_1}(t)$ for $t > 0$ and then, letting $t \to 0$ we conclude that $a = a_1$, which is a contradiction (see the proof of Theorem 2.3, [9]).

Then we can conclude that $\mathcal{U}$ is a global compact attractor for $\{T(\cdot)\}$.

We can apply Theorem 2.1, [17], to guarantee that there exists $\bar{\nu} \in \mathbb{P}(\mathcal{G})$, whose support $|\bar{\nu}|$ is such that $|\bar{\nu}| \subset \mathcal{U}$, $\bar{\nu}$ is invariant for $\{T(\cdot)\}$ and satisfies

$$\mathcal{F}_\xi(\bar{\varphi}) = \int_{\mathcal{G}} \bar{\varphi}(\xi)\bar{\nu}(d\xi), \text{ for any } \bar{\varphi} \in \mathcal{C}(\mathcal{G}).$$

Define $\nu \in \mathbb{P}(\mathcal{K})$ by $\nu = \pi_{+*}(\bar{\nu})$. Then ν is invariant for $T(t)$, $t \in \mathbb{R}$ and, from Theorem 3.1 (**Definition 3.3** $\Leftrightarrow$ **Definition 3.8**), it follows that $\mu = \pi_{0*}(\nu)$ is an invariant measure for $\{V(t)\}_{t\in\mathbb{R}^+}$. To end the proof it is enough to note that, given $\varphi \in \mathcal{C}(X)$, if we define $\bar{\varphi}(\xi) = \varphi(\pi_0(\xi))$, then $\bar{\varphi} \in \mathcal{C}(\mathcal{G})$ and

$$\mathrm{LIM}_{t\to\infty}\frac{1}{t}\int_0^t \varphi(\xi(\tau))d\tau = \mathrm{LIM}_{t\to\infty}\frac{1}{t}\int_0^t \bar{\varphi}(T(\tau)\xi)d\tau = \int_{\mathcal{G}} \bar{\varphi}(\xi)\bar{\nu}(d\xi) = \int_X \varphi(x)\mu(dx).$$

Uniqueness follows from Theorem 15.1, [2]. $\square$

Now it is natural to ask if it is possible to obtain an invariant measure for $\{V(t)\}_{t\in\mathbb{R}^+}$ from a given initial measure. The next theorem gives a positive answer for this question.

Theorem 4.3. *Let X be a complete separable metric space and $\{V(t)\}_{t\in\mathbb{R}^+}$ a multivalued semigroup determined by a continuous generalized semiflow $\mathcal{G}$ on X which possesses a global attractor $\mathcal{A}$. Given a generalized Banach limit $LIM_{t\to\infty}$ and $\mu_0 \in \mathbb{P}(X)$, there exists $\nu_0 \in \mathbb{P}(\mathcal{G})$ such that $\mu_0 = \pi_{0*}(\nu_0)$ and a unique $\mu \in \mathbb{P}(X)$, μ invariant for $\{V(t)\}_{t\in\mathbb{R}^+}$, such that*

$$LIM_{t\to\infty}\frac{1}{t}\int_0^t\int_{\mathcal{G}}\varphi\circ\pi_0(T(\tau)\xi)\nu_0(d\xi)d\tau = \int_X \varphi(x)\mu(dx),\ \textit{for any } \varphi\in\mathcal{C}_b(X), \tag{4.2}$$

where $\mathcal{C}_b(X)$ stands for the set of continuous and bounded real valued functions on X.

Proof. Given $\mu_0 \in \mathbb{P}(X)$, there exists $\nu_0 \in \mathbb{P}(\mathcal{G})$ such that $\mu_0 = \pi_{0*}(\nu_0)$. This is a consequence of Theorem 3.1, [14] (see also Theorem 3.1 [12]). Then, as we have done in the proof of the above theorem, we apply Theorem 2.2, [17], to conclude that there exists a unique $\nu \in \mathbb{P}(\mathcal{G})$ such that $|\nu| \subset \mathcal{U}$, ν is invariant for $\{T(t)\}_{t\in\mathbb{R}^+}$, and

$$\mathrm{LIM}_{t\to\infty}\frac{1}{t}\int_0^t\int_{\mathcal{G}}\bar\varphi(T(\tau)\xi)\nu_0(d\xi)d\tau = \int_{\mathcal{G}}\bar\varphi(\xi)\nu(d\xi),\ \text{for any } \bar\varphi\in\mathcal{C}_b(\mathcal{G}).$$

Let $\mu := \pi_{0*}(\nu)$. From Theorem 3.1, Definition 3.3 $\Leftrightarrow$ Definition 3.8, we know that μ is invariant for $\{V(t)\}_{t\in\mathbb{R}^+}$. Given $\varphi \in \mathcal{C}_b(X)$, consider $\bar\varphi := \varphi\circ\pi_0 \in \mathcal{C}_b(\mathcal{G})$. Then 4.2 follows by observing that

$$\int_X \varphi(x)\mu(dx) = \int_X \varphi(x)\pi_{0*}(\nu)(dx) = \int_{\mathcal{G}}\varphi(\pi_0(\xi))\nu(d\xi) = \int_{\mathcal{G}}\bar\varphi(\xi)\nu(d\xi). \quad \square$$

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