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Note

On the determination of the grad-div criterion

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ABSTRACT

Grad-div stabilization, adding a term $-\gamma \operatorname{grad} \operatorname{div} u$ to penalize violation of incompressibility, has proven to be a useful tool in the simulation of incompressible flows. Such a term requires a choice of the coefficient γ and studies have begun appearing with various suggestions for its value. We give an analysis herein that provides a restricted range of possible values for the coefficient in $3d$ turbulent flows away from walls. If U, L denote the large scale velocity and length respectively and κ is the *signal to noise ratio* of the body force, estimates suggest that γ should be restricted to the range

$$\frac{\kappa^2}{24} LU \leq \gamma \leq \frac{\kappa^2}{4} \mathcal{R}eLU, \text{ mesh independent case,}$$
$$\frac{\kappa^2}{24} LU \leq \gamma \leq \frac{\kappa^2}{4} \left(\frac{h}{L}\right)^{-\frac{4}{3}} LU, \text{ mesh dependent case.}$$

1. Introduction

In the numerical simulation of incompressible flows including an additive term $-\gamma \operatorname{grad} \operatorname{div} u$ is reported to increase accuracy [17], [20], [23] and improve performance of iterative solvers [12], [26], [5], [3] in addition to enhancing conservation of mass under the assumption of constant density by penalization of violation of incompressibility, as pointed out by many, e.g., [21]. In all cases performance depends on the value of γ chosen, [19]. Suggested values include $\gamma = \mathcal{O}(1)$, [4], [24], [25], [26] Remark 3.6 p. 308, $\gamma = \mathcal{O}(\nu)$ suggested by estimate (3.17) p. 306 in [26], $\gamma = \mathcal{O}(\Delta x^{1 \text{ or } 2})$, [17], [26] Remark 3.7 p. 308, $\gamma = \mathcal{O}(10^3)$, [11], $\gamma = \infty$

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when the body force is a potential, [21] (who also analyzes consequences for space discretization) and global or local ratios of semi-norms $\gamma = \mathcal{O}(|p|_k/|u|_{k+1})$, [16], [24], [17], [22], [15]. Values of γ have been derived by optimizing numerical errors for selected exact solutions, by balancing error estimates in various norms for the Stokes or Oseen problem and by optimizing solver performance. Among other interesting work on grad-div stabilization, John and Kindl [18] include a grad-div term with a variational multiscale discretization and give numerical tests with $\gamma = 1/2$ (from [4]) and $\gamma = 0.5\left[\nu^2 + 0.25(h||u||)^2\right]^{1/2}$ (from [6], [13], [14]). One finding was (p. 850) *"... using only grad-div stabilization gave reasonable results"*. A novel extension to discontinuous Galerkin methods via edge jumps occurs in [1]. Its effect on long time stability was studied in [2]. Grad-div stabilization of only the fluctuating component of the velocity was studied in [8].

The mesh dependent influence of γ on total energy dissipation was tested in [7]. We consider herein a (complementary) mesh independent approach limiting γ to values where the additional dissipation introduced does not disturb statistical equilibrium. Denote time averaging by $\langle\cdot\rangle = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \cdot dt$. For $3d$, fully developed, turbulent flows away from walls, it is known that the total energy dissipation rate balances energy input, $\langle\varepsilon(u)\rangle = \mathcal{O}(U^3/L)$, where the energy dissipation rate (per unit volume) $\varepsilon(u)$ is

$$\varepsilon(u) = \frac{1}{|\Omega|}\int_\Omega \nu|\nabla u(x,t)|^2 + \gamma|\nabla\cdot u(x,t)|^2 dx \text{ so } \langle\varepsilon\rangle = \lim\sup_{T\to\infty}\frac{1}{T}\int_0^T \varepsilon(u)dt.$$

This balance is one of the *two laws of experimental turbulence,* [10] Ch. 5. Building on [9], we analyze the dependence of $\langle\varepsilon\rangle$ on γ for the *simplest system arising when incompressibility is relaxed by* $-\gamma\,\mathrm{grad}\,\mathrm{div}\,u$, given by

$$u_t + \mathrm{div}(u\otimes u) - \frac{1}{2}(\nabla\cdot u)u - \nu\Delta u - \gamma\nabla\nabla\cdot u = f(x). \tag{1}$$

The domain $\Omega = (0, L_\Omega)^3$ is a $3d$ periodic box, $f(x)$ and $u(x,0)$ are periodic, satisfy $\nabla\cdot u(x,0) = \nabla\cdot f = 0$ and have zero mean:

$$u(x + L_\Omega e_j, t) = u(x,t) \text{ and } \int_\Omega \phi dx = 0 \text{ for } \phi = u,\, u_0,\, f. \tag{2}$$

The body force $f(x)$ is assumed smooth so that it inputs energy only into large scales. Recalling $f(x)$ has mean zero, define the *signal to noise ratio* of the body force κ by

$$\kappa^2 = \frac{||f||^2_{L^\infty}}{\frac{1}{|\Omega|}\int_\Omega |f(x)|^2 dx}.$$

Since $\nabla\cdot u \neq 0$ the nonlinearity is explicitly skew symmetrized by adding $-\frac{1}{2}(\nabla\cdot u)u$.

Let $(\cdot,\cdot), ||\cdot||$ denote the $L^2(\Omega)$ inner product and norm. Let F, L, U denote

$$F = \left(\frac{1}{|\Omega|}||f||^2\right)^{\frac{1}{2}},\ L = \min\left\{L_\Omega, \frac{F}{||\nabla f||_{L^\infty}}, \frac{F}{(\frac{1}{|\Omega|}||\nabla f||^2)^{\frac{1}{2}}}\right\}, U = \left\langle\frac{1}{|\Omega|}||u||^2\right\rangle^{\frac{1}{2}}.$$

Non-dimensionalization by $t^* = t/T, x^* = x/L, U = L/T, u^* = u/U, p^* = p/U^2$ gives:

$$u_t^* + \mathrm{div}^*(u^*\otimes u^*) - \frac{1}{2}(\nabla^*\cdot u^*)u^* - \frac{\nu}{LU}\Delta^* u^* - \frac{\gamma}{LU}\nabla^*\nabla^*\cdot u^* = \frac{f(x)}{U^2}.$$

We recall $\mathcal{R}e = \frac{LU}{\nu}$ and define the non-dimensional parameter $\mathcal{R}_\gamma = \frac{LU}{\gamma}$. The following is proven in Section 2.[3]

Theorem 1. *Let $u(x,t)$ be a weak solution of (1). Then,*

$$\langle \varepsilon(u) \rangle \leq \left(6 + \mathcal{R}e^{-1} + \frac{1}{4}\kappa^2 \mathcal{R}_\gamma\right) \frac{U^3}{L}. \tag{3}$$

This estimate gives insight into γ by asking $grad - div$ dissipation be comparable to (respectively) the pumping rate of energy to small scales by the nonlinearity, U^3/L, and to the correction to the asymptotic, $\mathcal{R}e \to \infty$, rate due to energy dissipation in the inertial range, $\mathcal{R}e^{-1}\frac{U^3}{L}$. The cases $2 \simeq \kappa^2 \mathcal{R}_\gamma$ and $\mathcal{R}e^{-1} \simeq \kappa^2 \mathcal{R}_\gamma$ yield

$$\frac{\kappa^2}{24} \leq \frac{\gamma}{LU} \leq \frac{\kappa^2}{4}\mathcal{R}e, \text{ mesh independent case.}$$

Let $\eta \simeq \mathcal{R}e^{-3/4}L$ denote the Kolmogorov microscale so $\mathcal{R}e = (\eta/L)^{-4/3}$. When the model is solved on a spacial mesh with meshwidth $\eta << h$ the smallest scale available is $\mathcal{O}(h)$. Replacing η by h leads to an estimate of mesh dependence of

$$\frac{\kappa^2}{24} \leq \frac{\gamma}{LU} \leq \frac{\kappa^2}{4}\left(\frac{h}{L}\right)^{-\frac{4}{3}}, \text{ mesh dependent case.}$$

2. Proof of Theorem 1

Compared to the NSE case [9] the term $-\frac{1}{2}(\nabla \cdot u)u$ adds dependence on γ since $\operatorname{div} u \neq 0$. A smooth enough solution of (1) satisfies the same á priori energy equality as a discrete NSE system with $grad - div$ stabilization

$$\frac{1}{2}\frac{d}{dt}||u(t)||^2 + \left\{\nu||\nabla u(t)||^2 + \gamma||\nabla \cdot u(t)||^2\right\} = (f, u).$$

Definition 1. A weak solution of (1) is a distributional solution satisfying the energy inequality

$$\frac{1}{2}||u(T)||^2 + \int_0^T \nu||\nabla u(t)||^2 + \gamma||\nabla \cdot u(t)||^2 dt \leq \frac{1}{2}||u(0)||^2 + \int_0^T (f, u) dt. \tag{4}$$

Standard differential inequalities then establish that

$$\frac{1}{2}||u(T)||^2 + \frac{1}{T}\int_0^T \varepsilon(u) dt \leq C < \infty, \ C = C(data) \text{ independent of } T. \tag{5}$$

Indeed, applying the Cauchy–Schwarz inequality to the right hand side and the Poincaré–Friedrichs inequality to the left hand side of the above energy bound yields

$$\frac{d}{dt}||u(t)||^2 + a||u(x,t)||^2 \leq C||f||^2, \text{ for some } a > 0, C > 0.$$

[3] An expanded version of this report including background material and many details of the proof is available at https://arxiv.org/abs/1704.04171.

Using an integrating factor yields the uniform bound $\frac{1}{2}||u(T)||^2 \leq C < \infty$. For the bound on $\varepsilon(u)$, divide the energy inequality by T. This gives

$$\frac{1}{2}\frac{1}{T}||u(T)||^2 + \frac{1}{T}\int_0^T \varepsilon(u)dt \leq \frac{1}{2}\frac{1}{T}||u(0)||^2 + \frac{1}{T}\int_0^T (f,u)dt.$$

Thus,

$$\begin{aligned}\frac{1}{T}\int_0^T \varepsilon(u)dt &\leq \mathcal{O}\left(\frac{1}{T}\right) + \frac{1}{T}\int_0^T (f,u)dt \\ &\leq \mathcal{O}\left(\frac{1}{T}\right) + \sqrt{\frac{1}{T}\int_0^T ||f||_{-1}^2 dt}\sqrt{\frac{1}{T}\int_0^T ||\nabla u||^2 dt},\end{aligned}$$

and the claimed bound follows by the arithmetic-geometric mean inequality and the fact that $f = f(x)$.

From (5) $\langle\varepsilon\rangle$ is well defined and finite and $\frac{1}{T}||u(T)||^2 \to 0$ as $\mathcal{O}(\frac{1}{T})$. L has units of length and satisfies

$$||\nabla f||_{L^\infty} \leq \frac{F}{L} \text{ and } \frac{1}{|\Omega|}\int_\Omega |\nabla f(x)|^2 dx \leq \frac{F^2}{L^2}. \tag{6}$$

Dividing (4) by $T|\Omega|$ gives

$$\begin{aligned}&\frac{1}{2T|\Omega|}||u(T)||^2 + \frac{1}{T|\Omega|}\int_0^T \nu||\nabla u(t)||^2 + \gamma||\nabla\cdot u(t)||^2 dt \\ &\leq \frac{1}{2}\frac{1}{T|\Omega|}||u(0)||^2 + \frac{1}{T|\Omega|}\int_0^T (f,u)dt.\end{aligned} \tag{7}$$

Define $U_T := (\frac{1}{T}\int_0^T \frac{1}{|\Omega|}||u||^2 dt)^{1/2}$. Given (5) and the definition of F, this is

$$\begin{aligned}\frac{1}{T}\int_0^T \varepsilon(u)dt &\leq \mathcal{O}(\frac{1}{T}) + \frac{1}{T|\Omega|}\int_0^T (f,u)dt \\ &\leq \mathcal{O}(\frac{1}{T}) + F\sqrt{\frac{1}{T}\int_0^T \frac{1}{|\Omega|}||u||^2 dt} = \mathcal{O}(\frac{1}{T}) + FU_T\end{aligned} \tag{8}$$

To estimate F, take the inner product of (1) with f (recall $\nabla\cdot f = 0$), divide by $T|\Omega|$, and integrate from $(0,T)$

$$\begin{aligned}F^2 = \frac{(u(T)-u_0,f)}{T|\Omega|} &- \frac{1}{T|\Omega|}\int_0^T (u\otimes u,\nabla f) - (\frac{1}{2}(\nabla\cdot u)u,f)dt + \\ &+\frac{1}{T}\int_0^T \frac{\nu}{|\Omega|}(\nabla u,\nabla f)dt.\end{aligned} \tag{9}$$

Of the four terms on the RHS, by (5) the first is $\mathcal{O}(1/T)$. The second and fourth are bounded using Hölders and Young's inequalities by

$$\left|\frac{1}{T|\Omega|}\int_0^T (u\otimes u, \nabla f)dt\right| \leq ||\nabla f||_{L^\infty}\frac{3}{T|\Omega|}\int_0^T ||u||^2 dt \leq 3\frac{F}{L}U_T^2,$$

$$\left|\frac{1}{T}\int_0^T \frac{\nu}{|\Omega|}(\nabla u, \nabla f)dt\right| \leq \left(\frac{1}{T}\int_0^T \frac{\nu}{|\Omega|}||\nabla u||^2 dt\right)^{\frac{1}{2}} \left(\frac{1}{T}\int_0^T \frac{\nu}{|\Omega|}||\nabla f||^2 dt\right)^{\frac{1}{2}}$$

$$\leq \left(\frac{1}{T}\int_0^T \frac{\nu}{|\Omega|}||\nabla u||^2 dt\right)^{\frac{1}{2}} \frac{\sqrt{\nu}F}{L} \leq \frac{1}{2}\frac{F}{U}\frac{1}{T}\int_0^T \frac{\nu}{|\Omega|}||\nabla u||^2 dt + \frac{1}{2}UF\frac{\nu}{L^2}.$$

The third term is treated as follows:

$$\left|\frac{1}{T}\int_0^T \frac{1}{|\Omega|}(\frac{1}{2}(\nabla\cdot u)u, f)dt\right| \leq \frac{1}{2}||f||_{L^\infty}\sqrt{\frac{1}{T}\int_0^T \frac{1}{|\Omega|}||\nabla\cdot u||^2 dt}U_T$$

$$\leq \frac{\gamma}{2}\frac{F}{U}\frac{1}{T}\int_0^T \frac{1}{|\Omega|}||\nabla\cdot u||^2 dt + \frac{1}{8\gamma}UF\frac{||f||_{L^\infty}^2}{\frac{1}{|\Omega|}\int_\Omega |f(x)|^2 dx}U_T^2$$

$$\leq \frac{\gamma}{2}\frac{F}{U}\frac{1}{T}\int_0^T \frac{1}{|\Omega|}||\nabla\cdot u||^2 dt + \frac{1}{8\gamma}\kappa^2 UFU_T^2$$

Inserting these estimates in (9) and simplifying we obtain

$$F \leq \mathcal{O}(\frac{1}{T}) + \frac{3}{L}U_T^2 + \frac{1}{2U}\frac{1}{T}\int_0^T \varepsilon(t)dt + \frac{1}{2}\frac{\nu U}{L^2} + \frac{1}{8\gamma}U\kappa^2 U_T^2.$$

Inserting this in the RHS of (8) gives

$$\frac{1}{T}\int_0^T \varepsilon(u)dt \leq \mathcal{O}(\frac{1}{T}) + FU_T \leq \mathcal{O}(\frac{1}{T}) + 3\frac{U_T^3}{L} +$$

$$+\frac{U_T}{2U}\frac{1}{T}\int_0^T \varepsilon(u)dt + \frac{\nu U}{2L^2}U_T + \frac{1}{8\gamma}U\kappa^2 U_T^3.$$

Letting $T \to \infty$ we have, as claimed, that

$$\langle\varepsilon(u)\rangle \leq \left(6 + \mathcal{R}e^{-1} + \frac{1}{4}\kappa^2\mathcal{R}_\gamma\right)\frac{U^3}{L}.$$

3. Conclusions

This report considered parameter selection for 3d turbulence in a box driven by a persistent body force with periodic boundary conditions. The analysis does not apply to 2d flows, laminar flows, turbulence generated by shear flows and decaying turbulence. These cases are interesting open problems. For the case considered, the analysis herein suggests the following linkage. Weak imposition of $\nabla \cdot u = 0$ at higher Reynolds numbers means explicit skew symmetrization becomes necessary. Since $\nabla \cdot u \neq 0$, this leads to a second nonlinear term $-\frac{1}{2}(\nabla \cdot u)u$. The parameter γ affects the size of $||\nabla \cdot u||$ which affects the rate at which $-\frac{1}{2}(\nabla \cdot u)u$ pumps energy to smaller scales. This leads to restricting γ by aligning this energy transfer rate with that of the NSE:

Compressibility, however so slight
Doubles nonlinearity for skew-symmetry.
Cascades can stop by penalty, however light,
Unless its criterion is chosen with sagacity.

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