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Equivalence between almost-greedy and semi-greedy bases ☆

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ABSTRACT

In [3] it was proved that almost-greedy and semi-greedy bases are equivalent in the context of Banach spaces with finite cotype. In this paper we show this equivalence for general Banach spaces.

1. Introduction

Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space over $\mathbb{F}$ ($\mathbb{F}$ denotes the real field $\mathbb{R}$ or the complex field $\mathbb{C}$) and let $\mathscr{B} = (e_n)_{n=1}^\infty$ be a semi-normalized Schauder basis of $\mathbb{X}$ with constant K_b and with biorthogonal functionals $(e_n^*)_{n=1}^\infty$, i.e., $0 < \inf_n \|e_n\| \leq \sup_n \|e_n\| < \infty$ and $K_b = \sup_N \|S_N(x)\|/\|x\| < \infty$ $\forall x \in \mathbb{X}$, where $S_N(x) = \sum_{j=1}^N e_j^*(x)e_j$ denotes the algorithm of the partial sums.

As usual $\operatorname{supp}(x) = \{n \in \mathbb{N} : e_n^*(x) \neq 0\}$, given a finite set $A \subset \mathbb{N}$, $|A|$ denotes the cardinality of the set A, P_A is the projection operator, that is, $P_A(\sum_j a_j e_j) = \sum_{j\in A} a_j e_j$, $P_{A^c} = \mathrm{I}_{\mathbb{X}} - P_A$, $\mathbf{1}_{\varepsilon A} = \sum_{n\in A} \varepsilon_n e_n$ with $|\varepsilon_n| = 1$ (where ε_n could be real or complex), $\mathbf{1}_A = \sum_{n\in A} e_n$ and for $A, B \subset \mathbb{N}$, we write $A < B$ if $\max_{i\in A} i < \min_{j\in B} j$.

In 1999, S.V. Konyagin and V.N. Temlyakov introduced the *Thresholding Greedy Algorithm* (TGA) (see [7]): given $x = \sum_{i=1}^\infty e_i^*(x)e_i \in \mathbb{X}$, we define the *natural greedy ordering* for x as the map $\rho : \mathbb{N} \longrightarrow \mathbb{N}$ such that $\operatorname{supp}(x) \subset \rho(\mathbb{N})$ and so that if $j < k$ then either $|e_{\rho(j)}^*(x)| > |e_{\rho(k)}^*(x)|$ or $|e_{\rho(j)}^*(x)| = |e_{\rho(k)}^*(x)|$ and $\rho(j) < \rho(k)$. The *m-th greedy sum* of x is

$$\mathscr{G}_m(x) = \sum_{j=1}^m e_{\rho(j)}^*(x)e_{\rho(j)},$$

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and the sequence of maps $(\mathcal{G}_m)_{m=1}^{\infty}$ is known as the *Thresholding Greedy Algorithm* associated to $\mathscr{B}$ in $\mathbb{X}$. Alternatively we can write $\mathcal{G}_m(x) = \sum_{k\in A_m(x)} e_k^*(x)e_k$, where $A_m(x) = \{\rho(n) : n \leq m\}$ is the *greedy set* of x: $\min_{k\in A_m(x)} |e_k^*(x)| \geq \max_{k\notin A_m(x)} |e_k^*(x)|$.

To study the efficiency of the TGA, S.V. Konyagin and V.N. Temlyakov introduced in [7] the so called *greedy bases*.

Definition 1.1. We say that $\mathscr{B}$ is *greedy* if there exists a constant $C \geq 1$ such that

$$\|x - \mathcal{G}_m(x)\| \leq C\sigma_m(x),\ \forall x \in \mathbb{X}, \forall m \in \mathbb{N},$$

where $\sigma_m(x)$ is the m-th error of approximation with respect to $\mathscr{B}$, and it is defined as

$$\sigma_m(x, \mathscr{B})_{\mathbb{X}} = \sigma_m(x) := \inf\left\{\left\|x - \sum_{n\in C} a_n e_n\right\| : |C| = m, a_n \in \mathbb{F}\right\}.$$

Also, S.V. Konyagin and V.N. Temlyakov characterized greedy bases in terms of *unconditional* bases with the additional property of being *democratic*, i.e., $\|\mathbf{1}_A\| \leq C_d\|\mathbf{1}_B\|$ for any pair of finite sets A, B with $|A| \leq |B|$. Recall that a basis $\mathscr{B}$ in $\mathbb{X}$ is called unconditional if any rearrangement of the series $\sum_{n=1}^{\infty} e_n^*(x)e_n$ converges in norm to x for any $x \in \mathbb{X}$. This turns out to be equivalent the fact that the projections P_A are uniformly bounded on all finite sets A, i.e. there exists a constant $C \geq 1$ such that

$$\|P_A(x)\| \leq C\|x\|,\ \ \forall x \in \mathbb{X} \text{ and } \ \forall A \subset \mathbb{N}.$$

Another important concept in greedy approximation theory is the notion of *quasi-greedy* bases introduced in [7].

Definition 1.2. We say that $\mathscr{B}$ is *quasi-greedy* if there exists a constant $C \geq 1$ such that

$$\|x - \mathcal{G}_m(x)\| \leq C\|x\|,\ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}. \tag{1}$$

We denote by C_q the least constant that satisfies (1) and we say that $\mathscr{B}$ is C_q-quasi-greedy.

Subsequently, P. Wojtaszczyk proved in [8] that $\mathscr{B}$ is quasi-greedy in a quasi-Banach space $\mathbb{X}$ if and only if the algorithm converges, that is,

$$\lim_{m\to\infty} \|x - \mathcal{G}_m(x)\| = 0,\ \forall x \in \mathbb{X}.$$

One intermediate concept between greedy and quasi-greedy bases, *almost-greedy* bases, was introduced by S.J. Dilworth et al. in [5].

Definition 1.3. We say that $\mathscr{B}$ is *almost-greedy* if there exists a constant $C \geq 1$ such that

$$\|x - \mathcal{G}_m(x)\| \leq C\tilde{\sigma}_m(x),\ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}, \tag{2}$$

where $\tilde{\sigma}_m(x, \mathscr{B})_{\mathbb{X}} = \tilde{\sigma}_m(x) := \inf\{\|x - P_A(x)\| : |A| = m\}$. We denote by C_{al} the least constant that satisfies (2) and we say that $\mathscr{B}$ is C_{al}-almost-greedy.

In [5], the authors characterized the almost-greedy bases in terms of quasi-greedy and democratic bases.

Theorem 1.4 *([5, Theorem 3.3]). $\mathscr{B}$ is almost-greedy if and only if $\mathscr{B}$ is quasi-greedy and democratic.*

We will use the notion of *super-democracy* instead of democracy. This is a classical concept in this theory.

Definition 1.5. We say that $\mathscr{B}$ is *super-democratic* if there exists a constant $C \geq 1$ such that

$$\|\mathbf{1}_{\varepsilon A}\| \leq C\|\mathbf{1}_{\eta B}\|, \tag{3}$$

for any pair of finite sets A and B such that $|A| \leq |B|$ and any choice $|\varepsilon| = |\eta| = 1$. We denote by C_{sd} the least constant that satisfies (3) and we say that $\mathscr{B}$ is C_{sd}-super-democratic.

Remark 1.6. It is well known that in Theorem 1.4 we can replace democracy by super-democracy (see for instance [1, Theorem 1.3]).

On the other hand, S.J. Dilworth, N.J. Kalton and D. Kutzarova introduced in [3] the concept of *semi-greedy* bases. This concept was born as an enhancement of the TGA to improve the rate of convergence. To study the notion of semi-greediness, we need to define the *Thresholding Chebyshev Greedy Algorithm*: let $A_m(x)$ be the greedy set of x of cardinality m. Define the *m-th Chebyshev-greedy sum* as any element $\mathscr{CG}_m(x) \in span\{e_i : i \in A_m(x)\}$ such that

$$\|x - \mathscr{CG}_m(x)\| = \min\left\{ \left\| x - \sum_{n \in A_m(x)} a_n e_n \right\| : a_n \in \mathbb{F} \right\}.$$

The collection $\{\mathscr{CG}_m\}_{m=1}^{\infty}$ is the *Thresholding Chebyshev Greedy Algorithm*.

Definition 1.7. We say that $\mathscr{B}$ is *semi-greedy* if there exists a constant $C \geq 1$ such that

$$\|x - \mathscr{CG}_m(x)\| \leq C\sigma_m(x),\ \forall x \in \mathbb{X}, \forall m \in \mathbb{N}. \tag{4}$$

We denote by C_s the least constant that satisfies (4) and we say that $\mathscr{B}$ is C_s-semi-greedy.

In [3], the following theorem is proved:

Theorem 1.8 *([3, Theorem 3.2]). Every almost-greedy basis in a Banach space is semi-greedy.*

In this paper we study the converse of this theorem. In [3], the authors established the following "converse" theorem:

Theorem 1.9 *([3, Theorem 3.6]). Assume that $\mathscr{B}$ is a semi-greedy basis in a Banach space $\mathbb{X}$ which has finite cotype. Then, $\mathscr{B}$ is almost-greedy.*

The objective here is to show that the condition of the finite cotype in the last theorem is not necessary. The main result is the following:

Theorem 1.10. *Assume that $\mathscr{B}$ is a Schauder basis in a Banach space $\mathbb{X}$.*

a) *If $\mathscr{B}$ is C_q-quasi-greedy and C_{sd}-super-democratic, then $\mathscr{B}$ is C_s-semi-greedy with constant $C_s \leq C_q + 4C_qC_{sd}$.*
b) *If $\mathscr{B}$ is C_s-semi-greedy, then $\mathscr{B}$ is C_{sd}-super-democratic with constant $C_{sd} \leq 2(C_sK_b)^2$ and C_q-quasi-greedy with constant $C_q \leq K_b(2 + 3(K_bC_s)^2)$.*

Remark 1.11. S.J. Dilworth et al. ([3]) proved the item $a)$ with the bound $C_s = O(C_q^2 C_d)$, where C_d is the democracy constant. Here, we slightly relax this bound proving that $C_s = O(C_q C_{sd})$.

Corollary 1.12. *If $\mathscr{B}$ is a Schauder basis in $\mathbb{X}$, $\mathscr{B}$ is almost-greedy if and only if $\mathscr{B}$ is semi-greedy.*

2. Preliminary results

To prove Theorem 1.10, we need the following technical results that we can find in [1] and [5].

2.1. Convexity lemma

Lemma 2.1 *([1, Lemma 2.7]). For every finite set $A \subset \mathbb{N}$, we have*

$$co\left(\{\mathbf{1}_{\varepsilon A} : |\varepsilon| = 1\}\right) = \left\{\sum_{n\in A} z_n e_n : |z_n| \leq 1\right\},$$

where $co(S) = \{\sum_{j=1}^n \alpha_j x_j : x_j \in S, 0 \leq \alpha_j \leq 1, \sum_{j=1}^n \alpha_j = 1, n \in \mathbb{N}\}$.

As a consequence, for any finite sequence $(z_n)_{n\in A}$ with $z_n \in \mathbb{F}$ for all $n \in A$,

$$\left\|\sum_{n\in A} z_n e_n\right\| \leq \max_{n\in A} |z_n| \varphi(|A|),$$

where $\varphi(m) = \sup_{|A|=m, |\varepsilon|=1} \|\mathbf{1}_{\varepsilon A}\|$.

2.2. The truncation operator

For each $\alpha > 0$, we define the *truncation function* of $z \in \mathbb{F}$ as

$$T_\alpha(z) = \alpha \operatorname{sgn}(z),\ |z| > \alpha,\ \ T_\alpha(z) = z,\ |z| \leq \alpha.$$

We can extend T_α to an operator in $\mathbb{X}$ by

$$T_\alpha(x) = \sum_{i=1}^{\infty} T_\alpha(e_i^*(x)) e_i = \alpha \mathbf{1}_{\varepsilon \Gamma_\alpha} + P_{\Gamma_\alpha^c}(x),$$

where $\Gamma_\alpha = \{n : |e_n^*(x)| > \alpha\}$ and $\varepsilon_j = \operatorname{sgn}(e_j^*(x))$ with $j \in \Gamma_\alpha$. Hence, this is a well-defined operator for all $x \in \mathbb{X}$ since Γ_α is a finite set.

This operator was introduced in [3] to prove Theorem 1.8 showing that for quasi-greedy bases, this operator is uniformly bounded. A slight improvement of the boundedness constant was given in [1].

Proposition 2.2 *([1, Lemma 2.5]). Assume that $\mathscr{B}$ is C_q-quasi-greedy basis in a Banach space $\mathbb{X}$. Then, for every $\alpha > 0$,*

$$\|T_\alpha(x)\| \leq C_q \|x\|,\ \forall x \in \mathbb{X}.$$

We shall also use the following known inequality from [5].

Lemma 2.3 *([5, Lemma 2.2]). If $\mathscr{B}$ is a C_q-quasi-greedy basis in $\mathbb{X}$,*

$$\min_{j\in G}|e_j^*(x)|\|\mathbf{1}_{\varepsilon G}\| \leq 2C_q\|x\|, \ \forall x\in\mathbb{X}, \forall G \text{ greedy set of } x, \tag{5}$$

with $\varepsilon = \{\operatorname{sgn}(e_j^*(x))\}$.

3. Proof of the main result

Using the lemmas of Section 2, we prove Theorem 1.10.

Proof of Theorem 1.10. First, we show the proof of a). Suppose that $\mathscr{B}$ is C_q-quasi-greedy and C_{sd}-super-democratic. To show the semi-greediness, we will follow the same procedure as in the proof of [4, Theorem 4.1] and [3, Theorem 3.2]. Take $x\in\mathbb{X}$ and $z=\sum_{i\in B}a_ie_i$ with $|B|=m$ such that $\|x-z\|<\sigma_m(x)+\delta$, for $\delta>0$. Let $A_m(x)$ the greedy set of x of cardinality m. We write $x-z := \sum_{i=1}^\infty y_ie_i$, where $y_i=e_i^*(x)-a_i$ for $i\in B$ and $y_i=e_i^*(x)$ for $i\notin B$. To prove that $\mathscr{B}$ is semi-greedy we only have to show that there exists $w\in\mathbb{X}$ so that *supp*$(x-w)\subset A_m(x)$ and $\|w\|\leq c\|x-z\|$ for some positive constant c. If $\alpha=\max_{j\notin A_m(x)}|e_j^*(x)|$, we take the element w as is defined in [3]:

$$w := \sum_{i\in A_m(x)} T_\alpha(y_i)e_i + P_{A_m^c(x)}(x) = \sum_{i=1}^\infty T_\alpha(y_i)e_i + \sum_{i\in B\setminus A_m(x)}(e_i^*(x)-T_\alpha(y_i))e_i.$$

Of course, w satisfies that $\operatorname{supp}(x-w)\subset A_m(x)$ and we will prove that $\|w\|\leq(C_q+4C_qC_s)\|x-z\|$. To obtain this bound, using Proposition 2.2,

$$\|\sum_{i=1}^\infty T_\alpha(y_i)e_i\| \leq C_q\|x-z\|. \tag{6}$$

Taking into account that $|e_i^*(x)-T_\alpha(y_i)|\leq 2\alpha$ for $i\in B\setminus A_m(x)$, using Lemma 2.1,

$$\left\|\sum_{i\in B\setminus A_m(x)}(e_i^*(x)-T_\alpha(y_i))e_i\right\| \leq 2\alpha\varphi(|B\setminus A_m(x)|) \leq 2\min_{j\in A_m(x)\setminus B}|e_j^*(x-z)|\varphi(|A_m(x)\setminus B|). \tag{7}$$

To improve the bound of C_s as we have commented in Remark 1.11, based on ([6, Lemma 2.1]), we can find a greedy set Γ of $x-z$ with the following conditions:

- $|\Gamma|=|B\setminus A_m(x)|$,
- $\min_{j\in A_m(x)\setminus B}|e_j^*(x-z)|\leq\min_{j\in\Gamma}|e_j^*(x-z)|$.

Hence, using $\varepsilon=\{\operatorname{sgn}(e_j^*(x-z))\}$ and Lemma 2.3,

$$\min_{j\in A_m(x)\setminus B}|e_j^*(x-z)|\varphi(|B\setminus A_m(x)|) \leq C_{sd}\min_{j\in\Gamma}|e_j^*(x-z)|\|\mathbf{1}_{\varepsilon\Gamma}\| \leq 2C_qC_{sd}\|x-z\|. \tag{8}$$

Thus, using (6), (7), (8), the basis is C_s-semi-greedy with constant $C_s\leq(C_q+4C_qC_{sd})$.

Now, we prove b). Assume that $\mathscr{B}$ is C_s-semi-greedy.

Super-democracy can be proved using the technique of [3, Proposition 3.3]. Indeed, take A and B with $|A|\leq|B|$ and $|\varepsilon|=|\eta|=1$. Select now a set D such that $|D|=|A|$, $D>(A\cup B)$ and define $z := \mathbf{1}_{\varepsilon A}+(1+\delta)\mathbf{1}_D$ with $\delta>0$. It is clear that $\mathscr{G}_{|D|}(z)=(1+\delta)\mathbf{1}_D$. Then,

$$\|z - \mathscr{C}\mathscr{G}_{|D|}(z)\| = \left\|\mathbf{1}_{\varepsilon A} + \sum_{i\in D} c_i e_i\right\|,$$

where the scalars $(c_i)_{i\in D}$ are given by the Chebyshev approximation. Then,

$$\|\mathbf{1}_{\varepsilon A}\| \leq K_b\|\mathbf{1}_{\varepsilon A} + \sum_{i\in D} c_i e_i\| \leq K_b C_s \sigma_{|D|}(z) \leq K_b C_s\|(1+\delta)\mathbf{1}_D\|.$$

If δ goes to 0,

$$\|\mathbf{1}_{\varepsilon A}\| \leq C_s K_b\|\mathbf{1}_D\|. \tag{9}$$

The next step is to obtain that $\|\mathbf{1}_D\| \leq 2K_b C_s\|\mathbf{1}_{\eta B}\|$. For that, we take the element $y := (1+\delta)\mathbf{1}_{\eta B} + \mathbf{1}_D$ with $\delta > 0$. Then, $\mathscr{G}_{|B|}(y) = (1+\delta)\mathbf{1}_{\eta B}$. Hence,

$$\|y - \mathscr{C}\mathscr{G}_{|B|}(y)\| = \left\|\sum_{i\in B} d_i e_i + \mathbf{1}_D\right\|,$$

where as before, the scalars $(d_i)_{i\in B}$ are given by the Chebyshev approximation. Using again the semi-greediness,

$$\|\mathbf{1}_D\| \leq 2K_b\|\sum_{i\in B} d_i e_i + \mathbf{1}_D\| \leq 2C_s K_b \sigma_{|B|}(y) \leq 2C_s K_b\|(1+\delta)\mathbf{1}_{\eta B}\|.$$

Taking $\delta \to 0$, we obtain that

$$\|\mathbf{1}_D\| \leq 2C_s K_b\|\mathbf{1}_{\eta B}\|. \tag{10}$$

Using (9) and (10),

$$\|\mathbf{1}_{\varepsilon A}\| \leq 2(C_s K_b)^2\|\mathbf{1}_{\eta B}\|.$$

Hence, the basis is super-democratic with constant $C_{sd} \leq 2(C_s K_b)^2$.

To prove now the quasi-greediness, we will present a more elementary proof than in [3, Theorem 3.6] that works for general Banach spaces: take an element $x \in \mathbb{X}$ with finite support and $A_m(x)$ the greedy set of x with cardinality m, take $D > \operatorname{supp}(x)$ with $|D| = |A_m(x)| = m$ and define $z := x - \mathscr{G}_m(x) + (\delta+\alpha)\mathbf{1}_D$, where $\delta > 0$ and $\alpha = \min_{j\in A_m(x)} |e_j^*(x)|$. Then, since $A_m(z) = D$,

$$\|z - \mathscr{C}\mathscr{G}_m(z)\| = \left\|x - \mathscr{G}_m(x) + \sum_{i\in D} f_i e_i\right\|,$$

for some scalars $(f_i)_{i\in D}$ given by the Chebyshev approximation. Then,

$$\|x - \mathscr{G}_m(x)\| \leq K_b\left\|x - \mathscr{G}_m(x) + \sum_{i\in D} f_i e_i\right\| \leq K_b C_s \sigma_m(z) \leq K_b C_s\|x + (\delta+\alpha)\mathbf{1}_D\|.$$

Taking $\delta \to 0$,

$$\|x - \mathscr{G}_m(x)\| \leq K_b C_s\|x + \alpha\mathbf{1}_D\| \leq K_b C_s(\|x\| + \|\alpha\mathbf{1}_D\|). \tag{11}$$

Select now $y := \sum_{j\in A_m(x)}(e_j^*(x)+\delta\varepsilon_j)e_j + \sum_{j\in A_m^c(x)} e_j^*(x)e_j + \alpha\mathbf{1}_D$, with $\delta>0$ and $\varepsilon_j = \operatorname{sgn}(e_j^*(x))$ for $j\in A_m(x)$. Then, since $\mathcal{G}_m(y) = \sum_{j\in A_m(x)}(e_j^*(x)+\delta\varepsilon_j)e_j$, using Chebyshev approximation,

$$\|y - \mathcal{C}\mathcal{G}_m(y)\| = \left\| \sum_{j\in A_m(x)} a_i e_i + \sum_{j\in A_m^c(x)} e_j^*(x)e_j + \alpha\mathbf{1}_D \right\|.$$

Hence,

$$\begin{aligned}\|\alpha\mathbf{1}_D\| &\le 2K_b \left\| \sum_{j\in A_m(x)} a_i e_i + \sum_{j\in A_m^c(x)} e_j^*(x)e_j + \alpha\mathbf{1}_D \right\| \le 2K_bC_s\sigma_m(y)\\ &\le 2K_bC_s \left\| \sum_{j\in A_m(x)} (e_j^*(x)+\delta\varepsilon_j)e_j + \sum_{j\in A_m^c(x)} e_j^*(x)e_j \right\|.\end{aligned}$$

Taking $\delta\to 0$, $\|\alpha\mathbf{1}_D\| \le 2K_bC_s\|x\|$. Using the last inequality and (11),

$$\|x-\mathcal{G}_m(x)\| \le K_bC_s(\|x\| + 2K_bC_s\|x\|) \le 3(K_bC_s)^2\|x\|.$$

Thus, $\|x-\mathcal{G}_m(x)\| \le 3(K_bC_s)^2\|x\|$ for any finite $x\in\mathbb{X}$ and $m\le|\operatorname{supp}(x)|$.

For the general case, we take $x\in\mathbb{X}$ and $A_m(x)$ the greedy set of x with cardinality m. We can find a number $N\in\mathbb{N}$ such that $A_m(x)\subset\{1,...,N\}$. Then, since $\mathcal{G}_m(x) = \mathcal{G}_m(S_N(x))$, applying that $\mathscr{B}$ is Schauder and quasi-greedy for elements with finite support,

$$\begin{aligned}\|x-\mathcal{G}_m(x)\| &\le \|x-S_N(x)\| + \|S_N(x)-\mathcal{G}_m(x)\|\\ &= \|x-S_N(x)\| + \|S_N(x)-\mathcal{G}_m(S_N(x))\|\\ &\le 2K_b\|x\| + 3(K_bC_s)^2\|S_N(x)\|\\ &\le K_b(2+3(K_bC_s)^2)\|x\|.\end{aligned}$$

This completes the proof. □

Proof of Corollary 1.12. The proof follows using Theorem 1.10, Theorem 1.4 and Remark 1.6. □

Remark 3.1. In [2, Section 6-Question 3], the authors ask the following question: if a basis $\mathscr{B}$ satisfies Property (A) and the inequality (5), is $\mathscr{B}$ semi-greedy? We recall that $\mathscr{B}$ satisfies Property (A) if there is a positive constant C_a such that

$$\|x+\mathbf{1}_{\varepsilon A}\| \le C_a\|x+\mathbf{1}_{\eta B}\|,$$

for any $x\in\mathbb{X}$, A,B such that $|A|=|B|<\infty$, $A\cap B=\emptyset$, $(A\cup B)\cap\operatorname{supp}(x)=\emptyset$, $|\varepsilon|=|\eta|=1$ and $\max_j|e_j^*(x)|\le 1$. The answer is not due to the example in [1, Subsection 5.5] of a basis $\mathscr{B}$ in a Banach space such that $\mathscr{B}$ satisfies the Property (A) and (5), but is not quasi-greedy, hence is not almost-greedy and using Theorem 1.10, $\mathscr{B}$ is not semi-greedy.

4. Open questions

As discussed in [8] (see also [4]), one can define the Thresholding Greedy Algorithm and the Thresholding Chebyshev Greedy Algorithm in the context of Markushevich bases, that is, $\{e_i, e_i^*\}$ is a semi-normalized biorthogonal system, $\mathbb{X} = \overline{span\{e_i : i \in \mathbb{N}\}}^{\mathbb{X}}$ and $\mathbb{X}^* = \overline{span\{e_i^* : i \in \mathbb{N}\}}^{w^*}$. In section a) of Theorem 1.10, it is enough to work with Markushevich bases instead of Schauder bases. However, in the item b), seems to be necessarily to use that $\mathscr{B}$ is Schauder to prove the result.

Question 1: Is it possible to remove the condition to be Schauder in section b) of Theorem 1.10?

Another interesting problem is to establish if almost-greediness implies the condition to be Schauder. Of course, if $\mathscr{B}$ is greedy then $\mathscr{B}$ is Schauder since greediness implies unconditionality. As far as we know, all of examples of almost-greedy bases in the literature seem to be Schauder bases, but we don't know if almost-greediness implies that $\mathscr{B}$ is Schauder or not.

Question 2: If $\mathscr{B}$ is an almost-greedy Markushevich basis, is it necessarily Schauder in some order?

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