J. Math. Anal. Appl. 473 (2019) 786–802

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Models of Lie algebra $sl(2, \mathbb{C})$ and special matrix functions by means of a matrix integral transformation

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ARTICLE INFO

Article history:
Received 25 December 2017
Available online 7 January 2019
Submitted by L. Fialkow

Keywords:
Lie algebra $sl(2, \mathbb{C})$
Matrix functional calculus
Special matrix functions
Matrix integral transforms

ABSTRACT

In this paper, we present four models of irreducible representations of special complex Lie algebra $sl(2, \mathbb{C})$ from the special matrix functions point of view. These models, which involve differential operators, are transformed into matrix difference–differential operators using an integral transformation motivated by the integral representation of beta matrix function. We also obtain the matrix identities involving one or two variable special matrix functions.

1. Introduction

Lie theoretic techniques to find special functions identities are available in the literature. In particular, Govil and Manocha [4] have exploited models of Lie algebras $sl(2)$ and $\mathcal{G}(0,1)$ to find special function identities using an integral transformation, which is motivated by beta integral transformation. Also, Khan and Ali [8,9] used models of Lie algebras $\mathcal{G}(0,1)$ and $\mathcal{K}_5$ to obtain results involving Hermite polynomials and Laguerre–Hermite polynomials. In general, the use of Lie theoretic techniques provide a natural, unified and efficient framework for a general treatment of a wide range of special functions arising in mathematical physics.

Recently, models of certain Lie algebras have been studied from the special matrix polynomials point of view. Khan and her coworkers [10,11] have used the Lie algebraic techniques to find matrix identities involving the Hermite and Laguerre matrix polynomials. Jódar and Cortés [6] have considered the matrix analogue of Gauss hypergeometric function ${}_2F_1$ and gave its convergence conditions and integral representation. Authors [2] have found the convergence conditions of the generalized Gauss hypergeometric matrix function, Appell matrix functions and the two variable Kampé de Fériet matrix function. In addition, integral representations of Appell matrix functions were also determined. In the present paper, we apply the

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https://doi.org/10.1016/j.jmaa.2018.12.070

representation theory of the Lie algebra $sl(2,\mathbb{C})$ to matrix hypergeometric functions. In particular, we construct models of Lie algebra $sl(2,\mathbb{C})$ acting on the space of Gauss hypergeometric matrix function ${}_2F_1$. These models are in terms of differential operators. Using a matrix integral transformation, which is motivated by the integral representation of beta matrix function, we obtain another set of models of $sl(2,\mathbb{C})$ in terms of matrix difference–differential operators acting on the hypergeometric matrix function ${}_3F_2$. The whole exercise leads to new matrix identities involving one and two variable hypergeometric matrix functions. The section-wise treatment is as follows.

In Section 2, we review the part of matrix functional calculus that is needed in the sequel. In Section 3, we elaborate the representation theory of the special complex Lie algebra $sl(2,\mathbb{C})$. In Section 4, we introduce a matrix integral transformation that help us to upgrade the differential models of $sl(2,\mathbb{C})$ to matrix difference–differential models of $sl(2,\mathbb{C})$. In Sections 5–8, we present new models of Lie algebra $sl(2,\mathbb{C})$ and using the representation theory, find matrix identities, which are believed to be new.

2. Preliminaries

Let $\mathbb{C}^{r\times r}$ be the vector space of all $r\times r$ matrices with entries from $\mathbb{C}$. For $A\in\mathbb{C}^{r\times r}$, $\sigma(A)$ denotes the spectrum of A. The spectral abscissa of A is given by $\alpha(A)=\max\{\,\Re(z)\mid z\in\sigma(A)\,\}$, where $\Re(z)$ denotes the real part of $z\in\mathbb{C}$. If $\beta(A)=\min\{\,\Re(z)\mid z\in\sigma(A)\,\}$, then $\beta(A)=-\alpha(-A)$. A square matrix $A\in\mathbb{C}^{r\times r}$ is said to be positive stable if $\beta(A)>0$. If $f(z)$ and $g(z)$ are holomorphic functions of $z\in\mathbb{C}$, defined in an open set Ω of the complex plane, and $A\in\mathbb{C}^{r\times r}$ with $\sigma(A)\subset\Omega$, then using the matrix functional calculus [1], we have $f(A)g(A)=g(A)f(A)$. Furthermore, if $B\in\mathbb{C}^{r\times r}$ such that $\sigma(B)\subset\Omega$ and $AB=BA$, then $f(A)g(B)=g(B)f(A)$.

The reciprocal gamma function $\Gamma^{-1}(z)=1/\Gamma(z)$ is an entire function of $z\in\mathbb{C}$. The image of $\Gamma^{-1}(z)$ acting on A, denoted by $\Gamma^{-1}(A)$, is a well defined matrix. If $A+nI$ is invertible for all integers $n\geq 0$, where I denotes the $r\times r$ identity matrix, then the reciprocal gamma matrix function is defined as [5]

$$\Gamma^{-1}(A)=A(A+I)\cdots(A+(n-1)I)\Gamma^{-1}(A+nI),\ n\geq 1. \tag{2.1}$$

The matrix analogue of Pochhammer symbol $(A)_n$, $A\in\mathbb{C}^{r\times r}$, is given by

$$(A)_n=\begin{cases} I, & \text{if } n=0\\ A(A+I)\cdots(A+(n-1)I), & \text{if } n\geq 1.\end{cases} \tag{2.2}$$

This gives

$$(A)_n=\Gamma^{-1}(A)\,\Gamma(A+nI),\qquad n\geq 1. \tag{2.3}$$

We shall use the notation $\Gamma\begin{pmatrix}A_1,\dots,A_p\\ B_1,\dots,B_q\end{pmatrix}$ for $\Gamma(A_1)\cdots\Gamma(A_p)\,\Gamma^{-1}(B_1)\cdots\Gamma^{-1}(B_q)$.

If $A\in\mathbb{C}^{r\times r}$ be such that $\Re(z)>0$ for all eigenvalues z of A, then $\Gamma(A)$ can be expressed as [5]

$$\Gamma(A)=\int_0^\infty e^{-t}\,t^{A-I}\,dt. \tag{2.4}$$

Furthermore, if P and Q are positive stable matrices in $\mathbb{C}^{r\times r}$, then the beta matrix function is defined as

$$\mathfrak{B}(P,Q)=\int_0^1 t^{P-I}\,(1-t)^{Q-I}dt. \tag{2.5}$$

If $PQ = QP$, then we have [6]

$$\mathfrak{B}(P,Q) = \Gamma(P)\Gamma(Q)\Gamma^{-1}(P+Q). \tag{2.6}$$

Following special matrix functions will be used throughout this work. The generalized hypergeometric matrix function is defined as

$$ {}_pF_q(A_1,\ldots,A_p;B_1,\ldots,B_q;z) = \sum_{n\geq 0}(A_1)_n\cdots(A_p)_n\,(B_1)_n^{-1}\cdots(B_q)_n^{-1}\,\frac{z^n}{n!}. \tag{2.7}$$

The Appell matrix functions F_1, F_2, F_3, F_4 are defined by

$$F_1[A,B,B';C;x,y] = \sum_{m,n\geq 0}(A)_{m+n}(B)_m(B')_n(C)_{m+n}^{-1}\,\frac{x^m\,y^n}{m!\,n!}, \tag{2.8}$$

$$F_2[A,B,B';C,C';x,y] = \sum_{m,n\geq 0}(A)_{m+n}(B)_m(B')_n(C)_m^{-1}(C')_n^{-1}\,\frac{x^m\,y^n}{m!\,n!}, \tag{2.9}$$

$$F_3[A,A',B,B';C;x,y] = \sum_{m,n\geq 0}\frac{(A)_m(A')_n(B)_m(B')_n(C)_{m+n}^{-1}}{m!\,n!}\,x^m\,y^n, \tag{2.10}$$

$$F_4[A,B;C,C';x,y] = \sum_{m,n\geq 0}\frac{(A)_{m+n}(B)_{m+n}(C)_m^{-1}(C')_n^{-1}}{m!\,n!}\,x^m\,y^n. \tag{2.11}$$

The Kampé de Fériet matrix function is defined as

$$\begin{aligned} &F^{m_1:n_1,n_1'}_{m_2:n_2,n_2'}\left(\begin{matrix} A: & B, & C \\ D: & E, & F\end{matrix};x,\ y\right)\\ &= \sum_{m,n\geq 0}\prod_{i=1}^{m_1}(A_i)_{m+n}\prod_{i=1}^{n_1}(B_i)_m\prod_{i=1}^{n_1'}(C_i)_n\prod_{i=1}^{m_2}(D_i)_{m+n}^{-1}\prod_{i=1}^{n_2}(E_i)_m^{-1}\prod_{i=1}^{n_2'}(F_i)_n^{-1}\,\frac{x^m\,y^n}{m!\,n!}, \end{aligned} \tag{2.12}$$

where A abbreviates the sequence of matrices A_1, $\ldots$, A_{m_1}, etc. For convergence of these special matrix functions, see [2].

3. Lie algebra $sl(2,\mathbb{C})$

The complex Lie algebra $sl(2,\mathbb{C}) = L[SL(2,\mathbb{C})]$, the Lie algebra of the complex Lie group

$$SL(2,\mathbb{C}) = \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} : a,b,c,d\in\mathbb{C}, ad-bc=1\right\}, \tag{3.1}$$

consists of all 2×2 matrices

$$\alpha = \begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & -\alpha_1\end{pmatrix},\quad \alpha_1,\alpha_2,\alpha_3\in\mathbb{C}, \tag{3.2}$$

having [12]

$$g^+ = \begin{pmatrix}0 & -1\\ 0 & 0\end{pmatrix},\quad g^- = \begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix},\quad g^0 = \begin{pmatrix}1/2 & 0\\ 0 & -1/2\end{pmatrix} \tag{3.3}$$

as its basis.

These basis elements obey the commutation relations

$$[g^0, g^{\pm}] = \pm g^{\pm}, \qquad [g^+, g^-] = 2g^0. \tag{3.4}$$

Let ρ be a representation of $sl(2, \mathbb{C})$ on the complex vector space V and let

$$J^+ = \rho(g^+), \quad J^- = \rho(g^-), \quad J^0 = \rho(g^0). \tag{3.5}$$

Then the linear operators J^+, J^-, J^0 obey the commutation relations (3.4).

The operator

$$\mathcal{C} = J^+J^- + J^0J^0 - J^0 \tag{3.6}$$

commutes with every operator $\rho(\alpha)$, $\alpha \in sl(2, \mathbb{C})$.

For complex numbers α, β, γ where γ is neither zero nor a negative integer, Rainville [14] has shown that the hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; x)$ satisfy exactly 12 differential recurrence relations which allows one to raise or lower the parameters α, β, γ. For example

$$(x\partial_x + \alpha)\, {}_2F_1(\alpha, \beta; \gamma; x) = \alpha\ {}_2F_1(\alpha + 1, \beta; \gamma; x),$$

raise the parameter α. However, this operator depends upon α leading to the computation of quantities such as $D^n(\alpha)\ {}_2F_1$ or $[\exp D(\alpha)]\ {}_2F_1$ rather complicated. To overcome this, Miller [13] introduced new variables s, u, t associated with the parameters α, β, γ respectively and defined functions $f_{\alpha\beta\gamma}(s, u, t, x) = {}_2F_1(\alpha, \beta; \gamma; x)\, s^\alpha u^\beta t^\gamma$. This leads to 12 differential operators independent of the parameters α, β, γ.

Analogously, we define matrix functions f_{ABC} by

$$\begin{aligned} f_{ABC}(s, u, t, z) &= {}_2F_1(A, B; C; z)\ s^A\ u^B\ t^C \\ &= \sum_{n=0}^{\infty} \frac{(A)_n(B)_n(C)_n^{-1}}{n!}\ z^n\ s^A\ u^B\ t^C, \end{aligned}$$

where A, B, C are commuting matrices in $\mathbb{C}^{r\times r}$ and $C + kI$ is invertible for all integers $k \geq 0$. It can be easily verified that the differential operators

$$\begin{aligned} E^A &= s[z\partial_z + s\partial_s], \\ E_A &= s^{-1}[z(1-z)\partial_z + t\partial_t - s\partial_s - zu\partial_u], \\ E^B &= s[z\partial_z + u\partial_u], \\ E_B &= u^{-1}[z(1-z)\partial_z + t\partial_t - u\partial_u - zs\partial_s], \\ E^C &= t[(1-z)\partial_z - s\partial_s - u\partial_u + t\partial_t], \\ E_C &= t^{-1}[-z\partial_z - t\partial_t + 1], \\ E^{AC} &= st[(1-z)\partial_z - s\partial_s], \\ E_{AC} &= s^{-1}t^{-1}[z(1-z)\partial_z - zu\partial_u + t\partial_t - 1], \\ E^{BC} &= ut[(1-z)\partial_z - u\partial_u], \\ E_{BC} &= u^{-1}t^{-1}[z(1-z)\partial_z - zs\partial_s + t\partial_t - 1], \\ E^{ABC} &= sut\partial_z, \\ E_{ABC} &= s^{-1}u^{-1}t^{-1}[z(z-1)\partial_z + zs\partial_s + zu\partial_u - t\partial_t - z + 1], \end{aligned} \tag{3.7}$$

satisfy the relations

$$\begin{aligned}
&E^A f_{ABC} = A f_{A+I\,B\,C},\\
&E_A f_{ABC} = (C-A) f_{A-I\,B\,C},\\
&E^B f_{ABC} = B f_{A\,B+I\,C},\\
&E_B f_{ABC} = (C-B) f_{A\,B-I\,C},\\
&E^C f_{ABC} = (C-A)(C-B)C^{-1} f_{A\,B\,C+I},\\
&E_C f_{ABC} = -(C-I) f_{A\,B\,C-I},\\
&E^{AC} f_{ABC} = A(B-C)C^{-1} f_{A+I\,B\,C+I},\\
&E_{AC} f_{ABC} = (C-I) f_{A-I\,B\,C-I},\\
&E^{BC} f_{ABC} = (A-C)BC^{-1} f_{A\,B+I\,C+I},\\
&E_{BC} f_{ABC} = (C-I) f_{A\,B-I\,C-I},\\
&E^{ABC} f_{ABC} = ABC^{-1} f_{A+I\,B+I\,C+I},\\
&E_{ABC} f_{ABC} = -(C-I) f_{A-I\,B-I\,C-I}.
\end{aligned} \tag{3.8}$$

Finally the operators

$$J_A = s\partial_s, \quad J_B = u\partial_u, \quad J_C = t\partial_t \tag{3.9}$$

satisfy

$$J_A\, f_{ABC} = A\, f_{ABC}, \quad J_B\, f_{ABC} = B\, f_{ABC}, \quad J_C\, f_{ABC} = C\, f_{ABC}. \tag{3.10}$$

Let V be the vector space having basis

$$\{f_{A+kI\,B+lI\,C+mI} \mid k,l,m \in \mathbb{Z}\}.$$

We note that from the operators (3.7) and (3.9), each of the triplet *viz.*

$$\begin{aligned}
&\{J^+, J^-, J^0\} \equiv \{E^A, E_A,\, J_A - \frac{1}{2}J_C\},\ \{E^B, E_B,\, J_B - \frac{1}{2}J_C\}\\
&\{E^C, E_C,\, J_C - \frac{1}{2}J_A - \frac{1}{2}J_B - \frac{1}{2}\},\ \{E^{AC}, E_{AC},\, \frac{1}{2}J_C + \frac{1}{2}J_A - \frac{1}{2}J_B - \frac{1}{2}\}\\
&\{E^{BC}, E_{BC},\, \frac{1}{2}J_C - \frac{1}{2}J_A + \frac{1}{2}J_B - \frac{1}{2}\},\ \{E^{ABC}, E_{ABC},\, \frac{1}{2}J_A + \frac{1}{2}J_B - \frac{1}{2}\}
\end{aligned} \tag{3.11}$$

satisfy the commutation relations (3.4) and hence give us a representation of the Lie algebra $sl(2,\mathbb{C})$. Among these six models of representation of $sl(2,\mathbb{C})$ given in (3.11), precisely four models *viz.* [15]

$$\begin{aligned}
&\{E^A, E_A,\, J_A - \frac{1}{2}J_C\}, \quad \{E^C, E_C,\, J_C - \frac{1}{2}J_A - \frac{1}{2}J_B - \frac{1}{2}\},\\
&\{E^{AC}, E_{AC},\, \frac{1}{2}J_C + \frac{1}{2}J_A - \frac{1}{2}J_B - \frac{1}{2}\}, \quad \{E^{ABC}, E_{ABC},\, \frac{1}{2}J_A + \frac{1}{2}J_B - \frac{1}{2}\}
\end{aligned} \tag{3.12}$$

are distinct and are capable of obtaining interesting matrix function identities.

4. Integral transforms of matrix functions

In this section, we introduce an integral transformation based on the integral representation of beta matrix function [5].

For positive stable matrices B', C' and $C' - B'$, define

$$h(B', C', x) = \mathbf{I}[f(vx)] = \Gamma\left(\begin{matrix} C' \\ B', C' - B' \end{matrix}\right) \int_0^1 v^{B'-I} (1-v)^{C'-B'-I} f(vx)\, dv. \tag{4.1}$$

From (4.1) we have the following transforms:

$$\begin{aligned}
\mathbf{I}[vf(vx)] &= B'\, C'^{-1}\, E_{B'C'}\; h(B', C', x), \\
\mathbf{I}[\partial_v f(vx)] &= (C' - I)\; L_{C'} \bigtriangledown h(B', C', x) \\
&= (B' - I)^{-1}(C' - I)\, L_{B'C'}\, x\partial_x\, h(B', C', x), \text{ provided } B' - I \text{ is invertible,} \\
\mathbf{I}[v\partial_v f(vx)] &= B' \bigtriangleup h(B', C', x) \\
&= x\, \partial_x\, h(B', C', x), \\
\mathbf{I}[v^2\partial_v f(vx)] &= B'\, (B' + I)\, C'^{-1} \bigtriangleup E_{B'C'}\, h(B', C', x) \\
&= B'\, C'^{-1}\, E_{B'C'}\, x\partial_x\; h(B', C', x),
\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}
E_{B'} h(B', C', x) &= h(B' + I, C', x), \\
L_{B'} h(B', C', x) &= h(B' - I, C', x), \\
E_{B'C'} &= E_{B'}\, E_{C'}, \\
L_{B'C'} &= L_{B'}\, L_{C'}, \\
\bigtriangleup &= E_{B'} - I, \\
\bigtriangledown &= I - L_{B'}.
\end{aligned} \tag{4.3}$$

5. Models of representation $D\left(-\frac{1}{2}C, A\right)$

In this section, we have given two models of irreducible representation of $sl(2, \mathbb{C})$ connected via an integral transformation defined in (4.1). We also obtain the matrix identities involving the Kampé de Fériet matrix function and hypergeometric matrix functions ${}_3F_2$ and ${}_2F_1$.

Let A, B, B', B'', C, C' and C'' be commuting matrices in $\mathbb{C}^{r\times r}$ such that each $C + kI$, $C' + kI$ and $C'' + kI$ are invertible for all integers $k \geq 0$. Then, as suggested by (3.12), we give models of the irreducible representations of $sl(2, \mathbb{C})$, explicitly, along with the group action of $SL(2, \mathbb{C})$. We begin with the model arising from the triplet $\{E^A, E_A,\ J_A - \frac{1}{2}J_C\}$. Consider the following model of representation $D\left(-\frac{1}{2}C, A\right)$ of $sl(2, \mathbb{C})$:

$$\begin{aligned}
J^+ &= s(z\partial_z + s\partial_s), \\
J^- &= s^{-1}(z(1 - z)\partial_z + t\partial_t - s\partial_s - zu\partial_u), \\
J^0 &= s\partial_s - \frac{1}{2}t\partial_t
\end{aligned} \tag{5.1}$$

with basis functions as

$$f_\lambda = {}_2F_1(\lambda, B; C; z)\, s^\lambda u^B t^C, \quad \lambda = A,\, A \pm I,\, \ldots. \tag{5.2}$$

The action of the J-operators on f_λ is given by

$$\begin{aligned} &J^+ f_\lambda = \lambda f_{\lambda+I}, \qquad J^- f_\lambda = (C-\lambda) f_{\lambda-I}, \\ &J^0 f_\lambda = \left(\lambda - \frac{1}{2}C\right) f_\lambda, \qquad (J^+J^- + J^0J^0 - J^0) f_\lambda = \frac{1}{2}C\left(\frac{1}{2}C - I\right) f_\lambda. \end{aligned} \tag{5.3}$$

Indeed,

$$[J^0, J^\pm] = \pm J^\pm, \qquad [J^+, J^-] = 2J^0. \tag{5.4}$$

We now determine the multiplier representation $[T(g)f](s,u,t,z)$ on the space $\mathcal{F}$ of all matrix functions analytic in the neighborhood of $(s_0, u_0, t_0, 0)$. The group action of $SL(2,\mathbb{C})$ is given in terms of the Lie algebra action by

$$T(g) = \exp\left(-\frac{b}{d}J^+\right)\exp\left(-cdJ^-\right)\exp\left(\tau J^0\right), \quad e^{\tau/2} = d^{-1}, \tag{5.5}$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}), \quad ad - bc = 1.$$

The multiplier representation induced by the J-operators (5.1) is given by

$$T_1(g)f(s,u,t,z) = f\left(\frac{as+c}{d+bs}, \frac{u(as+c)}{as+c(1-z)}, \frac{ts}{as+c}, \frac{zs}{(d+bs)(as+c(1-z))}\right). \tag{5.6}$$

To obtain a model of $\mathrm{D}\left(-\frac{1}{2}C, A\right)$ in terms of matrix difference–differential operators, we put $z = vx$, $v \in \mathbb{R}$. As a result, (5.1) turn into

$$\begin{aligned} &J^+ = s(v\partial_v + s\partial_s), \\ &J^- = s^{-1}\left(v(1-vx)\partial_v + t\partial_t - s\partial_s - vxu\partial_u\right), \\ &J^0 = s\partial_s - \frac{1}{2}t\partial_t, \quad f_\lambda(vx) = {}_2F_1(\lambda, B; C; vx)\, s^\lambda u^B t^C, \quad \lambda = A,\, A \pm I,\, \ldots. \end{aligned} \tag{5.7}$$

Theorem 5.1. *Let ρ be the irreducible representation of $sl(2,\mathbb{C})$ on the representation space V having basis functions $\{f_\lambda \mid \lambda \in S\}$ in terms of Lie algebra operators $\{J^+, J^-, J^0\}$. Then the transformation $\boldsymbol{I}$ induces another irreducible representation ρ of $sl(2,\mathbb{C})$ on the representation space $W = \boldsymbol{I}V$ having basis functions $\{h_\lambda = \boldsymbol{I}\, f_\lambda \mid \lambda \in S\}$ in terms of Lie algebra operators $\{K^+ = \boldsymbol{I}J^+\boldsymbol{I}^{-1},\ K^- = \boldsymbol{I}J^-\boldsymbol{I}^{-1},\ K^0 = \boldsymbol{I}J^0\boldsymbol{I}^{-1}\}$.*

The proof of the above theorem follows from the fact that if the operators $\{J^+, J^-, J^0\}$ and matrix functions f_λ satisfy equations (5.3) and (5.4), then so does the operators $\{K^+, K^-, K^0\}$ and matrix functions h_λ.

Using Theorem 5.1 we can transform the model (5.7) as

$$K^+ = \mathbf{I}J^+\mathbf{I}^{-1} = s(x\partial_x + s\partial_s),$$

$$K^{-} = \mathbf{I}J^{-}\mathbf{I}^{-1} = s^{-1}(x(I - B'C'^{-1}xE_{B'C'})\partial_x + t\partial_t - s\partial_s - B'C'^{-1}xE_{B'C'}u\partial_u),$$
$$K^{0} = \mathbf{I}J^{0}\mathbf{I}^{-1} = s\partial_s - \frac{1}{2}t\partial_t, \tag{5.8}$$

with basis functions

$$h_\lambda(B', C', x) = \mathbf{I}[f_\lambda(vx)] = {}_3F_2\,(\lambda, B, B'; C, C'; x)\, s^\lambda u^B t^C, \quad \lambda = A,\ A \pm I, \ldots. \tag{5.9}$$

The K-operators (5.8) and basis functions h_λ satisfy (5.3) as well as (5.4).

The multiplier representation induced by the K-operators on the space $\mathcal{W} = \mathbf{I}\mathcal{F}$ is

$$[T_1'(g)h](B', C', x) = [\mathbf{I}\ T_1(g)\ \mathbf{I}^{-1} f](s, u, t, z). \tag{5.10}$$

To obtain the identities, it is easy to verify that the matrix function

$$F = F_2(A, B'', B; C'', C, s, z)\, s^A u^B t^C \tag{5.11}$$

satisfies

$$(J^+J^- + J^0J^0 - J^0)F = \frac{1}{2}C\left(\frac{1}{2}C - I\right)F \tag{5.12}$$

as well as

$$[J^0J^0 - J^+J^0 - (B'' + \frac{1}{2}C - A)J^+ - (I{+}2A - C - C'')J^0]F$$
$$= (A - \frac{1}{2}C)(C'' + \frac{1}{2}C - A - I)F. \tag{5.13}$$

In addition, the function

$$G = (1 - s)^{-A}\ {}_2F_1\left(A, B; C; \frac{z}{1 - s}\right) s^A u^B t^C \tag{5.14}$$

satisfies both

$$(J^+J^- + J^0J^0 - J^0)G = \frac{1}{2}C\left(\frac{1}{2}C - I\right)G \tag{5.15}$$

and

$$\left[J^0J^0 - J^+J^0 - \left(\frac{1}{2}C - A\right)J^+ - (I + 2A - C)J^0\right]G = (A - \frac{1}{2}C)(\frac{1}{2}C - A - I)G. \tag{5.16}$$

Therefore the matrix function

$$H_1 = F^{1:1;2}_{0:1;2}\left(\begin{matrix} A : B''; B, B'; \\ - : C''; C, C'; \end{matrix}\ s, x\right) s^A u^B t^C \tag{5.17}$$

satisfies both

$$(K^+K^- + K^0K^0 - K^0)H_1 = \frac{1}{2}C\left(\frac{1}{2}C - I\right)H_1 \tag{5.18}$$

as well as

$$[K^0K^0 - K^+K^0 - (B'' + \frac{1}{2}C - A)K^+ - (I + 2A - C - C'')K^0]H_1$$
$$= (A - \frac{1}{2}C)(C'' + \frac{1}{2}C - A - I)H_1, \tag{5.19}$$

and the function

$$H_2 = (1-s)^{-A}\,{}_3F_2\left(A, B, B'; C, C'; \frac{x}{1-s}\right) s^A u^B t^C \tag{5.20}$$

satisfies

$$(K^+K^- + K^0K^0 - K^0)H_2 = \frac{1}{2}C\left(\frac{1}{2}C - I\right) H_2 \tag{5.21}$$

and

$$[K^0K^0 - K^+K^0 - (\frac{1}{2}C - A)K^+ - (I{+}2A - C)J^0]H_2$$
$$= (A - \frac{1}{2}C)(\frac{1}{2}C - A - I)H_2. \tag{5.22}$$

This leads to the expansions [7,16]

$$T_1'(g)H_i = \sum_{n=-\infty}^{\infty} R_{in}(g)h_{A+nI},\ \ i = 1, 2. \tag{5.23}$$

Theorem 5.2. *Let A, B, B', B'', C, C' and C'' be commuting matrices in $\mathbb{C}^{r\times r}$ such that $\alpha(A) < 1$, $\alpha(B'') < \beta(C''), \alpha(B) + \alpha(B') < \beta(C) + \beta(C')$. Then the following matrix generating function holds*

$$F^{1:2;1}_{0:2;1}\left(\begin{matrix} A : B, B'; B'' \\ - : C, C'; C''\end{matrix}; \frac{x}{1-s}, \frac{-ws}{1-s}\right)(1-s)^{-A}$$
$$= \sum_{n=0}^{\infty} \frac{(A)_n}{n!}\ {}_2F_1\left(-nI, B''; C''; w\right)\ {}_3F_2\left(A+nI, B, B'; C, C'; x\right) s^n,$$
$$|s| < 1,\ \ |x/(1-s)| + |ws/(1-s)| < 1. \tag{5.24}$$

Proof. Using (5.10) and (5.23) for $i = 1$, we get

$$F^{1:2;1}_{0:2;1}\left(\begin{matrix} A : B, B'; B'' \\ - : C, C'; C''\end{matrix}; \frac{xs}{(d+bs)(as+c(1-x))}, \frac{as+c}{d+bs}\right)\left(\frac{as+c}{d+bs}\right)^A$$
$$\times \left(\frac{u(as+c)}{as+c(1-x)}\right)^B \left(\frac{ts}{as+c}\right)^C$$
$$= \sum_{n=0}^{\infty} R_{1n}(g)\, h_{A+nI}. \tag{5.25}$$

To obtain the matrix element $R_{1n}(g)$, put $x = 0$, $a = 1$, $d = 1$, $c = 0$ and using the commutativity of matrices, we get

$$R_{1n}(g) = \frac{(A)_n}{n!}\,{}_2F_1(-nI, B''; C''; w), \tag{5.26}$$

where $w = \frac{1}{b}$. Putting $R_{1n}(g)$ in (5.25) gives the required identity (5.24). □

Theorem 5.3. *Let A, B, B', C, C' be commuting matrices in $\mathbb{C}^{r\times r}$ such that $\alpha(A) < \beta(C), \alpha(B) + \alpha(B') < \beta(C')$. Then the following matrix generating function holds*

$$\begin{aligned}
&\left(1+\frac{c}{s}\right)^{A-C} (1-c-s)^{-A}\; F^{2:\,1;\,0}_{1:\,1;\,0}\left(\begin{matrix} B,\ B' : A; - \\ C' : C; - \end{matrix}\,;\, \frac{xs}{(s+c)(1-c-s)},\, \frac{cx}{s+c}\right) \\
&= \sum_{n=-\infty}^{\infty} \frac{\Gamma(A+nI)\Gamma^{-1}(A)}{\Gamma(n+1)}\; {}_2F_1\left(A+nI,\ A-C+(n+1)I; (n+1)I;\ c\right) \\
&\quad \times\; {}_3F_2\left(A+nI, B, B'; C, C'; x\right) s^n, \\
&\qquad \left|\frac{c}{s}\right| < 1,\ |c+s| < 1,\ \left|\frac{xs}{(s+c)(1-c-s)}\right| + \left|\frac{xc}{(s+c)}\right| < 1.
\end{aligned} \tag{5.27}$$

Proof. Using (5.10), (5.23) for $i = 2$, and proceeding as in the above theorem, with $a = 1$, $d = 1$, $b = 0$, the identity (5.27) can be easily obtained. □

In (5.27), the terms corresponding to $n = -1, -2, \ldots,$ are well defined in view of the relation

$$\begin{aligned}
&\lim_{k\to -l} \frac{1}{\Gamma(k+1)}\; {}_2F_1\left(A+kI,\ A-C+(k+1)I; (k+1)I;\ c\right) \\
&\quad = \frac{(A-lI)_l(A-C-(l-1)I)_l}{l!}\; {}_2F_1\left(A,\ A-C+I; (l+1)I;\ c\right) c^l.
\end{aligned} \tag{5.28}$$

6. Models of representation $D(C, -\frac{1}{2}(A+B+I))$

This section consists of two models of representation $D(C, -\frac{1}{2}(A+B+I))$ of Lie algebra $sl(2,\mathbb{C})$ connected via the integral transformation (4.1). We also give the corresponding matrix generating functions.

Consider the model arising from the triplet $\{E^C, E_C,\ J_C - \frac{1}{2}J_A - \frac{1}{2}J_B - \frac{1}{2}\}$, given by

$$\begin{aligned}
J^+ &= t((1-z)\partial_z + t\partial_t - s\partial_s - u\partial_u), \\
J^- &= -t^{-1}(z\partial_z + t\partial_t - 1), \\
J^0 &= t\partial_t - \frac{1}{2}s\partial_s - \frac{1}{2}u\partial_u - \frac{1}{2}, \\
f_\lambda &= {}_2F_1\left(A, B; \lambda; z\right) s^A u^B t^\lambda, \quad \lambda = C,\ C \pm I, \ldots.
\end{aligned} \tag{6.1}$$

The action of the J-operators on f_λ is

$$\begin{aligned}
&J^+ f_\lambda = \lambda^{-1}(\lambda - A)(\lambda - B) f_{\lambda+I}, \qquad J^- f_\lambda = -(\lambda - I) f_{\lambda - I}, \\
&J^0 f_\lambda = \left(\lambda - \frac{1}{2}(A+B+I)\right) f_\lambda, \\
&(J^+J^- + J^0J^0 - J^0) f_\lambda = \frac{1}{4}(A-B+I)(A-B-I) f_\lambda.
\end{aligned} \tag{6.2}$$

Indeed the J-operators satisfy (5.4).

The operators (6.1) induce the following multiplier representation:

$$T_2(g)f(s,u,t,z) \\ = \left(a+\frac{c}{t}\right)^{-1} f\left(s(d+bt), u(d+bt), \frac{c+at}{d+bt}, \frac{(c+at)(d+bt)}{t}\left(z-\frac{bt}{d+bt}\right)\right). \tag{6.3}$$

Putting $z = vx$ in (6.1) and using Theorem 5.1, the following K-model is obtained:

$$\begin{aligned} K^+ &= t((C'-I)(B'-I)^{-1}L_{B'C'}\partial_x - x\partial_x + t\partial_t - s\partial_s - u\partial_u), \\ K^- &= -t^{-1}(x\partial_x + t\partial_t - 1), \\ K^0 &= t\partial_t - \frac{1}{2}s\partial_s - \frac{1}{2}u\partial_u - \frac{1}{2}, \end{aligned} \tag{6.4}$$

$$h_\lambda(B',C',x) = {}_3F_2\,(A,B,B';\lambda,C';x)\,s^A u^B t^\lambda, \quad \lambda = C,\ C\pm I,\dots. \tag{6.5}$$

The matrix operators (6.4) with basis functions (6.5) satisfy (5.4) and

$$\begin{aligned} &K^+h_\lambda = \lambda^{-1}(\lambda-A)(\lambda-B)h_{\lambda+I}, \qquad K^-h_\lambda = -(\lambda-I)h_{\lambda-I}, \\ &K^0h_\lambda = \left(\lambda - \frac{1}{2}(A+B+I)\right)h_\lambda, \\ &(K^+K^- + K^0K^0 - K^0)h_\lambda = \frac{1}{4}(A-B+I)(A-B-I)h_\lambda. \end{aligned} \tag{6.6}$$

The multiplier representation induced by the matrix operators (6.4) on the space $\mathcal{W} = \mathbf{I}\mathcal{F}$ is

$$[T_2'(g)h](B',C',x) = [\mathbf{I}\ T_2(g)\ \mathbf{I}^{-1}f](s,u,t,z). \tag{6.7}$$

The matrix function

$$F = F_3(A,A'',B,B'';C;z,t)s^A u^B t^C \tag{6.8}$$

satisfies

$$(J^+J^- + J^0J^0 - J^0)F = \frac{1}{4}(A-B+I)(A-B-I)F \tag{6.9}$$

as well as

$$\left[J^0J^0 + J^-J^0 + \left(\frac{A+B+I}{2} - C\right)J^- - (2C-A-B-A''-B''-I)J^0\right]F \\ = \left[\frac{A+B+I}{2}(C+I) - \left(\frac{A+B+I}{2}-C\right)\left(A''+B''+I-\frac{A+B+I}{2} - A''B''\right)\right]F. \tag{6.10}$$

Therefore

$$H = F^{0:3;2}_{1:1;0}\left(\begin{matrix} -: A,B,B';A'',B''; \\ C:C';-; \end{matrix}\ x,t\right) s^A u^B t^C \tag{6.11}$$

satisfies both

$$(K^+K^- + K^0K^0 - K^0)H = \frac{1}{4}(A-B+I)(A-B-I)H \tag{6.12}$$

and

$$\begin{aligned}&\left[K^0K^0+K^-K^0+\left(\frac{A+B+I}{2}-C\right)K^- -(2C-A-B-A''-B''-I)K^0\right]H\\&=\left[\frac{A+B+I}{2}(C+I)-\left(\frac{A+B+I}{2}-C\right)\left(A''+B''+I-\frac{A+B+I}{2}-A''B''\right)\right]H. \qquad (6.13)\end{aligned}$$

Using the expansions

$$T_2(g)F=\sum_{n=-\infty}^{\infty} i_n(g) f_{C+nI}, \qquad (6.14)$$

$$T_2'(g)H=\sum_{n=-\infty}^{\infty} i_n(g) h_{C+nI} \qquad (6.15)$$

we can obtain matrix generating functions, stated in the following theorems.

Theorem 6.1. *Let A, A'', B, B'', C be commuting matrices in $\mathbb{C}^{r\times r}$ such that $\alpha(A'')+\alpha(B'')<2$, $\beta(C)>1$, $\alpha(A)+\alpha(B)<2$. Then the following matrix generating function holds*

$$\begin{aligned}&\left(1+\frac{c}{t}\right)^{C-I} F_3\left(A,A'',B,B'';C;\left(1+\frac{c}{t}\right)z,c+t\right)\\&=\sum_{n=-\infty}^{\infty}\Gamma\begin{pmatrix}A''+nI,B''+nI,C\\A'',B'',C+nI,(n+1)I\end{pmatrix}{}_2F_1\left(A''+nI,B''+nI;(n+1)I;c\right)\\&\quad\times{}_2F_1\left(A,B;C+nI;z\right)t^n,\ |c/t|<1,\ |c+t|<1,\ \max\{|(1+c/t)z|,\ |c+t|\}<1. \qquad (6.16)\end{aligned}$$

Theorem 6.2. *Let A, A'', B, B', B'', C, C' be commuting matrices in $\mathbb{C}^{r\times r}$ such that $\alpha(A'')+\alpha(B'')<2$, $\beta(C)>1$, $\alpha(A)+\alpha(B)+\alpha(B')<\beta(C')$. Then the following matrix generating function holds*

$$\begin{aligned}&\left(1+\frac{c}{t}\right)^{C-I} F^{0:3;2}_{1:1;0}\left(\begin{matrix}-:A,B,B';A'',B''\\C:C';-\end{matrix}\ \left(1+\frac{c}{t}\right)x,c+t\right)\\&=\sum_{n=-\infty}^{\infty}\Gamma\begin{pmatrix}A''+nI,B''+nI,C\\A'',B'',C+nI,(n+1)I\end{pmatrix}{}_2F_1\left(A''+nI,B''+nI;(n+1)I;c\right)\\&\quad\times{}_3F_2\left(A,B,B';C+nI,C';x\right)t^n,\\&\quad |c/t|<1,\ |c+t|<1,\ \max\{|(1+c/t)x|,\ |c+t|\}<1. \qquad (6.17)\end{aligned}$$

Using (6.3), (6.7), (6.14), (6.15) and proceeding exactly in the same manner as in Theorem 5.2, the matrix identities given in (6.16) and (6.17) can be easily obtained. The terms corresponding to $n=-1,-2,\dots$, are well defined in view of the relation (5.28).

7. Models of representation $D(A,C,-\frac{1}{2}A,-\frac{1}{2}(B+C+I))$

We now consider the model arising from the triplet $\{E^{AC},E_{AC},\ \frac{1}{2}J_C+\frac{1}{2}J_A-\frac{1}{2}J_B-\frac{1}{2}\}$. We have

$$\begin{aligned}J^+&=st((1-z)\partial_z-s\partial_s),\\J^-&=s^{-1}t^{-1}(z(1-z)\partial_z-zu\partial_u+t\partial_t-1),\end{aligned}$$

$$J^0 = \frac{1}{2}t\partial_t + \frac{1}{2}s\partial_s - \frac{1}{2}u\partial_u - \frac{1}{2}, \tag{7.1}$$

along with the basis functions

$$f_\lambda = {}_2F_1(A - C + \lambda,\, B;\, \lambda;\, z)s^{A-C+\lambda}\, u^B\, t^\lambda, \quad \lambda = C, C \pm I, \dots, \tag{7.2}$$

as a model of representation $D(A, C, -\frac{1}{2}A, -\frac{1}{2}(B + C + I))$.

Action of the J-operators (7.1) on basis functions (7.2) is given by

$$\begin{aligned}
&J^+ f_\lambda = \lambda^{-1}\,(A - C + \lambda)\,(B - \lambda) f_{\lambda+I},\\
&J^- f_\lambda = (\lambda - I) f_{\lambda-I},\\
&J^0 f_\lambda = \left(\lambda + \frac{1}{2}(A - B - C - I)\right) f_\lambda,\\
&(J^+J^- + J^0J^0 - J^0) f_\lambda = \frac{1}{4}(A + B - C + I)\,(A + B - C - I) f_\lambda.
\end{aligned} \tag{7.3}$$

The computation of the group action from the Lie derivatives (7.1) gives

$$[T_3(g)f](s, u, t, z) = \left(a - \frac{c}{st}\right)^{-1} f\left(\frac{s}{d - bst}, \frac{sut}{ast - cz}, at - \frac{c}{s}, \frac{(dz - bst)(ast - c)}{(d - bst)(ast - cz)}\right). \tag{7.4}$$

Using Theorem 5.1, the J-model (7.1) induces the following K-model:

$$\begin{aligned}
K^+ &= st\left((B' - I)^{-1}\,(C' - I)\, L_{B'C'}\partial_x - x\partial_x - s\partial_s\right),\\
K^- &= s^{-1}t^{-1}\left(x(I - B'C'^{-1}xE_{B'C'})\partial_x - B'C'^{-1}xE_{B'C'}u\partial_u + t\partial_t - 1\right),\\
K^0 &= \frac{1}{2}t\partial_t + \frac{1}{2}s\partial_s - \frac{1}{2}u\partial_u - \frac{1}{2},
\end{aligned} \tag{7.5}$$

along with the basis functions

$$h_\lambda(B', C', x) = {}_3F_2(A - C + \lambda,\, B,\, B';\, \lambda, C';\, z)s^{A-C+\lambda}\, u^B\, t^\lambda,\ \ \lambda = C, C \pm I, \dots. \tag{7.6}$$

Indeed the K-operators (7.5) satisfy (7.3). The multiplier representation induced by the K-operators (7.5) on the space $\mathcal{W} = \mathbf{I}\mathcal{F}$ is

$$[T_3'(g)h](B', C', x) = [\mathbf{I}\; T_3(g)\; \mathbf{I}^{-1} f](s, u, t, z). \tag{7.7}$$

The function

$$F = F_1(A, B, B''; C; z, st)s^A u^B t^C \tag{7.8}$$

satisfies

$$(J^+J^- + J^0J^0 - J^0)F = \frac{1}{4}((A + B - C)^2 - I)F. \tag{7.9}$$

It therefore follows that

$$H = F^{1:2;1}_{1:1;0}\left(\begin{array}{lccc} A: & B, B'; & B''; & \\ C: & C'; & -; & x, st \end{array}\right) s^A u^B t^C \tag{7.10}$$

satisfies

$$(K^+K^- + K^0K^0 - K^0)H = \frac{1}{4}((A+B-C)^2 - I)H. \tag{7.11}$$

The expansions

$$T_3(g)F = \sum_{n=-\infty}^{\infty} i_n(g) f_{C+nI}, \tag{7.12}$$

$$T_3'(g)H = \sum_{n=-\infty}^{\infty} i_n(g) h_{C+nI} \tag{7.13}$$

lead to the following matrix identities under special cases:

Theorem 7.1. *Let A, B, B'' and C be commuting matrices in $\mathbb{C}^{r\times r}$ such that $\alpha(A) < \beta(C)$, $\alpha(B) < 1$, $\alpha(B'') < 1$. Then the following matrix generating function holds*

$$\begin{aligned}
&(1-w)^{C-I}(1-wz)^{-B} F_1\left(A, B, B''; C; \frac{z(1-w)}{1-wz}, (1-w)y\right) \\
&= \sum_{n=-\infty}^{\infty} \Gamma\left(\begin{array}{c} A+nI, B''+nI, C \\ A, B'', C+nI, (n+1)I \end{array}\right) {}_2F_1(A+nI, B''+nI; (n+1)I; -wy) \\
&\quad \times {}_2F_1(A+nI, B; C+nI; z) y^n, \\
&\quad |w| < 1, |wz| < 1, \left|\frac{z(1-w)}{1-wz}\right| < 1, \ |(1-w)y| < 1.
\end{aligned} \tag{7.14}$$

Proof. Using Equations (7.4), (7.12) and particular values $a = 1$, $d = 1$, $b = 0$, the matrix elements $i_n(g)$ are given by

$$i_n(g) = \Gamma\left(\begin{array}{c} A+nI, B''+nI, C \\ A, B'', C+nI, (n+1)I \end{array}\right) {}_2F_1(A+nI, B''+nI; (n+1)I; -wy), \tag{7.15}$$

which leads to the matrix identity (7.14). □

Theorem 7.2. *Let A, B, B', B'', C and C' be commuting matrices in $\mathbb{C}^{r\times r}$ such that $\alpha(A) < \beta(C)$, $\alpha(B) + \alpha(B') < \beta(C') + 1$, $\alpha(B'') < 1$. Then the following matrix generating function holds*

$$\begin{aligned}
&(1-w)^{C-I} F^{(3)}\left(\begin{array}{cccc} - :: & A; -; & B, B' : & -; B''; - \\ - :: & C; -; & C' : & -; -; - \end{array}; x(1-w), y(1-w), wx\right) \\
&= \sum_{n=-\infty}^{\infty} \Gamma\left(\begin{array}{c} A+nI, B''+nI, C \\ A, B'', C+nI, (n+1)I \end{array}\right) {}_2F_1(A+nI, B''+nI; (n+1)I; -wy) \\
&\quad \times {}_3F_2(A+nI, B, B'; C+nI, C'; x) y^n, \\
&|w| < 1, \ |x(1-w)| + |wx| < 1, \ |y(1-w)| < 1,
\end{aligned} \tag{7.16}$$

where $F^{(3)}$ is a matrix analogue of general triple hypergeometric function, defined by

$$F^{(3)}\left(\begin{matrix} - :: & A;-; & B,B': & -;B'';- \\ - :: & C;-; & C': & -;-;- \end{matrix} ; x(1-w), y(1-w), wx\right)$$
$$= \sum_{m,n,p\geq 0} (A)_{m+n}(B)_{m+p}(B')_{m+p}(B'')_n(C)^{-1}_{m+n}(C')^{-1}_{m+p}$$
$$\times \frac{(x(1-w))^m(y(1-w))^n(wx)^p}{m!\,n!\,p!}. \tag{7.17}$$

Proof. Using Equations (7.7), (7.13) and the result

$$\mathbf{I}\left[(1-wz)^{-B}F_1(A,B,B'';C;\frac{z(1-w)}{1-wz},(1-w)y)\right] = F^{(3)}, \tag{7.18}$$

the matrix identity (7.16) can be obtained easily. □

The terms corresponding to $n = -1, -2, \dots,$ are well defined in view of the relation (5.28). For convergence of hypergeometric matrix functions of several variables, see [3].

8. Models of representation $D(A, B, C, -\frac{1}{2}(A + B + C))$

In this section, we give a model of irreducible representation of $sl(2,\mathbb{C})$ arising from the triplet $\{E^{ABC}, E_{ABC}, \frac{1}{2}J_A + \frac{1}{2}J_B - \frac{1}{2}\}$ and find the corresponding matrix generating function. Consider

$$\begin{aligned}
&J^+ = sut\partial_z,\\
&J^- = s^{-1}u^{-1}t^{-1}(z(z-1)\partial_z - t\partial_t + zs\partial_s + zu\partial_u - z + 1),\\
&J^0 = \frac{1}{2}(s\partial_s + u\partial_u - 1),\\
&f_\lambda = {}_2F_1(A-C+\lambda, B-C+\lambda;\lambda;z)s^{A-C+\lambda}\,u^{B-C+\lambda}\,t^\lambda,\ \lambda = C, C\pm I, \dots.
\end{aligned} \tag{8.1}$$

The action of the J-operators on f_λ is given by

$$\begin{aligned}
&J^+ = \lambda^{-1}(A-C+\lambda)(B-C+\lambda)f_{\lambda+I},\\
&J^- = -(\lambda - I)f_{\lambda-I},\\
&J^0 = \frac{1}{2}(2\lambda + A + B - 2C - I)f_\lambda,\\
&(J^+J^- + J^0J^0 - J^0)f_\lambda = \frac{1}{4}(A-B+I)(A-B-I)f_\lambda.
\end{aligned} \tag{8.2}$$

The multiplier representation T_4 of $SL(2,\mathbb{C})$ induced by the J-operators (8.1) is:

$$[T_4(g)f](s,u,t,z) = \left(a + \frac{c(1-z)}{sut}\right)^{-1}$$
$$\times f\left(as - \frac{cz}{ut}, au - \frac{cz}{st}, t\left(\frac{asut + c - cz}{asut - cz}\right), (zd - bsut)\left(a - \frac{c(z-1)}{sut}\right)\right). \tag{8.3}$$

Putting $z = vx$ in the model (8.1) and using the transformation $\mathbf{I}$, we get the following transformed model:

$$\begin{aligned}
&K^+ = sut(B'-I)^{-1}(C'-I)L_{B'C'}\partial_x,\\
&K^- = s^{-1}u^{-1}t^{-1}(x(B'C'^{-1}xE_{B'C'} - I)\partial_x + B'C'^{-1}xE_{B'C'}(s\partial_s + u\partial_u - 1) - t\partial_t + 1),
\end{aligned}$$

$$K^0 = \frac{1}{2}(s\partial_s + u\partial_u - 1),$$
$$h_\lambda(B', C', x) = {}_3F_2(A - C + \lambda, B - C + \lambda, B'; \lambda, C'; x)\, s^{A-C+\lambda}\, u^{B-C+\lambda}\, t^\lambda. \tag{8.4}$$

Indeed, the K-operators (8.4) satisfy (8.2). The function

$$F = {}_2F_1(A, B; C; z + sut)s^A u^B t^C \tag{8.5}$$

satisfies

$$(J^+J^- + J^0J^0 - J^0)F = \frac{1}{4}(A - B + I)(A - B - I)F.$$

It therefore follows that

$$H = F^{2:1;0}_{1:1;0}\left(\begin{matrix} A, B : & B'; & -; \\ C : & C'; & -; \end{matrix}\, x, sut\right) s^A u^B t^C \tag{8.6}$$

satisfies

$$(K^+K^- + K^0K^0 - K^0)H = \frac{1}{4}(A - B + I)(A - B - I)H. \tag{8.7}$$

The expansion

$$[T_4(g)F](s, u, t, z) = \sum_{n=-\infty}^{\infty} p_n(g) f_{C+nI} \tag{8.8}$$

gives a matrix generating function relation stated in the following theorem.

Theorem 8.1. *Let A, B, C be commuting matrices in $\mathbb{C}^{r\times r}$ such that $\alpha(A) + \alpha(B) < \beta(C)$. Then the following matrix generating function holds*

$$\begin{aligned}&\left(1 + \frac{c(1-z)}{y}\right)^{C-I}\left(1 - \frac{cz}{y}\right)^{A+B-C} {}_2F_1\left(A, B; C; \left(1 + \frac{c(1-z)}{y}\right)(z + y - cz)\right)\\ &= \sum_{n=-\infty}^{\infty} \Gamma(A + nI)\Gamma(B + nI)\Gamma(C)\Gamma^{-1}(A)\Gamma^{-1}(B)\Gamma^{-1}(C + nI)\Gamma^{-1}(1 + n)I\\ &\quad \times {}_2F_1(A + nI, B + nI; (1 + n)I; c){}_2F_1(A + nI, B + nI; C + nI; z)y^n,\\ &\qquad |cz/y| < 1, |c(1 - z)/y| < 1, \left|\left(1 + \frac{c(1-z)}{y}\right)(z + y - cz)\right| < 1.\end{aligned} \tag{8.9}$$

The terms corresponding to $n = -1, -2, \ldots,$ are well defined in view of the relation (5.28).

The expansion arising from the K-operators does not lead to an elegant identity and is therefore not included.

Acknowledgments

The authors thank the referee for valuable suggestions that led to a better presentation of the paper. This work is inspired by and is dedicated to Professor H.L. Manocha. The authors are thankful to him for providing a preprint. The financial assistance provided to the first author in the form of a Senior Research Fellowship (Grant No. 09/107(0371)/2015-EMR-I) from Council of Scientific and Industrial Research, India is gratefully acknowledged.

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