J. Math. Anal. Appl. ••• (••••) ••••••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Sampling numbers of a class of infinitely differentiable functions ☆

Heping Wang [a], Guiqiao Xu [b,*]

[a] School of Mathematical Sciences, Capital Normal University, Beijing 100048, China
[b] Department of Mathematics, Tianjin Normal University, Tianjin, 300387, China

ARTICLE INFO

Article history:
Received 15 January 2019
Available online xxxx
Submitted by S. Tikhonov

Keywords:
Sampling numbers
Optimal algorithm
Lagrange interpolation

ABSTRACT

This paper investigates the optimal recovery of a class F_∞ of infinitely differentiable functions on $[-1,1]$ in $L_p[-1,1]$. We obtain strong equivalences of the sampling numbers of F_∞ in $L_p[-1,1]$, $1 \le p < \infty$, and even equality in $L_\infty[-1,1]$. We prove that the Lagrange interpolation algorithms $I_{n,p}$, $1 \le p \le \infty$, based on the zeros of the polynomial of degree n with the leading coefficient 1 of the least deviation from zero in $L_p[-1,1]$ are strongly asymptotically optimal for $1 \le p < \infty$, and optimal for $p = \infty$.

1. Introduction and main results

Let F be a Banach space of functions defined on a compact set D that can be continuously embedded in $C(D)$, BF is the unit ball of F, and $G\,(\supseteqq F)$ is a normed linear space with norm $\|\cdot\|_G$. We want to approximate functions f from BF by using a finite number of arbitrary function values $f(t)$ (standard information) for some $t \in D$. We consider only nonadaptive information. For $\mathbf{x} = (\xi_1, \dots, \xi_n) \in D^n$, we use $I_{\mathbf{x}}$ the nonadaptive information operator, i.e.,

$$I_{\mathbf{x}}(f) := (f(\xi_1), f(\xi_2), \dots, f(\xi_n)) \in \mathbb{R}^n, \quad f \in F.$$

We say that $A_n = \varphi \circ I_{\mathbf{x}}$ is an algorithm based on the information operator $I_{\mathbf{x}}$, where φ is an arbitrary mapping from $\mathbb{R}^n$ to G. We also consider linear algorithms, i.e., algorithms of the form

$$A_n^{\rm lin}(f) = \varphi^{\rm lin} \circ I_{\mathbf{x}}(f) := \sum_{j=1}^n f(\xi_j) h_j,\ h_j \in G,\ \xi_j \in D,\ j = 1, \dots, n.$$

☆ The first author was supported by the National Natural Science Foundation of China (Project no. 11671271) and the Beijing Natural Science Foundation (1172004). The second author was supported by the National Natural Science Foundation of China (11671271, 11871006).
* Corresponding author.
E-mail addresses: wanghp@cnu.edu.cn (H. Wang), Xuguiqiao@aliyun.com (G. Xu).

https://doi.org/10.1016/j.jmaa.2019.123689
0022-247X/© 2019 Elsevier Inc. All rights reserved.

We use an algorithm A_n to reconstruct functions from BF. The worst case error of the algorithm A_n for BF in G is defined by

$$e(BF, A_n, G) := \sup_{f \in BF} \|f - A_n(f)\|_G.$$

For a given $\mathbf{x} = (\xi_1, \dots, \xi_n) \in D^n$, the worst case error for BF in G based on the information operator $I_{\mathbf{x}}$ is defined by

$$e(BF, I_{\mathbf{x}}, G) := \inf_{\varphi} \sup_{f \in BF} \|f - \varphi \circ I_{\mathbf{x}}(f)\|_G,$$

where the infimum is taken over all mappings φ from $\mathbb{R}^n$ to G.

We define the linear sampling numbers and the sampling numbers of BF in G by

$$g_n^{\mathrm{lin}}(BF, G) := \inf_{A_n^{\mathrm{lin}}} e(BF, A_n^{\mathrm{lin}}, G),$$

and

$$g_n(BF, G) := \inf_{A_n} e(BF, A_n, G) = \inf_{\mathbf{x} \in D^n} e(BF, I_{\mathbf{x}}, G),$$

respectively. If there exists an information operator $I_{\mathbf{x}^*}$ and a mapping φ^* such that the algorithm $A_n^* = \varphi^* \circ I_{\mathbf{x}^*}$ satisfies

$$e(BF, A_n^*, G) = g_n(BF, G),$$

then we call $I_{\mathbf{x}^*}$ the nth optimal information and A_n^* the nth optimal algorithm.

The sampling numbers are closely related to many classical approximation problems such as width and information-based complexity, and they have a wide range of applications in numerical analysis. The aim of studying the sampling numbers is to find optimal or nearly optimal information, construct optimal or nearly optimal algorithms according to the known standard information, and determine orders (or values) of the sampling numbers.

In recent years, the study of sampling numbers has attracted much interest, and a great number of interesting results have been obtained (see [1–3,7–12,17,19–21,23–30]).

This paper investigates the sampling numbers of a class F_∞ of infinitely differentiable functions on $[-1, 1]$. Approximation of infinitely differentiable multivariate functions has been investigated in [5,13,14,16,18,22,24,31,32].

We remark that, in most cases, we can achieve only weak equivalences (orders) of the sampling numbers. In this paper, we obtain strong equivalences of the sampling numbers of F_∞ in $L_p := L_p[-1, 1]$, $1 \le p < \infty$, and even equality in $L_\infty[-1, 1]$.

Denote the spaces of functions with nth order continuous derivative on $[-1, 1]$ using $C^n[-1, 1]$, $n = 1, 2, \dots$, respectively. For $1 \le p \le \infty$, let $L_p \equiv L_p[-1, 1]$ be the space of measurable functions defined on $[-1, 1]$, for which the norm

$$\|f\|_p := \begin{cases} \left(\int_{-1}^{1} |f(x)|^p \, dx \right)^{1/p}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}_{x \in [-1,1]} |f(x)|, & p = \infty \end{cases}$$

is finite. The space F_∞ of infinitely differentiable functions on $[-1, 1]$ is defined as

$$F_\infty = \left\{ f \in C^\infty[-1, 1] \;\middle|\; \|f\|_{F_\infty} = \sup_{n \in \mathbb{N}_0} \|f^{(n)}\|_\infty < \infty \right\}.$$

We set

$$E_{n,p} := \inf_{g\in\mathcal{P}_{n-1}} \|x^n - g(x)\|_p, \quad 1 \le p \le \infty,$$

where $\mathcal{P}_n$ represents the space of all algebraic polynomials of degree at most n. Furthermore, let $W_{n,p} \in \mathcal{P}_n$ $(1 \le p \le \infty)$ satisfy

$$\|W_{n,p}\|_p = E_{n,p} \quad \text{and} \quad W_{n,p}(x) = x^n + c_1x^{n-1} + \cdots + c_n.$$

$W_{n,p}$ is unique (see [4, Chapter 3, Theorem 6.1, p. 75, Theorems 10.8 and 10.9, p. 86]) and has n zeros

$$-1 < \xi_{1,p} < \xi_{2,p} < \cdots < \xi_{n,p} < 1 \tag{1.1}$$

in $(-1, 1)$ (see [4, Chapter 3, Theorems 6.1, 10.8 and 10.9]), which means that

$$W_{n,p}(x) = \prod_{k=1}^{n} (x - \xi_{k,p}). \tag{1.2}$$

Let $I_{n,p}f$ be the Lagrange interpolation polynomial of a function $f : [-1, 1] \to \mathbb{R}$ based on the nodes $\xi_{1,p}, \ldots, \xi_{n,p}$, i.e.,

$$I_{n,p}f \in \mathcal{P}_{n-1}, \quad I_{n,p}f(\xi_{k,p}) = f(\xi_{k,p}),\ k = 1, 2, \ldots, n.$$

It is known that $I_{n,p}f$ has the explicit expression

$$I_{n,p}f(x) = \sum_{k=1}^{n} f(\xi_{k,p})\ell_{k,p}(x), \tag{1.3}$$

where

$$\ell_{k,p}(x) = \frac{W_{n,p}(x)}{(x - \xi_{k,p})W'_{n,p}(\xi_{k,p})},$$

and $W_{n,p}$ is given by (1.2).

For $\mathbf{x} = (\xi_1, \ldots, \xi_n),\ -1 \le \xi_1 < \xi_2 < \cdots < \xi_n \le 1$, if

$$\xi_k = -\xi_{n+1-k},\ k = 1, 2, \ldots, n,$$

then we call the information operator $I_{\mathbf{x}}$ the symmetric information. We define

$$\begin{aligned}\tilde{g}_n(BF_\infty, L_p) &:= \inf_{I_{\mathbf{x}}\ \text{symmetric}} e(BF_\infty, I_{\mathbf{x}}, L_p) \\ &= \inf_{I_{\mathbf{x}}\ \text{symmetric}} \inf_{\varphi} \|f - \varphi \circ I_{\mathbf{x}}(f)\|_p, \end{aligned} \tag{1.4}$$

where the first infimum is taken over all symmetric information. We can define $\tilde{g}_n^{\text{lin}}(BF_\infty, L_p)$ similarly.

Now we can formulate our main results as follows.

Theorem 1.1. *For $1 \le p < \infty$, we have*

$$\lim_{n\to\infty} \frac{n!}{E_{n,p}} g_n^{\text{lin}}(BF_\infty, L_p) = \lim_{n\to\infty} \frac{n!}{E_{n,p}} g_n(BF_\infty, L_p) = 1. \tag{1.5}$$

The Lagrange interpolation algorithm $I_{n,p}$ *given by* (1.3) *is a strongly asymptotically optimal (linear) algorithm.*

Theorem 1.2. *For* $1 \le p \le \infty$ *and* $n \in \mathbb{N}$ *we have*

$$\tilde{g}_n(BF_\infty, L_p) = \tilde{g}_n^{\mathrm{lin}}(BF_\infty, L_p) = \frac{E_{n,p}}{n!}. \tag{1.6}$$

The Lagrange interpolation algorithm $I_{n,p}$ *given by* (1.3) *is an optimal (linear) algorithm for* $\tilde{g}_n(BF_\infty, L_p)$.

Theorem 1.3. *For* $p = \infty$ *and* $n \in \mathbb{N}$ *we have*

$$g_n(BF_\infty, L_\infty) = g_n^{\mathrm{lin}}(BF_\infty, L_\infty) = \frac{E_{n,\infty}}{n!}. \tag{1.7}$$

The Lagrange interpolation algorithm $I_{n,\infty}$ *given by* (1.3) *is an optimal (linear) algorithm.*

We use the Lagrange interpolation algorithms $I_{n,p}$, $1 \le p \le \infty$, based on the zeros of $W_{n,p}$, the polynomial of degree n with the leading term x^n of the least deviation from zero in $L_p[-1,1]$, to give the upper bounds of $g_n^{\mathrm{lin}}(BF_\infty, L_p)$. The key of proving Theorems 1.1–1.3 is to prove the lower bounds. We shall show that for any $\xi_1, \xi_2, \dots, \xi_n \in [-1,1]$, there exists a function $f_0 \in BF_\infty$ for which $f_0(\xi_1) = \cdots = f_0(\xi_n) = 0$ and

$$\|f_0\|_p \ge \frac{E_{n,p}}{n!(1+\frac{1}{n})},$$

which gives the lower bound of $g_n(BF_\infty, L_p)$. If $\xi_1, \xi_2, \dots, \xi_n \in [-1,1]$ are symmetrical, then we have a better estimate

$$\|f_0\|_p \ge \frac{E_{n,p}}{n!},$$

which gives the lower bound of $\tilde{g}_n(BF_\infty, L_p)$. For $p = \infty$ we can use the Chebyshev alternation theorem to obtain the equality of $g_n(BF_\infty, L_p)$.

The paper is organized as follows. In Section 2, we give some necessary lemmas. The proofs of Theorems 1.1, 1.2 and 1.3 will be given in Section 3.

2. Lemmas

Let $x_1, x_2, \dots, x_n$ be n distinct points in $[-1,1]$. Then, the Lagrange interpolation polynomial $I_n f$ of a function $f : [-1,1] \to \mathbb{R}$ based on the knots $\{x_k\}_{k=1}^n$ is defined by

$$I_n f \in \mathcal{P}_{n-1}, \quad \text{and} \quad I_n f(x_k) = f(x_k), \ k = 1, 2, \dots, n.$$

The classical Lagrange interpolation formula gives

$$I_n f(x) = \sum_{k=1}^n f(x_k)\ell_k(x),$$

where

$$\ell_k(x) = \frac{W_n(x)}{(x - x_k)W_n'(x_k)}, \quad W_n(x) = \prod_{k=1}^n (x - x_k).$$

Please cite this article in press as: H. Wang, G. Xu, Sampling numbers of a class of infinitely differentiable functions, J. Math. Anal. Appl. (2019), https://doi.org/10.1016/j.jmaa.2019.123689

Lemma 2.1. *(See [15].) Let $f \in C^n[-1,1]$. Then, the remainder $R_n f(x) := f(x) - I_n f(x)$ for the Lagrange interpolation polynomial based on knots $x_1, x_2, \dots, x_n \in [-1,1]$ can be represented in the form*

$$R_n f(x) = f(x) - I_n f(x) = \frac{f^{(n)}(\xi)}{n!} W_n(x), \ \ x \in [-1,1], \tag{2.1}$$

for some $\xi \in [-1,1]$ depending on x and the knots $x_1, \dots, x_n$.

Let $-1 < \xi_{1,p} < \xi_{2,p} < \cdots < \xi_{n,p} < 1$ be given by (1.1) and $W_{n,p}$ is given by (1.2). Then, $W_{n,p}$ is odd if n is odd, and even if n is even. We obtain that (here $[x]$ represents the integer part of x)

$$W_{n,p}(x) = x^n + c_2 x^{n-2} + c_4 x^{n-4} + \cdots + c_{n-2[n/2]} x^{n-2[n/2]}, \tag{2.2}$$

and the zeros of $W_{n,p}$ is symmetrical, i.e.,

$$\xi_{k,p} = -\xi_{n+1-k,p}, \quad k = 1, 2, \dots, n.$$

Hence, the Lagrange interpolation algorithm $I_{n,p}$ given by (1.3) is a linear algorithm based on symmetric information. It follows that

$$g_n(BF_\infty, L_p) \le g_n^{\text{lin}}(BF_\infty, L_p) \le e(BF_\infty, I_{n,p}, L_p), \tag{2.3}$$

and

$$\tilde{g}_n(BF_\infty, L_p) \le \tilde{g}_n^{\text{lin}}(BF_\infty, L_p) \le e(BF_\infty, I_{n,p}, L_p). \tag{2.4}$$

The following lemma gives the upper bound of the worst case error $e(BF_\infty, I_{n,p}, L_p)$ of the Lagrange interpolation algorithm $I_{n,p}$ for $1 \le p \le \infty$.

Lemma 2.2. *For $1 \le p \le \infty$, we have*

$$e(BF_\infty, I_{n,p}, L_p) \le \frac{E_{n,p}}{n!}. \tag{2.5}$$

Proof. Let $f \in BF_\infty$. Then, we have $\|f^{(n)}\|_\infty \le 1$. It follows from (2.1) that, for $x \in [-1,1]$,

$$|f(x) - I_{n,p} f(x)| = \frac{|f^{(n)}(\xi)|}{n!} |W_{n,p}(x)|, \ \ x \in [-1,1], \tag{2.6}$$

for some $\xi \in [-1,1]$ depending on x and $\xi_{1,p}, \xi_{2,p}, \dots, \xi_{n,p}$. Since $\|f^{(n)}\|_\infty \le 1$, we obtain that

$$|f(x) - I_{n,p} f(x)| \le \frac{|W_{n,p}(x)|}{n!}, \ \ x \in [-1,1].$$

It follows that

$$\|f - I_{n,p} f\|_p \le \frac{\|W_{n,p}\|_p}{n!} = \frac{E_{n,p}}{n!}. \tag{2.7}$$

Lemma 2.2 is proved. □

Now let $\xi_1, \xi_2, \dots, \xi_n$ be n arbitrary distinct points in $[-1,1]$, $-1 \le \xi_1 < \xi_2 < \cdots < \xi_n \le 1$. Consider the function

$$g(x) = \prod_{k=1}^{n}(x - \xi_k) = x^n - Ax^{n-1} + c_2x^{n-2} + \cdots + c_n. \tag{2.8}$$

The next lemma gives the exact value of $\|g\|_{F_\infty}$.

Lemma 2.3. *Let g be given by (2.8). Then, we have*

$$\|g\|_{F_\infty} = n!\left(1 + \frac{|A|}{n}\right), \tag{2.9}$$

where $A = \xi_1 + \cdots + \xi_n$. Specifically, if $\xi_1, \xi_2, \dots, \xi_n$ are symmetrical, i.e.,

$$\xi_{n+1-k} = -\xi_k, \;\; k = 1, 2, \dots, n,$$

then $A = 0$ and hence

$$\|g\|_{F_\infty} = n!.$$

Proof. First, we note that

$$g^{(n)}(x) = n!, \quad g^{(n-1)}(x) = n!\left(x - \frac{A}{n}\right).$$

The above two equalities mean that

$$\|g^{(n)}\|_\infty = n!, \quad \|g^{(n-1)}\|_\infty = n!\left(1 + \frac{|A|}{n}\right).$$

Hence, to show (2.9), it suffices to prove that, for $k = 2, 3, \dots, n$, we have

$$\|g^{(n-k)}\|_\infty \le n!\left(1 + \frac{|A|}{n}\right).$$

Since g has n different zeros, then by the Rolle theorem and the Vieta theorem, we know that $g^{(n-k)}$, $k = 1, 2, \dots, n$ has k different zeros

$$-1 < \xi_1^{(n-k)} < \xi_2^{(n-k)} < \cdots < \xi_k^{(n-k)} < 1, \quad k = 1, 2, \dots, n,$$

which satisfy the following equality

$$\frac{1}{k}\sum_{j=1}^{k}\xi_j^{(n-k)} = \frac{A}{n}.$$

By $\xi_1^{(n-k)} < \cdots < \xi_k^{(n-k)}$ for $k \ge 2$ and the above equality, it follows that

$$\xi_1^{(n-k)} < \frac{A}{n} < \xi_k^{(n-k)}, \quad k = 2, 3, \dots, n.$$

It is easy to show that

$$g^{(n-k)}(x) = n(n-1)\cdots(k+1)\prod_{j=1}^{k}(x-\xi_j^{(n-k)}).$$

For arbitrary $x \in [-1,1]$, we have

$$\left|x-\xi_j^{(n-k)}\right| \le 2,\ j=2,\dots,k-1, \quad \text{and} \quad \left|\left(x-\xi_1^{(n-k)}\right)\left(x-\xi_k^{(n-k)}\right)\right| \le 2\left(1+\frac{|A|}{n}\right),$$

which means that, for $x \in [-1,1]$,

$$\left|\prod_{j=1}^{k}(x-\xi_j^{(n-k)})\right| = \left|\left(x-\xi_1^{(n-k)}\right)\left(x-\xi_k^{(n-k)}\right)\right|\prod_{j=2}^{k-1}\left|x-\xi_j^{(n-k)}\right| \le 2^{k-1}\left(1+\frac{|A|}{n}\right).$$

It follows that, for $k=2,3,\dots,n$,

$$\|g^{(n-k)}\|_\infty \le \frac{n!}{k!}\,2^{k-1}\left(1+\frac{|A|}{n}\right).$$

Since $2^{k-1} \le k\,!$, we have for $k=2,\dots,n$,

$$\left\|g^{(n-k)}\right\|_\infty \le n!\left(1+\frac{|A|}{n}\right).$$

Lemma 2.3 is proved. □

Concerning the worst case error for BF_∞ in L_p based on a given information operator $I_{\mathbf{x}}$, $\mathbf{x} = (x_1,\dots,x_n) \in [-1,1]^n$, we have the following lemma.

Lemma 2.4. *(See [21, Lemma 4.3] or [27, p. 71].) For $\mathbf{x} = (\xi_1,\xi_2,\dots,\xi_n) \in [-1,1]^n$, we have*

$$\begin{aligned} e(BF_\infty, I_{\mathbf{x}}, L_p) &:= \inf_{\varphi}\sup_{\|f\|_{F_\infty}\le 1}\|f-\varphi\circ I_{\mathbf{x}}(f)\|_p \\ &= \sup_{\substack{\|f\|_{F_\infty}\le 1,\\ f(\xi_1)=f(\xi_2)=\cdots=f(\xi_n)=0}}\|f\|_p. \end{aligned} \tag{2.10}$$

3. Proofs of Theorems 1.1–1.3 and some remarks

Proof of Theorem 1.1. Let $1 \le p \le \infty$. We shall show the following inequalities

$$\frac{E_{n,p}}{n!(1+\frac{1}{n})} \le g_n(BF_\infty, L_p) \le g_n^{\mathrm{lin}}(BF_\infty, L_p) \le \frac{E_{n,p}}{n!}, \tag{3.1}$$

from which (1.5) follows directly. To prove (3.1), due to (2.3) and (2.5), it suffices to prove that

$$g_n(BF_\infty, L_p) \ge \frac{E_{n,p}}{n!(1+\frac{1}{n})}. \tag{3.2}$$

For arbitrary $\xi_1,\dots,\xi_n \in [-1,1]$, we set $A = \xi_1+\cdots+\xi_n$ and consider the function $g(x)$ given by (2.8). Then, it follows from Lemma 2.3 that (2.9) holds. Now, we set

Please cite this article in press as: H. Wang, G. Xu, Sampling numbers of a class of infinitely differentiable functions, J. Math. Anal. Appl. (2019), https://doi.org/10.1016/j.jmaa.2019.123689

$$f_0(x) = \frac{g(x)}{\|g\|_{F_\infty}} = \frac{g(x)}{n!(1+\frac{|A|}{n})}.$$

Then, $f_0 \in BF_\infty$ and

$$f_0(\xi_1) = f_0(\xi_2) = \cdots = f_0(\xi_n) = 0.$$

It follows from Lemma 2.4 that, for $\mathbf{x} = (\xi_1, \xi_2, \ldots, \xi_n) \in [-1,1]^n$,

$$e(BF_\infty, I_{\mathbf{x}}, L_p) \geq \|f_0\|_p = \frac{\|g\|_p}{n!(1+\frac{|A|}{n})}. \tag{3.3}$$

Let

$$E_{n,A,p} := \inf_{h \in \mathcal{P}_{n-2}} \|x^n - Ax^{n-1} - h(x)\|_p. \tag{3.4}$$

Clearly, we have

$$E_{n,A,p} \geq \inf_{h \in \mathcal{P}_{n-1}} \|x^n - h(x)\|_p = E_{n,p}, \tag{3.5}$$

and

$$\|g\|_p = \|x^n - Ax^{n-1} + c_2 x^{n-2} + \cdots + c_n\|_p \geq E_{n,A,p}. \tag{3.6}$$

It follows from the definition of $g_n(BF_\infty, L_p)$, (3.3), and (3.6) that

$$g_n(BF_\infty, L_p) = \inf_{I_{\mathbf{x}}} e(BF_\infty, I_{\mathbf{x}}, L_p) \geq \inf_{A \in \mathbb{R}} \frac{E_{n,A,p}}{n!(1+\frac{|A|}{n})}. \tag{3.7}$$

To prove (3.2), by (3.7), it suffices to show that, for all $A \in \mathbb{R}$,

$$\frac{E_{n,A,p}}{1+\frac{|A|}{n}} \geq \frac{E_{n,p}}{1+\frac{1}{n}}. \tag{3.8}$$

If $A = 0$, then by (3.4), we have

$$E_{n,0,p} = \inf_{h \in \mathcal{P}_{n-2}} \|x^n - h(x)\|_p = \|W_{n,p}\|_p = E_{n,p} \geq \frac{E_{n,p}}{1+\frac{1}{n}}. \tag{3.9}$$

For $A \neq 0$, according to [4, Chapter 3, Theorem 6.1, p. 75, Theorems 10.8 and 10.9, p. 86], there exists a unique polynomial $h_0 \in \mathcal{P}_{n-2}$ of best approximation to the continuous function $x^n - Ax^{n-1}$ in L_p such that

$$E_{n,A,p} = \|x^n - Ax^{n-1} - h_0(x)\|_p. \tag{3.10}$$

Clearly, by the unicity of the best approximation polynomial, we obtain that

$$E_{n,-A,p} = E_{n,A,p} = \|x^n + Ax^{n-1} - (-1)^n h_0(-x)\|_p. \tag{3.11}$$

Hence, for $A \neq 0$, we have

Please cite this article in press as: H. Wang, G. Xu, Sampling numbers of a class of infinitely differentiable functions, J. Math. Anal. Appl. (2019), https://doi.org/10.1016/j.jmaa.2019.123689

$$
\begin{aligned}
E_{n,A,p} &= \frac{1}{2}\big(E_{n,A,p} + E_{n,-A,p}\big) \\
&= \frac{1}{2}\big(\|x^n - Ax^{n-1} - h_0(x)\|_p + \|x^n + Ax^{n-1} - (-1)^n h_0(-x)\|_p\big) \\
&\geq \frac{1}{2}\big\|\big(x^n - Ax^{n-1} - h_0(x)\big) - \big(x^n + Ax^{n-1} - (-1)^n h_0(-x)\big)\big\|_p \\
&= |A| \left\| x^{n-1} + \frac{h_0(x) - (-1)^n h_0(-x)}{2A} \right\|_p \\
&\geq |A| \inf_{h \in \mathcal{P}_{n-2}} \|x^{n-1} - h(x)\|_p = |A| E_{n-1,p}.
\end{aligned} \tag{3.12}
$$

We note that, for some $h_1 \in \mathcal{P}_{n-2}$

$$
E_{n-1,p} = \|x^{n-1} - h_1(x)\|_p \geq \|x^n - xh_1(x)\|_p \geq E_{n,p}. \tag{3.13}
$$

It follows from (3.5), (3.12), and (3.13) that

$$
E_{n,A,p} \geq \max\{|A|, 1\}\, E_{n,p}.
$$

Since the inequality

$$
\frac{\max\{|A|, 1\}}{1 + \frac{|A|}{n}} \geq \frac{1}{1 + \frac{1}{n}}
$$

holds for all $A \neq 0$, we obtain (3.8). Theorem 1.1 is proved. $\square$

Proof of Theorem 1.2. Let $1 \leq p \leq \infty$. To prove (1.6), due to (2.4) and (2.5), it suffices to prove that

$$
\tilde{g}_n(BF_\infty, L_p) \geq \frac{E_{n,p}}{n!}. \tag{3.14}
$$

For any symmetric $\xi_1, \ldots, \xi_n \in [-1, 1]$, we consider the function $g(x)$ given by (2.8). Then, it follows from Lemma 2.3 that $\|g\|_{F_\infty} = n!$. We set

$$
f_0(x) = \frac{g(x)}{\|g\|_{F_\infty}} = \frac{g(x)}{n!}.
$$

Then, $f_0 \in BF_\infty$ and

$$
f_0(\xi_1) = f_0(\xi_2) = \cdots = f_0(\xi_n) = 0.
$$

It follows from Lemma 2.4 and (3.3) that, for symmetric $\mathbf{x} = (\xi_1, \xi_2, \ldots, \xi_n) \in [-1, 1]^n$,

$$
\begin{aligned}
e(BF_\infty, I_{\mathbf{x}}, L_p) &\geq \|f_0\|_p = \frac{1}{n!}\|g\|_p \\
&\geq \frac{1}{n!} \inf_{h \in \mathcal{P}_{n-2}} \|x^n - h(x)\|_p = \frac{E_{n,0,p}}{n!} = \frac{E_{n,p}}{n!}.
\end{aligned} \tag{3.15}
$$

By the definition of $\tilde{g}_n(BF_\infty, L_p)$ and (3.15), we have that

$$
\tilde{g}_n(BF_\infty, L_p) = \inf_{I_{\mathbf{x}} \text{ symmetric}} e(BF_\infty, I_{\mathbf{x}}, L_p) \geq \frac{E_{n,p}}{n!},
$$

proving (3.14). Theorem 1.2 is proved. $\square$

Proof of Theorem 1.3. To prove Theorem 1.3, due to (2.3) and (2.5), it suffices to prove that

$$g_n(BF_\infty, L_\infty) \geq \frac{E_{n,\infty}}{n!}. \tag{3.16}$$

It follows from (3.7) that

$$g_n(BF_\infty, L_\infty) \geq \inf_{A\in\mathbb{R}} \frac{E_{n,A,\infty}}{n!(1+\frac{|A|}{n})}, \tag{3.17}$$

where $E_{n,A,\infty}$ is given by (3.4). Hence, to prove (3.16), by (3.17), it suffices to prove that, for arbitrary $A \in \mathbb{R}$,

$$E_{n,A,\infty} \geq \left(1 + \frac{|A|}{n}\right) E_{n,\infty}. \tag{3.18}$$

For $n = 1$, it is easy to verify that (3.18) holds with equality. In the following, we consider $n \geq 2$. If $A = 0$, then from (3.9), we know that (3.18) holds with equality. For $A \neq 0$, it follows from [4, p. 75] that

$$E_{n,\infty} = \frac{1}{2^{n-1}}, \quad W_{n,\infty}(x) = \frac{T_n(x)}{2^{n-1}}, \quad \xi_{k,\infty} = \cos\frac{(2k-1)\pi}{2n}, \ k = 1, \dots, n, \tag{3.19}$$

where T_n is the nth Chebyshev polynomial of the first kind, i.e.,

$$T_n(x) = \cos(n \arccos x).$$

From (3.19) it follows that

$$E_{n,\infty} = 2^{1-n}, \quad E_{n-1,\infty} = 2^{2-n} = 2E_{n,\infty}.$$

From above equalities and (3.12), it follows that,

$$E_{n,A,\infty} \geq |A| E_{n-1,\infty} = 2|A| E_{n,\infty}.$$

If $|A| \geq \frac{1}{2-\frac{1}{n}}$, then

$$E_{n,A,\infty} \geq 2|A|E_{n,\infty} \geq \left(1 + \frac{|A|}{n}\right) E_{n,\infty},$$

showing (3.18).

Next, we assume $0 < |A| < \frac{1}{2-\frac{1}{n}}$. Without loss of generality, we assume $A > 0$. Consider the function

$$\tilde{T}_n(x) = \cos n \arccos \frac{x - \frac{A}{n}}{1 + \frac{A}{n}} = T_n\left(\frac{x - \frac{A}{n}}{1 + \frac{A}{n}}\right) \in \mathcal{P}_n.$$

Then, by (3.19), $\tilde{T}_n$ has the expression

$$\begin{aligned}\tilde{T}_n(x) &= 2^{n-1}\left(\left(\frac{x - \frac{A}{n}}{1 + \frac{A}{n}}\right)^n + c_2\left(\frac{x - \frac{A}{n}}{1 + \frac{A}{n}}\right)^{n-2} + \cdots + c_{2[n/2]}\left(\frac{x - \frac{A}{n}}{1 + \frac{A}{n}}\right)^{n-2[n/2]}\right) \\ &= \frac{2^{n-1}}{(1+\frac{A}{n})^n}\left(x^n - Ax^{n-1} + c_2 x^{n-2} + \cdots + c_n\right).\end{aligned}$$

Please cite this article in press as: H. Wang, G. Xu, Sampling numbers of a class of infinitely differentiable functions, J. Math. Anal. Appl. (2019), https://doi.org/10.1016/j.jmaa.2019.123689

If $0 < A \le n\tan^2\frac{\pi}{2n}$, then the extreme points of $\tilde{T}_n$ given by

$$x_k = \frac{A}{n} + \cos\frac{k\pi}{n} + \frac{A}{n}\cos\frac{k\pi}{n}, \quad k = 1, \dots, n$$

belong to $[-1, 1]$. In fact, for $n = 2$,

$$0 < x_1 = \frac{A}{2} \le \tan^2\frac{\pi}{4} = 1, \text{ and } x_2 = -1.$$

For $n > 2$, we have $x_n = -1$, and for $k = 1, \dots, n-1$,

$$\begin{aligned} |x_k| &\le \frac{A}{n} + \cos\frac{\pi}{n} + \frac{A}{n}\cos\frac{\pi}{n} = \frac{A}{n}\left(1 + \cos\frac{\pi}{n}\right) + \cos\frac{\pi}{n} \\ &\le \tan^2\frac{\pi}{2n}\, 2\cos^2\frac{\pi}{2n} + \cos\frac{\pi}{n} = 1. \end{aligned}$$

We note that

$$\frac{x_k - \frac{A}{n}}{1 + \frac{A}{n}} = \cos\frac{k\pi}{n}, \ k = 1, 2, \dots, n,$$

which means that

$$\tilde{T}_n(x_k) = (-1)^k \|\tilde{T}_n\|_\infty = (-1)^k, \quad k = 1, 2, \dots, n.$$

According to the Chebyshev alternation theorem (see [4, Chapter 3, Theorem 5.1]), we obtain that

$$\frac{(1 + \frac{A}{n})^n}{2^{n-1}}\tilde{T}_n(x)$$

is the polynomial of degree n with the leading two terms $x^n - Ax^{n-1}$ of the least deviation from zero in $C[-1, 1]$. It follows that

$$E_{n,A,\infty} = \frac{(1 + \frac{A}{n})^n}{2^{n-1}} \ge \left(1 + \frac{A}{n}\right) 2^{1-n} = \left(1 + \frac{A}{n}\right) E_{n,\infty}, \tag{3.20}$$

which leads to (3.18).

Finally, assume that $n\tan^2\frac{\pi}{2n} < A < \frac{1}{2 - \frac{1}{n}}$. Let α, β satisfy

$$\alpha > \beta > 0, \ \alpha + \beta = 1, \quad (\alpha - \beta)A = n\tan^2\frac{\pi}{2n}.$$

α, β are unique. Indeed,

$$\alpha = \frac{1}{2}\left(1 + \frac{n\tan^2\frac{\pi}{2n}}{A}\right), \quad \beta = \frac{1}{2}\left(1 - \frac{n\tan^2\frac{\pi}{2n}}{A}\right).$$

By (3.11), (3.19), and (3.20), we have

$$\begin{aligned} E_{n,A,\infty} &= \alpha E_{n,A,\infty} + \beta E_{n,-A,\infty} \\ &\ge \|(\alpha + \beta)x^n - (\alpha - \beta)Ax^{n-1} - \alpha h_0(x) - \beta(-1)^n h_0(-x)\|_\infty \end{aligned}$$

$$
\begin{aligned}
&\geq E_{n,n\tan^2\frac{\pi}{2n},\infty} = 2^{1-n}\left(1+\tan^2\frac{\pi}{2n}\right)^n \\
&\geq 2^{1-n}\left(1+n\tan^2\frac{\pi}{2n}\right) \geq 2^{1-n}\left(1+\frac{\pi^2}{4n}\right) \\
&\geq 2^{1-n}\left(1+\frac{A}{n}\right) = \left(1+\frac{A}{n}\right)E_{n,\infty},
\end{aligned}
$$

where in the second-to-last inequality we used the inequality $\tan x > x$ for $x \in (0, \frac{\pi}{2})$, and in the last inequality we used $\frac{\pi^2}{4} \geq \frac{1}{2-\frac{1}{n}} > A$.

Hence, (3.18) holds for all $A \in \mathbb{R}$. The proof of Theorem 1.3 is completed. □

Remark 3.1. One can rephrase (1.5) as strong equivalences

$$
g_n^{\mathrm{lin}}(BF_\infty, L_p) \sim g_n(BF_\infty, L_p) \sim \frac{E_{n,p}}{n!} \quad \text{as} \quad n \to \infty,
$$

for arbitrary fixed p, $1 \leq p < \infty$. The novelty of Theorems 1.1 and 1.3 is that they give strong equivalences of $g_n(BF_\infty, L_p)$ as $n \to \infty$ for arbitrary fixed p, $1 \leq p < \infty$, and even the equality of $g_n(BF_\infty, L_p)$ for $p = \infty$.

Remark 3.2. Now, we give three classical examples to show our results.

For $p = \infty$, the Lagrange interpolation polynomial $I_{n,\infty}f$ of a function f is given by

$$
I_{n,\infty}f(x) = \sum_{k=1}^{n} f(\xi_{k,\infty})\ell_{k,\infty}(x),
$$

where

$$
\ell_{k,\infty}(x) = \frac{(-1)^{k+1}\sqrt{1-\xi_{k,\infty}^2}\,T_n(x)}{n(x-\xi_{k,\infty})}, \qquad k = 1, \dots, n.
$$

From Theorems 1.2 and 1.3, and (3.19), it follows that

$$
e(BF_\infty, I_{n,\infty}, L_\infty) = g_n(BF_\infty, L_\infty) = \tilde{g}_n(BF_\infty, L_\infty) = \frac{1}{2^{n-1}n!}.
$$

For $p = 1$, it follows from [4, pp. 87-88] that

$$
E_{n,1} = \frac{1}{2^{n-1}}, \quad W_{n,1}(x) = \frac{U_n(x)}{2^n}, \quad \xi_{k,1} = \cos\frac{k\pi}{n+1}, \quad k = 1, \dots, n,
$$

where U_n is the nth Chebyshev polynomial of the second kind, i.e.,

$$
U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta.
$$

In this case, the Lagrange interpolation polynomial $I_{n,1}$ is given by

$$
I_{n,1}f(x) = \sum_{k=1}^{n} f(\xi_{k,1})\ell_{k,1}(x),
$$

where

$$\ell_{k,1}(x) = \frac{(-1)^{k+1}(1-\xi_{k,1}^2)U_n(x)}{(n+1)(x-\xi_{k,1})}, \qquad k=1,\ldots,n.$$

Furthermore, from Theorem 1.2, it follows that

$$e(BF_\infty, I_{n,1}, L_1) = \tilde{g}_n(BF_\infty, L_1) = \frac{E_{n,1}}{n!} = \frac{1}{2^{n-1}n!}.$$

For $p=2$, we have (see [15, p. 205])

$$W_{n,2}(x) = \frac{2^n(n!)^2}{(2n)!}P_n(x) = \prod_{k=1}^{n}(x-\xi_{k,2}), \tag{3.21}$$

where P_n is the nth Legendre polynomial, i.e.,

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

In this case, the Lagrange interpolation polynomial $I_{n,2}f$ is given by

$$I_{n,2}f(x) = \sum_{k=1}^{n} f(\xi_{k,2})\ell_{k,2}(x),$$

where

$$\ell_{k,2}(x) = \frac{P_n(x)}{(x-\xi_{k,2})P_n'(\xi_{k,2})}, \qquad k=1,\ldots,n.$$

From [6, p. 18], one obtains

$$\|P_n\|_2 = \frac{2^{1/2}}{\sqrt{2n+1}}. \tag{3.22}$$

By Theorem 1.2, (3.21) and (3.22), we have

$$e(BF_\infty, I_{n,2}, L_2) = \tilde{g}_n(BF_\infty, L_2) = \frac{E_{n,2}}{n!} = \frac{2^{n+1/2}n!}{(2n)!\sqrt{2n+1}}.$$

Remark 3.3. One may conjecture that, for $1 \le p < \infty$,

$$g_n(BF_\infty, L_p) = \frac{E_{n,p}}{n!}.$$

However, this conjecture is likely wrong. As fact, the inequality

$$E_{n,A,p} \ge \left(1 + \frac{|A|}{n}\right) E_{n,p}$$

does not hold for all $A \in \mathbb{R}$ and $n \in \mathbb{N}$. Indeed, if $n=1$ and $A \in (0,1)$, then

$$E_{1,p} = \left(\frac{2}{p+1}\right)^{1/p}, \quad (1+A)E_{1,p} = \left(\frac{2}{p+1}\right)^{1/p} + \left(\frac{2}{p+1}\right)^{1/p} A,$$

and

Please cite this article in press as: H. Wang, G. Xu, Sampling numbers of a class of infinitely differentiable functions, J. Math. Anal. Appl. (2019), https://doi.org/10.1016/j.jmaa.2019.123689

$$E_{1,A,p} = \left(\frac{(1+A)^{p+1}}{p+1} + \frac{(1-A)^{p+1}}{p+1} \right)^{1/p} = \left(\frac{2}{p+1} \right)^{1/p} + O(A^2), \text{ as } A \to 0+.$$

Hence, for sufficiently small $A > 0$, we have

$$E_{1,A,p} < (1+A)E_{1,p}.$$

References

[1] G. Byrenheid, D. Dũng, W. Sickel, T. Ullrich, Sampling on energy-norm based sparse grids for the optimal recovery of Sobolev type functions in H^γ, J. Approx. Theory 207 (2016) 207–231.
[2] G. Byrenheid, L. Kämmerer, T. Ullrich, T. Volkmer, Tight error bounds for rank-1 lattice sampling in spaces of hybrid mixed smoothness, Numer. Math. 136 (4) (2017) 993–1034.
[3] G. Byrenheid, T. Ullrich, Optimal sampling recovery of mixed order Sobolev embeddings via discrete Littlewood-Paley type characterizations, Anal. Math. 43 (2) (2017) 133–191.
[4] R.A. Devore, G.G. Lorentz, Constructive Approximation, Springer-Verlag, New York, 1993.
[5] J. Dick, P. Kritzer, F. Pillichshammer, H. Woźniakowski, Approximation of analytic functions in Korobov spaces, J. Complexity 30 (2014) 2–28.
[6] C.F. Dunki, Y. Xu, Orthogonal Polynomial of Several Variables, Cambridge University Press, 2001.
[7] D. Dũng, B-spline quasi-interpolant representations and sampling recovery of functions with mixed smoothness, J. Complexity 27 (2011) 541–567.
[8] D. Dũng, Optimal adaptive sampling recovery, Adv. Comput. Math. 34 (2011) 1–41.
[9] D. Dũng, Sampling and cubature on sparse grids based on a B-spline quasi-interpolation, Found. Comput. Math. 16 (2016) 1193–1240.
[10] D. Dũng, B-spline quasi-interpolation sampling representation and sampling recovery in Sobolev spaces of mixed smoothness, Acta Math. Vietnam. 43 (2018) 83–110.
[11] D. Dũng, V. Temlyakov, T. Ullrich, Hyperbolic Cross Approximation, Advanced Courses in Mathematics CRM Barcelona, Birkhäuser, 2018.
[12] G.S. Fang, F.J. Hickernell, H. Li, Approximation on anisotropic Besov classes with mixed norms by standard information, J. Complexity 21 (2005) 294–313.
[13] F.L. Huang, S. Zhang, Approximation of infinitely differentiable multivariate functions is not strongly tractable, J. Complexity 23 (2007) 73–81.
[14] C. Irrgeher, P. Kritzer, F. Pillichshammer, H. Woźniakowski, Approximation in Hermite spaces of smooth functions, J. Approx. Theory 207 (2016) 98–126.
[15] R. Kress, Numerical Analysis, Springer-Verlag, New York, 2003.
[16] P. Kritzer, F. Pillichshammer, H. Woźniakowski, $\mathbb{L}_\infty$-approximation in Korobov spaces with exponential weights, J. Complexity 31 (2015) 380–404.
[17] S.N. Kudryavtsev, The best accuracy of reconstruction of finitely smooth functions from their values at a given number of points, Izv. Math. 62 (1) (1998) 19–53.
[18] Y.P. Liu, G.Q. Xu, J. Zhang, Exponential convergence of an approximation problem for infinitely differentiable multivariate functions, Math. Notes 103 (5) (2018) 769–779.
[19] C.A. Micchelli, T.J. Rivlin, Lecture on Optimal Recovery, Lecture Notes in Mathematics, vol. 1129, Springer, Berlin, New York, 1985, pp. 21–93.
[20] E. Novak, H. Triebel, Function spaces in Lipschitz domains and optimal rates of convergence for sampling, Constr. Approx. 23 (2006) 325–350.
[21] E. Novak, H. Woźniakowski, Tractability of Multivariate Problems, vol. I: Linear Information, EMS, Zürich, 2008.
[22] E. Novak, H. Woźniakowski, Approximation of infinitely differentiable multivariate functions is intractable, J. Complexity 25 (2009) 398–404.
[23] E. Novak, H. Woźniakowski, Tractability of Multivariate Problems, vol. II: Standard Information for Functionals, EMS, Zürich, 2010.
[24] E. Novak, H. Woźniakowski, Tractability of Multivariate Problems, vol. III: Standard Information for Operators, EMS, Zürich, 2012.
[25] W. Sickel, T. Ullrich, Spline interpolation on sparse grids, Appl. Anal. 90 (3–4) (2011) 337–383.
[26] V. Temlyakov, Multivariate Approximation, Cambridge University Press, 2018.
[27] J.F. Traub, G.W. Wasilkowski, H. Woźniakowski, Information-Based Complexity, Academic Press, New York, 1988.
[28] H. Triebel, Bases in Function Spaces, Sampling, Discrepancy, Numerical Integration, EMS, Zürich, 2010.
[29] J. Vybíral, Sampling numbers and function spaces, J. Complexity 23 (2007) 773–792.
[30] H. Wang, K. Wang, Optimal recovery of Besov classes of generalized smoothness and Sobolev classes on the sphere, J. Complexity 32 (2016) 40–52.
[31] M. Weimar, Tractability results for weighted Banach spaces of smooth functions, J. Complexity 28 (2012) 59–75.
[32] G.Q. Xu, On weak tractability of the Smolyak algorithm for approximation problems, J. Approx. Theory 192 (2015) 347–361.

Please cite this article in press as: H. Wang, G. Xu, Sampling numbers of a class of infinitely differentiable functions, J. Math. Anal. Appl. (2019), https://doi.org/10.1016/j.jmaa.2019.123689