Doctopic: Partial Differential Equations ARTICLE IN PRESS YJMAA:123674

J. Math. Anal. Appl. ••• (••••) ••••••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Asymptotic problems for nonlinear ordinary differential equations with φ-Laplacian

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A R T I C L E I N F O

Article history:
Received 24 April 2019
Available online xxxx
Submitted by R. Popovych

Keywords:
Oscillation
Asymptotic behavior
Unbounded solutions
Weakly increasing solutions
Extremal solutions
Prescribed mean curvature equations

A B S T R A C T

This paper deals with the asymptotic problems for the nonlinear differential equation $(a(t)\varphi(x'))' + b(t)|x|^\gamma \operatorname{sgn} x = 0$ involving φ-Laplacian. Necessary and sufficient conditions are given for the oscillation of solutions of this equation. Moreover, we study the existence of unbounded solutions with different asymptotic behavior, in particular, weakly increasing solutions and extremal solutions. Examples for prescribed mean curvature equation are given to illustrate our results.

1. Introduction

Consider second-order nonlinear differential equations of the form

$$(a(t)\varphi(x'))' + b(t)|x|^\gamma \operatorname{sgn} x = 0, \quad t \ge t_0, \tag{1.1}$$

where $\varphi\colon \mathbb{R} \to (-\sigma, \sigma)$ with $0 < \sigma \le \infty$ is odd, continuous, strictly increasing, and bijective, $\varphi^{-1}\colon (-\sigma,\sigma) \to \mathbb{R}$ is the inverse function of φ with $\varphi^{-1}(\sigma) = \infty$, the real-valued functions $a(t)$ and $b(t)$ are positive and continuous on (t_0, ∞), and $\gamma \ne 1$ is a positive constant. Throughout this paper, we assume that

$$\int_{t_0}^{\infty} \frac{1}{a(t)}\, \mathrm{d}t = \infty \tag{1.2}$$

and

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https://doi.org/10.1016/j.jmaa.2019.123674

$$\int_{t_0}^{\infty} b(t)\,\mathrm{d}t < \infty \tag{1.3}$$

hold, and there exists a positive constant c such that $\varphi(u)$ satisfies

$$\lim_{u\to 0} \frac{\varphi(u)}{u} = c. \tag{1.4}$$

A function $x(t)$ is said to be a *solution* of equation (1.1), if $x(t)$ and its quasiderivative

$$x^{[1]}(t) = a(t)\varphi(x'(t))$$

are continuously differentiable, and $x(t)$ satisfies equation (1.1) on (t_x, ∞). In this paper, we assume that equation (1.1) possesses such a solution. A nontrivial solution $x(t)$ of equation (1.1) is said to be *oscillatory*, if there exists a sequence $\{t_n\}$ tending to ∞ such that $x(t_n) = 0$. Otherwise, that is, if $x(t)$ is eventually positive or eventually negative, it is said to be *nonoscillatory*.

The differential operator in equation (1.1) is called φ-Laplacian, which is a generalization of Laplacian, p-Laplacian, and mean curvature operator. Asymptotic problems associated to equation (1.1) arise in the search for radial solutions to partial differential equations which model fluid mechanics problems. In recent years, there has been an increasing interest in the study of asymptotic behavior for solutions of equations with φ-Laplacian (see [2,3,6–12]). In particular, Cecchi et al. [3] gave the sufficient conditions for equation (1.1) to have unbounded solutions under the assumption that the range of φ is bounded, for instance, when φ-Laplacian is the one-dimensional mean curvature operator. Furthermore, in [6], the oscillation criteria for equation (1.1) are obtained in the case when (1.3) does not hold.

Let $\varphi(u) = \varphi_p(u)$, where $\varphi_p(u) = |u|^{p-2}u$ $(p > 1)$. Then, the range of φ is unbounded, and φ-Laplacian becomes p-Laplacian. In this case, oscillation problems and asymptotic problems for equation (1.1) have been studied in various papers (see [8,9,19] and references therein). Moreover, if $p = 2$, then equation (1.1) becomes the so-called generalized Emden–Fowler equation

$$(a(t)x')' + b(t)|x|^{\gamma} \operatorname{sgn} x = 0, \quad \gamma \neq 1. \tag{1.5}$$

The study of equation (1.5) originates from gas dynamics in astrophysics. Equation (1.5) is also related to the model of the concentration of a substance disappearing according to an isothermal reaction in an finite slab of catalyst (see [15]). We note that the asymptotic properties of the solutions correspond to the case when the ratio of the characteristic reaction rate to the characteristic diffusion rate is infinite. Hence, a great deal of articles has been devoted to the study of equation (1.5), for example, those results can be found in [1,15,16,20] and the references cited therein. Especially, the following necessary and sufficient conditions are known for all nontrivial solutions of equation (1.5) to be oscillatory.

Theorem A *([16, Corollaries 11.1 and 11.3]). Suppose that (1.2) and (1.3) are satisfied. Then all nontrivial solutions of equation (1.5) are oscillatory if and only if either*

(I) $\gamma > 1$ *and*

$$\int_{t_0}^{\infty} \frac{1}{a(t)} \left(\int_{t}^{\infty} b(s)\,\mathrm{d}s \right) \mathrm{d}t = \infty \tag{1.6}$$

hold, or

Please cite this article in press as: Z. Došlá, K. Fujimoto, Asymptotic problems for nonlinear ordinary differential equations with φ-Laplacian, J. Math. Anal. Appl. (2019), https://doi.org/10.1016/j.jmaa.2019.123674

(II) $0 < \gamma < 1$ *and*

$$\int_{t_0}^{\infty} b(t) \left(\int_{t_0}^{t} \frac{1}{a(s)} \, \mathrm{d}s \right)^{\gamma} \mathrm{d}t = \infty \tag{1.7}$$

hold.

Remark 1.1. From [16, Corollary 6.1], if $\int_{t_0}^{\infty} 1/a(t)\,\mathrm{d}t$ and $\int_{t_0}^{\infty} b(t)\,\mathrm{d}t$ diverge, then all nontrivial solutions of equation (1.5) are oscillatory. On the other hand, if both integrals converge, then all nonoscillatory solutions and their quasiderivatives are bounded (see [14]). Hence, the assumptions (1.2) and (1.3), which include the classical case that $a(t) \equiv 1$, are suitable in a certain sense.

If $x(t)$ is an eventually negative solution of equation (1.1), then $-x(t)$ is an eventually positive solution of equation (1.1). Therefore, we will restrict our attention only to eventually positive solutions of equation (1.1) when we discuss the nonoscillatory solutions. According to Lemma 2.1 below, we see that all eventually positive solutions of equation (1.1) are increasing. In this paper, we focus on the unbounded positive increasing solution x such that $\lim_{t\to\infty} x'(t)$ exists. We have the following three types of such solutions. It is said to be a *weakly increasing solution*, if $x'(t) \to 0$ as $t \to \infty$. It is said to be an *asymptotically linear solution*, if there exists $0 < c_x < \infty$ such that $x'(t) \to c_x$ as $t \to \infty$. It is said to be an *extremal solution*, if $x'(t) \to \infty$ as $t \to \infty$.

Naito [17] gave sufficient conditions for the existence of weakly increasing solutions of equation (1.5) with $\gamma > 1$ and $a(t) \equiv 1$. Moreover, in [8,9], sufficient conditions for the existence of weakly increasing solutions of equation (1.1) with $\varphi(u) = \varphi_p(u)$ were presented. On the other hand, from Lemma 2.2 below, there are no extremal solutions of equation (1.1), if the range of φ is unbounded and

$$\liminf_{t\to\infty} a(t) > 0 \tag{1.8}$$

is satisfied. Here, a natural question now arises. There exist extremal solutions for equation (1.1) if the range of φ is bounded? The purpose of this paper is to answer the question. Moreover, we discuss the oscillation problem and the existence of weakly increasing solutions of equation (1.1). To be precise, we give an analogue of Theorem A, and sufficient conditions for the existence of weakly increasing solutions and extremal solutions of equation (1.1). Furthermore, we consider the coexistence of other types of unbounded solutions of equation (1.1). Our results are also motivated by [4,5], in which the oscillation criteria and the asymptotic behavior of solutions of corresponding difference equations are considered.

This paper is organized as follows. In Section 2, we give equivalence theorems, which are analogues of Theorem A. In Section 3, we show the existence of weakly increasing solutions of equation (1.1). In Section 4, we give sufficient conditions for the existence of extremal solutions of equation (1.1). Our proofs are based on the Tychonov fixed point theorem. Finally, in Section 5, we consider a special case and give some examples. Moreover, we propose some open problems which include the coexistence of weakly increasing solutions and extremal solutions.

2. Oscillation

In this section, we establish oscillation criteria for solutions of equation (1.1) by comparing with solutions of equation (1.5).

Theorem 2.1. *Assume* (1.2), (1.3), (1.4), *and* (1.8). *Suppose that* $\gamma > 1$ *is satisfied. Then the following statements are equivalent.*

(i) *All nontrivial solutions of equation* (1.1) *are oscillatory.*
(ii) *All nontrivial solutions of equation* (1.5) *are oscillatory.*
(iii) (1.6) *holds.*

Theorem 2.2. *Assume* (1.2), (1.3), (1.4), *and* (1.8). *Suppose that* $0 < \gamma < 1$ *is satisfied. Then the following statements are equivalent.*

(i) *All nontrivial solutions of equation* (1.1) *are oscillatory.*
(ii) *All nontrivial solutions of equation* (1.5) *are oscillatory.*
(iii) (1.7) *holds.*

In order to give the proofs of these results, the following lemmas are required.

Lemma 2.1. *Assume* (1.2), (1.3), *and* (1.4). *Then, for any positive solution* $x(t)$ *of equation* (1.1), $x^{[1]}(t)$ *is bounded and* x *is increasing.*

Proof. From (1.4), we can choose $h > 0$ and $0 < \delta < \sigma$ such that

$$\varphi^{-1}(u) \geq hu \tag{2.1}$$

for $u \in [0, \delta]$, where $\varphi^{-1} \colon (-\sigma, \sigma) \to \mathbb{R}$ with $\varphi^{-1}(\sigma) = \infty$ is the inverse function of φ. Let $x(t)$ be a positive solution of equation (1.1). From equation (1.1), $x^{[1]}(t)$ is decreasing for t sufficiently large. If $x^{[1]}(t)$ is eventually negative, then there exists $t_1 \geq t_0$ such that $x(t) > 0$ and $x'(t) < 0$ for $t \geq t_1$. Integrating both sides of equation (1.1), we have

$$x^{[1]}(t) \geq x^{[1]}(t_1) - x(t_1) \int_{t_1}^{t} b(s)\, \mathrm{d}s.$$

From (1.3), we see that $x^{[1]}(t)$ is bounded.

Suppose that there exists $t_2 \geq t_0$ such that $x^{[1]}(t_2) \leq 0$. Then there exists $t_3 \geq t_2$ such that $x^{[1]}(t) \leq x^{[1]}(t_3) < 0$ for $t \geq t_3$ because $x^{[1]}(t)$ is decreasing. Hence, we get

$$\varphi(x'(t)) < \frac{x^{[1]}(t_3)}{a(t)} < 0 \tag{2.2}$$

for $t \geq t_3$. In the case when $\liminf_{t\to\infty} a(t) = 0$, we obtain a contradiction to $\sigma < \infty$. We consider the case of (1.8). In this case, there exist $t_4 \geq t_3$ and $0 < \lambda_0 < |x^{[1]}(t_3)|$ such that $\lambda_0/a(t) < \delta$ for $t \geq t_4$. Then, from (2.1) and (2.2), we have

$$x'(t) < -\varphi^{-1}\left(\frac{\lambda_0}{a(t)}\right) \leq -\frac{h\lambda_0}{a(t)}$$

for $t \geq t_4$. Integrating both sides of this inequality from t_4 to t, we get

$$x(t) < x(t_4) - h\lambda_0 \int_{t_4}^{t} \frac{1}{a(s)}\, \mathrm{d}s$$

for $t \geq t_4$. From (1.2), we obtain $x(t) \to -\infty$ as $t \to \infty$, which is a contradiction to the positivity of $x(t)$. Thus, we see that $x^{[1]}(t)$ is positive, and therefore, $x'(t)$ is also positive. □

Lemma 2.2. *Assume* (1.2), (1.3), (1.4), *and* (1.8). *Then, for any positive solution $x(t)$ of equation* (1.1), *there exists a positive constant c_0 such that*

$$\frac{x'(t)}{\varphi(x'(t))} \geq c_0 \tag{2.3}$$

for t sufficiently large. Moreover, if $\sigma = \infty$, then equation (1.1) *has no extremal solutions.*

Proof. From Lemma 2.1, $x'(t)$ is positive for t sufficiently large and $x^{[1]}(t)$ is bounded from above. Hence, it follows from (1.8) that $\varphi(x'(t))$ is bounded from above.

We first consider the case when $\limsup_{t\to\infty} x'(t) < \infty$. It is easy to check that (2.3) holds if $\liminf_{t\to\infty} x'(t) > 0$. Moreover, from (1.4), it also holds if $\liminf_{t\to\infty} x'(t) = 0$.

We next consider the case when $\limsup_{t\to\infty} x'(t) = \infty$. In this case, since $\varphi(x'(t))$ is bounded from above, we have $\sigma < \infty$, which implies

$$\lim_{u\to\infty} \frac{u}{\varphi(u)} > \lim_{u\to\infty} \frac{u}{\sigma} = \infty.$$

In the same manner as above, we obtain (2.3). Moreover, we see that if $\sigma = \infty$, then $\limsup_{t\to\infty} x'(t) < \infty$, and therefore, equation (1.1) has no extremal solutions. □

To prove Theorem 2.1, we need the following lemma in addition to Lemmas 2.1 and 2.2.

Lemma 2.3. *Assume* (1.2), (1.3), (1.4), *and* (1.8). *If*

$$\int_{t_0}^{\infty} \frac{1}{a(t)} \left(\int_t^{\infty} b(s)\,\mathrm{d}s \right) \mathrm{d}t < \infty \tag{2.4}$$

holds, then equation (1.1) *has a bounded nonoscillatory solution.*

Proof. From (1.4), we can find $H > 0$ and $0 < \delta < \sigma$ such that

$$\varphi^{-1}(u) \leq Hu \tag{2.5}$$

for $u \in [0, \delta]$. By using (1.3) and (1.8), we get

$$\lim_{t\to\infty} \frac{1}{a(t)} \left(\int_t^{\infty} b(s)\,\mathrm{d}s \right) = 0.$$

Hence there exists $t_1 \geq t_0$ such that

$$\frac{1}{a(t)} \left(\int_t^{\infty} b(s)\,\mathrm{d}s \right) < \delta$$

for $t \geq t_1$. Using (2.4) and (2.5), we obtain

$$\int_{t_1}^{\infty} \varphi^{-1} \left(\frac{1}{a(t)} \left(\int_t^{\infty} b(s)\,\mathrm{d}s \right) \right) \mathrm{d}t < H \int_{t_1}^{\infty} \frac{1}{a(t)} \left(\int_t^{\infty} b(s)\,\mathrm{d}s \right) \mathrm{d}t < \infty.$$

According to Cecchi et al. [3, Theorem 4.1 (i_2)], we see that equation (1.1) has a bounded nonoscillatory solution. □

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. Theorem A shows that (ii) is equivalent to (iii). From the contrapositive of Lemma 2.3, it is easy to show that (i) implies (iii).

We prove that (iii) implies (i) by contradiction. Suppose that there exists a positive solution $x(t)$ of equation (1.1). Then, from Lemma 2.1, there exists $t_1 \geq t_0$ such that $x'(t) > 0$ for $t \geq t_1$. Consider the equation

$$(A(t)z')' + b(t)|z|^\gamma \operatorname{sgn} z = 0, \tag{2.6}$$

where $A(t) = a(t)\varphi(x'(t))/x'(t)$. From Lemma 2.2, (1.2), and (1.6), we have

$$\int_{t_1}^{\infty} \frac{1}{A(t)}\,\mathrm{d}t = \int_{t_1}^{\infty} \frac{1}{a(t)} \cdot \frac{x'(t)}{\varphi(x'(t))}\,\mathrm{d}t \geq c_0 \int_{t_1}^{\infty} \frac{1}{a(t)}\,\mathrm{d}t = \infty$$

and

$$\int_{t_1}^{\infty} \frac{1}{A(t)} \left(\int_t^{\infty} b(s)\,\mathrm{d}s \right) \mathrm{d}t \geq c_0 \int_{t_1}^{\infty} \frac{1}{a(t)} \left(\int_t^{\infty} b(s)\,\mathrm{d}s \right) \mathrm{d}t = \infty.$$

Using Theorem A again, we see that all nontrivial solutions of equation (2.6) are oscillatory. However, $x(t)$ is a positive solution of equation (2.6) because it is one of equation (1.1). This is a contradiction. □

To prove Theorem 2.2, we need the following lemma in addition to Lemmas 2.1 and 2.2.

Lemma 2.4. *Assume* (1.4) *and* (1.8). *If*

$$\int_{t_0}^{\infty} b(t) \left(\int_{t_0}^{t} \frac{1}{a(s)}\,\mathrm{d}s \right)^{\gamma} \mathrm{d}t < \infty \tag{2.7}$$

holds, then equation (1.1) *has a unbounded nonoscillatory solution* $x(t)$ *such that* $x^{[1]}(t) \to d_x$ *as* $t \to \infty$ *with* $0 < d_x < \infty$.

Proof. From (1.8), for any small $\varepsilon > 0$, there exists $t_1 \geq t_0$ such that $a(t) \geq \varepsilon$ for $t \geq t_1$. Moreover, from (1.4), we can find $H > 0$ and $0 < \delta < \sigma$ satisfying (2.5) for $u \in [0, \delta]$. By using (2.7), we see that the left-hand side of (2.7) tends to 0 as $t_0 \to \infty$, and therefore, there exists $t_2 \geq t_1$ such that

$$\int_{t_2}^{\infty} b(t) \left(\int_{t_2}^{t} \frac{1}{a(s)}\,\mathrm{d}s \right)^{\gamma} \mathrm{d}t \leq \frac{1}{(HW)^\gamma} \frac{W}{2}, \tag{2.8}$$

where $W = \varepsilon\delta$.

Let $\mathbb{X}$ be a Fréchet space of all continuous functions defined for any $t \geq t_2$ endowed with the topology of uniform convergence on compact subintervals of $[t_2, \infty)$, and we put $\Omega \subset \mathbb{X}$ be

$$\Omega = \left\{ u \in \mathbb{X} \mid \frac{W}{2} \le u(t) \le W \right\}.$$

Let $\mathscr{T} : \Omega \to \mathbb{X}$ be an operator defined by

$$\mathscr{T}(u)(t) = \frac{W}{2} + \int_t^\infty b(s) \left(\int_{t_2}^s \varphi^{-1}\left(\frac{u(\tau)}{a(\tau)} \right) \mathrm{d}\tau \right)^\gamma \mathrm{d}s.$$

Then $\mathscr{T}$ is well defined because

$$\frac{u(t)}{a(t)} \le \frac{W}{a(t)} \le \frac{W}{\varepsilon} = \delta < \sigma \tag{2.9}$$

for $t \ge t_2$. From (2.5) and (2.8), we have

$$\begin{aligned} \mathscr{T}(u)(t) &\le \frac{W}{2} + \int_t^\infty b(s) \left(\int_{t_2}^s \varphi^{-1}\left(\frac{W}{a(\tau)} \right) \mathrm{d}\tau \right)^\gamma \mathrm{d}s \le \frac{W}{2} + \int_t^\infty b(s) \left(\int_{t_2}^s \frac{HW}{a(\tau)} \mathrm{d}\tau \right)^\gamma \mathrm{d}s \\ &= \frac{W}{2} + (HW)^\gamma \int_t^\infty b(s) \left(\int_{t_2}^s \frac{1}{a(\tau)} \mathrm{d}\tau \right)^\gamma \mathrm{d}s \le \frac{W}{2} + (HW)^\gamma \frac{1}{(HW)^\gamma} \frac{W}{2} = W. \end{aligned} \tag{2.10}$$

Hence we see that $\mathscr{T}(\Omega) \subset \Omega$ and it is uniformly bounded.

We prove that $\mathscr{T}(\Omega)$ is relatively compact. Let $u \in \Omega$. Then, for any $t \ge t_2$ and $\tilde{t} \ge t_2$, we have

$$|\mathscr{T}(u)(t) - \mathscr{T}(u)(\tilde{t})| = \left| \int_t^{\tilde{t}} b(s) \left(\int_{t_2}^s \varphi^{-1}\left(\frac{u(\tau)}{a(\tau)} \right) \mathrm{d}\tau \right)^\gamma \mathrm{d}s \right| \le \left| (HW)^\gamma \int_t^{\tilde{t}} b(s) \left(\int_{t_2}^s \frac{1}{a(\tau)} \mathrm{d}\tau \right)^\gamma \mathrm{d}s \right|.$$

From (2.7), for any $\tilde{\varepsilon} > 0$, there exists $\delta_{\tilde{\varepsilon}} > 0$ such that $|\mathscr{T}(u)(t) - \mathscr{T}(u)(\tilde{t})| < \tilde{\varepsilon}$ if $|t - \tilde{t}| < \delta_{\tilde{\varepsilon}}$, that is, $\mathscr{T}(\Omega)$ is equicontinuous. From the Ascoli theorem, we see that $\mathscr{T}(\Omega)$ is relatively compact.

We next give the proof of the continuity of $\mathscr{T}$ in Ω. Let $\{u_n\}$ $(n \in \mathbb{N})$ be a sequence in Ω which uniformly converges on every compact subinterval of $[t_2, \infty)$ to $\bar{u} \in \Omega$. Since $\mathscr{T}(\Omega)$ is relatively compact, the sequence $\{\mathscr{T}(u_n)\}$ admits a subsequence which converges to $\bar{z}_u \in \overline{\mathscr{T}(\Omega)}$ in the topology of $\mathbb{X}$. For simplicity, let $\{u_n\}$ be such a sequence and let

$$z_n(t) = b(t) \left(\int_{t_2}^t \varphi^{-1}\left(\frac{u_n(s)}{a(s)} \right) \mathrm{d}s \right)^\gamma.$$

Then, from (2.9), we get

$$\int_{t_2}^t \varphi^{-1}\left(\frac{u_n(s)}{a(s)} \right) \mathrm{d}s < \varphi^{-1}(\delta)(t - t_2)$$

for each fixed $t \ge t_2$. Hence, $\{z_n\}$ is an uniformly integrable sequence on $[t_2, t]$, and using the Vitali convergence theorem, we get

$$\lim_{n\to\infty} z_n(t) = b(t) \left(\int_{t_2}^t \varphi^{-1}\left(\frac{\bar{u}(s)}{a(s)} \right) \mathrm{d}s \right)^\gamma$$

(see [18]). Moreover, in view of $|u_n| < W$, proceeding as in (2.10), we have

$$b(t)\left(\int_{t_2}^{t} \varphi^{-1}\left(\frac{u_n(s)}{a(s)}\right) \mathrm{d}s\right)^{\gamma} \le (HW)^{\gamma} b(t)\left(\int_{t_2}^{t} \frac{1}{a(s)} \mathrm{d}s\right)^{\gamma}.$$

From (2.7) and the Lebesgue dominated convergence theorem, we see that $\{\mathscr{T}(u_n)\}$ pointwise converges to $\mathscr{T}(\bar{u})(t)$. Since $\mathscr{T}(\bar{u}) = \bar{z}_u$ is the only one cluster point of the compact sequence $\{\mathscr{T}(u_n)\}$, $\mathscr{T}$ is continuous in the topology of $\mathbb{X}$.

From the Tychonov fixed point theorem, there exist $\tilde{u} \in \Omega$ such that $\mathscr{T}(\tilde{u}) = \tilde{u}$ (see [13]). Let

$$x(t) = \int_{t_2}^{t} \varphi^{-1}\left(\frac{\tilde{u}(s)}{a(s)}\right) \mathrm{d}s.$$

Then we have

$$x'(t) = \varphi^{-1}\left(\frac{\tilde{u}(t)}{a(t)}\right),$$

which implies $a(t)\varphi(x'(t)) = \tilde{u}(t) = \mathscr{T}(\tilde{u})(t)$. Thus we get

$$(a(t)\varphi(x'(t)))' = (\mathscr{T}(\tilde{u})(t))' = -b(t)\left(\int_{t_2}^{t} \varphi^{-1}\left(\frac{\tilde{u}(s)}{a(s)}\right) \mathrm{d}s\right)^{\gamma} = -b(t)(x(t))^{\gamma},$$

and therefore, $x(t)$ is a positive solution of equation (1.1). Since $x^{[1]}(t)$ is decreasing and satisfies

$$x^{[1]}(t) = a(t)\varphi(x'(t)) = \tilde{u}(t) \ge \frac{W}{2},$$

there exists $d_x > 0$ such that $x^{[1]}(t) \to d_x$ as $t \to \infty$. $\square$

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2. Theorem A shows that (ii) is equivalent to (iii). From the contrapositive of Lemma 2.4, it is easy to show that (i) implies (iii).

We prove that (iii) implies (i) by contradiction. Suppose that there exists a positive solution $x(t)$ of equation (1.1). Then, from Lemma 2.1, there exists $t_1 \ge t_0$ such that $x'(t) > 0$ for $t \ge t_1$. Consider the equation (2.6). From Lemma 2.2, (1.2), and (1.7), we have

$$\int_{t_1}^{\infty} \frac{1}{A(t)} \mathrm{d}t = \int_{t_1}^{\infty} \frac{1}{a(t)} \cdot \frac{x'(t)}{\varphi(x'(t))} \mathrm{d}t \ge c_0 \int_{t_1}^{\infty} \frac{1}{a(t)} \mathrm{d}t = \infty$$

and

$$\int_{t_1}^{\infty} b(t)\left(\int_{t_1}^{t} \frac{1}{A(s)} \mathrm{d}s\right)^{\gamma} \mathrm{d}t \ge c_0 \int_{t_1}^{\infty} b(t)\left(\int_{t_1}^{t} \frac{1}{a(s)} \mathrm{d}s\right)^{\gamma} \mathrm{d}t = \infty.$$

Using Theorem A again, we see that all nontrivial solutions of equation (2.6) are oscillatory. However, $x(t)$ is a nonoscillatory solution of equation (2.6) because it is one of equation (1.1). This is a contradiction. $\square$

Remark 2.1. We can relax the condition (1.4) in Theorems 2.1 and 2.2, and Lemmas 2.1, 2.3, and 2.4 as

$$0 < \liminf_{u \to 0} \frac{\varphi(u)}{u} \quad \text{and} \quad \limsup_{u \to 0} \frac{\varphi(u)}{u} < \infty. \tag{2.11}$$

3. Weakly increasing solutions

In this section, we give sufficient conditions for the existence of weakly increasing solutions of equation (1.1), that is, solutions $x(t)$ such that $x(t) \to \infty$ and $x'(t) \to 0$ as $t \to \infty$.

Theorem 3.1. *Assume that $a(t) \to \alpha > 0$ as $t \to \infty$. Suppose that*

$$\int_{t_0}^{\infty} b(t) t^{\gamma} \, \mathrm{d}t < \infty \tag{3.1}$$

and

$$\int_{t_0}^{\infty} \varphi^{-1} \left(\int_{t}^{\infty} b(s) \, \mathrm{d}s \right) \mathrm{d}t = \infty \tag{3.2}$$

are satisfied, where $\varphi^{-1} \colon (-\sigma, \sigma) \to \mathbb{R}$ with $\varphi^{-1}(\sigma) = \infty$ is the inverse function of φ. Then there exists a weakly increasing solution of equation (1.1).

Remark 3.1. From (1.3), we can choose t_0 so large that $\int_{t_0}^{\infty} b(s) \, \mathrm{d}s < \sigma$ holds.

For the proof of Theorem 3.1, we need the following result.

Lemma 3.1. *Assume that $a(t) \to \alpha > 0$ as $t \to \infty$. Suppose that (3.1) is satisfied. Then, for all $L \geq 0$, there exists a solution $x(t)$ of equation (1.1) satisfying $x'(t) \to L$ as $t \to \infty$ and*

$$x(t) = (\alpha + 1)^{1/\gamma} + \int_{T}^{t} \varphi^{-1} \left(\frac{1}{a(s)} \left(\alpha \varphi(L) + \int_{s}^{\infty} b(\tau) (x(\tau))^{\gamma} \, \mathrm{d}\tau \right) \right) \mathrm{d}s$$

for T sufficiently large.

Proof. Let $h > L$. Then there exists $0 < \eta < 1$ such that $\varphi(L) < (1 - \eta)\varphi(h)$ because φ is increasing. Hence we have $\alpha(1 - \eta)\varphi(h) - \alpha\varphi(L) > 0$. From (3.1) and $a(t) \to \alpha$ as $t \to \infty$, there exists $t_1 \geq t_0$ such that

$$1 + \frac{(\alpha + 1)^{1/\gamma}}{h} < t_1, \tag{3.3}$$

$$\int_{t_1}^{\infty} b(t) (ht)^{\gamma} \, \mathrm{d}t < \alpha(1 - \eta)\varphi(h) - \alpha\varphi(L), \tag{3.4}$$

and

$$\alpha(1 - \eta) \leq a(t) \leq \alpha(1 + \eta) \tag{3.5}$$

hold for $t \geq t_1$. Let $\mathbb{X}$ be a Fréchet space of all continuous functions defined for any $t \geq t_1$ endowed with the topology of uniform convergence on compact subintervals of $[t_1, \infty)$, and we put $\Omega \subset \mathbb{X}$ be

$$\Omega = \left\{ u \in \mathbb{X} \mid (\alpha+1)^{1/\gamma} \leq u(t) \leq (\alpha+1)^{1/\gamma} + h(t-t_1) \right\}.$$

Let $\mathscr{T} : \Omega \to \mathbb{X}$ be an operator defined by

$$\mathscr{T}(u)(t) = (\alpha+1)^{1/\gamma} + \int_{t_1}^{t} \varphi^{-1}\left(\frac{1}{a(s)} \left(\alpha\varphi(L) + \int_{s}^{\infty} b(\tau)(u(\tau))^{\gamma}\, \mathrm{d}\tau \right) \right) \mathrm{d}s.$$

In order to prove that $\mathscr{T}$ is well defined, we show that the positive function

$$z(t) = \frac{1}{a(t)} \left(\alpha\varphi(L) + \int_{t}^{\infty} b(s)(u(s))^{\gamma}\, \mathrm{d}s \right)$$

satisfies $z(t) < \sigma$ for $t \geq t_1$. By using (3.3) and (3.4), we have

$$\begin{aligned} \int_{t}^{\infty} b(s)(u(s))^{\gamma}\, \mathrm{d}s &\leq \int_{t_1}^{\infty} b(s)(u(s))^{\gamma}\, \mathrm{d}s \leq \int_{t_1}^{\infty} b(s) \left((\alpha+1)^{1/\gamma} + h(s-t_1) \right)^{\gamma} \mathrm{d}s \\ &< \int_{t_1}^{\infty} b(s) \left((\alpha+1)^{1/\gamma} + h\left(s - \left(1 + \frac{(\alpha+1)^{1/\gamma}}{h} \right) \right) \right)^{\gamma} \mathrm{d}s \\ &= \int_{t_1}^{\infty} b(s)(hs-h)^{\gamma}\, \mathrm{d}s \leq \int_{t_1}^{\infty} b(s)(hs)^{\gamma}\, \mathrm{d}s < \alpha(1-\eta)\varphi(h) - \alpha\varphi(L). \end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we get

$$z(t) < \frac{1}{\alpha(1-\eta)} (\alpha\varphi(L) + \alpha(1-\eta)\varphi(h) - \alpha\varphi(L)) = \varphi(h), \tag{3.7}$$

and therefore, $\mathscr{T}$ is well defined. Moreover, from (3.7), we get

$$\mathscr{T}(u)(t) = (\alpha+1)^{1/\gamma} + \int_{t_1}^{t} \varphi^{-1}(z(s))\, \mathrm{d}s < (\alpha+1)^{1/\gamma} + h(t-t_1),$$

which implies $\mathscr{T}(\Omega) \subset \Omega$ and it is uniformly bounded on every compact subinterval of $[t_1, \infty)$.

We prove that $\mathscr{T}(\Omega)$ is relatively compact. Let $u \in \Omega$. Then, for any $t \geq t_1$ and $\tilde{t} \geq t_1$, we have

$$|\mathscr{T}(u)(t) - \mathscr{T}(u)(\tilde{t})| = \left| \int_{\tilde{t}}^{t} \varphi^{-1}(z(s))\, \mathrm{d}s \right| \leq \left| h(t-\tilde{t}) \right|$$

because of (3.7). Hence, for any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that $|\mathscr{T}(u)(t) - \mathscr{T}(u)(\tilde{t})| < \varepsilon$ if $|t - \tilde{t}| < \delta_\varepsilon$, that is, $\mathscr{T}(\Omega)$ is equicontinuous on every compact subinterval of $[t_1, \infty)$. Using the Ascoli theorem, we see that $\mathscr{T}(\Omega)$ is relatively compact. Moreover, we can show that $\mathscr{T}$ is continuous in the topology of $\mathbb{X}$ as in the proof of Lemma 2.4.

Please cite this article in press as: Z. Došlá, K. Fujimoto, Asymptotic problems for nonlinear ordinary differential equations with φ-Laplacian, J. Math. Anal. Appl. (2019), https://doi.org/10.1016/j.jmaa.2019.123674

From the Tychonov fixed point theorem, there exists $\tilde{u} \in \Omega$ such that $\mathscr{T}(\tilde{u}) = \tilde{u}$. Then we have

$$\tilde{u}'(t) = \varphi^{-1}\left(\frac{1}{a(t)}\left(\alpha\varphi(L) + \int_t^\infty b(s)(\tilde{u}(s))^\gamma \,\mathrm{d}s\right)\right). \tag{3.8}$$

Thus we get

$$a(t)\varphi(\tilde{u}'(t)) = \alpha\varphi(L) + \int_t^\infty b(s)(\tilde{u}(s))^\gamma \,\mathrm{d}s,$$

and therefore, we obtain $(a(t)\varphi(\tilde{u}'(t)))' = -b(t)(\tilde{u}(t))^\gamma$. Hence $\tilde{u}(t)$ is a solution of equation (1.1). Since $\tilde{u} \in \Omega$, we have

$$0 < \int_t^\infty b(s)(\tilde{u}(s))^\gamma \,\mathrm{d}s \le \int_t^\infty b(s)((\alpha+1)^{1/\gamma} + h(s-t_1))^\gamma \,\mathrm{d}s \to 0$$

as $t \to \infty$. Together with (3.8) and $a(t) \to \alpha$ as $t \to \infty$, we see that $\tilde{u}'(t) \to L$ as $t \to \infty$. $\square$

Proof of Theorem 3.1. From Lemma 3.1 with $L = 0$, there exists a solution $x(t)$ of equation (1.1) satisfying $x'(t) \to 0$ as $t \to \infty$ and

$$x(t) = (\alpha+1)^{1/\gamma} + \int_T^t \varphi^{-1}\left(\frac{1}{a(s)}\int_s^\infty b(\tau)(x(\tau))^\gamma \,\mathrm{d}\tau\right)\mathrm{d}s$$

for T sufficiently large.

We prove that $x(t)$ is an unbounded solution. There exists $t_1 \ge T$ such that $\alpha + 1 > a(t)$ and $x(t) \ge (\alpha+1)^{1/\gamma}$ for $t \ge t_1$. Hence, we have

$$x(t) \ge (\alpha+1)^{1/\gamma} + \int_{t_1}^t \varphi^{-1}\left(\frac{\alpha+1}{a(s)}\int_s^\infty b(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s > (\alpha+1)^{1/\gamma} + \int_{t_1}^t \varphi^{-1}\left(\int_s^\infty b(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s.$$

From (3.2), we see that $x(t) \to \infty$ as $t \to \infty$. This completes the proof of Theorem 3.1. $\square$

Remark 3.2. We observe that neither (1.4) nor (2.11) are required in Theorem 3.1. However, if assumptions of Theorem 3.1 hold together with

$$\liminf_{u\to 0} \frac{\varphi(u)}{u} > 0, \tag{3.9}$$

then, from Fubini's theorem, we can find positive constants $t_1 \ge t_0$ and c_1 such that

$$\int_{t_1}^\infty \varphi^{-1}\left(\int_t^\infty b(s)\,\mathrm{d}s\right)\mathrm{d}t < c_1\int_{t_1}^\infty \left(\int_t^\infty b(s)\,\mathrm{d}s\right)\mathrm{d}t = c_1\int_{t_1}^\infty \left(b(t)\int_{t_1}^t \mathrm{d}s\right)\mathrm{d}t = c_1\int_{t_1}^\infty b(t)(t-t_1)\,\mathrm{d}t.$$

Hence, from (3.1) and (3.2), it is necessary that $0 < \gamma < 1$ holds. Consequently, if (3.9) holds, then assumptions of Theorem 3.1 implies $0 < \gamma < 1$.

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Remark 3.3. In [19], Theorem 3.1 was proved for equation (1.1) with $\varphi(u) = \varphi_p(u)$ and $a(t) \equiv 1$. We see that (3.9) holds if $p \geq 2$. Hence, in particular, the existence of weakly increasing solutions for (1.5) with $0 < \gamma < 1$ follows from [19, Theorem 1.3].

We also obtain the following result, which extends [3, Theorem 3.1 (i_1)].

Theorem 3.2. *Assume that $a(t) \to \alpha > 0$ as $t \to \infty$. Then,* (3.1) *is necessary and sufficient condition for the existence of asymptotically linear solutions of equation* (1.1)*.*

Proof. From Lemma 3.1, (3.1) implies the existence of asymptotically linear solutions of equation (1.1).

Let $x(t)$ be an asymptotically linear solution of equation (1.1) satisfying $x'(t) \to L > 0$ as $t \to \infty$. Then there exist $t_1 \geq t_0$ and $0 < L_0 \leq L$ such that $x(t) \geq L_0 t$ for $t \geq t_1$. Integrating both sides of equation (1.1) from t_1 to ∞, we get

$$a(t_1)\varphi(x'(t_1)) = \alpha\varphi(L) + \int_{t_1}^{\infty} b(s)(x(s))^\gamma \,\mathrm{d}s \geq \alpha\varphi(L) + \int_{t_1}^{\infty} b(s)(L_0 t)^\gamma \,\mathrm{d}s,$$

which implies (3.1). □

From Theorems 3.1 and 3.2, we obtain the following corollary.

Corollary 3.1. *Assume $a(t) \to \alpha > 0$ as $t \to \infty$,* (3.1)*, and* (3.2)*. Then asymptotically linear solutions and weakly increasing solutions of* (1.1) *coexist.*

4. Extremal solutions

In this section, we give the sufficient conditions for the existence of extremal solutions of equation (1.1), that is, solution $x(t)$ such that $x(t) \to \infty$ and $x'(t) \to \infty$ as $t \to \infty$. In view of Lemma 2.2, such solutions may exist only for bounded φ-Laplacian.

Theorem 4.1. *Assume that $\sigma < \infty$, $a(t)$ is decreasing, and $a(t) \to \alpha > 0$ as $t \to \infty$. Suppose that there exists $0 < \varepsilon < \alpha\sigma$ satisfying*

$$\frac{1}{a(t) - \alpha} \int_t^{\infty} b(s) \left(a(t_0)\Phi(s)\right)^\gamma \,\mathrm{d}s < \frac{\varepsilon^\gamma}{\sigma^{\gamma-1}} \tag{4.1}$$

and

$$\varphi^{-1}\left(\frac{1}{a(t)}\left(\alpha\sigma + \int_t^{\infty} b(s)\left(\frac{a(t_0)\sigma}{\varepsilon}\Phi(s)\right)^\gamma \,\mathrm{d}s\right)\right) < \frac{1}{\varepsilon}\left(\alpha\sigma + \int_t^{\infty} b(s)\left(\Phi(s)\right)^\gamma \,\mathrm{d}s\right)\varphi^{-1}\left(\frac{\alpha\sigma}{a(t)}\right) \tag{4.2}$$

for any $t \geq t_0$, where

$$\Phi(t) = \int_{t_0}^{t} \varphi^{-1}\left(\frac{\alpha\sigma}{a(s)}\right) \mathrm{d}s.$$

Then equation (1.1) *has an extremal solution.*

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Proof. We note that $a(t_0)\sigma/\varepsilon > 1$ and $\Phi(t) \to \infty$ as $t \to \infty$. Let $\mathbb{X}$ be a Fréchet space of all continuous functions defined for any $t \geq t_0$ endowed with the topology of uniform convergence on compact subintervals of $[t_0, \infty)$, and we put $\Omega \subset \mathbb{X}$ be

$$\Omega = \left\{ u \in \mathbb{X} \mid \Phi(t) \leq u(t) \leq \frac{a(t_0)\sigma}{\varepsilon} \Phi(t) \right\}.$$

Let $\mathscr{T} : \Omega \to \mathbb{X}$ be an operator defined by

$$\mathscr{T}(u)(t) = \int_{t_0}^{t} \varphi^{-1} \left(\frac{1}{a(s)} \left(\alpha\sigma + \int_{s}^{\infty} b(\tau)(u(\tau))^\gamma \,\mathrm{d}\tau \right) \right) \mathrm{d}s.$$

We first prove that $\mathscr{T}$ is well defined. From $u \in \Omega$ and (4.1), we get

$$\int_{t}^{\infty} b(s)\,(u(s))^\gamma \,\mathrm{d}s < \int_{t}^{\infty} b(s) \left(\frac{a(t_0)\sigma}{\varepsilon} \Phi(s) \right)^\gamma \mathrm{d}s < \sigma(a(t) - \alpha),$$

and therefore, we obtain

$$\frac{1}{a(t)} \left(\alpha\sigma + \int_{t}^{\infty} b(s)\,(u(s))^\gamma \,\mathrm{d}s \right) < \sigma. \tag{4.3}$$

Hence $\mathscr{T}$ is well defined.

We next show that $\mathscr{T}(\Omega) \subset \Omega$. By using $u \in \Omega$ and (4.2), we have

$$\begin{aligned} \varphi^{-1} \left(\frac{1}{a(t)} \left(\alpha\sigma + \int_{t}^{\infty} b(s)\,(u(s))^\gamma \,\mathrm{d}s \right) \right) &\leq \varphi^{-1} \left(\frac{1}{a(t)} \left(\alpha\sigma + \int_{t}^{\infty} b(s) \left(\frac{a(t_0)\sigma}{\varepsilon} \Phi(s) \right)^\gamma \mathrm{d}s \right) \right) \\ &< \frac{1}{\varepsilon} \left(\alpha\sigma + \int_{t}^{\infty} b(s)\,(\Phi(s))^\gamma \,\mathrm{d}s \right) \varphi^{-1} \left(\frac{\alpha\sigma}{a(t)} \right) \\ &\leq \frac{1}{\varepsilon} \left(\alpha\sigma + \int_{t}^{\infty} b(s)\,(u(s))^\gamma \,\mathrm{d}s \right) \varphi^{-1} \left(\frac{\alpha\sigma}{a(t)} \right). \end{aligned} \tag{4.4}$$

Since $a(t)$ is decreasing and (4.3) holds, we have

$$\alpha\sigma + \int_{t}^{\infty} b(s)\,(u(s))^\gamma \,\mathrm{d}s < a(t)\sigma < a(t_0)\sigma.$$

From (4.4), we obtain

$$\varphi^{-1} \left(\frac{1}{a(t)} \left(\alpha\sigma + \int_{t}^{\infty} b(s)\,(u(s))^\gamma \,dd \right) \right) < \frac{a(t_0)\sigma}{\varepsilon} \varphi^{-1} \left(\frac{\alpha\sigma}{a(t)} \right).$$

Hence we get

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$$\Phi(t) \le \mathscr{T}(u)(t) < \int_{t_0}^{t} \frac{a(t_0)\sigma}{\varepsilon} \varphi^{-1}\left(\frac{\alpha\sigma}{a(s)}\right) \mathrm{d}s = \frac{a(t_0)\sigma}{\varepsilon}\Phi(t),$$

which implies $\mathscr{T}(\Omega) \subset \Omega$ and it is uniformly bounded on every compact subinterval of $[t_0, \infty)$.

Proceeding as in the proof of Lemma 3.1, we see that $\mathscr{T}(\Omega)$ is relatively compact and $\mathscr{T}$ is continuous in the topology of $\mathbb{X}$. Hence, from the Tychonov fixed point theorem, there exists $\tilde{u} \in \Omega$ such that $\mathscr{T}(\tilde{u}) = \tilde{u}$. Then we have

$$\tilde{u}'(t) = \varphi^{-1}\left(\frac{1}{a(t)}\left(\alpha\sigma + \int_{t}^{\infty} b(s)\left(\tilde{u}(s)\right)^{\gamma} \mathrm{d}s\right)\right). \tag{4.5}$$

Thus we get

$$a(t)\varphi(\tilde{u}'(t)) = \alpha\sigma + \int_{t}^{\infty} b(s)\left(\tilde{u}(s)\right)^{\gamma} \mathrm{d}s,$$

and therefore, we obtain $(a(t)\varphi(\tilde{u}'(t)))' = -b(t)\left(\tilde{u}(t)\right)^{\gamma}$. Furthermore, from (4.5) and $a(t) \to \alpha$ as $t \to \infty$, we get

$$\tilde{u}'(t) \ge \varphi^{-1}\left(\frac{1}{a(t)}\left(\alpha\sigma + \int_{t}^{\infty} b(s)\left(\Phi(s)\right)^{\gamma} \mathrm{d}s\right)\right) > \varphi^{-1}\left(\frac{\alpha\sigma}{a(t)}\right) \to \infty$$

as $t \to \infty$, which implies that $x(t)$ is an extremal solution of equation (1.1). $\square$

Remark 4.1. We see that neither (1.4) nor (2.11) are needed in Theorem 4.1.

Remark 4.2. Let $x(t)$ be an extremal solution of (1.1) such that $x'(t)$ is increasing for $t \ge T$ for some $T \ge t_0$. Then, from equation (1.1), we see that $x^{[1]}(t)$ is decreasing and

$$x^{[1]}(t) = a(t)\varphi(x'(t)) < a(T)\varphi(x'(T)) = x^{[1]}(T).$$

Hence we have

$$a(t) < a(T)\frac{\varphi(x'(T))}{\varphi(x'(t))} < a(T)$$

because $x'(t)$ is increasing. Thus, the condition that $a(t)$ is decreasing is a necessary for the existence of extremal solutions whose derivatives are increasing for t sufficiently large.

Remark 4.3. Assume $a(t) > \alpha$, $a(t) \to \alpha$ as $t \to \infty$, and (4.1). Then we see that (3.1) holds. In fact, from (4.1), we have

$$\int_{t}^{\infty} b(s)\left(\int_{t_0}^{s} \varphi^{-1}\left(\frac{\alpha\sigma}{a(\tau)}\right) \mathrm{d}\tau\right)^{\gamma} \mathrm{d}s < \infty.$$

Since

$$\lim_{t\to\infty} \varphi^{-1}\left(\frac{\alpha\sigma}{a(t)}\right) = \infty,$$

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we see that

$$t < \int_{t_0}^{t} \varphi^{-1}\left(\frac{\alpha\sigma}{a(s)}\right) \mathrm{d}s$$

for t sufficiently large, which implies (3.1).

In view of Theorems 3.2 and 4.1, and Remark 4.3, we get the following corollary.

Corollary 4.1. *Assume that $\sigma < \infty$, $a(t)$ is decreasing, and $a(t) \to \alpha > 0$ as $t \to \infty$. Suppose that there exists $0 < \varepsilon < \alpha$ satisfying (4.1) and (4.2) for any $t \geq t_0$. Then asymptotically linear solutions and extremal solutions of equation (1.1) coexist.*

The prototype of bounded φ-Laplacian is the one-dimensional mean curvature operator. We consider the prescribed mean curvature equation

$$\left(a(t)\frac{x'}{\sqrt{1+(x')^2}}\right)' + b(t)|x|^{\gamma} \operatorname{sgn} x = 0, \tag{4.6}$$

which is a special case of equation (1.1). Note that φ-Laplacian becomes the one-dimensional mean curvature operator in this case. We simply denote that $\varphi_C(u) = u/\sqrt{1+u^2}$ and $\varphi_C^{-1}(u) = u/\sqrt{1-u^2}$. It is easy to show that $\varphi_C\colon \mathbb{R} \to (-1,1)$ and φ_C^{-1} is the inverse function of φ_C. For equation (4.6), we can relax the conditions (4.1) and (4.2) in Theorem 4.1.

Theorem 4.2. *Assume that $a(t)$ is decreasing and $a(t) \to \alpha > 0$ as $t \to \infty$. Suppose that there exists $0 < \varepsilon < \alpha$ satisfying*

$$\frac{1}{a(t)-\alpha}\int_t^{\infty} b(s)\left(\int_{t_0}^{s} \varphi_C^{-1}\left(\frac{\alpha}{a(\tau)}\right) \mathrm{d}\tau\right)^{\gamma} \mathrm{d}s < (1-\varepsilon^2)\left(\frac{\varepsilon}{a(t_0)}\right)^{\gamma} \tag{4.7}$$

for any $t \geq t_0$. Then equation (4.6) has an extremal solution.

Proof. Let Φ, $\mathbb{X}$, Ω, and $\mathscr{T}\colon \Omega \to \mathbb{X}$ be defined as in the proof of Theorem 4.1. Then, using $u \in \Omega$ and (4.7), we have

$$\int_t^{\infty} b(s)\,(u(s))^{\gamma}\, \mathrm{d}s < (1-\varepsilon^2)(a(t)-\alpha). \tag{4.8}$$

Since $a(t) > \alpha$ for any $t \geq t_0$, we see that

$$\frac{1}{a(t)}\left(\alpha + \int_t^{\infty} b(s)\,(u(s))^{\gamma}\, \mathrm{d}s\right) < 1 - \varepsilon^2\left(1 - \frac{\alpha}{a(t)}\right) < 1 \tag{4.9}$$

holds and $\mathscr{T}$ is well defined.

We show that $\mathscr{T}(\Omega) \subset \Omega$. From (4.8), we get

$$a(t) - \alpha - \int_t^{\infty} b(s)\,(u(s))^{\gamma}\, \mathrm{d}s > \varepsilon^2(a(t)-\alpha).$$

Hence we have

$$1-\left(\frac{\alpha}{a(t)}+\frac{1}{a(t)}\int_t^{\infty} b(s)\,(u(s))^{\gamma}\,\mathrm{d}s\right)>\frac{\varepsilon^2}{a(t)}(a(t)-\alpha). \tag{4.10}$$

Moreover, since $\int_t^{\infty} b(s)\,(u(s))^{\gamma}\,\mathrm{d}s$ is nonnegative, we get

$$1+\left(\frac{\alpha}{a(t)}+\frac{1}{a(t)}\int_t^{\infty} b(s)\,(u(s))^{\gamma}\,\mathrm{d}s\right)\geq 1+\frac{\alpha}{a(t)}=\frac{1}{a(t)}(a(t)+\alpha). \tag{4.11}$$

From (4.10) and (4.11), we obtain

$$1-\left(\frac{\alpha}{a(t)}+\frac{1}{a(t)}\int_t^{\infty} b(s)\,(u(s))^{\gamma}\,\mathrm{d}s\right)^2>\frac{\varepsilon^2((a(t))^2-\alpha^2)}{(a(t))^2},$$

that is,

$$\left(1-\left(\frac{\alpha}{a(t)}+\frac{1}{a(t)}\int_t^{\infty} b(s)\,(u(s))^{\gamma}\,\mathrm{d}s\right)^2\right)^{-1/2}<\frac{a(t)}{\varepsilon\sqrt{(a(t))^2-\alpha^2}}.$$

Hence we have

$$\begin{aligned}\varphi_C^{-1}\left(\frac{1}{a(t)}\left(\alpha+\int_t^{\infty} b(s)\,(u(s))^{\gamma}\,\mathrm{d}s\right)\right)&<\frac{1}{\varepsilon\sqrt{(a(t))^2-\alpha^2}}\left(\alpha+\int_t^{\infty} b(s)\,(u(s))^{\gamma}\,\mathrm{d}s\right)\\&=\frac{1}{\alpha\varepsilon}\left(\alpha+\int_t^{\infty} b(s)\,(u(s))^{\gamma}\,\mathrm{d}s\right)\varphi_C^{-1}\left(\frac{\alpha}{a(t)}\right).\end{aligned} \tag{4.12}$$

Since $a(t)$ is decreasing and (4.9) holds, we have

$$\alpha+\int_t^{\infty} b(s)\,(u(s))^{\gamma}\,\mathrm{d}s<a(t)\leq a(t_0).$$

From (4.12), we obtain

$$\varphi_C^{-1}\left(\frac{1}{a(t)}\left(\alpha+\int_t^{\infty} b(s)\,(u(s))^{\gamma}\,\mathrm{d}s\right)\right)<\frac{a(t_0)}{\alpha\varepsilon}\varphi_C^{-1}\left(\frac{\alpha}{a(t)}\right).$$

Hence we get

$$\Phi(t)\leq\mathscr{T}(u)(t)<\int_{t_0}^{t}\frac{a(t_0)}{\alpha\varepsilon}\varphi_C^{-1}\left(\frac{\alpha}{a(s)}\right)\mathrm{d}s=\frac{a(t_0)}{\alpha\varepsilon}\Phi(t),$$

which implies $\mathscr{T}(\Omega)\subset\Omega$ and it is uniformly bounded on every compact subinterval of $[t_0,\infty)$.

Proceeding as in the proof of Theorem 4.1, we can show that $\mathscr{T}$ has a fixed point which is an extremal solution of equation (1.1). □

Remark 4.4. We observe that the right-hand side of (4.7) takes the maximum value at $\varepsilon = \sqrt{\gamma/(\gamma+2)}$. Hence, in the case when $\alpha > \sqrt{\gamma/(\gamma+2)}$, we can replace (4.7) with

$$\frac{1}{a(t)-\alpha}\int_t^\infty b(s)\left(\int_{t_0}^s \varphi_C^{-1}\left(\frac{\alpha}{a(\tau)}\right)\mathrm{d}\tau\right)^\gamma \mathrm{d}s < \frac{2}{\gamma+2}\left(\frac{1}{a(t_0)}\sqrt{\frac{\gamma}{\gamma+2}}\right)^\gamma, \tag{4.13}$$

which does not contain the parameter ε.

5. Examples and discussion

The following examples illustrate our results.

Example 5.1. Consider the equation

$$(\varphi_C(x'))' + \frac{1}{(t+1)^{3/2}}|x|^\gamma \operatorname{sgn} x = 0, \quad \gamma > 0, \quad t \geq 0. \tag{5.1}$$

It is easy to check that (1.2), (1.3), and (1.8) are satisfied.

In the case when $\gamma > 1$, from

$$\int_t^\infty \frac{1}{(s+1)^{3/2}}\,\mathrm{d}s = \frac{2}{\sqrt{t+1}},$$

we get

$$\int_0^t \left(\int_s^\infty \frac{1}{(\tau+1)^{3/2}}\,\mathrm{d}\tau\right)\mathrm{d}s = 4\sqrt{t+1} - 4 \to \infty$$

as $t \to \infty$, which implies (1.6). From Theorem 2.1, all nontrivial solutions of equation (5.1) are oscillatory.

In the case when $1/2 \leq \gamma < 1$, we have

$$\begin{aligned}\int_0^t b(s)\left(\int_0^s \frac{1}{a(\tau)}\,\mathrm{d}\tau\right)^\gamma \mathrm{d}s &= \int_0^t \frac{s^\gamma}{(s+1)^{3/2}}\,\mathrm{d}s > \int_1^t \frac{\sqrt{s}}{(s+1)^{3/2}}\,\mathrm{d}s \\ &= \int_1^t \frac{1}{s+1}\sqrt{1-\frac{1}{s+1}}\,\mathrm{d}s > \frac{1}{\sqrt{2}}\int_1^t \frac{1}{s+1}\,\mathrm{d}s \to \infty\end{aligned}$$

as $t \to \infty$, which implies (1.7). From Theorem 2.2, all nontrivial solutions of equation (5.1) are oscillatory.

In the case when $0 < \gamma < 1/2$, we get

$$\int_0^\infty b(s)\left(\int_0^s \frac{1}{a(\tau)}\,\mathrm{d}\tau\right)^\gamma \mathrm{d}s = \int_0^\infty \frac{s^\gamma}{(s+1)^{3/2}}\,\mathrm{d}s < \int_0^\infty \frac{1}{(s+1)^{3/2-\gamma}}\,\mathrm{d}s < \infty,$$

which implies that (1.7) does not hold. From Theorem 2.2, there exists a nonoscillatory solution of equation (5.1).

Example 5.2. Consider the equation

$$(\varphi_C(x'))' + \frac{1}{(t+1)^2}|x|^{1/2}\operatorname{sgn} x = 0, \quad t \geq 0. \tag{5.2}$$

We have

$$\int_{t_0}^{\infty} b(t)t^{\gamma}\,\mathrm{d}t = \int_0^{\infty} \frac{t^{1/2}}{(t+1)^2}\,\mathrm{d}t < \int_0^{\infty} \frac{1}{(t+1)^{3/2}}\,\mathrm{d}t < \infty,$$

which implies (3.1). Moreover, we have

$$\int_{t_0}^{\infty} \varphi_C^{-1}\left(\int_t^{\infty} b(s)\,\mathrm{d}s\right)\mathrm{d}t = \int_0^{\infty} \varphi_C^{-1}\left(\frac{1}{t+1}\right)\mathrm{d}t > \int_0^{\infty} \frac{1}{t+1}\,\mathrm{d}t = \infty,$$

and therefore, (3.2) is satisfied. From Theorem 3.1, there exists a weakly increasing solution of equation (5.2). Moreover, from Remark 4.2, equation (5.2) has no extremal solutions.

Example 5.3. Consider the equation

$$\left(\left(1+\frac{1}{t}\right)\varphi_C(x')\right)' + \frac{\lambda}{t^5}|x|^2\operatorname{sgn} x = 0, \quad t \geq 1, \tag{5.3}$$

where $0 < \lambda < 3/16$. For any $t \geq 1$,

$$t^5 > \frac{16}{9}\lambda t^4(2t+1)$$

and

$$\int_1^t \varphi_C^{-1}\left(\frac{s}{s+1}\right)\mathrm{d}s = \int_1^t \frac{s}{\sqrt{2s+1}}\,\mathrm{d}s = \frac{(t-1)\sqrt{2t+1}}{3} - \frac{\sqrt{5}}{3} < \frac{t\sqrt{2t+1}}{3}$$

hold, and therefore, we have

$$\frac{1}{1+1/t-1}\int_t^{\infty} \frac{\lambda}{s^5}\left(\int_0^s \varphi_C^{-1}\left(\frac{\tau}{\tau+1}\right)\mathrm{d}\tau\right)^2 \mathrm{d}s < t\int_t^{\infty} \frac{9}{16s^4(2s+1)}\left(\frac{s\sqrt{2s+1}}{3}\right)^2 \mathrm{d}s = \frac{t}{16}\int_t^{\infty} \frac{1}{s^2}\,\mathrm{d}s = \frac{1}{16},$$

which implies (4.13). From Theorem 4.2 and Remark 4.4, we see that equation (5.3) has an extremal solution.

We next refer to the result for the corresponding difference equation

$$\Delta\left(a_n\varphi(\Delta x_n)\right) + b_n|x_{n+1}|^{\gamma}\operatorname{sgn} x_{n+1} = 0, \tag{5.4}$$

where $\{a_n\}$ and $\{b_n\}$ are positive sequences, and Δ is the forward difference operator. In [5], oscillation criteria for equation (5.4) was established. Theorem 2.2 is a continuous counterpart of [5, Theorems 5], and Theorem 2.1 corresponds to [5, Theorems 4] under the strong assumption $\liminf_{n\to\infty} a_n > 0$. Moreover, in [4], the asymptotic behavior of solutions of the difference equation which is a generalization of equation (5.4) was considered. The pair of Theorems 3.1 and 3.2, and Theorem 4.2 correspond to [4, Theorems 1 and 3], respectively.

We finally propose the following open problems.

(1) From Lemma 2.2, it is an open problem whether extremal solutions of equation (1.1) with $\sigma = \infty$ can exist if $\liminf_{t\to\infty} a(t) = 0$.
(2) From Corollaries 3.1 and 4.1, asymptotically linear solutions of equation (1.1) can coexist with either weakly increasing solutions or extremal solutions. However, it is an open problem whether solutions of all these types can coexist. Noting that, from numerical computation, we guess that equation (5.3) has all these solutions.
(3) Suppose that $a(t) \to \alpha > 0$ as $t \to \infty$ holds and (3.1) does not hold. Then, from Theorem 3.2, there are no asymptotically linear solutions of equation (1.1). Moreover, from Theorem 2.2, we see that all nontrivial solutions are oscillatory if $0 < \gamma < 1$ in this case. On the other hand, it is an open problem to show the existence of weakly increasing solutions and extremal solutions if $\gamma > 1$.
(4) It is an open problem to give sufficient conditions for the existence of positive increasing solution x such that $\lim_{t\to\infty} x'(t)$ does not exist. For example, equation (1.1) has the solution $x(t) = 2t + \sin t$.

Acknowledgments

The first author was supported by the Czech Science Foundation under grant 17-03224S. The second author was supported by JSPS KAKENHI Grant number JP17J00259. The authors thank to the associate editor and two anonymous reviewers for their valuable suggestions that helped to improve this paper.

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Please cite this article in press as: Z. Došlá, K. Fujimoto, Asymptotic problems for nonlinear ordinary differential equations with φ-Laplacian, J. Math. Anal. Appl. (2019), https://doi.org/10.1016/j.jmaa.2019.123674