



The weak* density in operator ideals [☆]

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ABSTRACT

Given a finitely generated tensor norm α , we investigate weak* densities of adjoint operators in the Banach operator ideal \mathcal{A} associated with α . We also study relations between the α -approximation properties and weak* densities of finite rank adjoint operators in \mathcal{A} . Some examples are given satisfying that all adjoint operators are not weak* dense in the ideal of compact operators.

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1. Introduction

Let X and Y be Banach spaces. We denote by $X \otimes Y$ the algebraic tensor product of X and Y . The normed space $X \otimes Y$ equipped with a norm α is denoted by $X \otimes_{\alpha} Y$ and its completion is denoted by $X \hat{\otimes}_{\alpha} Y$. The most classical two norms on $X \otimes Y$ are the *injective norm* and *projective norm*, which are denoted by ε and π , respectively. For $u \in X \otimes Y$,

$$\varepsilon(u; X, Y) := \sup \left\{ \left| \sum_{n=1}^l x^*(x_n) y^*(y_n) \right| : x^* \in B_{X^*}, y^* \in B_{Y^*} \right\},$$

where $\sum_{n=1}^l x_n \otimes y_n$ is any representation of u and we denote by B_Z the closed unit ball of a Banach space Z , and

$$\pi(u; X, Y) := \inf \left\{ \sum_{n=1}^l \|x_n\| \|y_n\| : u = \sum_{n=1}^l x_n \otimes y_n, l \in \mathbb{N} \right\}.$$

It is well known that ε and π are finitely generated tensor norms and that, if $u \in X \hat{\otimes}_{\pi} Y$, then there exist sequences $(x_n)_n$ in X and $(y_n)_n$ in Y with $\sum_{n=1}^{\infty} \|x_n\| \|y_n\| < \infty$ such that

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$$u = \sum_{n=1}^{\infty} x_n \otimes y_n$$

converges in $X \hat{\otimes}_{\pi} Y$ (cf. [4,17]). The starting point of the paper comes from the following question (cf. [4, p. 65, Corollary 3 below] and [15, Section 1]).

Question. Let X and Y be Banach spaces. For $u = \sum_{n=1}^{\infty} y_n^* \otimes x_n \in Y^* \hat{\otimes}_{\pi} X$, if

$$\sum_{n=1}^{\infty} y_n^*(Tx_n) = 0$$

for every $T \in \mathcal{L}(X, Y)$, then $u = 0$ in $Y^* \hat{\otimes}_{\pi} X$?

Here \mathcal{L} is the ideal of all operators with the operator norm $\|\cdot\|$. On the other hand, since

$$\mathcal{L}(Y^*, X^*) = (Y^* \hat{\otimes}_{\pi} X)^* = (X \hat{\otimes}_{\pi} Y^*)^* = \mathcal{L}(X, Y^{**})$$

hold isometrically with the natural dual action (see Lemma 1.1 below), $u = 0$ in $Y^* \hat{\otimes}_{\pi} X$ if

$$\sum_{n=1}^{\infty} (Ty_n^*)(x_n) = 0$$

for every $T \in \mathcal{L}(Y^*, X^*)$, or if

$$\sum_{n=1}^{\infty} (Tx_n)(y_n^*) = 0$$

for every $T \in \mathcal{L}(X, Y^{**})$. Reinov [15, Corollary 3.6] gave a negative answer to **Question** and more general facts can be found in [16].

We consider general finitely generated tensor norms and their associated Banach operator ideals. For tensor norms and operator ideals, one may refer to [4,17]. A tensor norm α is said to be *associated with* a Banach operator ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ if the canonical map $(\mathcal{A}(M, N), \|\cdot\|_{\mathcal{A}}) \rightarrow M^* \otimes_{\alpha} N$ is an isometry for all finite-dimensional normed spaces M and N . If α is finitely generated, then its associated maximal Banach operator ideal is uniquely determined (cf. [4, Sections 17.1, 17.2 and 17.3]). The injective norm ε is associated with \mathcal{L} and the projective norm π is associated with the ideal \mathcal{I} of integral operators. Let α' be the dual tensor norm of a tensor norm α . The following lemma is called *Representation Theorem for Maximal Operator Ideals*, which is the main tool in this paper.

Lemma 1.1 ([4, Theorem 17.5]). *Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be the maximal Banach operator ideal associated with a finitely generated tensor norm α . Then for all Banach spaces X and Y , $\mathcal{A}(X, Y^*) = (X \otimes_{\alpha'} Y)^*$ holds isometrically and $\mathcal{A}(X, Y)$ is isometrically imbedded in $(X \otimes_{\alpha'} Y^*)^*$ by the natural dual actions.*

Since $\varepsilon' = \pi$ and $\pi' = \varepsilon$,

$$\mathcal{L}(X, Y^*) = (X \hat{\otimes}_{\pi} Y)^*, \quad \mathcal{I}(X, Y^*) = (X \hat{\otimes}_{\varepsilon} Y)^*$$

hold isometrically and $\mathcal{L}(X, Y)$ is isometrically imbedded in $(X \hat{\otimes}_{\pi} Y^*)^*$ and $\mathcal{I}(X, Y)$ is isometrically imbedded in $(X \hat{\otimes}_{\varepsilon} Y^*)^*$.

Throughout this paper, α is the finitely generated tensor norm associated with a maximal Banach operator ideal \mathcal{A} . Suppose that \mathcal{B} is a linear subspace of \mathcal{A}^{dual} . In Section 2, we define a natural map

$$D_\alpha : Y^* \hat{\otimes}_{\alpha'} X \longrightarrow (\mathcal{B}(X, Y), \|\cdot\|_{\mathcal{A}^{dual}})^*$$

and show that D_α is injective if and only if

$$\mathcal{A}(Y^*, X^*) = \overline{\{R^* : R \in \mathcal{B}(X, Y)\}}^{\tau_{w^*}}$$

in $(Y^* \hat{\otimes}_{\alpha'} X)^*$, where τ_{w^*} is the *weak** topology. For instance, the map

$$D_\varepsilon : Y^* \hat{\otimes}_\pi X \longrightarrow (\mathcal{L}(X, Y), \|\cdot\|)^*$$

is defined by

$$D_\varepsilon(u)(T) = \sum_{n=1}^{\infty} y_n^*(Tx_n)$$

for $u = \sum_{n=1}^{\infty} y_n^* \otimes x_n \in Y^* \hat{\otimes}_\pi X$ and $T \in \mathcal{L}(X, Y)$. Consequently, **Question** can be reformulated as follows.

Question. For every Banach spaces X and Y ,

$$\mathcal{L}(Y^*, X^*) = \overline{\{R^* : R \in \mathcal{L}(X, Y)\}}^{\tau_{w^*}}$$

in $(Y^* \hat{\otimes}_\pi X)^*$?

Let $\lambda \geq 1$. We show in Section 2 that the map D_α is λ -isomorphic if and only if

$$\{T \in \mathcal{A}(Y^*, X^*) : \|T\|_{\mathcal{A}} \leq 1\} \subset \overline{\{R^* : R \in \mathcal{B}(X, Y), \|R\|_{\mathcal{A}^{dual}} \leq \lambda\}}^{\tau_{w^*}}$$

in $(Y^* \hat{\otimes}_{\alpha'} X)^*$. Consequently, the map

$$D_\varepsilon : Y^* \hat{\otimes}_\pi X \longrightarrow (\mathcal{L}(X, Y), \|\cdot\|)^*$$

is λ -isomorphic if and only if

$$\{T \in \mathcal{L}(Y^*, X^*) : \|T\| \leq 1\} \subset \overline{\{R^* : R \in \mathcal{L}(X, Y), \|R\| \leq \lambda\}}^{\tau_{w^*}}$$

in $(Y^* \hat{\otimes}_\pi X)^*$.

In Section 3, we study the approximation properties and the weak* density of finite rank operators in operator ideals. Let $\lambda \geq 1$. It is shown that a Banach space X has the λ -bounded α' -approximation property if and only if for every Banach space Y ,

$$\{T \in \mathcal{A}(Y^*, X^*) : \|T\|_{\mathcal{A}} \leq 1\} \subset \overline{\{R^* : R \in \mathcal{F}(X, Y), \alpha^t(v_R; X^*, Y) \leq \lambda\}}^{\tau_{w^*}}$$

in $(Y^* \hat{\otimes}_{\alpha'} X)^*$ and that X has the α' -approximation property if and only if for every Banach space Y ,

$$\mathcal{A}(Y^*, X^*) = \overline{\{R^* : R \in \mathcal{F}(X, Y)\}}^{\tau_{w^*}}$$

in $(Y^* \hat{\otimes}_{\alpha'} X)^*$, where \mathcal{F} is the ideal of finite rank operators and α^t is the transposed tensor norm of a tensor norm α . From a result of Godefroy and Ozawa [7], it follows that a separable Banach space X has the λ -bounded approximation property if and only if the map D_ε is λ -isomorphic.

In Section 4, we provide some Banach spaces Z satisfying that $\{R^* : R \in \mathcal{L}(Z, Z)\}$ is not weak* dense in $\mathcal{K}(Z^*, Z^*) \hookrightarrow (Z^* \hat{\otimes}_\pi Z)^*$, where \mathcal{K} is the ideal of compact operators.

2. The weak* density in operator ideals

Suppose that \mathcal{B} is a linear subspace of the *dual ideal* $\mathcal{A}^{dual} := \{T \in \mathcal{L} : T^* \in \mathcal{A}\}$, where $\|T\|_{\mathcal{A}^{dual}} := \|T^*\|_{\mathcal{A}}$ for $T \in \mathcal{A}^{dual}$. Let X and Y be Banach spaces. For $T \in \mathcal{A}(Y^*, X^*) = (Y^* \hat{\otimes}_{\alpha'} X)^*$ and $u \in Y^* \hat{\otimes}_{\alpha'} X$, we denote by $\langle T, u \rangle$ the dual action.

The map

$$\widehat{D}_\alpha : Y^* \otimes_{\alpha'} X \longrightarrow (\mathcal{B}(X, Y), \|\cdot\|_{\mathcal{A}^{dual}})^*$$

is defined by

$$\widehat{D}_\alpha(u)(T) = \sum_{n=1}^m y_n^*(Tx_n)$$

for $u = \sum_{n=1}^m y_n^* \otimes x_n \in Y^* \otimes_{\alpha'} X$ and $T \in \mathcal{B}(X, Y)$. If $\sum_{n=1}^m z_n^* \otimes w_n = u = \sum_{n=1}^m y_n^* \otimes x_n \in Y^* \otimes_{\alpha'} X$, then, for every $T \in \mathcal{B}(X, Y)$,

$$\sum_{n=1}^m y_n^*(Tx_n) = \left\langle T^*, \sum_{n=1}^m y_n^* \otimes x_n \right\rangle = \left\langle T^*, \sum_{n=1}^m z_n^* \otimes w_n \right\rangle = \sum_{n=1}^m z_n^*(Tw_n).$$

If $u = \sum_{n=1}^m y_n^* \otimes x_n \in Y^* \otimes_{\alpha'} X$ and $T \in \mathcal{B}(X, Y)$, then

$$|\widehat{D}_\alpha(u)(T)| = \left| \sum_{n=1}^m (T^* y_n^*)(x_n) \right| = \left| \left\langle T^*, \sum_{n=1}^m y_n^* \otimes x_n \right\rangle \right| \leq \|T^*\|_{\mathcal{A}\alpha'} \left(\sum_{n=1}^m y_n^* \otimes x_n; Y^*, X \right).$$

Therefore \widehat{D}_α is well defined and linear, and $\|\widehat{D}_\alpha(u)\| \leq \alpha'(u; Y^*, X)$ for every $u \in Y^* \otimes X$. Let

$$D_\alpha : Y^* \hat{\otimes}_{\alpha'} X \longrightarrow (\mathcal{B}(X, Y), \|\cdot\|_{\mathcal{A}^{dual}})^*$$

be the linear continuous extension of \widehat{D}_α .

Lemma 2.1. *Suppose that \mathcal{B} is a linear subspace of \mathcal{A}^{dual} . Let X and Y be Banach spaces. For every $u \in Y^* \hat{\otimes}_{\alpha'} X$ and every $R \in \mathcal{B}(X, Y)$,*

$$D_\alpha(u)(R) = \langle R^*, u \rangle.$$

Proof. Let $u \in Y^* \hat{\otimes}_{\alpha'} X$ and $R \in \mathcal{B}(X, Y)$. Let $(u_k)_k$ be a sequence in $Y^* \otimes_{\alpha'} X$ such that $\lim_{k \rightarrow \infty} \alpha'(u_k - u; Y^*, X) = 0$. For each k , let

$$u_k := \sum_{n=1}^{m_k} y_{n,k}^* \otimes x_{n,k}.$$

Since

$$\lim_{k \rightarrow \infty} D_\alpha(u_k) = D_\alpha(u)$$

in $(\mathcal{B}(X, Y), \|\cdot\|_{\mathcal{A}^{dual}})^*$, we have

$$D_\alpha(u)(R) = \lim_{k \rightarrow \infty} D_\alpha(u_k)(R) = \lim_{k \rightarrow \infty} \sum_{n=1}^{m_k} y_{n,k}^*(Rx_{n,k}) = \lim_{k \rightarrow \infty} \langle R^*, u_k \rangle = \langle R^*, u \rangle. \quad \square$$

Theorem 2.2. Suppose that \mathcal{B} is a linear subspace of \mathcal{A}^{dual} . Let X and Y be Banach spaces. The map

$$D_\alpha : Y^* \hat{\otimes}_{\alpha'} X \longrightarrow (\mathcal{B}(X, Y), \|\cdot\|_{\mathcal{A}^{dual}})^*$$

is injective if and only if

$$\mathcal{A}(Y^*, X^*) = \overline{\{R^* : R \in \mathcal{B}(X, Y)\}}^{\tau_{w^*}}$$

in $(Y^* \hat{\otimes}_{\alpha'} X)^*$.

Proof. To show the “if” part, suppose that D_α is not injective. Then there exists $u \in Y^* \hat{\otimes}_{\alpha'} X$ such that $u \neq 0$ but

$$D_\alpha(u) = 0.$$

Thus there exists a $T \in \mathcal{A}(Y^*, X^*)$ such that

$$\langle T, u \rangle \neq 0.$$

Since $D_\alpha(u) = 0$ in $(\mathcal{B}(X, Y), \|\cdot\|_{\mathcal{A}})^*$, by Lemma 2.1

$$\langle R^*, u \rangle = D_\alpha(u)(R) = 0$$

for every $R \in \mathcal{B}(X, Y)$. By the separation theorem, $T \notin \overline{\{R^* : R \in \mathcal{B}(X, Y)\}}^{\tau_{w^*}}$.

In order to show the converse, suppose that there exists a $T \in \mathcal{A}(Y^*, X^*)$ such that

$$T \notin \overline{\{R^* : R \in \mathcal{B}(X, Y)\}}^{\tau_{w^*}}.$$

Then there exists an $u \in Y^* \hat{\otimes}_{\alpha'} X$ such that

$$D_\alpha(u)(R) = \langle R^*, u \rangle = 0$$

for every $R \in \mathcal{B}(X, Y)$, but

$$\langle T, u \rangle \neq 0.$$

This means that D_α is not injective. \square

Let $1 \leq p, q \leq \infty$ with $1/p + 1/q \leq 1$ and let $1 \leq r \leq \infty$ with $1/p + 1/q + 1/r^* = 1$, where $1/r + 1/r^* = 1$. Let X and Y be Banach spaces. We denote by $\ell_p^w(X)$ the Banach space with the norm $\|\cdot\|_p^w$ of all X -valued weakly p -summable sequences. A linear map $T : X \rightarrow Y$ is called (p, q) -dominated if there exists a $C > 0$ such that

$$\|(y_n^*(Tx_n))_n\|_{\ell_r} \leq C \|(x_n)_n\|_p^w \|(y_n^*)_n\|_q^w$$

for every $(x_n)_n \in \ell_p^w(X)$ and $(y_n^*)_n \in \ell_q^w(Y^*)$. We denote by $\mathcal{D}_{p,q}(X, Y)$ the collection of all (p, q) -dominated operators from X to Y and for $T \in \mathcal{D}_{p,q}(X, Y)$, let $\|T\|_{\mathcal{D}_{p,q}}$ be the infimum C satisfying all such inequalities. Then $[\mathcal{D}_{p,q}, \|\cdot\|_{\mathcal{D}_{p,q}}]$ is a maximal Banach operator ideal (cf. [4, Section 19]). $\mathcal{P}_p := \mathcal{D}_{p,\infty}$ is well known as the ideal of absolutely p -summing operators (cf. [4,5,13,19]) and $\mathcal{D}_p := \mathcal{D}_{p,p^*}$ is the ideal of p -dominated operators.

Let $1 \leq r \leq \infty$ with $1/r = 1/p^* + 1/q^* - 1$. For $u \in X \otimes Y$, let

$$\alpha_{p^*,q^*}(u) := \inf \left\{ \|(\lambda_j)_{j=1}^n\|_r \| (x_j)_{j=1}^n \|_q^w \| (y_j)_{j=1}^n \|_p^w : u = \sum_{j=1}^n \lambda_j x_j \otimes y_j, n \in \mathbb{N} \right\}.$$

Then α_{p^*,q^*} is a finitely generated tensor norm and $\alpha_{p^*,q^*}^t = \alpha_{q^*,p^*}$ (cf. [4, Proposition 12.5]). The special cases $g_{p^*} := \alpha_{p^*,1}$ and $d_{p^*} := \alpha_{1,p^*}$ are called the *Chevet-Saphar tensor norms* [2,18] and $\alpha_{1,1} = \pi$. The tensor norm $w_{p^*} := \alpha_{p^*,p}$ is also well known.

Since $\mathcal{D}_{p,q}$ is associated with $\alpha_{p^*,q^*}^* := (\alpha_{p^*,q^*}^t)' = (\alpha_{q^*,p^*}')^t$ (cf. [4, Section 17]) and

$$(\alpha_{p^*,q^*}^*)' = \alpha_{q^*,p^*},$$

by Theorem 2.2, we have:

Corollary 2.3. *Let $1 \leq p, q \leq \infty$ with $1/p + 1/q \leq 1$. Let X and Y be Banach spaces. The map*

$$D_{\alpha_{p^*,q^*}^*} : Y^* \hat{\otimes}_{\alpha_{q^*,p^*}} X \longrightarrow (\mathcal{D}_{p,q}^{dual}(X, Y), \|\cdot\|_{\mathcal{D}_{p,q}^{dual}})^*$$

is injective if and only if

$$\mathcal{D}_{p,q}(Y^*, X^*) = \overline{\{R^* : R \in \mathcal{D}_{p,q}^{dual}(X, Y)\}}^{\tau_{w^*}}$$

in $(Y^ \hat{\otimes}_{\alpha_{q^*,p^*}} X)^*$.*

For instance, for $1 \leq p \leq \infty$, the map

$$D_{g_{p^*}} : Y^* \hat{\otimes}_{d_{p^*}} X \longrightarrow (\mathcal{P}_p^{dual}(X, Y), \|\cdot\|_{\mathcal{P}_p^{dual}})^*$$

is injective if and only if

$$\mathcal{P}_p(Y^*, X^*) = \overline{\{R^* : R \in \mathcal{P}_p^{dual}(X, Y)\}}^{\tau_{w^*}}$$

in $(Y^* \hat{\otimes}_{d_{p^*}} X)^*$. The most special case of Corollary 2.3 is:

Corollary 2.4. *Let X and Y be Banach spaces. The map*

$$D_\pi : Y^* \hat{\otimes}_\pi X \longrightarrow (\mathcal{L}(X, Y), \|\cdot\|)^*$$

is injective if and only if

$$\mathcal{L}(Y^*, X^*) = \overline{\{R^* : R \in \mathcal{L}(X, Y)\}}^{\tau_{w^*}}$$

in $(Y^ \hat{\otimes}_\pi X)^*$.*

We now consider the bounded version of Theorem 2.2.

Theorem 2.5. *Suppose that \mathcal{B} is a linear subspace of \mathcal{A}^{dual} . Let X and Y be Banach spaces and let $\lambda \geq 1$. The map*

$$D_\alpha : Y^* \hat{\otimes}_{\alpha'} X \longrightarrow (\mathcal{B}(X, Y), \|\cdot\|_{\mathcal{A}^{dual}})^*$$

is λ -isomorphic if and only if

$$\{T \in \mathcal{A}(Y^*, X^*) : \|T\|_{\mathcal{A}} \leq 1\} \subset \overline{\{R^* : R \in \mathcal{B}(X, Y), \|R\|_{\mathcal{A}^{dual}} \leq \lambda\}}^{T_{w^*}}$$

in $(Y^* \hat{\otimes}_{\alpha'} X)^*$.

Proof. Suppose that D_{α} is λ -isomorphic. Let $T \in \mathcal{A}(Y^*, X^*)$ with $\|T\|_{\mathcal{A}} \leq 1$. To use the separation theorem, let $u \in Y^* \hat{\otimes}_{\alpha'} X$. Using Lemma 2.1,

$$\begin{aligned} |\langle T, u \rangle| &\leq \alpha'(u; Y^*, X) \\ &\leq \lambda \|D_{\alpha}(u)\| \\ &= \lambda \sup\{|\langle R^*, u \rangle| : R \in \mathcal{B}(X, Y), \|R\|_{\mathcal{A}^{dual}} \leq 1\} \\ &= \sup\{|\langle R^*, u \rangle| : R \in \mathcal{B}(X, Y), \|R\|_{\mathcal{A}^{dual}} \leq \lambda\}. \end{aligned}$$

Hence $T \in \overline{\{R^* : R \in \mathcal{B}(X, Y), \|R\|_{\mathcal{A}^{dual}} \leq \lambda\}}^{T_{w^*}}$.

In order to show the converse, let $u \in Y^* \hat{\otimes}_{\alpha'} X$. Then

$$\begin{aligned} \alpha'(u; Y^*, X) &= \sup\{|\langle T, u \rangle| : T \in \mathcal{A}(Y^*, X^*), \|T\|_{\mathcal{A}} \leq 1\} \\ &\leq \sup\{|\langle R^*, u \rangle| : R \in \mathcal{B}(X, Y), \|R\|_{\mathcal{A}^{dual}} \leq \lambda\} \\ &= \lambda \sup\{|\langle R^*, u \rangle| : R \in \mathcal{B}(X, Y), \|R\|_{\mathcal{A}^{dual}} \leq 1\} \\ &= \lambda \|D_{\alpha}(u)\|. \end{aligned}$$

Hence D_{α} is λ -isomorphic. \square

Corollary 2.6. Let $1 \leq p, q \leq \infty$ with $1/p + 1/q \leq 1$. Let X and Y be Banach spaces and let $\lambda \geq 1$. The map

$$D_{\alpha_{p^*, q^*}} : Y^* \hat{\otimes}_{\alpha_{q^*, p^*}} X \longrightarrow (\mathcal{D}_{p, q}^{dual}(X, Y), \|\cdot\|_{\mathcal{D}_{p, q}^{dual}})^*$$

is λ -isomorphic if and only if

$$\{T \in \mathcal{D}_{p, q}(Y^*, X^*) : \|T\|_{\mathcal{D}_{p, q}} \leq 1\} \subset \overline{\{R^* : R \in \mathcal{D}_{p, q}^{dual}(X, Y), \|R\|_{\mathcal{D}_{p, q}^{dual}} \leq \lambda\}}^{T_{w^*}}$$

in $(Y^* \hat{\otimes}_{\alpha_{q^*, p^*}} X)^*$.

For instance, for $1 \leq p \leq \infty$ and $\lambda \geq 1$, the map

$$D_{w_{p^*}} : Y^* \hat{\otimes}_{w_p} X \longrightarrow (\mathcal{D}_p^{dual}(X, Y), \|\cdot\|_{\mathcal{D}_p^{dual}})^*$$

is λ -isomorphic if and only if

$$\{T \in \mathcal{D}_p(Y^*, X^*) : \|T\|_{\mathcal{D}_p} \leq 1\} \subset \overline{\{R^* : R \in \mathcal{D}_p^{dual}(X, Y), \|R\|_{\mathcal{D}_p^{dual}} \leq \lambda\}}^{T_{w^*}}$$

in $(Y^* \hat{\otimes}_{w_p} X)^*$.

Corollary 2.7. Let X and Y be Banach spaces and let $\lambda \geq 1$. The map

$$D_{\varepsilon} : Y^* \hat{\otimes}_{\pi} X \longrightarrow (\mathcal{L}(X, Y), \|\cdot\|)^*$$

is λ -isomorphic if and only if

$$\{T \in \mathcal{L}(Y^*, X^*) : \|T\| \leq 1\} \subset \overline{\{R^* : R \in \mathcal{L}(X, Y), \|R\| \leq \lambda\}}^{\tau_{w^*}}$$

in $(Y^* \hat{\otimes}_{\pi} X)^*$.

3. The approximation properties

A Banach space X is said to have the *approximation property* (AP) if

$$id_X \in \overline{\mathcal{F}(X, X)}^{\tau_c},$$

where id_X is the identity map on X and τ_c is the topology of uniform convergence on compact sets. It is well known that X has the AP if and only if for every Banach space Y , the natural map

$$J_{\pi} : Y^* \hat{\otimes}_{\pi} X \longrightarrow Y^* \hat{\otimes}_{\varepsilon} X$$

is injective (cf. [4, Theorem 5.6 and Proposition 21.7(4)]). This equivalent statement can be naturally extended to a more general concept. For a finitely generated tensor norm α , X is said to have the α -AP if for every Banach space Y , the natural map

$$J_{\alpha} : Y^* \hat{\otimes}_{\alpha} X \longrightarrow Y^* \hat{\otimes}_{\varepsilon} X$$

is injective (cf. [4, Section 21.7]).

Let us consider Theorem 2.2 when $\mathcal{B} = \mathcal{F}$. Then we have:

Corollary 3.1. *A Banach space X has the α' -AP if and only if for every Banach space Y ,*

$$\mathcal{A}(Y^*, X^*) = \overline{\{R^* : R \in \mathcal{F}(X, Y)\}}^{\tau_{w^*}}$$

in $(Y^* \hat{\otimes}_{\alpha'} X)^*$.

Proof. We show that the map $J_{\alpha'}$ is injective if and only if the map

$$D_{\alpha} : Y^* \hat{\otimes}_{\alpha'} X \longrightarrow (\mathcal{F}(X, Y), \|\cdot\|_{\mathcal{A}^{dual}})^*$$

is injective for every Banach spaces X and Y . To show this, it is enough to show that for $u \in Y^* \hat{\otimes}_{\alpha'} X$, $u = 0$ in $Y^* \hat{\otimes}_{\varepsilon} X$ if and only if $D_{\alpha}(u)(x^* \otimes y) = 0$ for every $x^* \in X^*$ and $y \in Y$.

Let $u \in Y^* \hat{\otimes}_{\alpha'} X$. Let $T_u \in \overline{\mathcal{F}(Y, X)}^{\|\cdot\|}$ be the corresponding operator to u in $Y^* \hat{\otimes}_{\varepsilon} X$ which is isometric to $\overline{\mathcal{F}(Y, X)}^{\|\cdot\|}$. Then we see that

$$D_{\alpha}(u)(x^* \otimes y) = x^*(T_u y)$$

for every $x^* \in X^*$ and $y \in Y$. Thus for every $x^* \in X^*$ and $y \in Y$, $D_{\alpha}(u)(x^* \otimes y) = 0$ if and only if $T_u = 0$ which is equivalent to that $u = 0$ in $Y^* \hat{\otimes}_{\varepsilon} X$. \square

Recall that $\mathcal{D}_{p,q}$ is associated with α_{p^*,q^*}^* and $(\alpha_{p^*,q^*}^*)' = \alpha_{q^*,p^*}$. Then we have:

Corollary 3.2. Let $1 \leq p, q \leq \infty$ with $1/p + 1/q \leq 1$. A Banach space X has the α_{q^*, p^*} -AP if and only if for every Banach space Y ,

$$\mathcal{D}_{p,q}(Y^*, X^*) = \overline{\{R^* : R \in \mathcal{F}(X, Y)\}}^{\tau_{w^*}}$$

in $(Y^* \hat{\otimes}_{\alpha_{q^*, p^*}} X)^*$.

For instance, for $1 \leq p \leq \infty$, a Banach space X has the w_p -AP if and only if for every Banach space Y ,

$$\mathcal{D}_p(Y^*, X^*) = \overline{\{R^* : R \in \mathcal{F}(X, Y)\}}^{\tau_{w^*}}$$

in $(Y^* \hat{\otimes}_{w_p} X)^*$.

Corollary 3.3. A Banach space X has the AP if and only if for every Banach space Y ,

$$\mathcal{L}(Y^*, X^*) = \overline{\{R^* : R \in \mathcal{F}(X, Y)\}}^{\tau_{w^*}}$$

in $(Y^* \hat{\otimes}_{\pi} X)^*$.

Let $\lambda \geq 1$. A Banach space X is said to have the λ -bounded AP if

$$id_X \in \overline{\{S \in \mathcal{F}(X, X) : \|S\| \leq \lambda\}}^{\tau_c}.$$

It is well known that a Banach space X has the λ -bounded AP if and only if for every Banach space Y , the natural map

$$I_{\pi} : Y \hat{\otimes}_{\pi} X \longrightarrow (Y^* \otimes_{\varepsilon} X^*)^*$$

is λ -isomorphic (cf. [4, Corollary 16.3.2]). More generally, for a finitely generated tensor norm α , a Banach space X is said to have the λ -bounded α -AP if for every Banach space Y , the natural map

$$I_{\alpha} : Y \hat{\otimes}_{\alpha} X \longrightarrow (Y^* \otimes_{\alpha'} X^*)^*$$

is λ -isomorphic.

From [4, Proposition 21.7(4)] and $\alpha^* := (\alpha')^t = (\alpha^t)'$, a Banach space X has the λ -bounded α -AP if and only if for every Banach space Y , the natural map

$$I_{\alpha} : Y^* \hat{\otimes}_{\alpha} X \longrightarrow (X^* \otimes_{\alpha^*} Y)^*$$

is λ -isomorphic.

Now, $X^* \otimes Y$ can be identified with $\mathcal{F}(X, Y)$ and for $v = \sum_{k=1}^m x_k^* \otimes y_k \in X^* \otimes Y$, denote by $\bar{v} := \sum_{k=1}^m x_k^* \underline{\otimes} y_k$ the corresponding operator to v , and for $R \in \mathcal{F}(X, Y)$, denote by v_R the corresponding tensor to R .

Consider the map

$$I_{\alpha'} : Y^* \hat{\otimes}_{\alpha'} X \longrightarrow (X^* \otimes_{\alpha^*} Y)^*.$$

Let $u \in Y^* \hat{\otimes}_{\alpha'} X$. Then we show that for every $v \in X^* \otimes Y$,

$$I_{\alpha'}(u)(v) = \langle \bar{v}^*, u \rangle,$$

which is the dual action in $(Y^* \hat{\otimes}_{\alpha'} X)^*$. Let $(u_n := \sum_{i=1}^{l_n} y_{ni}^* \otimes x_{ni})_n$ be a sequence in $Y^* \otimes X$ such that $\lim_{n \rightarrow \infty} \alpha'(u_n - u; Y^*, X) = 0$ and let $v = \sum_{k=1}^m x_k^* \otimes y_k \in X^* \otimes Y$. Then we have

$$\begin{aligned} I_{\alpha'}(u)(v) &= \lim_{n \rightarrow \infty} I_{\alpha'}(u_n)(v) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^{l_n} \sum_{k=1}^m x_k^*(x_{ni}) y_{ni}^*(y_k) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^{l_n} y_{ni}^*(\bar{v} x_{ni}) \\ &= \lim_{n \rightarrow \infty} \langle \bar{v}^*, u_n \rangle = \langle \bar{v}^*, u \rangle. \end{aligned}$$

Now, we obtain the bounded version of Corollary 3.1.

Theorem 3.4. Let $\lambda \geq 1$. A Banach space X has the λ -bounded α' -AP if and only if for every Banach space Y ,

$$\{T \in \mathcal{A}(Y^*, X^*) : \|T\|_{\mathcal{A}} \leq 1\} \subset \overline{\{R^* : R \in \mathcal{F}(X, Y), \alpha^t(v_R; X^*, Y) \leq \lambda\}}^{\tau_{w^*}}$$

in $(Y^* \hat{\otimes}_{\alpha'} X)^*$.

Proof. Suppose that X has the λ -bounded α' -AP. Let Y be a Banach space. Then the map

$$I_{\alpha'} : Y^* \hat{\otimes}_{\alpha'} X \longrightarrow (X^* \otimes_{\alpha^t} Y)^*$$

is λ -isomorphic. Let $T \in \mathcal{A}(Y^*, X^*)$ with $\|T\|_{\mathcal{A}} \leq 1$. To use the separation theorem, let $u \in Y^* \hat{\otimes}_{\alpha'} X$. Then

$$\begin{aligned} |\langle T, u \rangle| &\leq \alpha'(u; Y^*, X) \\ &\leq \lambda \|I_{\alpha'}(u)\| \\ &= \lambda \sup\{|\langle I_{\alpha'}(u)(v_R) \rangle| : R \in \mathcal{F}(X, Y), \alpha^t(v_R; X^*, Y) \leq 1\} \\ &= \sup\{|\langle R^*, u \rangle| : R \in \mathcal{F}(X, Y), \alpha^t(v_R; X^*, Y) \leq \lambda\}. \end{aligned}$$

Hence $T \in \overline{\{R^* : R \in \mathcal{F}(X, Y), \alpha^t(v_R; X^*, Y) \leq \lambda\}}^{\tau_{w^*}}$.

In order to show the converse, let Y be a Banach space and let $u \in Y^* \hat{\otimes}_{\alpha'} X$. Then

$$\begin{aligned} \alpha'(u; Y^*, X) &= \sup\{|\langle T, u \rangle| : T \in \mathcal{A}(Y^*, X^*), \|T\|_{\mathcal{A}} \leq 1\} \\ &\leq \sup\{|\langle R^*, u \rangle| : R \in \mathcal{F}(X, Y), \alpha^t(v_R; X^*, Y) \leq \lambda\} \\ &= \lambda \sup\{|\langle I_{\alpha'}(u)(v_R) \rangle| : R \in \mathcal{F}(X, Y), \alpha^t(v_R; X^*, Y) \leq 1\} \\ &= \lambda \|I_{\alpha'}(u)\|. \end{aligned}$$

Hence $I_{\alpha'}$ is λ -isomorphic. \square

For the maximal Banach operator ideal \mathcal{A} associated with a finitely generated tensor norm α , it is well known that $\|R\|_{\mathcal{A}} \leq \alpha(v_R; X^*, Y)$ for $R \in \mathcal{F}(X, Y)$, and $\|R\|_{\mathcal{A}} = \alpha(v_R; X^*, Y)$ when α is totally accessible. Also, for every $1 \leq p, q \leq \infty$ with $1/p + 1/q \leq 1$, the tensor norm $(\alpha_{p^*, q^*}^*)^t$ is totally accessible (cf. [4, Theorem 21.5]), which is associated with $\mathcal{D}_{p, q}^{dual}$. Then by Theorem 3.4, we have:

Corollary 3.5. Let $1 \leq p, q \leq \infty$ with $1/p + 1/q \leq 1$ and let $\lambda \geq 1$. A Banach space X has the λ -bounded α_{q^*, p^*} -AP if and only if for every Banach space Y ,

$$\{T \in \mathcal{D}_{p,q}(Y^*, X^*) : \|T\|_{\mathcal{D}_{p,q}} \leq 1\} \subset \overline{\{R^* : R \in \mathcal{F}(X, Y), \|R\|_{\mathcal{D}_{p,q}^{dual}} \leq \lambda\}}^{\tau_{w^*}}$$

in $(Y^* \hat{\otimes}_{\alpha_{q^*, p^*}} X)^*$.

For instance, for $1 \leq p \leq \infty$ and $\lambda \geq 1$, a Banach space X has the λ -bounded d_{p^*} -AP if and only if for every Banach space Y ,

$$\{T \in \mathcal{P}_p(Y^*, X^*) : \|T\|_{\mathcal{P}_p} \leq 1\} \subset \overline{\{R^* : R \in \mathcal{F}(X, Y), \|R\|_{\mathcal{P}_p^{dual}} \leq \lambda\}}^{\tau_{w^*}}$$

in $(Y^* \hat{\otimes}_{d_{p^*}} X)^*$.

Corollary 3.6. Let $\lambda \geq 1$. A Banach space X has the λ -bounded AP if and only if for every Banach space Y ,

$$\{T \in \mathcal{L}(Y^*, X^*) : \|T\| \leq 1\} \subset \overline{\{R^* : R \in \mathcal{F}(X, Y), \|R\| \leq \lambda\}}^{\tau_{w^*}}$$

in $(Y^* \hat{\otimes}_{\pi} X)^*$.

Now, we introduce a result of Godefroy and Ozawa [7] on a relation between the λ -bounded AP and the weak* density in the ideal of all operators. The following theorem is the linear version of [7, Corollary 3] (see below of [7, Corollary 3]) and its proof is due to [7, Theorem 2].

Theorem 3.7 (Godefroy and Ozawa). Suppose that X is a separable Banach space. Then X has the λ -bounded AP if and only if for every Banach space Y and every $T \in \mathcal{L}(X, Y^{**})$ with $\|T\| \leq 1$, there exists a net $(R_\alpha)_\alpha$ in $\{R \in \mathcal{L}(X, Y) : \|R\| \leq \lambda\}$ such that

$$\lim_{\alpha} |y^*(R_\alpha x) - (Tx)(y^*)| = 0$$

for every $x \in X$ and $y^* \in Y^*$.

Corollary 3.8. Suppose that X is a separable Banach space. Let $\lambda \geq 1$. Then the following statements are equivalent.

- (a) X has the λ -bounded AP.
- (b) For every Banach space Y ,

$$\{T \in \mathcal{L}(Y^*, X^*) : \|T\| \leq 1\} \subset \overline{\{R^* : R \in \mathcal{L}(X, Y), \|R\| \leq \lambda\}}^{\tau_{w^*}}$$

in $(Y^* \hat{\otimes}_{\pi} X)^*$.

- (c) For every Banach space Y and every $T \in \mathcal{L}(Y^*, X^*)$ with $\|T\| \leq 1$, there exists a net $(R_\alpha)_\alpha$ in $\{R \in \mathcal{L}(X, Y) : \|R\| \leq \lambda\}$ such that

$$\lim_{\alpha} |y^*(R_\alpha x) - (Ty^*)(x)| = 0$$

for every $x \in X$ and $y^* \in Y^*$.

Proof. (b) \Rightarrow (c) is trivial and (a) \Rightarrow (b) follows from Corollary 3.6.

(c) \Rightarrow (a): In order to use Theorem 3.7, let Y be a Banach space and let $T \in \mathcal{L}(X, Y^{**})$ with $\|T\| \leq 1$. Since $T^*i_{Y^*} \in \mathcal{L}(Y^*, X^*)$ and $\|T^*i_{Y^*}\| \leq 1$, by (c) there exists a net $(R_\alpha)_\alpha$ in $\{R \in \mathcal{L}(X, Y) : \|R\| \leq \lambda\}$ such that

$$\lim_{\alpha} |y^*(R_\alpha x) - (Tx)(y^*)| = \lim_{\alpha} |y^*(R_\alpha x) - (T^*i_{Y^*}y^*)(x)| = 0$$

for every $x \in X$ and $y^* \in Y^*$. Hence X has the λ -bounded AP. \square

Consequently, by Corollary 2.7, a separable Banach space X has the λ -bounded AP if (and only if) the map

$$D_\varepsilon : Y^* \hat{\otimes}_\pi X \longrightarrow (\mathcal{L}(X, Y), \|\cdot\|)^*$$

is λ -isomorphic for every Banach space Y . But we do not know whether X has the AP if the map D_ε is injective for every Banach space Y when X is separable.

4. The space of adjoint operators is not weak* dense in the space of compact operators in general

Examples of this section give negative answers of the question in the introduction as well as the question in [3, Section 3].

Lemma 4.1 ([10, Theorem]). *A Banach space X has the AP if and only if for every separable reflexive Banach space Y , $\mathcal{K}(X, Y) \subset \overline{\mathcal{F}(X, Y)}^{\tau_c}$.*

Lemma 4.2 ([8] and cf. [12, Proposition 1.e.3]). *Let X and Y be Banach spaces. Then the dual space $(\mathcal{L}(X, Y), \tau_c)^*$ consists of all functionals f of the form*

$$f(T) = \sum_{n=1}^{\infty} y_n^*(Tx_n),$$

where $\sum_{n=1}^{\infty} y_n^* \otimes x_n \in Y^* \hat{\otimes}_\pi X$.

Example 4.3. It is well known that there exists a Banach space X_0 having a basis such that X_0^* is separable but does not have the AP (cf. [1, Proposition 1.4]). Then by Lemma 4.1, there exists a separable reflexive Banach space Y_0 such that $\mathcal{K}(X_0^*, Y_0^*) \not\subset \overline{\mathcal{F}(X_0^*, Y_0^*)}^{\tau_c} = \overline{\{S^* : S \in \mathcal{F}(Y_0, X_0)\}}^{\tau_c}$ (cf. [12, Lemma 1.e.17]). Thus there exists a $T_0 \in \mathcal{K}(X_0^*, Y_0^*)$ such that $T_0 \notin \overline{\{S^* : S \in \mathcal{F}(Y_0, X_0)\}}^{\tau_c}$.

Now, since Y_0 is reflexive and X_0 has the AP, by Lemma 4.2, we have

$$\begin{aligned} T_0 &\notin \overline{\{S^* : S \in \mathcal{F}(Y_0, X_0)\}}^{\tau_c} \\ &= \overline{\{S^* : S \in \mathcal{F}(Y_0, X_0)\}}^{\tau_{w^*}} \hookrightarrow (X_0^* \hat{\otimes}_\pi Y_0)^* \\ &= \overline{\{S^* : S \in \mathcal{L}(Y_0, X_0)\}}^{\tau_{w^*}}. \end{aligned}$$

Lemma 4.4 ([20]). *There exists a separable reflexive Banach space X_W such that*

$$id_W \in \overline{\mathcal{K}(X_W, X_W)}^{\tau_c}$$

but X_W does not have the AP. Consequently,

$$\mathcal{K}(X_W, X_W) \not\subset \overline{\mathcal{F}(X_W, X_W)}^{\tau_c}.$$

The second example is due to Reinov [15], which needs the following crucial lemma.

Lemma 4.5 ([11]). *For every separable Banach space X , there exist a Banach space Y whose dual has a basis, operators $q : Y^* \rightarrow X$ and $u : Y^{**} \rightarrow X^*$ such that $uq^* = id_{X^*}$.*

Example 4.6. Let X_W be the Banach space in Lemma 4.4. Then by Lemma 4.2 and the separation theorem, there exist an $R \in \mathcal{K}(X_W, X_W)$ and $\sum_{n=1}^{\infty} x_n^* \otimes x_n \in X_W^* \hat{\otimes}_{\pi} X_W$ such that

$$\sum_{n=1}^{\infty} x_n^*(Rx_n) = 1 \text{ but } \sum_{n=1}^{\infty} x_n^*(Sx_n) = 0$$

for every $S \in \mathcal{F}(X_W, X_W)$. Since X_W is separable, we can take the objects in Lemma 4.5. Consider $\sum_{n=1}^{\infty} x_n^* q \otimes x_n \in Y^{**} \hat{\otimes}_{\pi} X_W$ and $u^* i_{X_W} R \in \mathcal{K}(X_W, Y^{***})$. Then

$$\sum_{n=1}^{\infty} (u^* i_{X_W} Rx_n)(x_n^* q) = \sum_{n=1}^{\infty} u(x_n^* q)(Rx_n) = \sum_{n=1}^{\infty} uq^*(x_n^*)(Rx_n) = \sum_{n=1}^{\infty} x_n^*(Rx_n) = 1$$

but

$$\sum_{n=1}^{\infty} x_n^* q(U)x_n = 0$$

for every $U \in \mathcal{F}(X_W, Y^*)$. Let $V := u^* i_{X_W} R$. Then $V^* i_{Y^{**}} \in \mathcal{K}(Y^{**}, X_W^*)$ and we see that

$$\sum_{n=1}^{\infty} (V^* i_{Y^{**}} x_n^* q)(x_n) = 1.$$

Since Y^* has the AP, for every $T \in \mathcal{L}(X_W, Y^*)$, we have

$$\sum_{n=1}^{\infty} (T^* x_n^* q)(x_n) = \sum_{n=1}^{\infty} x_n^* q(Tx_n) = 0.$$

Consequently, $V^* i_{Y^{**}} \notin \overline{\{T^* : T \in \mathcal{L}(X_W, Y^*)\}}^{T_{w^*}} \hookrightarrow (Y^{**} \hat{\otimes}_{\pi} X_W)^*$.

The final example is due to Johnson [9], which gives a direct argument of the one in Example 4.6 without Lemma 4.5.

Example 4.7. Let X_W be the Banach space in Lemma 4.4. Then there exist an $R \in \mathcal{K}(X_W, X_W)$ and $\sum_{n=1}^{\infty} x_n^* \otimes x_n \in X_W^* \hat{\otimes}_{\pi} X_W$ such that

$$\sum_{n=1}^{\infty} x_n^*(Rx_n) = 1 \text{ but } \sum_{n=1}^{\infty} x_n^*(Sx_n) = 0$$

for every $S \in \mathcal{F}(X_W, X_W)$. Since X_W is separable, for every $n \in \mathbb{N}$, we can choose a finite-dimensional subspace E_n of X_W such that $E_1 \subset E_2 \subset \cdots \subset E_n \subset \cdots$ and $X_W = \overline{\bigcup_{n=1}^{\infty} E_n}$. Let

$$X_1 := \left(\sum_{n=1}^{\infty} E_n \right)_{\ell_1}$$

and define the map $q : X_1 \rightarrow X_W$ by

$$q(e_n)_n = \sum_{n=1}^{\infty} e_n.$$

We also define the map $J : X_W \rightarrow X_1^{**} = (\sum_{n=1}^{\infty} E_n^*)_{\ell_{\infty}}^*$ by

$$(Jx)((e_n^*)_n) = \lim_{\mathcal{U}} \hat{e}_n^*(x),$$

where $\hat{e}_n^* \in X_W^*$ is a Hahn-Banach extension of e_n^* and \mathcal{U} is an ultra filter on \mathbb{N} . Then one may check that the map J is well defined, linear and bounded, moreover,

$$(Jx)(q^*x^*) = x^*(x)$$

for every $x \in X_W$ and $x^* \in X_W^*$.

Now, consider $R^*J^*i_{X_1^*} \in \mathcal{K}(X_1^*, X_W^*)$ and $\sum_{n=1}^{\infty} q^*x_n^* \otimes x_n \in X_1^* \hat{\otimes}_{\pi} X_W$. Then we have

$$\sum_{n=1}^{\infty} (R^*J^*i_{X_1^*}q^*x_n^*)(x_n) = \sum_{n=1}^{\infty} (JR x_n)(q^*x_n^*) = \sum_{n=1}^{\infty} x_n^*(R x_n) = 1.$$

We claim that $\sum_{n=1}^{\infty} (T^*q^*x_n^*)(x_n) = 0$ for every $T \in \mathcal{L}(X_W, X_1)$.

Let $T \in \mathcal{L}(X_W, X_1)$. For each k , let $P_k : X_1 \rightarrow (\sum_{n=1}^k E_n)_{\ell_1}$ be the canonical projection. Then

$$\lim_{k \rightarrow \infty} P_k T = T$$

in $(\mathcal{L}(X_W, X_1), \tau_c)$. Thus by Lemma 4.2,

$$\lim_{k \rightarrow \infty} \sum_{n=1}^{\infty} q^*x_n^*(P_k T x_n) = \sum_{n=1}^{\infty} q^*x_n^*(T x_n).$$

But for every k , we have

$$\sum_{n=1}^{\infty} q^*x_n^*(P_k T x_n) = \sum_{n=1}^{\infty} x_n^*(q P_k T x_n) = 0.$$

Thus

$$0 = \sum_{n=1}^{\infty} q^*x_n^*(T x_n) = \sum_{n=1}^{\infty} (T^*q^*x_n^*)(x_n).$$

Consequently, $R^*J^*i_{X_1^*} \notin \overline{\{T^* : T \in \mathcal{L}(X_W, X_1)\}}^{\tau_{w^*}} \hookrightarrow (X_1^* \hat{\otimes}_{\pi} X_W)^*$.

The proof of the following lemma is due to the proof of [15, Corollary 3.6].

Lemma 4.8. *Let X and Y be Banach spaces such that*

$$\mathcal{K}(Y^*, X^*) \not\subset \overline{\{T^* : \mathcal{L}(X, Y)\}}^{\tau_{w^*}} \hookrightarrow (Y^* \hat{\otimes}_{\pi} X)^*.$$

Then

$$\mathcal{K}(Z^*, Z^*) \not\subset \overline{\{T^* : \mathcal{L}(Z, Z)\}}^{\tau_{w^*}} \hookrightarrow (Z^* \hat{\otimes}_\pi Z)^*,$$

where $Z := Y \oplus X$.

Examples 4.3, 4.6 and 4.7 using Lemma 4.8 give:

Corollary 4.9. *There exists a separable Banach space Z such that*

$$\mathcal{K}(Z^*, Z^*) \not\subset \overline{\{T^* : \mathcal{L}(Z, Z)\}}^{\tau_{w^*}} \hookrightarrow (Z^* \hat{\otimes}_\pi Z)^*.$$

Now, we introduce a well known open problem. Reinov [14] constructed Banach spaces failing to have the g_p -AP and the d_p -AP ($1 \leq p \leq \infty, p \neq 2$). It was shown in [6, Proposition 2] that, if a Banach space has the g_p -AP, then it has the w_p -AP. So we are naturally led to the following question.

Problem. For $1 < p < \infty, p \neq 2$, does every Banach space have the w_p -AP?

In fact, we do not know whether every Banach space has the 1-bounded w_p -AP for $1 < p < \infty, p \neq 2$ (cf. [4, Section 21.12]). In view of Corollary 3.2 below, **Problem** can be reformulated by the following.

Problem. Let $1 < p < \infty, p \neq 2$. For every Banach spaces X and Y ,

$$\mathcal{D}_p(Y^*, X^*) = \overline{\{R^* : R \in \mathcal{F}(X, Y)\}}^{\tau_{w^*}}$$

in $(Y^* \hat{\otimes}_{w_p} X)^*$?

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