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On the strong regularity of degenerate additive noise driven stochastic differential equations with respect to their initial values

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A R T I C L E I N F O

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A B S T R A C T

Recently in [M. Hairer, M. Hutzenthaler, and A. Jentzen, Ann. Probab. 43 (2) (2015) 468–527] and [A. Jentzen, T. Müller-Gronbach, and L. Yaroslavtseva, Commun. Math. Sci. 14 (6) (2016) 1477–1500] stochastic differential equations (SDEs) with smooth coefficient functions have been constructed which have an arbitrarily slowly converging modulus of continuity in the initial value. In these SDEs it is crucial that some of the first order partial derivatives of the drift coefficient functions grow at least exponentially and, in particular, quicker than any polynomial. However, in applications SDEs do typically have coefficient functions whose first order partial derivatives are polynomially bounded. In this article we study whether arbitrarily bad regularity phenomena in the initial value may also arise in the latter case and we partially answer this question in the negative. More precisely, we show that every additive noise driven SDE which admits a Lyapunov-type condition (which ensures the existence of a unique solution of the SDE) and which has a drift coefficient function whose first order partial derivatives grow at most polynomially is at least logarithmically Hölder continuous in the initial value.

Contents

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1. Introduction

The regularity analysis of nonlinear stochastic differential equations (SDEs) with respect to their initial values is an active research topic in stochastic analysis (cf., e.g., [2,3,5,6,8,9,17–19,21–23,28] and the references mentioned therein). In particular, establishing sufficient regularity in the initial value for SDEs is of fundamental importance in determining convergence rates for numerical approximation schemes for such SDEs (see, e.g., [11]). It has recently been revealed in the literature that there exist SDEs with smooth coefficient functions which have very poor regularity properties in the initial value. More precisely, it has been shown in [6] that there exist additive noise driven SDEs with infinitely often differentiable drift coefficient functions which have a modulus of continuity in the initial value that converges to zero slower than with any polynomial rate. Moreover, in [15] additive noise driven SDEs with infinitely often differentiable drift coefficient functions have been constructed which even have an arbitrarily slowly converging modulus of continuity in the initial value. In these SDEs it is crucial that the first order partial derivatives of the drift coefficient functions grow at least exponentially and, in particular, quicker than any polynomial. However, in applications SDEs do typically have coefficient functions whose first order partial derivatives grow at most polynomially (cf., e.g., [1,4,7,20,24–27], [16, Chapter 7], and [10, Chapter 4] for examples). In particular, in many applications the coefficient functions of the SDEs under consideration are polynomials (cf., e.g., [1,4,24,26,27], [16, Chapter 7], and [10, Chapter 4] for examples). In view of this, the natural question arises whether such arbitrarily bad regularity phenomena in the initial value may also arise in the case of SDEs with coefficient functions whose first order partial derivatives grow at most polynomially. It is the subject of the main result of this article to partially answer this question in the negative. More precisely, the main result of this article, Theorem 1.1 below, shows that every additive noise driven SDE which admits a Lyapunov-type condition (which ensures the existence of a unique solution of the SDE) and which has a drift coefficient function whose first order partial derivatives grow at most polynomially is at least logarithmically Hölder continuous in the initial value.

Theorem 1.1. *Let $d, m \in \mathbb{N}$, $T, \kappa \in [0, \infty)$, $\alpha \in [0, 2)$, $\mu \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathbb{R}^{d\times m}$, $V \in C^1(\mathbb{R}^d, [0, \infty))$, let $\|\cdot\| \colon \mathbb{R}^d \to [0, \infty)$ and $|\!|\!|\cdot|\!|\!| \colon \mathbb{R}^m \to [0, \infty)$ be norms, assume for all $x, h \in \mathbb{R}^d$, $z \in \mathbb{R}^m$ that $\|\mu'(x)h\| \leq \kappa\big(1 + \|x\|^\kappa\big)\|h\|$, $V'(x)\mu(x + \sigma z) \leq \kappa(1 + |\!|\!|z|\!|\!|^\alpha)V(x)$, and $\|x\| \leq V(x)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $W \colon [0, T] \times \Omega \to \mathbb{R}^m$ be a standard Brownian motion with continuous sample paths. Then*

(i) *there exist unique stochastic processes $X^x \colon [0, T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, with continuous sample paths such that for all $x \in \mathbb{R}^d$, $t \in [0, T]$, $\omega \in \Omega$ it holds that*

$$X^x(t, \omega) = x + \int_0^t \mu(X^x(s, \omega))\, \mathrm{d}s + \sigma W(t, \omega) \tag{1}$$

and

(ii) *it holds for all* $R, q \in [0, \infty)$ *that there exists* $c \in (0, \infty)$ *such that for all* $x, y \in \{v \in \mathbb{R}^d : \|v\| \leq R\}$ *with* $0 < \|x - y\| \neq 1$ *it holds that*

$$\sup_{t \in [0,T]} \mathbb{E}\big[\|X^x(t) - X^y(t)\|\big] \leq c \,\big|\ln(\|x - y\|)\big|^{-q}. \tag{2}$$

Theorem 1.1 is proved as Theorem 7.4 in Subsection 7.3 below. Let us add some explanatory comments on the Lyapunov-type assumption in Theorem 1.1 that for all $x \in \mathbb{R}^d$, $z \in \mathbb{R}^m$ it holds that $V'(x)\mu(x+\sigma z) \leq \kappa(1 + \|z\|^\alpha)V(x)$. In particular, let us illustrate that this assumption is indeed a Lyapunov-type condition and that $V : \mathbb{R}^d \to [0, \infty)$ indeed deserves to be called a Lyapunov-type function. To do so, let us consider $\mathfrak{d} \in \mathbb{N}$, $\mathfrak{T} \in [0, \infty)$, $F \in C^{0,1}([0, \mathfrak{T}] \times \mathbb{R}^\mathfrak{d}, \mathbb{R}^\mathfrak{d})$, $\mathfrak{V} \in C^1(\mathbb{R}^\mathfrak{d}, [0, \infty))$, $\mathfrak{a} \in C([0, \mathfrak{T}], [0, \infty))$, and functions $\mathfrak{X}^x \in C^1([0, \mathfrak{T}], \mathbb{R}^\mathfrak{d})$, $x \in \mathbb{R}^d$, such that for all $t \in [0, \mathfrak{T}]$, $x \in \mathbb{R}^\mathfrak{d}$ it holds that

$$\mathfrak{V}'(x)F(t, x) \leq \mathfrak{a}(t)\mathfrak{V}(x), \quad (\mathfrak{X}^x)'(t) = F(t, \mathfrak{X}^x(t)), \quad \text{and} \quad \mathfrak{X}^x(0) = x. \tag{3}$$

The assumption that for all $t \in [0, \mathfrak{T}]$, $x \in \mathbb{R}^\mathfrak{d}$ it holds that $\mathfrak{V}'(x)F(t, x) \leq \mathfrak{a}(t)\mathfrak{V}(x)$ is a classical Lyapunov-type condition for the ordinary differential equation (ODE) in (3) above. Indeed, observe that the assumption that for all $t \in [0, \mathfrak{T}]$, $x \in \mathbb{R}^\mathfrak{d}$ it holds that $\mathfrak{V}'(x)F(t, x) \leq \mathfrak{a}(t)\mathfrak{V}(x)$ and the fundamental theorem of calculus ensure that for all $x \in \mathbb{R}^\mathfrak{d}$, $t \in [0, \mathfrak{T}]$ it holds that

$$\mathfrak{V}(\mathfrak{X}^x(t)) = \mathfrak{V}(x) + \int_0^t \mathfrak{V}'(\mathfrak{X}^x(s))F(s, \mathfrak{X}^x(s))\,\mathrm{d}s \leq \mathfrak{V}(x) + \int_0^t \mathfrak{a}(s)\mathfrak{V}(\mathfrak{X}^x(s))\,\mathrm{d}s \tag{4}$$

and Gronwall's inequality hence shows that for all $t \in [0, \mathfrak{T}]$, $x \in \mathbb{R}^\mathfrak{d}$ it holds that $\mathfrak{V}(\mathfrak{X}^x(t)) \leq \exp\big(\int_0^t \mathfrak{a}(s)\mathrm{d}s\big)\mathfrak{V}(x)$. A priori estimates of this type can then be used to deduce global existence of solutions of the ODE in (3) above. Next we show that the condition in Theorem 1.1 that for all $x \in \mathbb{R}^d$, $z \in \mathbb{R}^m$ it holds that $V'(x)\mu(x + \sigma z) \leq \kappa(1 + \|z\|^\alpha)V(x)$ reduces to a special case of the general Lyapunov condition in (3). Indeed, let $\mathbb{X}^x : [0, T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, satisfy for all $x \in \mathbb{R}^d$, $t \in [0, T]$, $\omega \in \Omega$ that $\mathbb{X}^x(t, \omega) = X^x(t, \omega) - \sigma W(t, \omega)$ and observe that (1) in Theorem 1.1 above ensures that for all $x \in \mathbb{R}^d$, $t \in [0, T]$, $\omega \in \Omega$ it holds that

$$\tfrac{\partial}{\partial t}\mathbb{X}^x(t, \omega) = \mu(\mathbb{X}^x(t, \omega) + \sigma W(t, \omega)) \qquad \text{and} \qquad \mathbb{X}^x(0, \omega) = x. \tag{5}$$

The assumption in Theorem 1.1 that for all $x \in \mathbb{R}^d$, $z \in \mathbb{R}^m$ it holds that $V'(x)\mu(x+\sigma z) \leq \kappa(1+\|z\|^\alpha)V(x)$ now ensures for every $\omega \in \Omega$ that the Lyapunov condition in (3) is satisfied for the ODE in (5) with the Lyapunov-type function $\mathfrak{V} = V$. For further reading on Lyapunov-type functions and concrete examples of nonlinear (stochastic) differential equations satisfying Lyapunov-type conditions, we refer, e.g., to [10, Chapter 4].

Next let us add some comments regarding the statement of Theorem 1.1 above. Inequality (2) in Theorem 1.1 proves, roughly speaking, only Hölder continuity in the initial value in a logarithmic sense but does neither prove local Lipschitz continuity nor prove local Hölder continuity in the initial value in the usual sense. In view of this, the question arises whether the statement of Theorem 1.1 can be strengthened to ensure local Hölder continuity in the initial value in the usual sense. In [13] we show that this is not the case and specify a concrete additive noise driven SDE which satisfies the hypotheses of Theorem 1.1 but whose solution fails for every arbitrarily small $\alpha \in (0, 1]$ to be locally α-Hölder continuous in the initial value. In particular, we show in [13] that under the hypotheses of Theorem 1.1 the upper bound in (2) can not be substantially improved in general. For the reader's convenience, we include in Theorem 1.2 below, a special case of the central result of [13].

Theorem 1.2. *Let $m \in \mathbb{N}$, $d \in \{5, 6, \dots\}$, $T \in (0,\infty)$, $\tau \in (0,T)$, $v \in \mathbb{R}^d$, $\delta \in \mathbb{R}^d \setminus \{0\}$ let $\|\cdot\| \colon \mathbb{R}^d \to [0,\infty)$ be the standard norm on $\mathbb{R}^d$, let $|\!|\!|\cdot|\!|\!| \colon \mathbb{R}^m \to [0,\infty)$ be a norm, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $W \colon [0,T] \times \Omega \to \mathbb{R}^m$ be a standard Brownian motion with continuous sample paths. Then there exist $\mu \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathbb{R}^{d \times m}$, $V \in C^\infty(\mathbb{R}^d, [0,\infty))$, $\kappa \in (0,\infty)$, and stochastic processes $X^x \colon [0,T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, with continuous sample paths such that it holds for all $x, h \in \mathbb{R}^d$, $z \in \mathbb{R}^m$, $t \in [0,T]$, $\omega \in \Omega$, $s \in (\tau, T)$, $\alpha \in (0,\infty)$ that $\|\mu'(x)h\| \leq \kappa\big(1 + \|x\|^\kappa\big)\|h\|$, $V'(x)\mu(x + \sigma z) \leq \kappa(1 + |\!|\!|z|\!|\!|)V(x)$, $\|x\| \leq V(x)$, $X^x(t,\omega) = x + \int_0^t \mu(X^x(s,\omega))\,\mathrm{d}s + \sigma W(t,\omega)$, and*

$$\exists\, c \in (0,\infty) \colon \forall\, w \in \{v + r\delta \colon r \in [0,1]\} \colon \mathbb{E}\big[\|X^v(s) - X^w(s)\|\big] \geq c\,\|v - w\|^\alpha. \tag{6}$$

Note that the objects whose existence is asserted in Theorem 1.2 satisfy the assumptions from Theorem 1.1 so that Theorem 1.1 can be applied to them. In particular, the stochastic processes $X^x \colon [0,T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, whose existence is asserted in Theorem 1.2 satisfy condition (2) in Theorem 1.1 and hence are logarithmically Hölder continuous in the initial value. By contrast, we have that (6) in Theorem 1.2 asserts that for every $\alpha \in (0,1]$ it holds that these stochastic processes are not α-Hölder continuous in the initial value.

In the following we briefly sketch the key ideas of our proof of inequality (2) in Theorem 1.1. A straightforward approach to estimating the expectation of the Euclidean distance between two solutions of the SDE (1) with different initial values (cf. the left hand side of (2)) would be (i) to apply the fundamental theorem of calculus to the difference of the two solutions with the derivative being taken with respect to the initial value, thereafter, (ii) to employ the triangle inequality to get the Euclidean norm inside of the Riemann integral which has appeared due to the application of the fundamental theorem of calculus, and, finally, (iii) to try to provide a finite upper bound for the expectation of the Euclidean operator norm of the derivative processes of solutions of (1) with respect to the initial value. This approach, however, fails to work in general under the hypotheses of Theorem 1.1 as the derivative processes of solutions may have very poor integrability properties and, in particular, may have infinite absolute moments (cf., e.g., [6, Sections 2 and 3], [13, Theorem 1.1], and Theorem 1.2 above). A key idea in this article for overcoming the latter obstacle is to estimate the expectation of the Euclidean distance between the two solutions in terms of the expectation of a new distance between the two solutions, which is induced from a very slowly growing norm-type function. As in the approach above, we then also apply the fundamental theorem of calculus to the difference of the two solutions. However, in the latter approach the derivative processes of solutions appear only inside of the argument of the very slowly growing norm-type function and the expectation of the resulting random variable is finite. We then estimate the expectation of this random variable by employing properties of the derivative processes of solutions and the assumption that the first order partial derivatives of the drift coefficient function grow at most polynomially and, thereby, finally establish inequality (2).

The remainder of this article is organized as follows. In Section 2 we present an a priori estimate and an existence and uniqueness result for a certain class of perturbed ODEs. In Section 3 we provide measurability properties for certain space-time maxima of stochastic processes and pathwise solutions of certain additive noise driven SDEs. In Section 4 we establish an existence, uniqueness and regularity result for solutions of certain additive noise driven SDEs. Section 5 is devoted to integrability properties for standard Brownian motions and solutions of certain additive noise driven SDEs. In Section 6 we study regularity properties of solutions of certain additive noise driven SDEs with respect to their initial values, conditional on growth properties of their derivative processes. Finally, in Section 7 we combine the results from Sections 4–6 to establish our main result Theorem 7.4.

2. Existence of solutions of perturbed ODEs

In this section we employ suitable Lyapunov-type functions to establish in Lemma 2.2 in Subsection 2.2 below an essentially well-known existence and uniqueness result for a certain class of perturbed ODEs. Lemma 2.2 follows in a straightforward manner from the essentially well-known a priori estimate in Lemma 2.1 in Subsection 2.1 below in combination with standard results on local existence of maximal mild solutions of evolution equations from the scientific literature (cf., e.g., [14, Section 8]). Lemma 2.1 is also used to establish in Lemma 5.2 in Subsection 5.2 integrability properties for SDEs. A detailed proof of Lemma 2.1 can be found, e.g., in the arXiv version of this article [12, Lemma 2.1]. Lemma 2.2 is employed to establish in Lemma 4.4 in Subsection 4.3 differentiability properties for SDEs with respect to their initial values.

2.1. A priori estimates for solutions of perturbed ODEs

Lemma 2.1. *Let $d, m \in \mathbb{N}$, $T \in [0, \infty)$, $\xi \in \mathbb{R}^d$, $\mu \in C(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathbb{R}^{d\times m}$, $\varphi \in C(\mathbb{R}^m, [0, \infty))$, $V \in C^1(\mathbb{R}^d, [0, \infty))$, let $\|\cdot\| \colon \mathbb{R}^d \to [0, \infty)$ be a norm, let $J \subseteq [0, T]$ be an interval with $0 \in J$, and let $y \in C(J, \mathbb{R}^d)$, $w \in C([0, T], \mathbb{R}^m)$ satisfy for all $x \in \mathbb{R}^d$, $u \in \mathbb{R}^m$, $t \in J$ that $V'(x)\mu(x + \sigma u) \leq \varphi(u)V(x)$, $\|x\| \leq V(x)$, and*

$$y(t) = \xi + \int_0^t \mu(y(s))\,\mathrm{d}s + \sigma w(t). \tag{7}$$

Then it holds that $\sup_{t\in J}\big[\varphi(w(t)) + \|\sigma w(t)\|\big] < \infty$ and

$$\sup_{t\in J}\|y(t)\| \leq V(\xi)\exp\!\left(T\left[\sup_{t\in J}\varphi(w(t))\right]\right) + \left[\sup_{t\in J}\|\sigma w(t)\|\right]. \tag{8}$$

2.2. Existence of solutions of perturbed ODEs

Lemma 2.2. *Let $d, m \in \mathbb{N}$, $T \in [0, \infty)$, $\xi \in \mathbb{R}^d$, $\sigma \in \mathbb{R}^{d\times m}$, $\varphi \in C(\mathbb{R}^m, [0, \infty))$, $V \in C^1(\mathbb{R}^d, [0, \infty))$, $w \in C([0, T], \mathbb{R}^m)$, let $\|\cdot\| \colon \mathbb{R}^d \to [0, \infty)$ be a norm, let $\mu \colon \mathbb{R}^d \to \mathbb{R}^d$ be a locally Lipschitz continuous function, and assume for all $x \in \mathbb{R}^d$, $z \in \mathbb{R}^m$ that $V'(x)\mu(x + \sigma z) \leq \varphi(z)V(x)$ and $\|x\| \leq V(x)$. Then there exists a unique $y \in C([0, T], \mathbb{R}^d)$ such that for all $t \in [0, T]$ it holds that $y(t) = \xi + \int_0^t \mu(y(s))\,\mathrm{d}s + \sigma w(t)$.*

3. Measurability properties for stochastic processes and solutions of SDEs

In this section we provide in Lemma 3.1 well-known measurability properties of space-time maxima of stochastic processes, which is used as a technical tool in the proof of Lemma 6.3 in Subsection 6.2 on conditional sub-Hölder properties for SDEs. Furthermore, we present in Lemma 3.2 below the well-known fact that pathwise solutions of certain additive noise driven SDEs are stochastic processes. Lemma 3.2 is used in the proof of Lemma 4.4 in Subsection 4.3 on differentiability properties with respect to the initial value for SDEs. A detailed proof of Lemma 3.1 and Lemma 3.2 can be found, e.g., in the arXiv version of this article [12, Lemma 3.1, Lemma 3.2].

Lemma 3.1. *Let $d \in \mathbb{N}$, $T, R \in [0, \infty)$, let $\|\cdot\| \colon \mathbb{R}^d \to [0, \infty)$ be a norm, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $Y^x \colon [0, T] \times \Omega \to [0, \infty)$, $x \in \mathbb{R}^d$, be stochastic processes with continuous sample paths which satisfy for all $t \in [0, T]$, $\omega \in \Omega$ that $(\mathbb{R}^d \ni x \mapsto Y^x(t, \omega) \in [0, \infty)) \in C(\mathbb{R}^d, [0, \infty))$. Then it holds that*

$$\Omega \ni \omega \mapsto \sup_{x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R\}} \sup_{t \in [0,T]} Y^x(t,\omega) \in [0,\infty] \tag{9}$$

is an $\mathcal{F}/\mathcal{B}([0,\infty])$-measurable function.

Lemma 3.2. *Let $d \in \mathbb{N}$, $T \in [0,\infty)$, $f \in C([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$, $\xi \in \mathbb{R}^d$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W \colon [0,T] \times \Omega \to \mathbb{R}^d$ be a stochastic process with continuous sample paths, let $\|\cdot\| \colon \mathbb{R}^d \to [0,\infty)$ be a norm, assume for all $r \in (0,\infty)$ that*

$$\sup_{t \in [0,T]} \sup_{\substack{x,y \in \mathbb{R}^d,\, x \neq y, \\ \|x\|+\|y\| \leq r}} \frac{\|f(t,x) - f(t,y)\|}{\|x-y\|} < \infty, \tag{10}$$

and let $Y \colon [0,T] \times \Omega \to \mathbb{R}^d$ satisfy for all $t \in [0,T]$, $\omega \in \Omega$ that $\big([0,T] \ni s \mapsto Y(s,\omega) \in \mathbb{R}^d\big) \in C([0,T], \mathbb{R}^d)$ and

$$Y(t,\omega) = \xi + \int_0^t f(s, Y(s,\omega))\,\mathrm{d}s + W(t,\omega). \tag{11}$$

Then it holds that Y is a stochastic process.

4. Differentiability with respect to the initial value for SDEs

In this section we establish in Lemma 4.4 in Subsection 4.3 below an existence, uniqueness and regularity result for solutions of certain additive noise driven SDEs. Lemma 4.4 is a main tool to prove Proposition 7.1 in Subsection 7.1 on regularity properties with respect to the initial value for SDEs with general noise. Our proof of Lemma 4.4 exploits the related regularity results for solutions of certain ODEs in Lemmas 4.1–4.3 below. A detailed proof of Lemma 4.1 can be found, e.g., in the arXiv version of this article [12, Lemma 5.1]. For the reader's convenience we include in this section also proofs for Lemmas 4.2–4.4.

4.1. Local Lipschitz continuity with respect to the initial value for ODEs

Lemma 4.1. *Let $d \in \mathbb{N}$, $w \in \mathbb{R}^d$, $T \in [0,\infty)$, $f \in C^{0,1}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$, let $\|\cdot\| \colon \mathbb{R}^d \to [0,\infty)$ be a norm, and let $y^x \in C([0,T], \mathbb{R}^d)$, $x \in \mathbb{R}^d$, be functions which satisfy for all $x \in \mathbb{R}^d$, $t \in [0,T]$ that*

$$y^x(t) = x + \int_0^t f(s, y^x(s))\,\mathrm{d}s. \tag{12}$$

Then there exist $r, L \in (0,\infty)$ such that for all $v \in \{u \in \mathbb{R}^d \colon \|u - w\| \leq r\}$, $t \in [0,T]$ it holds that

$$\|y^v(t) - y^w(t)\| \leq L\|v - w\|. \tag{13}$$

4.2. Differentiability with respect to the initial value for ODEs

Lemma 4.2. *Let $d \in \mathbb{N}$, $T \in [0,\infty)$, $f \in C^{0,1}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ and let $y^x \in C([0,T], \mathbb{R}^d)$, $x \in \mathbb{R}^d$, be functions which satisfy for all $t \in [0,T]$, $x \in \mathbb{R}^d$ that*

$$y^x(t) = x + \int_0^t f(s, y^x(s))\,\mathrm{d}s. \tag{14}$$

Then

(i) *it holds that* $([0,T] \times \mathbb{R}^d \ni (t,x) \mapsto y^x(t) \in \mathbb{R}^d) \in C^{0,1}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ *and*
(ii) *it holds for all* $x, h \in \mathbb{R}^d$, $t \in [0,T]$ *that*

$$\tfrac{\partial}{\partial x} y^x(t)h = h + \int_0^t f^{(0,1)}(s, y^x(s))\big(\tfrac{\partial}{\partial x} y^x(s)h\big)\,\mathrm{d}s. \tag{15}$$

Proof of Lemma 4.2. Throughout this proof let $\|\cdot\|\colon \mathbb{R}^d \to [0,\infty)$ be the d-dimensional Euclidean norm, and for all $r \in [0,\infty)$ and $u \in \mathbb{R}^d$ let

$$\begin{aligned} C_r &= \sup_{t\in[0,T]} \sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\le r\}} \|f(t,x)\|, \\ K_r &= \sup_{\substack{(x,t,v)\in\mathbb{R}^d\times[0,T]\times\mathbb{R}^d,\\ \|x\|\le r,\, \|v\|=1}} \|f^{(0,1)}(t,x)v\|, \\ R_u &= \sup_{t\in[0,T]} \|y^u(t)\|. \end{aligned} \tag{16}$$

Observe that the continuity of the functions y^u, $u \in \mathbb{R}^d$, and the fact that $f \in C^{0,1}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ imply for all $u \in \mathbb{R}^d$ and all $r \in [0,\infty)$ that $R_u, C_r, K_r < \infty$. Note that Lemma 4.1 proves that there exist $L_w, r_w \in (0,\infty)$, $w \in \mathbb{R}^d$, such that for all $v, w \in \mathbb{R}^d$, $t \in [0,T]$ with $\|v - w\| < r_w$ it holds that

$$\|y^v(t) - y^w(t)\| \le L_w \|v - w\|. \tag{17}$$

Next observe that (14) implies that for all $w \in \mathbb{R}^d$, $t \in [0,T]$, $u \in [0,t]$ it holds that

$$\|y^w(t) - y^w(u)\| = \left\| \int_u^t f(s, y^w(s))\,\mathrm{d}s \right\| \le (t-u) C_{R_w}. \tag{18}$$

This and (17) prove that for all $v, w \in \mathbb{R}^d$, $t, u \in [0,T]$ with $\|v - w\| < r_w$ it holds that

$$\|y^v(t) - y^w(u)\| \le \|y^v(t) - y^w(t)\| + \|y^w(t) - y^w(u)\| \le L_w\|v - w\| + C_{R_w}|t - u|. \tag{19}$$

Therefore, we obtain that for all $v, w \in \mathbb{R}^d$, $t, u \in [0,T]$, $\varepsilon \in (0,\infty)$ with $\|v - w\| < \min\{r_w, (2L_w)^{-1}\varepsilon\}$ and $|t - u| < (2C_{R_w} + 1)^{-1}\varepsilon$ it holds that $\|y^v(t) - y^w(u)\| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. This establishes that

$$[0,T] \times \mathbb{R}^d \ni (t,x) \mapsto y^x(t) \in \mathbb{R}^d \tag{20}$$

is a continuous function. Next note that there exist unique $v^{x,h} \in C([0,T], \mathbb{R}^d)$, $x, h \in \mathbb{R}^d$, such that for all $x, h \in \mathbb{R}^d$, $t \in [0,T]$ it holds that

$$v^{x,h}(t) = h + \int_0^t f^{(0,1)}(s, y^x(s)) v^{x,h}(s)\,\mathrm{d}s. \tag{21}$$

This implies that for all $x, h, k \in \mathbb{R}^d$, $\lambda, \mu \in \mathbb{R}$, $t \in [0,T]$ it holds that

$$\lambda v^{x,h}(t) + \mu v^{x,k}(t) = \lambda h + \mu k + \left[\int_0^t f^{(0,1)}(s, y^x(s))\big(\lambda v^{x,h}(s) + \mu v^{x,k}(s)\big)\, \mathrm{d}s\right]. \tag{22}$$

Combining this with (21) proves that for all $x, h, k \in \mathbb{R}^d$, $\lambda, \mu \in \mathbb{R}$ it holds that $v^{x,\lambda h+\mu k} = \lambda v^{x,h} + \mu v^{x,k}$. This shows that for all $x \in \mathbb{R}^d$, $t \in [0,T]$ it holds that $\mathbb{R}^d \ni h \mapsto v^{x,h}(t) \in \mathbb{R}^d$ is a linear function. Next observe that the fact that $f^{(0,1)}$ is continuous implies that there exist $\delta^{\rho}_{\varepsilon} \in (0,\infty)$, $\rho, \varepsilon \in (0,\infty)$, such that for all $\rho, \varepsilon \in (0,\infty)$, $t \in [0,T]$, $\theta \in \{x \in \mathbb{R}^d : \|x\| \le \rho\}$, $\vartheta \in \{x \in \mathbb{R}^d : \|x - \theta\| \le \delta^{\rho}_{\varepsilon}\}$, $h \in \mathbb{R}^d$ it holds that

$$\|f^{(0,1)}(t,\vartheta)h - f^{(0,1)}(t,\theta)h\| \le \varepsilon \|h\|. \tag{23}$$

In addition, note that (17) implies that for all $\rho, \varepsilon \in (0,\infty)$, $z, x \in \mathbb{R}^d$, $t \in [0,T]$, $u \in [0,1]$ with $\|x - z\| < \min\{r_z, (L_z)^{-1}\delta^{\rho}_{\varepsilon}\}$ it holds that

$$\big\|\big[y^z(t) + u(y^x(t) - y^z(t))\big] - y^z(t)\big\| = u\|y^x(t) - y^z(t)\| \le uL_z\|x - z\| \le \delta^{\rho}_{\varepsilon}. \tag{24}$$

Combining this with (23) shows that for all $\varepsilon \in (0,\infty)$, $z, x, h \in \mathbb{R}^d$, $t \in [0,T]$, $u \in [0,1]$ with $\|x - z\| < \min\{r_z, (L_z)^{-1}\delta^{R_z}_{\varepsilon}\}$ it holds that

$$\big\|f^{(0,1)}\big(t, y^z(t) + u(y^x(t) - y^z(t))\big)h - f^{(0,1)}(t, y^z(t))h\big\| \le \varepsilon\|h\|. \tag{25}$$

Hence for all $\varepsilon \in (0,\infty)$, $z, x, h, \mathfrak{h} \in \mathbb{R}^d$, $t \in [0,T]$, $u \in [0,1]$ with $\|x - z\| < \min\{r_z, (L_z)^{-1}\delta^{R_z}_{\varepsilon}\}$ it holds that

$$\begin{aligned} &\big\|f^{(0,1)}\big(t, y^z(t) + u(y^x(t) - y^z(t))\big)h - f^{(0,1)}(t, y^z(t))\mathfrak{h}\big\| \\ &\le \big\|f^{(0,1)}\big(t, y^z(t) + u(y^x(t) - y^z(t))\big)h - f^{(0,1)}(t, y^z(t))h\big\| \\ &\quad + \|f^{(0,1)}(t, y^z(t))h - f^{(0,1)}(t, y^z(t))\mathfrak{h}\| \\ &\le \varepsilon\|h\| + \|f^{(0,1)}(t, y^z(t))(h - \mathfrak{h})\| \le \varepsilon\|h\| + K_{R_z}\|h - \mathfrak{h}\|. \end{aligned} \tag{26}$$

The fundamental theorem of calculus and (17) hence prove that for all $\varepsilon \in (0,\infty)$, $z, k \in \mathbb{R}^d$, $t \in [0,T]$ with $\|k\| < \min\{r_z, (L_z)^{-1}\delta^{R_z}_{\varepsilon}\}$ it holds that

$$\begin{aligned} &\big\|f(t, y^{z+k}(t)) - f(t, y^z(t)) - f^{(0,1)}(t, y^z(t))(v^{z,k}(t))\big\| \\ &= \left\|\int_0^1 f^{(0,1)}\big(t, y^z(t) + u(y^{z+k}(t) - y^z(t))\big)(y^{z+k}(t) - y^z(t)) - f^{(0,1)}(t, y^z(t))v^{z,k}(t)\, \mathrm{d}u\right\| \\ &\le \int_0^1 \varepsilon\|y^{z+k}(t) - y^z(t)\| + K_{R_z}\|y^{z+k}(t) - y^z(t) - v^{z,k}(t)\|\, \mathrm{d}u \\ &\le \varepsilon L_z\|k\| + K_{R_z}\|y^{z+k}(t) - y^z(t) - v^{z,k}(t)\|. \end{aligned} \tag{27}$$

Combining this with (14) and (21) shows that for all $\varepsilon \in (0,\infty)$, $z, k \in \mathbb{R}^d$, $t \in [0,T]$ with $\|k\| < \min\{r_z, (L_z)^{-1}\delta^{R_z}_{\varepsilon}\}$ it holds that

$$
\begin{aligned}
&\|y^{z+k}(t) - y^z(t) - v^{z,k}(t)\| \\
&= \left\| \int_0^t f(s, y^{z+k}(s)) - f(s, y^z(s)) - f^{(0,1)}(s, y^z(s)) v^{z,k}(s) \,\mathrm{d}s \right\| \\
&\leq \int_0^t \varepsilon L_z \|k\| + K_{R_z} \|y^{z+k}(s) - y^z(s) - v^{z,k}(s)\| \,\mathrm{d}s \\
&\leq T\varepsilon L_z \|k\| + \int_0^t K_{R_z} \|y^{z+k}(s) - y^z(s) - v^{z,k}(s)\| \,\mathrm{d}s.
\end{aligned}
\tag{28}
$$

Gronwall's inequality therefore shows that for all $\varepsilon \in (0, \infty)$, $z, k \in \mathbb{R}^d$, $t \in [0, T]$ with $\|k\| < \min\{r_z, (L_z)^{-1} \delta_\varepsilon^{R_z}\}$ it holds that

$$
\|y^{z+k}(t) - y^z(t) - v^{z,k}(t)\| \leq T\varepsilon L_z \|k\| \exp(K_{R_z} t). \tag{29}
$$

Therefore, we obtain that for all $z \in \mathbb{R}^d$, $t \in [0, T]$ it holds that

$$
\limsup_{\substack{k \to 0, \\ k \in \mathbb{R}^d \setminus \{0\}}} \left[\frac{\|y^{z+k}(t) - y^z(t) - v^{z,k}(t)\|}{\|k\|} \right] = 0. \tag{30}
$$

Combining this with the fact that for all $x \in \mathbb{R}^d$, $t \in [0, T]$ it holds that $\mathbb{R}^d \ni h \mapsto v^{x,h}(t) \in \mathbb{R}^d$ is a linear function shows that for all $t \in [0, T]$, $x, h \in \mathbb{R}^d$ it holds that $\mathbb{R}^d \ni z \mapsto y^z(t) \in \mathbb{R}^d$ is a differentiable function and

$$
\tfrac{\partial}{\partial x} y^x(t) h = v^{x,h}(t). \tag{31}
$$

This and (21) establish (ii). Next note that (21) implies that for all $x, h \in \mathbb{R}^d$, $t \in [0, T]$ it holds that

$$
\|v^{x,h}(t)\| \leq \|h\| + \int_0^t K_{R_x} \|v^{x,h}(s)\| \,\mathrm{d}s. \tag{32}
$$

Gronwall's inequality hence ensures that for all $x, h \in \mathbb{R}^d$, $t \in [0, T]$ it holds that

$$
\|v^{x,h}(t)\| \leq \|h\| \exp(K_{R_x} T). \tag{33}
$$

In addition, observe that (17) implies that for all $x, z \in \mathbb{R}^d$, $t \in [0, T]$ with $\|x - z\| < \min\{1, r_z\}$ it holds that

$$
\|y^x(t)\| \leq \|y^x(t) - y^z(t)\| + \|y^z(t)\| \leq L_z \|x - z\| + R_z \leq L_z + R_z. \tag{34}
$$

This ensures that for all $x, z \in \mathbb{R}^d$ with $\|x - z\| < \min\{1, r_z\}$ it holds that $R_x \leq R_z + L_z$. Combining this with (33) proves that for all $x, z, h \in \mathbb{R}^d$, $t \in [0, T]$ with $\|x - z\| < \min\{1, r_z\}$ it holds that

$$
\|v^{x,h}(t)\| \leq \|h\| \exp(K_{R_z + L_z} T). \tag{35}
$$

Next note that (26) and (35) imply that for all $\varepsilon \in (0, \infty)$, $x, z, h \in \mathbb{R}^d$, $t \in [0, T]$ with $\|x - z\| < \min\{1, r_z, (L_z)^{-1} \delta_\varepsilon^{R_z}\}$ it holds that

$$
\begin{aligned}
\|v^{x,h}(t) - v^{z,h}(t)\| &= \left\| \int_0^t \big(f^{(0,1)}(s, y^x(s)) v^{x,h}(s) - f^{(0,1)}(s, y^z(s)) v^{z,h}(s) \big) \, \mathrm{d}s \right\| \\
&\leq \int_0^t \varepsilon \|v^{x,h}(s)\| + K_{R_z} \|v^{x,h}(s) - v^{z,h}(s)\| \, \mathrm{d}s \qquad (36) \\
&\leq \varepsilon \|h\| \exp(K_{R_z + L_z} T) + \int_0^t K_{R_z} \|v^{x,h}(s) - v^{z,h}(s)\| \, \mathrm{d}s.
\end{aligned}
$$

This and Gronwall's inequality show that for all $\varepsilon \in (0, \infty)$, $z, x, h \in \mathbb{R}^d$, $t \in [0, T]$ with $\|x - z\| < \min\{1, r_z, (L_z)^{-1} \delta_\varepsilon^{R_z}\}$ it holds that

$$
\|v^{x,h}(t) - v^{z,h}(t)\| \leq \varepsilon \|h\| \exp(K_{R_z + L_z} T) \exp(K_{R_z} T) \leq \varepsilon \|h\| \exp(2K_{R_z + L_z} T). \tag{37}
$$

Moreover, (21) and (33) show that for all $z, h \in \mathbb{R}^d$, $s \in [0, T]$, $t \in [0, s]$ it holds that

$$
\begin{aligned}
\|v^{z,h}(s) - v^{z,h}(t)\| &= \left\| \int_t^s f^{(0,1)}(u, y^z(u)) v^{z,h}(u) \, \mathrm{d}u \right\| \\
&\leq \int_t^s K_{R_z} \|v^{z,h}(u)\| \, \mathrm{d}u \leq (s - t) K_{R_z} \|h\| \exp(K_{R_z} T).
\end{aligned} \tag{38}
$$

Combining this with (37) proves that for all $\varepsilon \in (0, \infty)$, $z, x, h \in \mathbb{R}^d$, $s, t \in [0, T]$ with $\|x - z\| < \min\big\{1, r_z, (L_z)^{-1} \delta^{R_z}_{\exp(-2K_{R_z + L_z} T) 2^{-1} \varepsilon}\big\}$ and $|s - t| < (2K_{R_z} \exp(K_{R_z} T) + 1)^{-1} \varepsilon$ it holds that

$$
\|v^{x,h}(s) - v^{z,h}(t)\| \leq \|v^{x,h}(s) - v^{z,h}(s)\| + \|v^{z,h}(s) - v^{z,h}(t)\| \leq \varepsilon \|h\|. \tag{39}
$$

Combining this with (31) implies that $[0, T] \times \mathbb{R}^d \ni (t, x) \mapsto \frac{\partial}{\partial x} y^x(t) \in \mathbb{R}^{d \times d}$ is a continuous function. This and (20) prove (i). The proof of Lemma 4.2 is thus completed. $\square$

Lemma 4.3. *Let $d, m \in \mathbb{N}$, $T \in [0, \infty)$, $\mu \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathbb{R}^{d \times m}$, $w \in C([0, T], \mathbb{R}^m)$ and let $y^x \in C([0, T], \mathbb{R}^d)$, $x \in \mathbb{R}^d$, be the functions which satisfy for all $x \in \mathbb{R}^d$, $t \in [0, T]$ that*

$$
y^x(t) = x + \int_0^t \mu(y^x(s)) \, \mathrm{d}s + \sigma w(t). \tag{40}
$$

Then

(i) *it holds that $([0, T] \times \mathbb{R}^d \ni (t, x) \mapsto y^x(t) \in \mathbb{R}^d) \in C^{0,1}([0, T] \times \mathbb{R}^d, \mathbb{R}^d)$ and*
(ii) *it holds for all $x, h \in \mathbb{R}^d$, $t \in [0, T]$ that*

$$
\tfrac{\partial}{\partial x} y^x(t) h = h + \int_0^t \mu'(y^x(s)) \big(\tfrac{\partial}{\partial x} y^x(s) h \big) \, \mathrm{d}s. \tag{41}
$$

Proof of Lemma 4.3. Let $z^x \colon [0, T] \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, be the functions which satisfy for all $x \in \mathbb{R}^d$, $t \in [0, T]$ that $z^x(t) = y^x(t) - \sigma w(t)$. Next use (40) to obtain that for all $x \in \mathbb{R}^d$, $t \in [0, T]$ it holds that

$$z^x(t) = x + \int_0^t \mu(z^x(s) + \sigma w(s))\,\mathrm{d}s. \tag{42}$$

The statement of Lemma 4.3 is thus a straightforward consequence of Lemma 4.2 applied with $y^x = z^x$, $x \in \mathbb{R}^d$, and the function $f\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ that satisfies for all $t \in [0,T]$, $x \in \mathbb{R}^d$ that $f(t,x) = \mu(x + \sigma w(t))$. □

4.3. Differentiability with respect to the initial value for SDEs

Lemma 4.4. *Let $d, m \in \mathbb{N}$, $T \in [0,\infty)$, $\mu \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathbb{R}^{d\times m}$, $\varphi \in C(\mathbb{R}^m, [0,\infty))$, $V \in C^1(\mathbb{R}^d, [0,\infty))$, let $\|\cdot\|\colon \mathbb{R}^d \to [0,\infty)$ be a norm, assume for all $x \in \mathbb{R}^d$, $z \in \mathbb{R}^m$ that $V'(x)\mu(x+\sigma z) \leq \varphi(z)V(x)$ and $\|x\| \leq V(x)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $W\colon [0,T] \times \Omega \to \mathbb{R}^m$ be a stochastic process with continuous sample paths. Then*

(i) *there exist unique stochastic processes $X^x\colon [0,T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, with continuous sample paths which satisfy for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ that*

$$X^x(t,\omega) = x + \int_0^t \mu(X^x(s,\omega))\,\mathrm{d}s + \sigma W(t,\omega), \tag{43}$$

(ii) *it holds for all $\omega \in \Omega$ that $([0,T] \times \mathbb{R}^d \ni (t,x) \mapsto X^x(t,\omega) \in \mathbb{R}^d) \in C^{0,1}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$, and*

(iii) *it holds for all $x, h \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ that*

$$\tfrac{\partial}{\partial x} X^x(t,\omega)h = h + \int_0^t \mu'(X^x(s,\omega))\big(\tfrac{\partial}{\partial x} X^x(s,\omega)h\big)\,\mathrm{d}s. \tag{44}$$

Proof of Lemma 4.4. First, observe that Lemma 2.2 proves that there exist unique $y^x_\omega \in C([0,T], \mathbb{R}^d)$, $x \in \mathbb{R}^d$, $\omega \in \Omega$, such that for all $x \in \mathbb{R}^d$, $\omega \in \Omega$, $t \in [0,T]$ it holds that

$$y^x_\omega(t) = x + \int_0^t \mu(y^x_\omega(s))\,\mathrm{d}s + \sigma W(t,\omega). \tag{45}$$

In addition, note that the hypothesis that $\mu \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ ensures that for all $r \in (0,\infty)$ it holds that

$$\sup_{\substack{x,y\in\mathbb{R}^d,\, x\neq y,\\ \|x\|+\|y\|\leq r}} \frac{\|\mu(x) - \mu(y)\|}{\|x-y\|} < \infty. \tag{46}$$

Combining this and (45) with Lemma 3.2 shows that for all $x \in \mathbb{R}^d$ it holds that $[0,T] \times \Omega \ni (t,\omega) \mapsto y^x_\omega(t) \in \mathbb{R}^d$ is a stochastic process. This and (45) establish (i). Next note that (45) and Lemma 4.3 establishes (ii) and (iii). The proof of Lemma 4.4 is thus completed. □

5. Integrability properties for solutions of SDEs

In this section we present well-known elementary integrability properties for standard Brownian motions in Lemma 5.1 in Subsection 5.1 and for solutions of certain additive noise driven SDEs in Lemma 5.2 in Subsection 5.2. These lemmas are used in Subsections 7.1 and 7.2 for the proof of Proposition 7.1 and

Corollary 7.2 on regularity properties with respect to the initial value for SDEs with general noise and Wiener noise, respectively. A detailed proof of Lemma 5.1 can be found, e.g., in the arXiv version of this article [12, Lemma 6.1]. For the reader's convenience, we provide a proof of Lemma 5.2.

5.1. Integrability properties for multi-dimensional Brownian motions

Lemma 5.1. *Let $m \in \mathbb{N}$, $T, C \in [0,\infty)$, $\alpha \in [0,2)$, let $\|\cdot\| \colon \mathbb{R}^m \to [0,\infty)$ be a norm, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard Brownian motion with continuous sample paths and let $\varphi\colon \mathbb{R}^m \to [0,\infty)$ satisfy for all $z \in \mathbb{R}^m$ that $\varphi(z) \le C(1+\|z\|^\alpha)$. Then it holds for all $c \in [0,\infty)$ that*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \exp\big(c\,\varphi(W(t))\big)\Big] < \infty. \tag{47}$$

5.2. Integrability properties for solutions of SDEs

Lemma 5.2. *Let $d, m \in \mathbb{N}$, $T \in [0,\infty)$, $\mu \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathbb{R}^{d\times m}$, $\varphi \in C(\mathbb{R}^m, [0,\infty))$, $V \in C^1(\mathbb{R}^d, [0,\infty))$, let $\|\cdot\|\colon \mathbb{R}^d \to [0,\infty)$ be a norm, assume for all $x \in \mathbb{R}^d$, $z \in \mathbb{R}^m$ that $V'(x)\mu(x+\sigma z) \le \varphi(z)V(x)$ and $\|x\| \le V(x)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a stochastic process with continuous sample paths, let $X^x\colon [0,T]\times\Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, be stochastic processes with continuous sample paths, assume for all $c \in [0,\infty)$ that $\mathbb{E}\big[\sup_{t\in[0,T]} \exp\big(c\,\varphi(W(t))\big)\big] + \mathbb{E}\big[\sup_{t\in[0,T]}(\|\sigma W(t)\|^c)\big] < \infty$, and assume for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ that*

$$X^x(t,\omega) = x + \int_0^t \mu(X^x(s,\omega))\,\mathrm{d}s + \sigma W(t,\omega). \tag{48}$$

Then

(i) *it holds for all $R, r \in [0,\infty)$ that*

$$\Omega \ni \omega \mapsto \Big[\sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\le R\}} \sup_{t\in[0,T]} \big(\|X^x(t,\omega)\|^r\big)\Big] \in [0,\infty] \tag{49}$$

is an $\mathcal{F}/\mathcal{B}([0,\infty])$-measurable function and

(ii) *it holds for all $R, r \in [0,\infty)$ that*

$$\mathbb{E}\Big[\sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\le R\}} \sup_{t\in[0,T]} \big(\|X^x(t)\|^r\big)\Big] < \infty. \tag{50}$$

Proof of Lemma 5.2. Note that Lemma 4.3 ensures that for all $\omega \in \Omega$ it holds that

$$([0,T]\times\mathbb{R}^d \ni (t,x) \mapsto X^x(t,\omega) \in \mathbb{R}^d) \in C^{0,1}([0,T]\times\mathbb{R}^d, \mathbb{R}^d). \tag{51}$$

Hence for all $R, r \in [0,\infty)$ and $\omega \in \Omega$ it holds that

$$\sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\le R\}} \sup_{t\in[0,T]} \big(\|X^x(t,\omega)\|^r\big) = \sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\le R\}\cap\mathbb{Q}^d} \sup_{t\in[0,T]\cap\mathbb{Q}} \big(\|X^x(t,\omega)\|^r\big), \tag{52}$$

which implies (i). Let Y, $Z\colon \Omega \to [0,\infty)$ be the functions which satisfy for all $\omega \in \Omega$ that

$$Y(\omega) = \sup_{t \in [0,T]} \exp\big(\varphi(W(t,\omega))\big) \qquad \text{and} \qquad Z(\omega) = \sup_{t \in [0,T]} \|\sigma W(t,\omega)\|. \tag{53}$$

Note that Lemma 2.1 ensures that for all $x \in \mathbb{R}^d$, $\omega \in \Omega$ it holds that

$$\begin{aligned} \sup_{t \in [0,T]} \|X^x(t,\omega)\| &\leq V(x) \exp\left(T\left[\sup_{t \in [0,T]} \varphi(W(t,\omega))\right]\right) + \left[\sup_{t \in [0,T]} \|\sigma W(t,\omega)\|\right] \\ &= V(x)[Y(\omega)]^T + Z(\omega). \end{aligned} \tag{54}$$

Hence for all $\omega \in \Omega$, $R, r \in [0,\infty)$ it holds that

$$\begin{aligned} &\sup_{x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R\}} \sup_{t \in [0,T]} \big(\|X^x(t,\omega)\|^r\big) \\ &\quad \leq 2^r \big(\big[\sup\nolimits_{x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R\}} V(x)\big]^r [Y(\omega)]^{Tr} + [Z(\omega)]^r\big). \end{aligned} \tag{55}$$

In the next step we observe that the assumption that for all $c \in [0,\infty)$ it holds that $\mathbb{E}\big[\sup_{t \in [0,T]} \exp\big(c\,\varphi(W(t))\big)\big] < \infty$, ensures that for all $r \in [0,\infty)$ it holds that

$$\mathbb{E}\big[Y^{Tr}\big] < \infty. \tag{56}$$

Note further that the continuity of V implies that for all $R \in [0,\infty)$ it holds that

$$\sup_{x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R\}} V(x) < \infty. \tag{57}$$

Finally, observe that the fact that for all $c \in [0,\infty)$ it holds that $\mathbb{E}\big[\sup_{t \in [0,T]}(\|\sigma W(t)\|^c)\big] < \infty$ shows that for all $r \in [0,\infty)$ it holds that $\mathbb{E}[Z^r] < \infty$. Combining this, (52), (56), and (57) with (55) implies that for all $R, r \in [0,\infty)$ it holds that

$$\begin{aligned} &\mathbb{E}\left[\sup_{x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R\}} \sup_{t \in [0,T]} \big(\|X^x(t)\|^r\big)\right] \\ &\quad \leq 2^r \big(\big[\sup\nolimits_{x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R\}} V(x)\big]^r \mathbb{E}\big[Y^{Tr}\big] + \mathbb{E}\big[Z^r\big]\big) < \infty. \end{aligned} \tag{58}$$

This completes the proof of Lemma 5.2. □

6. Conditional regularity with respect to the initial value for SDEs

In this section we study in Lemmas 6.3 and 6.4 in Subsection 6.2 below regularity properties of solutions of certain additive noise driven SDEs with respect to their initial values. In particular, in Lemma 6.4 we establish in inequality (99) a quantitative estimate for the mean difference of two solutions of certain additive noise driven SDEs in terms of the distance of the respective initial values. Lemma 6.4 is the main tool to establish Proposition 7.1 in Subsection 7.1 on regularity properties with respect to the initial value for SDEs with general noise. Our proof of Lemma 6.4 is based on an application of Lemma 6.3 which establishes a similar statement in wider generality. Our proof of Lemma 6.3, in turn, uses, besides other arguments, the auxiliary results in Lemma 6.1 in Subsection 6.1 below and in Lemma 6.2 in Subsection 6.2 below. For convenience of the reader we include in this section also a proof for the elementary result in Lemma 6.2.

6.1. Conditional local Lipschitz continuity for ODEs

Note that in (59) in Lemma 6.1 below, we use the fact that under the assumptions of the lemma it holds for all $x \in \mathbb{R}^d$, $h \in \mathbb{R}^d$ that the function $[0,T] \ni t \mapsto \frac{\partial}{\partial x} z^x(t)h \in \mathbb{R}^d$ is $\mathcal{B}([0,T])/\mathcal{B}(\mathbb{R}^d)$-measurable. A detailed proof of this assertion can be found, e.g., in the arXiv version of this article [12, Lemma 7.1].

Lemma 6.1. *Let $d \in \mathbb{N}$, $T \in [0,\infty)$, $\varphi \in C(\mathbb{R}^d,[0,\infty))$, let $\|\cdot\| \colon \mathbb{R}^d \to [0,\infty)$ be a norm, let $z^x \in C([0,T],\mathbb{R}^d)$, $x \in \mathbb{R}^d$, be functions which satisfy for all $t \in [0,T]$ that $(\mathbb{R}^d \ni x \mapsto z^x(t) \in \mathbb{R}^d) \in C^1(\mathbb{R}^d,\mathbb{R}^d)$, and assume for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $h \in \mathbb{R}^d$ that $\int_0^T \big\|\frac{\partial}{\partial x} z^x(s)h\big\| \, \mathrm{d}s < \infty$ and*

$$\big\|\tfrac{\partial}{\partial x} z^x(t)h\big\| \leq \|h\| + \int_0^t \varphi(z^x(s)) \, \big\|\tfrac{\partial}{\partial x} z^x(s)h\big\| \, \mathrm{d}s. \tag{59}$$

Then it holds for all $x,y \in \mathbb{R}^d$, $t \in [0,T]$ that

$$\|z^x(t) - z^y(t)\| \leq \sup_{u\in[0,1]} \left[\|x-y\| \exp\!\left(T \left[\sup_{s\in[0,T]} \varphi(z^{(1-u)y+ux}(s)) \right] \right) \right]. \tag{60}$$

Proof of Lemma 6.1. Throughout this proof let $D^x \colon [0,T] \to \mathbb{R}^{d\times d}$, $x \in \mathbb{R}^d$, be the functions which satisfy for all $x \in \mathbb{R}^d$, $t \in [0,T]$ that

$$D^x(t) = \tfrac{\partial}{\partial x} z^x(t). \tag{61}$$

Clearly, for all $t \in [0,T]$, $h \in \mathbb{R}^d$ it holds that $\mathbb{R}^d \ni x \mapsto D^x(t)h \in \mathbb{R}^d$ is continuous. The fundamental theorem of calculus hence implies that for all $x,y \in \mathbb{R}^d$, $t \in [0,T]$ it holds that

$$\begin{aligned} \|z^x(t) - z^y(t)\| &= \left\| \int_0^1 D^{(1-u)y+ux}(t)(x-y) \, \mathrm{d}u \right\| \\ &\leq \sup_{u\in[0,1]} \|D^{(1-u)y+ux}(t)(x-y)\|. \end{aligned} \tag{62}$$

By the properties of the mappings φ and z^x, $x \in \mathbb{R}^d$, it follows that for all $x,h \in \mathbb{R}^d$ it holds that

$$\int_0^T \|D^x(s)h\| \, \mathrm{d}s < \infty, \qquad \text{and} \qquad \sup_{s\in[0,T]} \varphi(z^x(s)) < \infty. \tag{63}$$

Moreover, using (59) we obtain that for all $x,h \in \mathbb{R}^d$, $t \in [0,T]$ it holds that

$$\begin{aligned} \|D^x(t)h\| &\leq \|h\| + \int_0^t \varphi(z^x(s)) \, \|D^x(s)h\| \, \mathrm{d}s \\ &\leq \|h\| + \left[\sup_{s\in[0,T]} \varphi(z^x(s)) \right] \int_0^t \|D^x(s)h\| \, \mathrm{d}s. \end{aligned} \tag{64}$$

Observing (63) we may thus employ Gronwall's inequality to conclude that for all $x,h \in \mathbb{R}^d$, $t \in [0,T]$ it holds that

$$\|D^x(t)h\| \leq \|h\| \exp\left(T\left[\sup_{s\in[0,T]} \varphi(z^x(s))\right]\right). \tag{65}$$

Combining this with (62) shows that for all $x, y \in \mathbb{R}^d$, $t \in [0, T]$ it holds that

$$\|z^x(t) - z^y(t)\| \leq \sup_{u\in[0,1]} \left[\|x - y\| \exp\left(T\left[\sup_{s\in[0,T]} \varphi(z^{(1-u)y+ux}(s))\right]\right)\right]. \tag{66}$$

This completes the proof of Lemma 6.1. □

6.2. Conditional sub-Hölder continuity for SDEs

Lemma 6.2. *Let $q \in [0, \infty)$. Then it holds for all $a, b \in [e^q, \infty)$ with $a \leq b$ that*

$$\frac{a^2}{(\ln(a))^{2q}} \leq \frac{b^2}{(\ln(b))^{2q}}. \tag{67}$$

Proof of Lemma 6.2. In the case $q = 0$ the statement is obviously true (note that for all $x \in \{1\}$, $q \in \{0\}$ it holds that $(\ln(x))^{2q} = (\ln(1))^0 = 0^0 = 1$). Hence we assume $q > 0$. Let $f\colon (1, \infty) \to (0, \infty)$ be the function which satisfies for all $z \in (1, \infty)$ that

$$f(z) = \frac{z^2}{(\ln(z))^{2q}}. \tag{68}$$

Note that f is a continuously differentiable function and for all $z \in [e^q, \infty)$ it holds that

$$f'(z) = \frac{2z[\ln(z)]^{2q} - 2qz^2[\ln(z)]^{2q-1}z^{-1}}{[\ln(z)]^{4q}} = \frac{2z\ln(z) - 2qz}{[\ln(z)]^{2q+1}} \geq 0. \tag{69}$$

Hence $f|_{[e^q,\infty)}$ is increasing. This establishes (67). The proof of Lemma 6.2 is thus completed. □

Lemma 6.3. *Let $d \in \mathbb{N}$, $T, R, q, K \in [0, \infty)$, $\varphi \in C(\mathbb{R}^d, [0, \infty))$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\|\cdot\|\colon \mathbb{R}^d \to [0, \infty)$ be a norm, let $X^x\colon [0, T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, be stochastic processes with continuous sample paths which satisfy for all $t \in [0, T]$, $\omega \in \Omega$ that $(\mathbb{R}^d \ni x \mapsto X^x(t, \omega) \in \mathbb{R}^d) \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, assume for all $x \in \mathbb{R}^d$, $t \in [0, T]$, $\omega \in \Omega$, $h \in \mathbb{R}^d$ that $\int_0^T \left\|\frac{\partial}{\partial x} X^x(s, \omega)h\right\| \mathrm{d}s < \infty$ and*

$$\left\|\tfrac{\partial}{\partial x} X^x(t, \omega)h\right\| \leq \|h\| + \int_0^t \varphi(X^x(s, \omega)) \left\|\tfrac{\partial}{\partial x} X^x(s, \omega)h\right\| \mathrm{d}s, \tag{70}$$

assume that

$$\mathbb{E}\left[\sup_{x\in\{z\in\mathbb{R}^d : \|z\|\leq R+1\}} \sup_{t\in[0,T]} \left([\varphi(X^x(t))]^{4q+4}\right)\right] \leq K, \tag{71}$$

and assume that

$$\sup_{x\in\{z\in\mathbb{R}^d : \|z\|\leq R+1\}} \sup_{t\in[0,T]} \mathbb{E}\left[\|X^x(t)\|^2\right] \leq K. \tag{72}$$

Let $\mathcal{K} = 1 + 2^{4q+4}\left(|\ln(2 + e^q)|^{4q+4} + T^{4q+4}K\right)$. Then it holds for all $x \in \{z \in \mathbb{R}^d : \|z\| \leq R\}$, $h \in \{v \in \mathbb{R}^d \setminus \{0\} : \|v\| < 1\}$, $t \in [0, T]$ that

$$\mathbb{E}\left[\|X^{x+h}(t) - X^x(t)\|\right] \leq 2\sqrt{(1 + 4K)\mathcal{K}}\, \left|\ln(\|h\|)\right|^{-q}. \tag{73}$$

Proof of Lemma 6.3. Let

$$Y = \sup_{x \in \{z \in \mathbb{R}^d : \, \|z\| \le R+1\}} \sup_{t \in [0,T]} \varphi(X^x(t)), \tag{74}$$

and note that Y is $\mathcal{F}/\mathcal{B}([0,\infty])$-measurable, due to Lemma 3.1. In particular, the left hand side in (71) is well-defined. Let $F, G\colon [0,\infty) \to [0,\infty)$ be the functions which satisfy for all $y \in [0,\infty)$ that

$$F(y) = \ln(1+y), \qquad G(y) = \begin{cases} 0 & \text{if } y = 0, \\ [\ln(1+y)]^{-1} y & \text{if } y \neq 0, \end{cases} \tag{75}$$

let $D^x\colon [0,T] \times \Omega \to \mathbb{R}^{d \times d}$, $x \in \mathbb{R}^d$, be the functions which satisfy for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ that

$$D^x(t,\omega) = \tfrac{\partial}{\partial x} X^x(t,\omega), \tag{76}$$

let

$$A = \{\omega \in \Omega \colon Y(\omega) < \infty\}, \tag{77}$$

and let $Z\colon \Omega \to [0,\infty)$ be the function which satisfies for all $\omega \in \Omega$ that

$$Z(\omega) = \begin{cases} \exp(T\,Y(\omega)) & \text{if } \omega \in A, \\ 1 & \text{if } \omega \in \Omega \setminus A. \end{cases} \tag{78}$$

Clearly, for all $y \in [0,\infty)$ it holds that

$$y = G(y) F(y). \tag{79}$$

Hence we obtain that for all $x, h \in \mathbb{R}^d$, $t \in [0,T]$ it holds that

$$\mathbb{E}\big[\|X^{x+h}(t) - X^x(t)\|\big] = \mathbb{E}\big[G(\|X^{x+h}(t) - X^x(t)\|) F(\|X^{x+h}(t) - X^x(t)\|)\big]. \tag{80}$$

Next observe that the fundamental theorem of calculus ensures that for all $y \in [0,\infty)$ it holds that

$$\ln(1+y) = \int_0^y \frac{1}{1+z} \, \mathrm{d}z \ge \frac{y}{1+y}. \tag{81}$$

This shows that for all $y \in [0,\infty)$ it holds that

$$G(y) \le 1 + y. \tag{82}$$

It follows that for all $x, h \in \mathbb{R}^d$, $t \in [0,T]$ it holds that

$$\begin{aligned} \mathbb{E}\big[\big|G(\|X^{x+h}(t) - X^x(t)\|)\big|^2\big] &\le \mathbb{E}\big[\big(1 + \|X^{x+h}(t) - X^x(t)\|\big)^2\big] \\ &\le 2\big(1 + 2\big(\mathbb{E}\big[\|X^{x+h}(t)\|^2\big] + \mathbb{E}\big[\|X^x(t)\|^2\big]\big)\big). \end{aligned} \tag{83}$$

The hypothesis that for all $x \in \{z \in \mathbb{R}^d \colon \|z\| \le R+1\}$, $t \in [0,T]$ it holds that $\mathbb{E}[\|X^x(t)\|^2] \le K$ hence implies that for all $x \in \{z \in \mathbb{R}^d \colon \|z\| \le R\}$, $h \in \{v \in \mathbb{R}^d \colon \|v\| < 1\}$, $t \in [0,T]$ it holds that

$$\mathbb{E}\big[\big|G(\|X^{x+h}(t) - X^x(t)\|)\big|^2\big] \leq 2 + 8K. \tag{84}$$

In the next step we note that (70), the hypothesis that for all $x \in \mathbb{R}^d$, $\omega \in \Omega$, $h \in \mathbb{R}^d$ it holds that $\int_0^T \big\|\frac{\partial}{\partial x} X^x(s,\omega)h\big\| \,\mathrm{d}s < \infty$, and Lemma 6.1 show that for all $x, h \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ it holds that

$$\|X^{x+h}(t,\omega) - X^x(t,\omega)\| \leq \sup_{u\in[0,1]} \left[\|h\| \exp\left(T\left[\sup_{s\in[0,T]} \varphi(X^{x+uh}(s,\omega))\right]\right)\right]. \tag{85}$$

Therefore, we obtain that for all $x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R\}$, $h \in \{v \in \mathbb{R}^d \colon \|v\| < 1\}$, $t \in [0,T]$, $\omega \in A$ it holds that

$$\begin{aligned}\|X^{x+h}(t,\omega) - X^x(t,\omega)\| &\leq \sup_{y\in\{z\in\mathbb{R}^d\colon \|z\|\leq R+1\}} \left[\|h\| \exp\left(T\left[\sup_{s\in[0,T]} \varphi(X^y(s,\omega))\right]\right)\right] \\ &= \|h\| \exp(T\,Y(\omega)) \\ &= \|h\|\, Z(\omega).\end{aligned} \tag{86}$$

By (71) we have

$$\mathbb{P}(A) = 1. \tag{87}$$

Combining this and (86), demonstrates that for all $x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R\}$, $h \in \{v \in \mathbb{R}^d \colon \|v\| < 1\}$, $t \in [0,T]$ it holds that

$$\mathbb{E}\big[\big|F(\|X^{x+h}(t) - X^x(t)\|)\big|^2\big] \leq \mathbb{E}\big[\big|\ln(1 + \|h\|Z)\big|^2\big]. \tag{88}$$

Moreover, note that for all $C \in [1,\infty)$, $r \in [0, 4q+4]$, $\omega \in A$ it holds that

$$|\ln(CZ(\omega))|^r \leq 1 + |\ln(CZ(\omega))|^{4q+4} \leq 1 + 2^{4q+4}\big(|\ln(C)|^{4q+4} + T^{4q+4}[Y(\omega)]^{4q+4}\big). \tag{89}$$

Combining this with (87) and (71) proves that for all $C \in [1, 2+e^q]$, $r \in [0, 4q+4]$ it holds that

$$\mathbb{E}\big[|\ln(CZ)|^r\big] \leq 1 + 2^{4q+4}\big(|\ln(2+e^q)|^{4q+4} + T^{4q+4}K\big) = \mathcal{K}. \tag{90}$$

Next note that $Z \geq 1$, the fact that for all $y \in [0,\infty)$ it holds that $\ln(1+y) \leq y$ and Lemma 6.2 show that for all $h \in \{v \in \mathbb{R}^d \setminus \{0\} \colon \|v\| < 1\}$ it holds that

$$\begin{aligned}\mathbb{E}\big[|\ln(1+\|h\|Z)|^2 \mathbb{1}_{\{Z\leq 1/\|h\|\}}\big] &\leq \mathbb{E}\big[\|h\|^2 Z^2 \mathbb{1}_{\{Z\leq 1/\|h\|\}}\big] \\ &= \|h\|^2 e^{-2q}\, \mathbb{E}\left[\frac{(e^q Z)^2}{|\ln(e^q Z)|^{2q}} |\ln(e^q Z)|^{2q} \mathbb{1}_{\{Z\leq 1/\|h\|\}}\right] \\ &\leq \|h\|^2 e^{-2q}\, \mathbb{E}\left[\frac{\big(\frac{e^q}{\|h\|}\big)^2}{\big|\ln\big(\frac{e^q}{\|h\|}\big)\big|^{2q}} |\ln(e^q Z)|^{2q} \mathbb{1}_{\{Z\leq 1/\|h\|\}}\right] \\ &\leq \big|\ln\big(\tfrac{e^q}{\|h\|}\big)\big|^{-2q}\, \mathbb{E}\big[|\ln(e^q Z)|^{2q}\big] \\ &\leq \big|\ln(\|h\|)\big|^{-2q}\, \mathbb{E}\big[|\ln(e^q Z)|^{2q}\big].\end{aligned} \tag{91}$$

This and (90) establish that for all $h \in \{v \in \mathbb{R}^d \setminus \{0\} \colon \|v\| < 1\}$ it holds that

$$\mathbb{E}\big[|\ln(1+\|h\|Z)|^2 \mathbb{1}_{\{Z\leq 1/\|h\|\}}\big] \leq \mathcal{K}\, \big|\ln(\|h\|)\big|^{-2q}. \tag{92}$$

In the next step we observe that (90) implies that for all $h \in \{v \in \mathbb{R}^d \setminus \{0\} \colon \|v\| < 1\}$ it holds that

$$\begin{aligned} \mathbb{E}\big[\mathbb{1}_{\{Z>1/\|h\|\}}\big] &\leq \mathbb{E}\big[|\ln(Z)|^{4q}|\ln(1/\|h\|)|^{-4q}\mathbb{1}_{\{Z>1/\|h\|\}}\big] \\ &\leq \mathbb{E}\big[|\ln(Z)|^{4q}\big]\big|\ln(\|h\|)\big|^{-4q} \\ &\leq \mathcal{K}\,\big|\ln(\|h\|)\big|^{-4q}. \end{aligned} \tag{93}$$

Furthermore, observe that (90) shows that

$$\mathbb{E}\big[|\ln(2Z)|^4\big] \leq \mathcal{K}. \tag{94}$$

Combining (93) with (94) establishes that for all $h \in \{v \in \mathbb{R}^d \setminus \{0\} \colon \|v\| < 1\}$ it holds that

$$\begin{aligned} \mathbb{E}\big[|\ln(1+\|h\|Z)|^2\mathbb{1}_{\{Z>1/\|h\|\}}\big] &\leq \mathbb{E}\big[|\ln(2Z)|^2\mathbb{1}_{\{Z>1/\|h\|\}}\big] \\ &\leq \Big(\mathbb{E}\big[|\ln(2Z)|^4\big]\mathbb{E}\big[\mathbb{1}_{\{Z>1/\|h\|\}}\big]\Big)^{1/2} \\ &\leq \mathcal{K}\,\big|\ln(\|h\|)\big|^{-2q}. \end{aligned} \tag{95}$$

Combining this, (88), and (92) proves that for all $x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R\}$, $h \in \{v \in \mathbb{R}^d \setminus \{0\} \colon \|v\| < 1\}$, $t \in [0,T]$ it holds that

$$\mathbb{E}\big[\big|F(\|X^{x+h}(t) - X^x(t)\|)\big|^2\big] \leq 2\mathcal{K}\,\big|\ln(\|h\|)\big|^{-2q}. \tag{96}$$

Combining this with (84) and (80) proves that for all $x \in \{z \in \mathbb{R}^d \colon \|z\| \leq R\}$, $h \in \{v \in \mathbb{R}^d \setminus \{0\} \colon \|v\| < 1\}$, $t \in [0,T]$ it holds that

$$\begin{aligned} &\mathbb{E}\big[\|X^{x+h}(t) - X^x(t)\|\big] \\ &\leq \Big(\mathbb{E}\big[\big|G(\|X^{x+h}(t) - X^x(t)\|)\big|^2\big]\mathbb{E}\big[\big|F(\|X^{x+h}(t) - X^x(t)\|)\big|^2\big]\Big)^{1/2} \\ &\leq \big((4+16K)\mathcal{K}\,\big|\ln(\|h\|)\big|^{-2q}\big)^{1/2} \\ &= 2\sqrt{(1+4K)\mathcal{K}}\,\big|\ln(\|h\|)\big|^{-q}. \end{aligned} \tag{97}$$

This completes the proof of Lemma 6.3. □

The following result is an immediate consequence of Lemma 6.3.

Lemma 6.4. *Let $d \in \mathbb{N}$, $T, \kappa \in [0,\infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\|\cdot\| \colon \mathbb{R}^d \to [0,\infty)$ be a norm, let $X^x \colon [0,T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, be stochastic processes which satisfy for all $\omega \in \Omega$ that $([0,T] \times \mathbb{R}^d \ni (t,x) \mapsto X^x(t,\omega) \in \mathbb{R}^d) \in C^{0,1}([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$, assume for all $R, r \in [0,\infty)$ that $\mathbb{E}\big[\sup_{x\in\{z\in\mathbb{R}^d \colon \|z\|\leq R\}} \sup_{t\in[0,T]}\big(\|X^x(t)\|^r\big)\big] < \infty$, and assume for all $x, h \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ that*

$$\big\|\tfrac{\partial}{\partial x}X^x(t,\omega)h\big\| \leq \|h\| + \kappa\int_0^t (1 + \|X^x(s,\omega)\|^\kappa)\,\big\|\tfrac{\partial}{\partial x}X^x(s,\omega)h\big\|\,\mathrm{d}s. \tag{98}$$

Then it holds for all $R, q \in [0,\infty)$ that there exists $c \in [0,\infty)$ such that for all $h \in \{v \in \mathbb{R}^d \setminus \{0\} \colon \|v\| < 1\}$ it holds that

$$\sup_{x\in\{z\in\mathbb{R}^d \colon \|z\|\leq R\}}\ \sup_{t\in[0,T]} \mathbb{E}\big[\|X^{x+h}(t) - X^x(t)\|\big] \leq c\,\big|\ln(\|h\|)\big|^{-q}. \tag{99}$$

7. Regularity with respect to the initial value for SDEs

In this section we establish in Theorem 7.4 in Subsection 7.3 below the main result of this article. Theorem 7.4 proves that every additive noise driven SDE with a drift coefficient function whose derivatives grow at most polynomially and which also admits a Lyapunov-type condition (which ensures the existence of a unique solution) is at least logarithmically Hölder continuous in the initial value (see (118) in Theorem 7.4 below for the precise statement). Our proof of Theorem 7.4 exploits Corollary 7.2 in Subsection 7.2 below and the auxiliary continuity-regularity result in Lemma 7.3 in Subsection 7.3 below. Our proof of Corollary 7.2 is based on an application of Proposition 7.1 below. Our proof of Proposition 7.1, in turn, is based on the differentiability result in Lemma 4.4 in Subsection 4.3, the integrability result in Lemma 5.2 in Subsection 5.2 and the regularity result in Lemma 6.4 in Subsection 6.2 above.

7.1. Regularity with respect to the initial value for SDEs with general noise

Proposition 7.1. *Let $d,m \in \mathbb{N}$, $T,\kappa \in [0,\infty)$, $\mu \in C^1(\mathbb{R}^d,\mathbb{R}^d)$, $\sigma \in \mathbb{R}^{d\times m}$, $\varphi \in C(\mathbb{R}^m,[0,\infty))$, $V \in C^1(\mathbb{R}^d,[0,\infty))$, let $\|\cdot\|\colon \mathbb{R}^d \to [0,\infty)$ be a norm, assume for all $x,h \in \mathbb{R}^d$, $z \in \mathbb{R}^m$ that $\|\mu'(x)h\| \leq \kappa\big(1+\|x\|^\kappa\big)\|h\|$, $V'(x)\mu(x+\sigma z) \leq \varphi(z)V(x)$, and $\|x\| \leq V(x)$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a stochastic process with continuous sample paths, and assume for all $c \in [0,\infty)$ that $\mathbb{E}\big[\sup_{t\in[0,T]}\exp\big(c\,\varphi(W(t))\big)\big] + \mathbb{E}\big[\sup_{t\in[0,T]}(\|\sigma W(t)\|^c)\big] < \infty$. Then*

(i) *there exist unique stochastic processes $X^x\colon [0,T]\times\Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, with continuous sample paths which satisfy for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ that*

$$X^x(t,\omega) = x + \int_0^t \mu(X^x(s,\omega))\,\mathrm{d}s + \sigma W(t,\omega), \tag{100}$$

(ii) *it holds for all $R,r \in [0,\infty)$ that $\sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\leq R\}}\sup_{t\in[0,T]}\mathbb{E}\big[\|X^x(t)\|^r\big] < \infty$, and*

(iii) *it holds for all $R,q \in [0,\infty)$ that there exists $c \in (0,\infty)$ such that for all $h \in \{v \in \mathbb{R}^d\setminus\{0\}\colon \|v\| < 1\}$ it holds that*

$$\sup_{x\in\{v\in\mathbb{R}^d\colon \|v\|\leq R\}}\ \sup_{t\in[0,T]} \mathbb{E}\big[\|X^{x+h}(t)-X^x(t)\|\big] \leq c\,\big|\ln(\|h\|)\big|^{-q}. \tag{101}$$

Proof of Proposition 7.1. First, observe that Lemma 4.4 shows

(a) that there exist unique stochastic processes $X^x\colon [0,T]\times\Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, with continuous sample paths which satisfy for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ that

$$X^x(t,\omega) = x + \int_0^t \mu(X^x(s,\omega))\,\mathrm{d}s + \sigma W(t,\omega), \tag{102}$$

(b) that for all $\omega \in \Omega$ it holds that

$$\big([0,T]\times\mathbb{R}^d \ni (t,x) \mapsto X^x(t,\omega) \in \mathbb{R}^d\big) \in C^{0,1}([0,T]\times\mathbb{R}^d,\mathbb{R}^d), \tag{103}$$

and

(c) that for all $x, h \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ it holds that

$$\tfrac{\partial}{\partial x} X^x(t,\omega)h = h + \int_0^t \mu'(X^x(s,\omega))\big(\tfrac{\partial}{\partial x} X^x(s,\omega)h\big)\,\mathrm{d}s. \tag{104}$$

In the next step we note that Lemma 5.2 ensures that for all $R, r \in [0,\infty)$ it holds that

$$\mathbb{E}\Big[\sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\leq R\}} \sup_{t\in[0,T]} \big(\|X^x(t)\|^r\big)\Big] < \infty, \tag{105}$$

which establishes (ii). In the next step we combine (104) and the hypothesis that for all $x, h \in \mathbb{R}^d$ it holds that $\|\mu'(x)h\| \leq \kappa\big(1 + \|x\|^\kappa\big)\|h\|$ to obtain that for all $x, h \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ it holds that

$$\begin{aligned}\big\|\tfrac{\partial}{\partial x} X^x(t,\omega)h\big\| &\leq \|h\| + \int_0^t \big\|\mu'(X^x(s,\omega))\big(\tfrac{\partial}{\partial x} X^x(s,\omega)h\big)\big\|\,\mathrm{d}s \\ &\leq \|h\| + \kappa\int_0^t (1 + \|X^x(s,\omega)\|^\kappa)\,\big\|\tfrac{\partial}{\partial x} X^x(s,\omega)h\big\|\,\mathrm{d}s.\end{aligned} \tag{106}$$

Combining this and (103), with Lemma 6.4 establishes (iii). The proof of Proposition 7.1 is thus completed. □

7.2. Regularity with respect to the initial value for SDEs with Wiener noise

Corollary 7.2. *Let $d, m \in \mathbb{N}$, $T, \kappa \in [0,\infty)$, $\alpha \in [0,2)$, $\mu \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathbb{R}^{d\times m}$, $V \in C^1(\mathbb{R}^d, [0,\infty))$, let $\|\cdot\|\colon \mathbb{R}^d \to [0,\infty)$ and $|\!|\!|\cdot|\!|\!|\colon \mathbb{R}^m \to [0,\infty)$ be norms, assume for all $x, h \in \mathbb{R}^d$, $z \in \mathbb{R}^m$ that $\|\mu'(x)h\| \leq \kappa\big(1 + \|x\|^\kappa\big)\|h\|$, $V'(x)\mu(x + \sigma z) \leq \kappa(1 + |\!|\!|z|\!|\!|^\alpha)V(x)$, and $\|x\| \leq V(x)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $W\colon [0,T] \times \Omega \to \mathbb{R}^m$ be a standard Brownian motion with continuous sample paths. Then*

(i) *there exist unique stochastic processes $X^x\colon [0,T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, with continuous sample paths such that for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $\omega \in \Omega$ it holds that*

$$X^x(t,\omega) = x + \int_0^t \mu(X^x(s,\omega))\,\mathrm{d}s + \sigma W(t,\omega), \tag{107}$$

(ii) *it holds for all $R, r \in [0,\infty)$ that $\sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\leq R\}} \sup_{t\in[0,T]} \mathbb{E}\big[\|X^x(t)\|^r\big] < \infty$, and*

(iii) *it holds for all $R, q \in [0,\infty)$ that there exists $c \in (0,\infty)$ such that for all $h \in \{v \in \mathbb{R}^d \setminus \{0\}\colon \|v\| < 1\}$ it holds that*

$$\sup_{x\in\{v\in\mathbb{R}^d\colon \|v\|\leq R\}} \sup_{t\in[0,T]} \mathbb{E}\big[\|X^{x+h}(t) - X^x(t)\|\big] \leq c\,\big|\ln(\|h\|)\big|^{-q}. \tag{108}$$

Proof of Corollary 7.2. Throughout this proof let $\varphi\colon \mathbb{R}^m \to [0,\infty)$ be the function which satisfies for all $z \in \mathbb{R}^m$ that

$$\varphi(z) = \kappa(1 + |\!|\!|z|\!|\!|^\alpha). \tag{109}$$

Note that Lemma 5.1 shows that for all $c \in [0, \infty)$ it holds that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \exp\bigl(c\,\varphi(W(t))\bigr)\right] < \infty. \tag{110}$$

Furthermore, it is well-known that for all $c \in [0, \infty)$ it holds that

$$\mathbb{E}\left[\sup_{t\in[0,T]} (\|\sigma W(t)\|^c)\right] < \infty. \tag{111}$$

Combining this and (110) with Proposition 7.1 establishes (i), (ii), and (iii). The proof of Corollary 7.2 is thus completed. □

7.3. Sub-Hölder continuity with respect to the initial value for SDEs

Lemma 7.3. *Let $d \in \mathbb{N}$, $T, R, q, c \in [0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $X^x\colon [0, T] \times \Omega \to \mathbb{R}^d$, $x \in \mathbb{R}^d$, be stochastic processes, let $\|\cdot\|\colon \mathbb{R}^d \to [0, \infty)$ be a norm, and assume for all $h \in \{v \in \mathbb{R}^d \setminus \{0\}\colon \|v\| < 1\}$ that*

$$\sup_{x\in\{v\in\mathbb{R}^d\colon \|v\|\le R\}} \sup_{t\in[0,T]} \mathbb{E}\bigl[\|X^{x+h}(t) - X^x(t)\|\bigr] \le c\,\bigl|\ln(\|h\|)\bigr|^{-q}. \tag{112}$$

Let $C = \sup_{x\in\{v\in\mathbb{R}^d\colon \|v\|\le R\}} \sup_{t\in[0,T]} \mathbb{E}\bigl[\|X^x(t)\|\bigr]$. Then it holds for all $x, y \in \{v \in \mathbb{R}^d\colon \|v\| \le R\}$ with $0 < \|x - y\| \ne 1$ that

$$\sup_{t\in[0,T]} \mathbb{E}\bigl[\|X^x(t) - X^y(t)\|\bigr] \le \max\{c, 2C\,|\ln(2R + 1)|^q\}\,\bigl|\ln(\|x - y\|)\bigr|^{-q}. \tag{113}$$

Proof of Lemma 7.3. First, note that (112) implies that for all $x, y \in \{v \in \mathbb{R}^d\colon \|v\| \le R\}$ with $x \ne y$ and $\|x - y\| < 1$ it holds that

$$\sup_{t\in[0,T]} \mathbb{E}\bigl[\|X^x(t) - X^y(t)\|\bigr] \le \frac{c}{|\ln(\|x - y\|)|^q}. \tag{114}$$

Furthermore, observe that for all $x, y \in \{v \in \mathbb{R}^d\colon \|v\| \le R\}$, $t \in [0, T]$ it holds that

$$\mathbb{E}\bigl[\|X^x(t) - X^y(t)\|\bigr] \le 2C. \tag{115}$$

The fact that for all $x, y \in \{v \in \mathbb{R}^d\colon \|v\| \le R\}$ it holds that $\|x - y\| \le 2R$ hence shows that for all $t \in [0, T]$, $x, y \in \{v \in \mathbb{R}^d\colon \|v\| \le R\}$ with $\|x - y\| > 1$ it holds that

$$\sup_{t\in[0,T]} \mathbb{E}\bigl[\|X^x(t) - X^y(t)\|\bigr] \le \frac{2C\,|\ln(\|x - y\|)|^q}{|\ln(\|x - y\|)|^q} \le \frac{2C\,|\ln(2R + 1)|^q}{|\ln(\|x - y\|)|^q}. \tag{116}$$

Combining this with (114) completes the proof of Lemma 7.3. □

Theorem 7.4. *Let $d, m \in \mathbb{N}$, $T, \kappa \in [0, \infty)$, $\alpha \in [0, 2)$, $\mu \in C^1(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in \mathbb{R}^{d\times m}$, $V \in C^1(\mathbb{R}^d, [0, \infty))$, let $\|\cdot\|\colon \mathbb{R}^d \to [0, \infty)$ and $\|\!|\cdot|\!\|\colon \mathbb{R}^m \to [0, \infty)$ be norms, assume for all $x, h \in \mathbb{R}^d$, $z \in \mathbb{R}^m$ that $\|\mu'(x)h\| \le \kappa\bigl(1 + \|x\|^\kappa\bigr)\|h\|$, $V'(x)\mu(x + \sigma z) \le \kappa(1 + \|\!|z|\!\|^\alpha)V(x)$, and $\|x\| \le V(x)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $W\colon [0, T] \times \Omega \to \mathbb{R}^m$ be a standard Brownian motion with continuous sample paths. Then*

(i) *there exist unique stochastic processes* $X^x\colon [0,T]\times\Omega\to\mathbb{R}^d$, $x\in\mathbb{R}^d$, *with continuous sample paths such that for all* $x\in\mathbb{R}^d$, $t\in[0,T]$, $\omega\in\Omega$ *it holds that*

$$X^x(t,\omega) = x + \int_0^t \mu(X^x(s,\omega))\,\mathrm{d}s + \sigma W(t,\omega) \tag{117}$$

and

(ii) *it holds for all* $R,q\in[0,\infty)$ *that there exists* $c\in(0,\infty)$ *such that for all* $x,y\in\{v\in\mathbb{R}^d\colon \|v\|\le R\}$ *with* $0<\|x-y\|\neq 1$ *it holds that*

$$\sup_{t\in[0,T]} \mathbb{E}\big[\|X^x(t)-X^y(t)\|\big] \le c\,\big|\ln(\|x-y\|)\big|^{-q}. \tag{118}$$

Proof of Theorem 7.4. Note that Corollary 7.2 establishes (i) as well as that for all $R\in[0,\infty)$ it holds that

$$\sup_{x\in\{z\in\mathbb{R}^d\colon \|z\|\le R\}}\ \sup_{t\in[0,T]} \mathbb{E}\big[\|X^x(t)\|\big] < \infty, \tag{119}$$

and that for all $R,q\in[0,\infty)$ there exists $c_{R,q}\in(0,\infty)$ such that for all $h\in\{v\in\mathbb{R}^d\setminus\{0\}\colon \|v\|<1\}$ it holds that

$$\sup_{x\in\{v\in\mathbb{R}^d\colon \|v\|\le R\}}\ \sup_{t\in[0,T]} \mathbb{E}\big[\|X^{x+h}(t)-X^x(t)\|\big] \le c_{R,q}\,\big|\ln(\|h\|)\big|^{-q}. \tag{120}$$

Combining (119) and (120) with Lemma 7.3 establishes (ii). The proof of Theorem 7.4 is thus completed. □

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