Bayliss–Turkel-like Radiation Conditions on Surfaces of Arbitrary Shape

X. Antoine and H. Barucq

*Laboratoire de Mathématiques Appliquées pour l'Industrie and CNRS UPRES-A 5033,
Université de Pau et des Pays de l'Adour, 64000 Pau, France*
E-mail: helene.barucq@univ-pau.fr

and

A. Bendali

*Département de Génie Mathématique, INSA, Laboratoire MIP UMR 5640,
CNRS-UPS-INSA, Complexe Scientifique de Rangueil, 31077 Toulouse Cedex 4, France*

    This paper addresses the extension of the Bayliss–Turkel second-order radiation condition to an arbitrarily shaped surface. The derivation is based mainly on the pseudo-differential calculus as well as on the introduction of a criterion providing a precise handling of the approximation process involved in the derivation of the radiation condition. The radiation condition then ranges among the most accurate of those of order two. As a by-product of the derivation, almost all known radiation conditions of order less than or equal to two are recovered and their respective accuracies are compared.   © 1999 Academic Press

    Key Words: Absorbing boundary conditions; Helmholtz equation; wave equation; pseudo-differential operators.

1. INTRODUCTION

   The field of a scalar time-harmonic scattered wave is generally governed by the following model problem (cf., e.g., [12] in electromagnetic and [7] in acoustic scattering). Find u defined in an exterior domain Ω_+ of $\mathbb{R}^N$, with N = 2, 3 (i.e., such that its complement $\overline{\Omega_-}$ is a compact set), satisfying the Helmholtz equation

$$\Delta u + k^2 u = 0 \qquad \text{in } \Omega_+, \tag{1}$$

the Sommerfeld radiation condition

$$\lim_{|x| \to \infty} |x|^{(N-1)/2} \left(\partial_{|x|} u - iku \right) = 0, \tag{2}$$

expressing that the energy propagates from the obstacle $\overline{\Omega_-}$ toward infinity, and a boundary condition

$$\mathscr{B}u = g := -\mathscr{B}u^{\text{inc}} \qquad \text{on } \Sigma,$$

expressed by a given boundary differential operator $\mathscr{B}$. u^{inc} stands for the incident wave and is a given solution to the Helmholtz equation in the vicinity of the obstacle; the time variation $\exp(-i2\pi\nu t)$ is suppressed by linearity; $\nu > 0$ is the frequency; and $k = 2\pi\nu/c$ is the related wave number, where c is the velocity of the propagation. Hereafter, we assume that $\overline{\Omega_-}$ is a $\mathscr{C}^\infty$ compact manifold with boundary Σ imbedded in $\mathbb{R}^N$. However, such a degree of smoothness is needed only for theoretical purposes. This is relative to the use of the pseudo-differential calculus and may largely be weakened at the level of the final obtained radiation conditions.

It is well known that a process limiting the computational domain has to be used before any attempt to numerically solve the above problem. A broadly used approach consists of truncating the infinite domain with a terminating closed boundary Γ enclosing the obstacle. A boundary condition is then introduced on Γ in such a way that the resulting boundary-value problem is well posed and its solution approximates the restriction of u to the bounded domain Ω limited by Σ and Γ. Two classes of such a boundary condition can be used. Exact conditions give exactly the restriction of u if no further approximation is made. The approximate (sometimes also called radiation or absorbing) boundary conditions only yield an approximation of this restriction. They can be designed from numerous standpoints, such as low-reflecting boundary conditions [12, 20], one-way propagating waves [9, 10], radiation conditions obtained from a Wilcox expansion far from the obstacle [5, 18, 19], the behavior of the solution near a wave front [13, 14], etc. These boundary conditions can be seen as an exact or an approximate way to deal with the following so-called Dirichlet-to-Neumann operator. Let Ω' be the exterior domain limited by the introduced boundary Γ. For a function φ defined on Γ, consider w, the solution to Eq. (1) in Ω' subject to (2) and to the boundary condition

$$w = \varphi \qquad \text{on } \Gamma.$$

The operator is then defined by

$$\Lambda \varphi = \partial_{\mathbf{n}} w \qquad \text{on } \Gamma,$$

where $\mathbf{n}$ is the unit normal to Γ inwardly directed to Ω'.

It is well known that Λ is a pseudo-differential operator of order $+1$ (see [6, 23] concerning this point and for standard notation as well as functional spaces, which will be used in the sequel without further comment). Hereafter we limit ourselves to the extreme case where Γ exactly coincides with Σ. The problem to be solved is then reduced to a system of two boundary equations,

$$\mathscr{B}u = g, \quad \Lambda u - \partial_{\mathbf{n}}u = 0 \qquad \text{on } \Gamma,$$

in the two unknowns $u_{|\Gamma}$ and $\partial_{\mathbf{n}}u_{|\Gamma}$, classically referred to as the Cauchy data of the problem. The determination of these unknowns yields the solution entirely from the classical Helmholtz integral representation formula. The above approach is exactly what is done in a boundary integral equation method even if the effective derivation of this equation follows a different path.

Unfortunately, the mapping Λ is not a differential operator. As a result, a standard discretization of Λ by a finite-element method leads to a full matrix. Somewhat obviously, this results in storage and computational difficulties, especially when high-frequency or three-dimensional problems are considered.

Rather surprisingly, Kriegsmann et al. [16] have succeeded in designing a differential approximation of the operator A in a heuristic way that yields a reasonably accurate solution in the case of a bidimensional convex-shaped obstacle. They called the resulting equation an *on surface radiation condition* (OSRC). The essential trick of the approach is to write a radiation condition intrinsically on a large circle of radius R by putting the curvature ζ in place of the term $1/R$ and substituting the derivative with respect to the curvilinear abscissa ∂_s to the differential operator $R^{-1}\partial_\theta$, where θ is the angular variable.

From numerical experiments [1, 3] we have observed that the most accurate second-order condition is based on the related Bayliss–Turkel radiation condition on the circle. Furthermore, since circular geometry has a constant curvature, a more accurate condition can be obtained by first writing the OSRC in the form of a symmetric operator (relative to the scalar product of $L^2(\Gamma)$) incorporating the derivative of the curvature before formally writing it on an arbitrarily shaped obstacle.

It has been known for a long time that incorporating curvature terms gives more accurate radiation conditions (see [11], for instance). Jones [13] and, more recently, Stupfel [22] have constructed absorbing conditions depending on the derivatives of the curvature but which do not lead to a symmetric operator. However, we have observed in [3] that Jones's conditions result in a degeneracy of the quality of the approximation when applied to problems involving polygonal scatterers through the OSRC technique.

The above heuristic derivation of the OSRC from the radiation condition of Bayliss and Turkel may also be questionable. Is it really the most accurate OSRC of the second order that can be designed? Since the curvature operator is scalar for an arbitrarily shaped boundary in the two-dimensional case, the heuristic transposition of the Bayliss−Turkel radiation condition can be made in a straightforward way. But what about such a transposition for three-dimensional problems where this operator is scalar only for spherical geometries? An attempt to answer these questions can be made by using the formalist background of microlocal analysis (the calculus of pseudo-differential operators). Such an approach has been initiated by Engquist and Majda [9]. In this paper, we develop a clear and precise handling of the construction process that explains why the above modified Bayliss−Turkel condition is the most accurate and results in a condition of the same accuracy in three dimensions. Let us briefly describe this approach. Using Nirenberg's decomposition theorem [21] and symbolic calculus, one easily derives an asymptotic expansion of the operator Λ in the form

$$\Lambda \sim \sum_{-1 \leq j \leq \infty} \Lambda_{-j}, \tag{3}$$

where the sign $\sim$ has the following meaning: for every nonnegative integer m, the operator $\Lambda - \sum_{-1 \leq j \leq m}\Lambda_{-j}$ is a pseudo-differential operator of order at most $-(m + 1)$. In this way, the expansion (3) gives an explicit expression of the operator Λ, since the symbol of Λ_{-j} can be obtained from an explicit recursive formula. Apparently two steps are necessary for the construction of a differential operator approximating Λ. First, an ''approximation'' of the operator Λ by a finite sum $\sum_{-1 \leq j \leq m}\Lambda_{-j}$ has to be made. Strictly speaking, this process results in an approximation of the operator Λ such that the remaining correction is a regularizing operator producing $(m + 1)$ supplemental orders of derivations every time it is applied to a function. The question is: How many terms in this expansion have to be retained for a fixed accuracy? Second, the explicit operators Λ_{-j} have themselves to be approximated by a differential operator. How can this approximation be made without discarding any nonnegligible term? The following property gives the precise criterion for the construction. Essentially every operator Λ_{-j} has a symbol λ_{-j} that behaves like k^{-j} for large k. The symbol of $\sum_{-1 \leq j \leq 2}\Lambda_{-j}$ is expanded by a Taylor expansion in powers of frequency up to the term $1/k^2$. The resulting expression differs only by terms in $1/k^3$ with the symbol relative to the accurate condition, heuristically designed in the two-dimensional case [3]. Clearly, the same procedure can be used in dimension 3 to design an extension of the Bayliss−Turkel boundary condition for an arbitrarily

shaped surface. Furthermore, we shall see that this approach gives a common framework from which almost all radiation conditions of order less than or equal to 2 can be recovered.

Here we choose not to complicate the exposition by considering a general frame for a space of dimension N equal to either 2 or 3. We concentrate on the three-dimensional case since the heuristic procedure cannot be used there to design the most accurate second-order condition. The radiation condition for $N = 2$ can then be obtained as a special form of the three-dimensional one.

This paper is organized as follows. We begin in Section 2 by reviewing the description of a coordinate system that is well adapted to the description of a thin layer in the vicinity of a surface. Then we give the expression of the Laplace operator in this coordinate system. The coordinate change reduces the study to the case where the problem is set in a half-plane but involves a variable coefficient wave equation. In Section 3, we define the Dirichlet-to-Neumann nonlocal pseudo-differential operator Λ as the out-going part of the solution to the wave equation. Several properties of this operator that are crucial to the method are briefly noted, as is the application of the Nirenberg factorization theorem, which yields a formula that permits recursive determination of the entire sequence $\{\lambda_{-j}\}_{-1 \leq j \leq \infty}$. Moreover, we prove that the leading term of a symbol λ_{-j} associated with an operator Λ_{-j} behaves like k^{-j} as k tends to infinity. In Section 4, we introduce a precise criterion based upon the combination of a symbolical truncation and a Taylor's expansion in $1/k$ of symbols, providing a hierarchy of surface radiation conditions of different orders. We then show how most of the well-known radiation conditions can be recovered. Finally, we design a second-order Bayliss–Turkel-like radiation condition for an arbitrarily shaped surface.

2. REVIEW OF DIFFERENTIAL GEOMETRY

In this section we intend to write the Laplace operator in a system of local coordinates defined in a so-called tubular neighborhood of the surface Γ. For $\delta > 0$, this type of neighborhood $\mathscr{U}_\delta$ is defined by

$$\mathscr{U}_\delta = \{x \in \mathbb{R}^3; \, d(x) < \delta\},$$

where the function d stands for the distance from x to Γ. When δ is chosen small enough, there exists a unique p in Γ such that $|x - p| = d(x)$ for any x in $\mathscr{U}_\delta$ (see, for instance, [8]). The vector p is the orthogonal projection of x onto Γ. The unit normal vector to Γ is pointing into Ω_+, that is, defining

$$\mathbf{n}(p) = \pm \nabla d(x) \qquad \text{if } x \in \Omega_\pm,$$

we get an extension of the field $\mathbf{n}$ to all of $\mathscr{U}_\delta$. Therefore $\mathscr{U}_\delta$ admits a natural parametrization by

$$\mathscr{U}_\delta = \{x = p + r\mathbf{n}(p); \; -\delta < r < \delta, \; p \in \Gamma\}.$$

More precisely, for sufficiently small $\delta > 0$, the mapping

$$(p, r) \mapsto x = p + r\mathbf{n}(p) \tag{4}$$

is a $\mathscr{C}^\infty$-diffeomorphism of manifolds between $\Gamma \times (-\delta, \delta)$ and $\mathscr{U}_\delta$. For $-\delta < r < \delta$, the surface Γ_r parallel to Γ can be defined by

$$\Gamma_r = \{x \in \mathbb{R}^3; \; x = p + r\mathbf{n}(p)\}. \tag{5}$$

The unit normal to Γ_r at a point x is then given by the previous extension of the unit normal $\mathbf{n}(x)$. According to (5), Γ is simply the surface Γ_0.

Now consider a coordinate patch $(\mathscr{V}, \Psi)$ of Γ where $\Psi \colon \mathscr{V} \subset \mathbb{R}^2 \to \Gamma$. Each point p of $\Psi(\mathscr{V}) \subset \Gamma$ is then given by $p = \Psi(s)$, and the normal vector to Γ can be defined by the following rule. The coordinate patch gives rise to a basis (τ_1, τ_2) of the tangent plane $T_p(\Gamma)$:

$$\boldsymbol{\tau}_i = \partial_{s_i} \Psi, \qquad i = 1, 2,$$

where s_1 and s_2 are the coordinates of s. We assume that this coordinate patch is compatible with the orientation of the unit normal $\mathbf{n}$, that is,

$$\mathbf{n}(p) = \frac{\boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2}{|\boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2|},$$

where $\times$ stands for the usual vector product.

Now consider a function v depending on x and define $\tilde{v}(s, r) = v(x)$. To write the Laplacian in the coordinates (s, r), we first use the two following relations given by the chain rule:

$$\partial_{s_j} \bar{v}(s, r) = \sum_{i=1}^{3} \partial_{x_i} v \, \partial_{s_j} x_i = \nabla v \cdot \partial_{s_j} x,$$

$$\partial_r \tilde{v}(s, r) = \sum_{i=1}^{3} \partial_{x_i} v \, \partial_r x_i = \nabla v \cdot \partial_r x, \qquad j = 1, 2.$$

$\mathbf{w}_1 \cdot \mathbf{w}_2$ denotes the usual scalar product of two vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ with three (possibly complex) components. Thus, since $x = \Psi(s) + r\mathbf{n}(\Psi(s))$, we get

$$\partial_r x = \mathbf{n}(\Psi(s)), \quad \partial_{s_j} x = \boldsymbol{\tau}_j + r\partial_{s_j}\mathbf{n}, \qquad j = 1, 2.$$

The derivatives

$$\partial_{s_j} \mathbf{n} = \mathscr{R} \boldsymbol{\tau}_j, \qquad j = 1, 2$$

can be expressed using a self-adjoint operator $\mathscr{R}$ of the tangent plane, thereby giving the curvature of Γ at the point $p = \Psi(s)$ (cf., e.g., [8]).

If we denote by $\mathbb{I}$ the identity of the tangent plane, we then get

$$\partial_{s_j} x = (\mathbb{I} + r\mathscr{R}) \boldsymbol{\tau}_j, \qquad j = 1, 2.$$

Since $\mathscr{R}$ is self-adjoint, the relation $\nabla v \cdot \mathscr{R} \boldsymbol{\tau}_j = \mathscr{R} \Pi \nabla v \cdot \boldsymbol{\tau}_j$ yields

$$\partial_{s_j} \tilde{v}(s, r) = (\mathbb{I} + r\mathscr{R}) \nabla v \cdot \boldsymbol{\tau}_j = (\mathbb{I} + r\mathscr{R})(\Pi \nabla v) \cdot \boldsymbol{\tau}_j, \qquad j = 1, 2,$$

where Π is the projection onto the tangent plane. The above equation can be written intrinsically as

$$\Pi \nabla v = (\mathbb{I} + r\mathscr{R})^{-1} \nabla_\Gamma \tilde{v},$$

where ∇_Γ is the surface gradient of the partial mapping $p \in \Gamma \to \tilde{v}(p, r)$. For the sake of brevity, we do not distinguish in the sequel between $\tilde{v}(s, r)$ and the function obtained from v by the variable change (4). For a regular function φ compactly supported in $\mathscr{U}_\delta$, we can write

$$\int_{\mathscr{U}_\delta} \nabla v \cdot \nabla \varphi \, d\mathscr{U}_\delta = \int_{\mathscr{U}_\delta} \{ (\Pi \nabla v) \cdot (\Pi \nabla \varphi) + (\nabla v \cdot \mathbf{n})(\nabla \varphi \cdot \mathbf{n}) \} \, d\mathscr{U}_\delta.$$

Thus, since the volume element on $\mathscr{U}_\delta$ is given by

$$d\mathscr{U}_\delta = \det(\mathbb{I} + r\mathscr{R}) \, d\Gamma \, dr,$$

we get

$$\int_{\mathscr{U}_\delta} \nabla v \cdot \nabla \varphi \, d\mathscr{U}_\delta = \int_{\Gamma \times (-\delta, \delta)} \left((\mathbb{I} + r\mathscr{R})^{-2} \nabla_\Gamma \tilde{v} \cdot \nabla_\Gamma \tilde{\varphi} + \partial_r \tilde{v} \, \partial_r \tilde{\varphi} \right)$$
$$\times \det(\mathbb{I} + r\mathscr{R}) \, d\Gamma \, dr.$$

Integrating by parts, we obtain

$$\int_\Omega \nabla v \cdot \nabla \varphi \, dx = \int_{\Gamma \times (-\delta, \delta)} \left(-\mathbf{div}_\Gamma \left((\mathbb{I} + r\mathscr{R})^{-2} \det(\mathbb{I} + r\mathscr{R}) \nabla_\Gamma \tilde{v} \right) \right.$$
$$\left. - \partial_r (\det(\mathbb{I} + r\mathscr{R}) \, \partial_r \tilde{v}) \right) \tilde{\varphi} \, d\Gamma \, dr, \quad (6)$$

where div_Γ is the surface divergence of a tangent vector field on Γ. For any function φ compactly supported in $\mathscr{U}_\delta$, the following Green formula:

$$-\int_{\mathscr{U}_\delta} \Delta v \, \varphi \, dx = \int_{\mathscr{U}_\delta} \nabla v \cdot \nabla \varphi \, dx,$$

according to (6), directly yields the expression of the Laplacian in variables s and r:

$$\Delta v = \det(\mathbb{I} + r\mathscr{R})^{-1}\Big(\text{div}_\Gamma\big(\det(\mathbb{I} + r\mathscr{R})(\mathbb{I} + r\mathscr{R})^{-2}\nabla_\Gamma\tilde{v}\big)$$
$$+ \partial_r\big(\det(\mathbb{I} + r\mathscr{R}) \, \partial_r\tilde{v}\big)\Big).$$

Now to get a more tractable form for this expression, we make a particular choice for the coordinate patch. In fact we can choose a particular coordinate patch so that the coordinate system (s, r) may be an orthonormal system (i.e., (τ_1, τ_2) is an orthonormal basis of the tangent plane) and, furthermore, such that τ_j is an eigenvector of $\mathscr{R}$. Such a basis is called the principal basis of $T_p(\Gamma)$, whereas the vectors τ_1 and τ_2 are known as the principal curvature directions of Γ and satisfy

$$\mathscr{R}\tau_j = \mathscr{C}_j\tau_j,$$

where $\mathscr{C}_1$ and $\mathscr{C}_2$ are the principal curvatures of Γ. The Gauss curvature and the mean curvature can now be respectively defined by $\mathscr{K} = \mathscr{C}_1\mathscr{C}_2$ and $\mathscr{H} = (\mathscr{C}_1 + \mathscr{C}_2)/2$. Defining

$$h_j = 1 + r\mathscr{C}_j, \qquad j = 1, 2,$$

we can write

$$\det(\mathbb{I} + r\mathscr{R}) = h_1 h_2 \quad \text{and} \quad (\mathbb{I} + r\mathscr{R})\tau_j = h_j\tau_j, \qquad j = 1, 2.$$

Hence, in the orthonormal coordinates system, the Laplace operator takes the following expression:

$$\Delta v = \partial_r^2\tilde{v} + \frac{\partial_r(h_1 h_2)}{h_1 h_2}\,\partial_r\tilde{v} + \frac{1}{h_1 h_2}\left(\partial_{s_1}\!\left(\frac{h_2}{h_1}\,\partial_{s_1}\tilde{v}\right) + \partial_{s_2}\!\left(\frac{h_1}{h_2}\,\partial_{s_2}\tilde{v}\right)\right).$$

Hereafter, we denote by $\mathscr{H}_r$ the mean curvature of the parallel surface Γ_r given by

$$\mathscr{H}_r = \frac{1}{2}\frac{\partial_r(h_1 h_2)}{h_1 h_2},$$

and we drop the symbol $\tilde{\ }$ to simplify the notation.

3. APPROXIMATION OF THE
DIRICHLET-TO-NEUMANN OPERATOR

3.1. *The Dirichlet-to-Neumann Operator*

To keep the convention on the time variation and use the formalism of pseudo-differential operators, we write the Helmholtz equation as a wave equation, hence depending explicitly on the time t:

$$Lu = \Delta u - \partial_t^2 u = 0,$$

where $u(x, t) = v(x) \exp(-ikt)$. Consequently, we do not consider the multiplication by k^2 as a zero-order operator. In the local coordinate system, the above equation can be rewritten as

$$Lu = \partial_r^2 u + 2\mathscr{H}_r \, \partial_r u + \frac{1}{h_1 h_2} \left(\partial_{s_1}\!\left(\frac{h_2}{h_1} \, \partial_{s_1} u \right) + \partial_{s_2}\!\left(\frac{h_1}{h_2} \, \partial_{s_2} u \right) \right) - \partial_t^2 u = 0.$$
(7)

The symbol $\mathscr{L}$ of L, with respect to the variables s and t and their respective dual variables σ and ω, depends smoothly on the variable r, which will hereafter play the role of a parameter. Thanks to the expression of (7) for the Helmholtz operator L in the variables s and r, this symbol has the following explicit expression:

$$\mathscr{L} = \partial_r^2 + 2\mathscr{H}_r \partial_r - h_1^{-2}\sigma_1^2 - h_2^{-2}\sigma_2^2$$

$$+ \frac{i}{h_1 h_2}\left(\partial_{s_1} \frac{h_2}{h_1} \right)\sigma_1 + \frac{i}{h_1 h_2}\left(\partial_{s_2} \frac{h_1}{h_2} \right)\sigma_2 + \omega^2.$$

The selection of waves propagating in the normal direction is based on the following decomposition resulting from Nirenberg's factorization theorem [21]:

There exist two classical pseudo-differential operators Λ^- and Λ^+ of order $+1$, depending smoothly on r, such that

$$Lu = (\partial_r + i\Lambda^-)(\partial_r + i\Lambda^+)u.$$
(8)

In addition, the uniqueness of the decomposition is ensured by the following characterization. Let λ^+ and λ^- be the respective symbols of Λ^+ and Λ^-. From the definition of pseudo-differential operators of order $+1$, the symbols λ^+ and λ^- belong to the symbol class $S_{1,0}^1$ (cf., e.g., Chazarain and Piriou [6] for the definition of this class of symbols). By classical pseudo-differential operators, we mean that the symbols $\lambda^\pm$

admit the following asymptotic expansion:

$$\lambda^{\pm} \sim \sum_{j=-1}^{+\infty} \lambda_{-j}^{\pm}, \tag{9}$$

where $\lambda_{-j}^{\pm}$ are homogeneous functions of degree $-j$ in (σ, ω). The expansion holds in the following sense:

$$\forall m \geq -1, \qquad \lambda^{\pm} - \sum_{j=-1}^{m} \lambda_{-j}^{\pm} \in S^{-(m+1)}.$$

Hereafter, as in [23], we more concisely denote by S^m the symbol class $S_{1,0}^m$. Hence the corresponding classical pseudo-differential operators Λ^- and Λ^+ are in OPS^1. According to [17] we define the cone of propagation as the set of dual variables (σ, ω) satisfying $(\omega^2 - h_1^{-2}\sigma_1^2 - h_2^{-2}\sigma_2^2) > 0$. The decomposition is then made unique by requiring that λ_1^+ be positive in the cone of propagation. The waves propagating along the normal direction (the so-called physical outgoing waves) satisfy the equation

$$(\partial_r + i\Lambda^+)u = 0,$$

in the microlocal sense (cf. Taylor [23]). Therefore, we explicitly obtain the Dirichlet-to-Neumann operator as the nonlocal operator $\tilde{\Lambda}^+$ such that $\tilde{\Lambda}^+ = \Lambda_{|r=0}^+$. The exact radiation condition on Γ is given by

$$(\partial_r + i\Lambda^+)u_{|r=0} = (\partial_{\mathbf{n}} + i\tilde{\Lambda}^+)u = 0.$$

Thus, the problem is to determine Λ^+. To this end, we first need to develop the right-hand side of (8):

$$Lu = \partial_r^2 u + i\Lambda^- \partial_r u + i \partial_r \Lambda^+ u + \Lambda^- \Lambda^+ u = 0. \tag{10}$$

On the basic of the integral representation of pseudo-differential operators from their symbol (cf., e.g., [6, 23]), an immediate calculation gives

$$\partial_r \Lambda^+ u(r,s,t) = \partial_r \int e^{i(\omega t + \sigma \cdot s)} \lambda^+(r,s,t,\sigma,\omega)\hat{u}(r,\sigma,\omega)\, d\omega\, d\sigma$$

$$= \int e^{i(\omega t + \sigma \cdot s)}(\partial_r \lambda^+ \,\hat{u} + \lambda^+ \,\partial_r \hat{u})\, d\omega\, d\sigma.$$

Then we introduce the following convenient notation. $\{\mu\}$ is the pseudo-differential operator with the symbol μ, and we get the following expression:

$$\partial_r \Lambda^+ u(r,s,t) = \{\partial_r \lambda^+\}u + \Lambda^+ \,\partial_r u.$$

Thus relation (10) can be equivalently written as

$$Lu = \partial_r^2 u + i(\Lambda^- + \Lambda^+)\,\partial_r u + i\{\partial_r \lambda^+\}u - \Lambda^-\Lambda^+ u. \qquad (11)$$

Identifying in (7) and (11) the coefficients of the derivatives with respect to r, we obtain the system

$$\begin{cases} \Lambda^- + \Lambda^+ = -2i\mathscr{H}_r, \\ \Lambda^-\Lambda^+ - i\{\partial_r \lambda^+\} = \partial_t^2 - \dfrac{1}{h_1 h_2}\left(\partial_{s_1}\!\left(\dfrac{h_2}{h_1}\partial_{s_1}\right) + \partial_{s_2}\!\left(\dfrac{h_1}{h_2}\partial_{s_2}\right)\right). \end{cases} \qquad (12)$$

Thanks to the integral representation formula of pseudo-differential operators, the operators Λ^- and λ^+ are explicitly known from the determination of their symbols. Using the calculus of pseudo-differential operators (see [23]), system (12) can be written at the symbol level:

$$\begin{cases} \lambda^- + \lambda^+ = -2i\mathscr{H}_r \\ \displaystyle\sum_{\alpha=0}^{+\infty} \frac{(-i)^{|\alpha|}}{|\alpha|!}\,\partial_\sigma^\alpha \lambda^-\;\partial_s^\alpha \lambda^+ - i\,\partial_r \lambda^+ \\ \qquad = -\left(\omega^2 - h_1^{-2}\sigma_1^2 - h_2^{-2}\sigma_2^2\right) \\ \qquad\quad - \dfrac{i}{h_1 h_2}\left(\partial_{s_1}\dfrac{h_2}{h_1}\right)\sigma_1 - \dfrac{i}{h_1 h_2}\left(\partial_{s_2}\dfrac{h_1}{h_2}\right)\sigma_2. \end{cases} \qquad (13)$$

To determine the symbols λ^- and λ^+, we then use asymptotic expansion (9). The first equation of system (13) yields

$$\lambda_{-j}^- + \lambda_{-j}^+ = 0 \qquad \text{if } j \neq 0 \quad \text{and} \quad \lambda_0^- + \lambda_0^+ = -2i\mathscr{H}_r. \qquad (14)$$

Since we only need to compute symbol λ^+, relations (14) will be used for the elimination of λ^-. Thus, the combination of the second equation of (13) and (14) completely determines the operator Λ^+. Finally, the Dirichlet-to-Neumann operator $\tilde{\Lambda}^+$ is obtained from the relation $\Lambda_{|r=0}^+ = \tilde{\Lambda}^+$.

Obviously, the computation of all of the symbols λ_{-j}^+ is not practicable. Furthermore, our aim is to construct local boundary conditions, whereas $\tilde{\Lambda}^+$ is a nonlocal operator. Fortunately, as will be shown, only the first few symbols λ_{-j}^+, $-1 \leq j \leq 2$, are needed for the derivation of radiation conditions of order ≤ 2.

3.2. Recursive Determination of the Asymptotic Expansion of the Symbol of the Dirichlet-to-Neumann Operator

Based upon an identification of symbols with the same degree of homogeneity, a recursive formula permits us to express all of the symbols

λ_{-j}^{+} from the first one, λ_{1}^{+}. More precisely, we remark that Eq. (14) enables us to readily express λ_{-j}^{-} from λ_{-j}^{+} and then to solve system (13) relative to the symbols λ_{-j}^{+}.

The computation is done in three steps. In the first two, λ_{1}^{+} and λ_{0}^{+} are obtained by a straightforward calculation. The last step is devoted to the derivation of a recursive formula that expresses λ_{-m-1}^{+}, for $m \geq 0$, in terms of previously determined symbols as well as their derivatives.

Starting with an identification of homogeneous symbols of the highest degree, we get

$$\lambda_{1}^{-}\lambda_{1}^{+} = -\left(\omega^{2} - h_{1}^{-2}\sigma_{1}^{2} - h_{2}^{-2}\sigma_{2}^{2}\right).$$

From (14) and the condition $\lambda_{1}^{+} > 0$, we directly obtain

$$\lambda_{1}^{+} = \left(\omega^{2} - h_{1}^{-2}\sigma_{1}^{2} - h_{2}^{-2}\sigma_{2}^{2}\right)^{1/2}.$$

Going further with an identification of the homogeneous symbols of the next higher degree, we find a relation between the unknowns λ_{0}^{-} and λ_{0}^{+}:

$$\lambda_{1}^{-}\lambda_{0}^{+} + \lambda_{0}^{-}\lambda_{1}^{+} - i\,\partial_{\sigma_{1}}\lambda_{1}^{-}\,\partial_{s_{1}}\lambda_{1}^{+} - i\,\partial_{\sigma_{2}}\lambda_{1}^{-}\,\partial_{s_{2}}\lambda_{1}^{+} - i\,\partial_{r}\lambda_{1}^{+}$$
$$= -ih_{1}^{-1}h_{2}^{-1}\left(\partial_{s_{1}}\left(h_{2}^{-1}h_{1}\right)\sigma_{1} + \partial_{s_{2}}\left(h_{1}^{-1}h_{2}\right)\sigma_{2}\right).$$

Eliminating λ_{1}^{-} and λ_{0}^{-}, we get

$$\lambda_{0}^{+} = \frac{i}{2\,\lambda_{1}^{+}}\Big(-2\mathscr{H}_{r}\lambda_{1}^{+} + \partial_{\sigma_{1}}\lambda_{1}^{+}\,\partial_{s_{1}}\lambda_{1}^{+} + \partial_{\sigma_{2}}\lambda_{1}^{+}\,\partial_{s_{2}}\lambda_{1}^{+}$$
$$- \partial_{r}\lambda_{1}^{+} + h_{1}^{-1}h_{2}^{-1}\left(\partial_{s_{1}}\left(h_{2}h_{1}^{-1}\right)\sigma_{1} + \partial_{s_{2}}\left(h_{1}h_{2}^{-1}\right)\sigma_{2}\right)\Big).$$

The recursive formula derived from similar relations for lower degrees of homogeneity. At first, the left-hand side of the second equation in (13) can be rewritten as

$$\sum_{|\alpha|=0}^{+\infty} \frac{(-i)^{|\alpha|}}{|\alpha|!} \sum_{j=-1}^{+\infty} \partial_{\sigma}^{\alpha}\lambda_{-j}^{-} \sum_{k=-1}^{+\infty} \partial_{s}^{\alpha}\lambda_{-k}^{+} - i\sum_{l=-1}^{+\infty} \partial_{r}\lambda_{-l}^{+}. \qquad (15)$$

Next, we make use of the following properties (see [23]):

$$\partial_{\sigma}^{\alpha}\lambda_{-j}^{-} \in S^{-j-|\alpha|}, \quad \partial_{s}^{\alpha}\lambda_{-k}^{+} \in S^{-k}.$$

This, in turn, gives

$$\partial_{\sigma}^{\alpha}\lambda_{-j}^{-}\,\partial_{s}^{\alpha}\lambda_{-k}^{+} \in S^{-(j+k+|\alpha|)}.$$

Thus, in (15) for any nonnegative integer m, the homogeneous part of degree $-m$ is

$$\sum_{|\alpha|=0}^{m+2} \frac{(-i)^{|\alpha|}}{|\alpha|!} \sum_{\substack{j+k=m-|\alpha| \\ j\geq -1,\, k\geq -1}} \partial_\sigma^\alpha \lambda_{-j}^- \,\partial_s^\alpha \lambda_{-k}^+ - i\,\partial_r\lambda_{-m}^+.$$

Since there is no homogeneous symbol of degree $-m$ on the right-hand side of the second equation of system (13), an identification of symbols of the same homogeneity leads to the equation

$$\sum_{|\alpha|=0}^{m+2} \frac{(-i)^{|\alpha|}}{|\alpha|!} \sum_{\substack{j+k=m-|\alpha| \\ j\geq -1,\, k\geq -1}} \partial_\sigma^\alpha \lambda_{-j}^- \,\partial_s^\alpha \lambda_{-k}^+ - i\,\partial_r\lambda_{-m}^+ = 0, \qquad \text{for } m\geq 0.$$

Moreover, we have $\lambda_{-m-1}^- = -\lambda_{-m-1}^+$. Hence, the previous equation, symbol λ_{-m-1}^+, $m\geq 0$, can be recursively expressed from homogeneous symbols of higher order by

$$\lambda_{-m-1}^+ = \frac{1}{2\lambda_1^+}\left(\sum_{\substack{j+k=m \\ j\geq 0,\, k\geq 0}} \lambda_{-j}^- \lambda_{-k}^+ \right.$$

$$\left. + \sum_{|\alpha|=1}^{m+2} \frac{(-i)^{|\alpha|}}{|\alpha|!} \sum_{\substack{j+k=m-|\alpha| \\ j\geq -1,\, k\geq -1}} \partial_\sigma^\alpha \lambda_{-j}^- \,\partial_s^\alpha \lambda_{-k}^+ - i\,\partial_r\lambda_{-m}^+ \right), \quad (16)$$

Now we intend to establish that the design of radiation conditions up to the second order needs the explicit computation of only the first few symbols $\{\lambda_{-j}^+\}_{-1\leq j\leq 2}$. This property will be obtained basically from the following result.

LEMMA 3.1. *Each homogeneous symbol of order m, for $m\leq 1$, has the following form*:

$$\lambda_m^+ = \left(\lambda_1^+\right)^m P(s,r;X), \qquad (17)$$

where $X = \sigma/\lambda_1^+$ and $P(s,r;X)$ is a polynomial in variable X with smooth functions of (s,r) as coefficients. More precisely, polynomial $P(s,r;X)$ is of the form

$$P(s,r;X) = \sum_{|\alpha|\leq d} a_\alpha(s,r)X^\alpha. \qquad (18)$$

Proof. Let $P(s, r; X)$ and $Q(s, r; X)$ designate various functions (not the same in all instances) of the form (18). The chain rule gives directly that

$$\partial_{\sigma_j} \lambda_1^+ = P(s, r; X), \quad \partial_{s_j} \lambda_1^+ = \lambda_1^+ P(s, r; X), \quad \partial_r \lambda_1^+ = \lambda_1^+ P(s, r; X).$$

Then an inductive argument on the order of derivation easily shows that the derivatives of a function of the type $(\lambda^+)^{\varpi} P(s, r; X)$ have the same form given by

$$\begin{cases} \partial_{\sigma}^{\alpha} \left[\left(\lambda_1^+ \right)^{\varpi} P(s, r; X) \right] = \left(\lambda_1^+ \right)^{\varpi - |\alpha|} Q(s, r; X), \\ \partial_s^{\alpha} \left[\left(\lambda_1^+ \right)^{\varpi} P(s, r; X) \right] = \left(\lambda_1^+ \right)^{\varpi} Q(s, r; X), \\ \partial_r \left[\left(\lambda_1^+ \right)^{\varpi} P(s, r; X) \right] = \left(\lambda_1^+ \right)^{\varpi} Q(s, r; X). \end{cases}$$

The end of the proof is then easily obtained by an inductive argument due to the fact that relation (14) implies that λ_m^+ and λ_m^-, are both of the form (17) whenever it holds for one of them. ∎

We can then link up the approximation of the symbol λ^+ of the operator Λ^+ through its asymptotic expansion (9) in a precise manner.

THEOREM 3.2.   *Let m be an integer ≥ -1. Any symbol given by*

$$\sum_{-1 \leq j \leq p} \lambda_{-j}^+ \tag{19}$$

for $p > m$ differs from $\sum_{-1 \leq j \leq m} \lambda_{-j}^+$ by a term at most of order $\omega^{-(m+1)}$.

Proof.   Immediate from the above lemma. ∎

4. SURFACE RADIATION BOUNDARY CONDITIONS

4.1. *The Approximation Criterion*

We are now in a position to give a precise criterion allowing the derivation of various radiation conditions and to compare their respective accuracy. We limit ourselves to radiation conditions that can be built through the following process. Let ℓ be a positive integer and μ be the symbol of a pseudo-differential operator $\{\mu\}$ defined on $\Gamma \times (-\delta, \delta)$ such that $\sum_{-1 \leq j \leq p} \lambda_{-j}^+ - \mu$ is of order $(1/\omega)^{\ell-1}$ for all sufficiently large p. Denoting by $\tilde{\mu}$ the symbol defined on Γ by $\tilde{\mu} := \mu_{|r=0}$, we write out the

radiation condition

$$\partial_{\mathbf{n}} u + i\{\tilde{u}\}u = 0 \qquad \text{on } \Gamma,$$

which we call a *complete radiation condition of order $\ell/2$*. Such a condition is considered to agree with standard definitions of the order of classical radiation conditions (cf., for instance, [12]).

Hence, in view of Theorem 3.2, the definition of a complete radiation condition is consistent, and the most straightforward one of order $\ell/2$ is obtained as

$$\partial_{\mathbf{n}} u + i\left\{ \sum_{-1 \leq j \leq \ell-2} \widetilde{\lambda_j^+} \right\}u = 0 \qquad \text{on } \Gamma.$$

Unfortunately, the approximate operator has the undesirable property of being nonlocal, just like the exact one. To obtain local radiation conditions (i.e., expressed through a local operator), we have to consider some approximation process of the symbols λ_{-j}^+. With this end in view, we follow an approach that is similar although slightly different from that considered by Engquist and Majda [9]. From the retained definition of a complete radiation condition, it is quite obvious to approximate each λ_{-j}^+ by its Taylor expansion in powers of $1/\omega$ up to the term $(1/\omega)^{\ell-2}$ denoted $(\lambda_j^+)_{\ell-2}$. A complete radiation condition of order $\ell/2$ can then be obtained as well by

$$\partial_{\mathbf{n}} u + i\left\{ \sum_{-1 \leq j \leq \ell-2} \widetilde{\left(\lambda_j^+\right)}_{\ell-2} \right\}u = 0 \qquad \text{on } \Gamma.$$

Surprisingly enough, as seen below, this construction always leads to a local radiation condition for $\ell \leq 4$ (i.e., for conditions of order ≤ 2). Actually, these conditions involve operators with a symbol polynomial in σ, ω, and $1/\omega$, with coefficients smoothly depending on s and r. Since the symbol $1/\omega$ corresponds to an integration in t, from the assumed time dependence of the solution, this symbol only corresponds to the multiplication by $-1/k$.

Engquist and Majda have considered the same kind of approximation (Taylor expansion) but have used the "angle of incidence" $|\sigma|/\omega$ rather than the "wavelength" $1/\omega$ as a small parameter. The same standpoint has been adopted by Hanouzet and Sesquès [11], who have kept $(h_1^{-2}\sigma_1^2 + h_2^{-2}\sigma_2^2)^{1/2}/\omega$ as a small parameter for the expansion. The approach that has been retained here can mainly be seen as a high-frequency approximation, as opposed to that of Engquist and Majda.

Incomplete radiation conditions can also be created by making a Taylor expansion at a given order $\ell - 2$ but by retaining fewer terms in the asymptotic expansion of the symbol λ^+ (19), i.e.,

$$\partial_{\mathbf{n}} u + i\left\{ \sum_{j=-1}^{m} \left(\widetilde{\lambda^+_{-j}}\right)_{\ell-2} \right\} u = 0 \qquad \text{on } \Gamma,$$

with $m < \ell - 2$. In this way, we will be able to recover almost all known radiation conditions "of order ≤ 2." Finally, approximating the symbol of the complete radiation condition of order 2 in a suitable manner, we derive another one that coincides with the Bayliss–Turkel second-order radiation condition when Γ is a sphere.

4.2. *Computation of Symbols*

Since we are planning to deal with radiation conditions of order ≤ 2, from the above discussion we see that we have to calculate λ^+_{-j} only for $-1 \leq j \leq 2$ and to expand them in powers of $1/\omega$ until the term $1/\omega^2$.

For symbol λ^+_1, we get

$$\lambda^+_1 = \omega\left(1 - \frac{h_1^{-2}\sigma_1^2 + h_2^{-2}\sigma_2^2}{2\omega^2}\right) + \text{terms in } \frac{1}{\omega^3}.$$

Hence, the approximate symbol involved in the construction of radiation conditions up to the order 2 is given by

$$\left(\widetilde{\lambda^+_1}\right)_2 = \omega - \frac{\sigma_1^2 + \sigma_2^2}{2\omega}. \tag{20}$$

Note that lower-order approximations $(\widetilde{\lambda^+_1})_l$ of $\widetilde{\lambda^+_1}$ with $l < 2$ are obtained simply by dropping terms in $1/\omega^i$ with $l < i \leq 2$ from $(\widetilde{\lambda^+_1})_2$.

The second function is determined recursively from the first and has the following expression:

$$\lambda^+_0 = -i\mathscr{H}_r + \frac{ir}{2h_1 h_2 \lambda^+_1}\left(\frac{\partial_{s_1}\mathscr{C}_2 - \partial_{s_1}\mathscr{C}_1}{h_1^2}\sigma_1 + \frac{\partial_{s_2}\mathscr{C}_1 - \partial_{s_2}\mathscr{C}_2}{h_2^2}\sigma_2\right)$$

$$- \frac{i}{2\left(\lambda^+_1\right)^2}\left(\mathscr{C}_1 h_1^{-3}\sigma_1^2 + \mathscr{C}_2 h_2^{-3}\sigma_2^2\right)$$

$$- \frac{ir\sigma_1}{2h_1^2\left(\lambda^+_1\right)^3}\left(\partial_{s_1}\mathscr{C}_1 h_1^{-3}\sigma_1^2 + \partial_{s_1}\mathscr{C}_2 h_2^{-3}\sigma_2^2\right)$$

$$- \frac{ir\sigma_2}{2h_2^2\left(\lambda^+_1\right)^3}\left(\partial_{s_2}\mathscr{C}_1 h_1^{-3}\sigma_1^2 + \partial_{s_2}\mathscr{C}_2 h_2^{-3}\sigma_2^2\right).$$

Making use of the previous expansion of λ_1^+, we get

$$
\begin{aligned}
\lambda_0^+ = {} & -i\mathscr{H}_r + \frac{ir}{2\,\omega h_1 h_2}\left(\frac{\left(\partial_{s_1}\mathscr{C}_2 - \partial_{s_2}\mathscr{C}_1\right)}{h_1^2}\sigma_1 + \frac{\left(\partial_{s_2}\mathscr{C}_1 - \partial_{s_1}\mathscr{C}_2\right)}{h_2^2}\sigma_2\right) \\
& -\frac{ir}{2\,\omega^3}\left(\left(h_1^{-2}\sigma_1\right)\left(\partial_{s_1}\mathscr{C}_1\,h_1^{-3}\sigma_1^2 + \partial_{s_1}\mathscr{C}_2\,h_2^{-3}\sigma_2^2\right)\right. \\
& \left.\qquad + \left(h_2^{-2}\sigma_2\right)\left(\partial_{s_2}\mathscr{C}_1\,h_1^{-3}\sigma_1^2 + \partial_{s_2}\mathscr{C}_2\,h_2^{-3}\sigma_2^2\right)\right) \\
& -\frac{i}{2\,\omega^2}\left(\mathscr{C}_1 h_1^{-3}\,\sigma_1^2 + \mathscr{C}_2 h_2^{-3}\,\sigma_2^2\right) + \text{terms in } \frac{1}{\omega^3}
\end{aligned}
$$

and then deduce the approximate symbol of index zero,

$$
\left(\widetilde{\lambda_0^+}\right)_2 = -i\mathscr{H} - \frac{i\left(\mathscr{C}_1\sigma_1^2 + \mathscr{C}_2\sigma_2^2\right)}{2\,\omega^2}.
$$

For the third function, we remark that since all of the terms of order ω^{-j} are dropped when $j > 2$ and $(\lambda_1^+)^{-j}$ behaves like ω^{-j}, only terms of order $(\lambda_1^+)^{-j}$ with $0 \leq j \leq 2$, occur in the expression of λ_{-1}^+. Hence we write λ_{-1}^+ in the following form:

$$
\begin{aligned}
\lambda_{-1}^+ = {} & -\frac{\mathscr{H}_r^2}{2\lambda_1^+} - \frac{\partial_r\mathscr{H}_r}{2\lambda_1^+} + \frac{\left(\partial_{s_1}\mathscr{C}_2 - \partial_{s_1}\mathscr{C}_1\right)\sigma_1}{4(\lambda_1^+)^2 h_1^3 h_2} + \frac{\left(\partial_{s_2}\mathscr{C}_1 - \partial_{s_2}\mathscr{C}_2\right)\sigma_2}{4(\lambda_1^+)^2 h_1 h_2^3} \\
& -\frac{\partial_{s_1}\mathscr{H}_r\,\sigma_1}{2(\lambda_1^+)^2 h_1 h_2} - \frac{\partial_{s_2}\mathscr{H}_r\,\sigma_2}{2(\lambda_1^+)^2 h_1 h_2} + \text{terms in } \frac{1}{(\lambda_1^+)^3}.
\end{aligned}
$$

We then deduce that

$$
\begin{aligned}
\lambda_{-1}^+ = {} & -\frac{\mathscr{H}_r^2}{2\,\omega} - \frac{\partial_r\mathscr{H}_r}{2\,\omega} + \frac{\left(\partial_{s_1}\mathscr{C}_2 - \partial_{s_1}\mathscr{C}_1\right)\sigma_1}{4\omega^2 h_1^3 h_2} - \frac{\left(\partial_{s_2}\mathscr{C}_1 - \partial_{s_2}\mathscr{C}_2\right)\sigma_2}{4\omega^2 h_1 h_2^3} \\
& -\frac{h_1^{-2}\left(\partial_{s_1}\mathscr{H}_r\right)\sigma_1}{2\,\omega^2} - \frac{h_2^{-2}\left(\partial_{s_2}\mathscr{H}_r\right)\sigma_2}{2\,\omega^2} + \text{terms in } \frac{1}{\omega^3},
\end{aligned}
$$

which leads, once more taking (20) into account, to the following symbol approximation:

$$
\left(\widetilde{\lambda_{-1}^+}\right)_2 = -\frac{1}{2\,\omega}\left(\mathscr{K} - \mathscr{H}^2\right) - \frac{1}{2\,\omega^2}\left(\partial_{s_1}\mathscr{C}_1\,\sigma_1 + \partial_{s_2}\mathscr{C}_2\,\sigma_2\right).
$$

Following the same approach again, we finally write symbol λ^+_{-2} in the form

$$\lambda^+_{-2} = \frac{ih_1^{-2}\left(\partial_{s_1}^2\mathscr{H}_r\right)}{4\left(\lambda_1^+\right)^2} + \frac{ih_2^{-2}\left(\partial_{s_2}^2\mathscr{H}_r\right)}{4\left(\lambda_1^+\right)^2} + \frac{i\mathscr{H}_r\left(\partial_r\mathscr{H}_r\right)}{2\left(\lambda_1^+\right)^2} + \frac{i\,\partial_r^2\mathscr{H}_r}{4\left(\lambda_1^+\right)^2}$$

$$+ \text{ terms in } \frac{1}{\left(\lambda_1^+\right)^3}.$$

The approximate symbol $\widetilde{(\lambda^+_{-2})}$ is then given by

$$\left(\widetilde{\lambda^+_{-2}}\right)_2 = \frac{i}{4\omega^2}\left(\partial_{s_1}^2\mathscr{H} + \partial_{s_2}^2\mathscr{H}\right) - \frac{i}{\omega^2}\mathscr{H}(\mathscr{K} - \mathscr{H}^2).$$

4.3. *Sommerfeld Radiation Condition*

The first approximation consists only of keeping terms of order ω. We then consider the complete radiation condition of order $1/2$ given by $\partial_{\mathbf{n}} u + i\{(\widetilde{\lambda_1^+})_1\}u = 0$. Making the operator $\{(\widetilde{\lambda_1^+})_1\}$ explicit and returning to a function u with a time dependence in e^{-ikt} yields the condition

$$\partial_{\mathbf{n}} u - iku = 0 \qquad \text{on } \Gamma.$$

This well-known condition, called the Sommerfeld radiation condition, only specifies that wave u is outgoing to infinity and may be seen as the crudest radiation condition that can be designated.

4.4. *First-Order Radiation Condition*

Going further with the same procedure, we derive a complete radiation condition of order 1. Dropping any term of order $1/\omega^j$ with $j > 0$, we are led to $\partial_{\mathbf{n}} u + i\{\sum_{j=-1}^0 (\widetilde{\lambda^+_{-j}})_0\}u = 0$, and then to

$$\partial_{\mathbf{n}} u - iku + \mathscr{H}u = 0. \tag{21}$$

This is again a classical condition that has already been obtained by many authors [5, 11, 13, 22] following very different paths.

4.5. *Radiation Condition of Order* $3/2$

Now keeping the terms of order $1/\omega^j$ with $j \leq 1$, i.e.,

$$\partial_{\mathbf{n}} u + i\left\{\sum_{j=-1}^1 \left(\widetilde{\lambda^+_{-j}}\right)_{-1}\right\}u = 0,$$

we obtain the complete radiation condition of order $3/2$:

$$\partial_{\mathbf{n}} u - iku + \mathscr{H} u + \frac{i}{2k}(\mathscr{K} - \mathscr{H}^2) u + \frac{1}{2ik} \Delta_\Gamma u = 0, \qquad (22)$$

where Δ_Γ stands for the Laplace–Beltrami operator on Γ. It is the same as Jones's first-order condition, which has been obtained in a different way. Note that this complete condition of order $3/2$ does not correspond to another condition designed by Jones in [14] (from an approach based upon successive approximations):

$$\partial_{\mathbf{n}} u - iku + \mathscr{H} u + \frac{1}{2ik} \Delta_\Gamma u = 0.$$

Nonetheless, from our point of view, the latter condition can be constructed from the approximation $\partial_{\mathbf{n}} u + i\{\sum_{j=-1}^{0} \widetilde{(\lambda_{-j}^+)}_{-1}\} u = 0$ and can be seen as an incomplete condition of order $3/2$. Note that incomplete conditions coincide with complete conditions for a flat boundary.

4.6. *Radiation Conditions of Order* 2

The most accurate radiation condition that can be considered in the above framework is given as $\partial_{\mathbf{n}} u + i\{\sum_{j=-1}^{2} \widetilde{(\lambda_{-j}^+)}_{-2}\} u = 0$. Proceeding as for the previous conditions, we get the following result.

THEOREM 4.1.   *A complete radiation condition of order* 2 *is given by*

$$\partial_{\mathbf{n}} u - iku + \mathscr{H} u + \frac{i}{2k}\left(1 - i\frac{2\mathscr{H}}{k}\right)(\mathscr{K} - \mathscr{H}^2) u - \frac{\Delta_\Gamma \mathscr{H}}{4k^2} u$$

$$+ \operatorname{div}_\Gamma\left(\frac{1}{2ik}\left(\mathbb{I} - \frac{i\mathscr{R}}{k}\right)\nabla_\Gamma\right) u = 0. \qquad (23)$$

The above complete second-order condition is different from the condition established by Engquist and Majda, who have used the following approximation, corresponding in our notation to $\partial_{\mathbf{n}} u + i\{\sum_{j=-1}^{0} \widetilde{(\lambda_{-j}^+)}_{-2}\} u = 0$. Making the above relation explicit, one obtains the incomplete condition of order 2:

$$\partial_{\mathbf{n}} u - iku + \mathscr{H} u + \frac{1}{2ik}\left(\mathbb{I} - \frac{i\mathscr{R}}{k}\right)\Delta_\Gamma u = 0, \qquad (24)$$

where terms of order $1/\omega^2$ coming from $\widetilde{\lambda_{-j}^+}$, with $j = +1$ or $+2$, are not considered. It is worth noting that condition (24), except in the case in which Γ is a sphere, does not lead to symmetric variational formulations

when it is used either to terminate a finite element formulation or as an OSRC condition, although it is an important advantage with condition (23).

Remark. Following the same path, we can construct an incomplete condition by retaining only $\widetilde{\lambda_1^+}$, $\widetilde{\lambda_0^+}$, and $\widetilde{\lambda_{-1}^+}$ in the asymptotic development and making a Taylor approximation up to the terms of order $1/\omega^2$ of the involved symbols. We get another incomplete second-order radiation condition,

$$\partial_{\mathbf{n}}u - iku + \mathscr{H}u + \frac{i}{2k}(\mathscr{K} - \mathscr{H}^2)u + \operatorname{div}_\Gamma\left(\frac{1}{2ik}\left(\mathbb{I} - \frac{i\mathscr{R}}{k}\right)\nabla_\Gamma\right)u = 0,$$

which is derived from the approximation $\partial_{\mathbf{n}}u + i\{(\widetilde{\lambda_1^+} + \widetilde{\lambda_0^+} + \widetilde{\lambda_{-1}^+})_{-2}\}u = 0$.

At first sight, condition (23) seems to be different from Jones's second-order condition. In fact, it is exactly the same. But it is necessary to write out some canceling terms explicitly to see that the two expressions correspond exactly. Indeed, in our notation, Jones's second-order condition is defined on the parallel surface Γ_r by

$$\partial_{\mathbf{n}}u - (ik - \mathscr{H}_r)u + \frac{1}{2ik}\left((\mathscr{H}_r^2 - \mathscr{K}_r)u + \operatorname{div}_\Gamma\nabla_\Gamma u\right)$$

$$+ \frac{1}{k^2}\left(\mathscr{H}_r(\mathscr{H}_r^2 - \mathscr{K}_r)u + \frac{1}{4}\mathscr{H}_r\operatorname{div}_\Gamma\nabla_\Gamma u\right) + \frac{1}{4}\operatorname{div}_\Gamma\nabla_\Gamma(\mathscr{H}_r u)$$

$$- \frac{1}{4h_1 h_2}\partial_{s_1}\left((\mathscr{C}_2 - \mathscr{C}_1)\frac{h_2}{h_1}\partial_{s_1}u\right)$$

$$- \frac{1}{4h_1 h_2}\partial_{s_2}\left((\mathscr{C}_1 - \mathscr{C}_2)\frac{h_1}{h_2}\partial_{s_2}u\right) = 0.$$

To find condition (23) on the surface Γ again, it is sufficient to develop the expressions involving derivatives of u and $\mathscr{H}_r$ and to remark that

$$\partial_{s_j}(h_1 h_2^{-1})_{|r=0} = \partial_{s_j}(h_2 h_1^{-1})_{|r=0} = 0, \qquad \text{for } j = 1, 2.$$

4.7. *Second-Order Bayliss−Turkel-like Radiation Conditions*

Now we come to one of the main motivations of this work. As we said before, it consists of the justification of our heuristic derivation of an accurate second-order radiation condition from the Bayliss−Turkel condition in the two-dimensional case [3] and its extension to a surface of

arbitrary shape in dimension 3. The result is embodied in the following theorem.

THEOREM 4.2.   *A complete second-order radiation condition is given by*

$$\partial_{\mathbf{n}} u - iku + \mathscr{H}u + \frac{i}{2k}\left(1 + i\frac{2\mathscr{H}}{k}\right)^{-1}(\mathscr{K} - \mathscr{H}^2)u - \frac{\Delta_\Gamma \mathscr{H}}{4k^2}u$$

$$+ \operatorname{div}_\Gamma\left(\frac{1}{2ik}\left(\mathbb{I} + \frac{i\mathscr{R}}{k}\right)^{-1}\nabla_\Gamma\right)u = 0. \tag{25}$$

When Γ is the sphere $S_r := \{x \in \mathbb{R}^3; |x| = r\}$, this condition coincides with the corresponding Bayliss–Turkel second-order radiation condition.

Proof.   At first, we recall a suitable form for the Bayliss–Turkel second-order radiation condition, which is really due in the present case to Bayliss et al. [5]. In its basic form, this condition is written in spherical coordinates (r, θ, φ) as

$$\left(\partial_r - ik + \frac{3}{r}\right)\left(\partial_r - ik + \frac{1}{r}\right)u = 0 \qquad \text{on } S_r.$$

Using the Helmholtz equation expressed in the coordinates (r, θ, φ) to eliminate ∂_r^2, we can write the above equation in an equivalent form as

$$\partial_r u - iku + \frac{u}{r} + \frac{1}{2ik(1 + i/kr)}\frac{1}{r^2 \sin\theta}\left(\partial_\theta(\sin\theta\,\partial_\theta u) + \partial_\varphi^2 u\right) = 0. \tag{26}$$

Now, observing that

$$\left(1 + \frac{i\mathscr{C}_\ell}{\omega}\right) = \left(1 - \frac{i\mathscr{C}_\ell}{\omega}\right)^{-1} + \text{terms in } \frac{1}{\omega^3},$$

$$\partial_{s_\ell}(i\mathscr{C}_\ell/\omega) = \partial_{s_\ell}\left(1 - \frac{i\mathscr{C}_\ell}{\omega}\right)^{-1} + \text{terms in } \frac{1}{\omega^3},$$

$$\left(1 + i\frac{2\mathscr{H}}{\omega}\right) = \left(1 - i\frac{2\mathscr{H}}{\omega}\right)^{-1} + \text{terms in } \frac{1}{\omega^3},$$

the above expression of $\sum_{j=-1}^{2}(\widetilde{\lambda^{+}_{-j}})_{-2}$ may be rewritten as

$$\sum_{j=-1}^{2}\left(\widetilde{\lambda^{+}_{-j}}\right)_{-2} = \omega - i\mathscr{H} - \frac{1}{2\,\omega}\left(1 - \frac{2\,i\mathscr{H}}{\omega}\right)^{-1}(\mathscr{K} - \mathscr{H}^{2}) + \frac{i}{4\omega^{2}}\,\Delta_{\Gamma}\mathscr{H}$$

$$- \frac{1}{2\,\omega}\sum_{\ell=1}^{2}\left(\frac{\sigma_{\ell}^{2}}{(1 - i\mathscr{C}_{\ell}/\omega)} - \partial_{s_{\ell}}\frac{1}{(1 - i\mathscr{C}_{\ell}/\omega)}(i\sigma_{\ell})\right)$$

$$+ \text{ terms in } \frac{1}{\omega^{3}}.$$

Dropping the terms in $1/\omega^{3}$ and reusing the previous procedure, the above relation establishes that condition (25) is another complete radiation condition of order 2 on a general surface Γ. Since for $\Gamma = S_{r}$ the operator $(\mathbb{I} + i\mathscr{R}/k)^{-1}$ is scalar and is given by $(\mathbb{I} + i\mathscr{R}/k)^{-1} = (1 + i/kr)^{-1}\mathbb{I}$ and $(\mathscr{K} - \mathscr{H}^{2}) = 0$, conditions (25) and (26) are clearly identical. ∎

Remark. Using a suitable notation that permits us to particularize the above formula to the two-dimensional case, we readily obtain the following complete second-order radiation condition:

$$\partial_{\mathbf{n}}u - iku + \frac{\mathscr{C}}{2}u - \frac{i\mathscr{C}^{2}}{8k(1 + i\mathscr{C}/k)}u - \frac{\partial_{s}^{2}\mathscr{C}}{8k^{2}}u$$

$$+ \partial_{s}\left(\frac{1}{2ik(1 + i\mathscr{C}/k)}\partial_{s}\right)u = 0. \qquad (27)$$

In the above formula, Γ is a curve of the plane, $\mathscr{C}$ is its curvature, and s is its curvilinear abscissa increasing in the counterclockwise direction.

The latter formula exactly coincides with the following form of the usual Bayliss−Turkel radiation condition of order 2 in the two-dimensional case:

$$\partial_{r}u - iku + \frac{1}{2r}u + \frac{1}{8ikr(1 + i/kr)} + \frac{1}{2ikr(1 + i/kr)}\frac{1}{r^{2}}\partial_{\theta}^{2}u = 0$$

$$\text{for } |x| = r. \quad (28)$$

Note that condition (25) contains the two special forms of the Bayliss−Turkel radiation conditions (26) and (28); this feature is not obvious from condition (26) only.

In fact, the two-dimensional case has already been considered in a previous work [4]. Radiation conditions were derived there, following the same approach as in this paper. An Engquist−Majda-like condition can be

obtained as an incomplete second-order condition given by

$$\partial_{\mathbf{n}} u - iku + \frac{\mathscr{C}}{2} u - \frac{i\mathscr{C}^2}{8k} u + \partial_s \left(\frac{1}{2ik} \left(1 - \frac{i\mathscr{C}}{k} \right) \partial_s \right) u = 0.$$

Numerical experiments have shown (see [3] and the following numerical examples) that this condition, considered as an on-surface radiation condition, is less accurate than (27). Therefore, from this point of view, the theoretical study of this paper finds one of its main justifications.

Remark. A high degree of smoothness of the surface is needed for the derivation of such conditions by the pseudo-differential analysis. However, some geometrical singularities can be considered as limiting cases of more regular geometries using a suitable scheme [1, 2]. This is, for example, the case for the treatment of the two-dimensional scattering problem by a polygonal convex obstacle in the OSRC context.

To conclude, we design another complete radiation condition of order 2 that is close to a condition recently obtained by Stupfel [22], yet with the advantage of leading to symmetric variational formulations. The result is summarized in the following theorem.

THEOREM 4.3. *A complete second-order radiation condition is given by*

$$\partial_{\mathbf{n}} u - iku + \mathscr{H} u + \frac{i}{2k} \left(1 + i\frac{2\mathscr{H}}{k} \right)^{-1} (\mathscr{K} - \mathscr{H}^2) u - \frac{\Delta_\Gamma \mathscr{H}}{4k^2} u$$

$$- \operatorname{div}_\Gamma \left(\frac{1}{2ik} \left(1 + i\frac{2\mathscr{H}}{k} \right)^{-1} \left(\mathbf{n} \times \left(\mathbb{I} + \frac{i}{k}\mathscr{R} \right) (\mathbf{n} \times \nabla_\Gamma) \right) \right) u = 0. \quad (29)$$

Proof. The only difference the previous radiation condition lies in writing the terms $(1 + i\mathscr{C}_\ell / \omega)$ and $\partial_{s_\ell} i\, \mathscr{C}_\ell / \omega$ in the following form:

$$\left(1 + \frac{i\mathscr{C}_\ell}{\omega} \right) = \left(1 - \frac{2i\mathscr{H}}{\omega} \right)^{-1} \left(1 - \frac{i\mathscr{C}_{\ell+1}}{\omega} \right) + \text{terms in } \frac{1}{\omega^3},$$

$$\partial_{s_\ell} \frac{i\mathscr{C}_\ell}{\omega} = = \partial_{s_\ell} \left(\left(1 - \frac{2i\mathscr{H}}{\omega} \right)^{-1} \left(1 - \frac{i\mathscr{C}_{\ell+1}}{\omega} \right) \right) + \text{terms in } \frac{1}{\omega^3},$$

where $\ell + 1$ stands for 1 when $\ell = 2$. Dropping the terms in $1/\omega^3$, we first obtain the following expression of the radiation condition:

$$\partial_r u - iku + \mathscr{H} u + \frac{i}{2k} \left(1 + i\frac{2\mathscr{H}}{k} \right)^{-1} (\mathscr{K} - \mathscr{H}^2) u - \sum_{\ell=1}^{2} \frac{\partial_{s_\ell}^2 \mathscr{H}}{4k^2} u$$

$$+ \frac{1}{2ik} \sum_{\ell=1}^{2} \partial_{s_\ell} (1 + 2i\mathscr{H}/k)^{-1} (1 + i\mathscr{C}_{\ell+1}/k)\, \partial_{s_\ell} u = 0.$$

Observing now that $\nabla_\Gamma u = \sum_{\ell=1}^{2} \partial_{s_\ell} u \, \boldsymbol{\tau}_\ell$, $(\mathbb{I} + i\mathscr{R}/k)\boldsymbol{\tau}_\ell = (1 + i\mathscr{C}_\ell/k)\boldsymbol{\tau}_\ell$, and $\mathbf{n} \times \boldsymbol{\tau}_\ell = (-1)^{\ell+1}\boldsymbol{\tau}_{\ell+1}$ we readily get (29).

Remark. Following the criterion used in this paper, the radiation conditions (23), (25), and (29) have the same accuracy. Numerical experiments (cf. [1, 3]) have shown that, even with incomplete conditions, all of the radiation conditions of order 2 give the same accuracy as long as the frequency is taken to be sufficiently high and symmetric formulations involving the derivatives of the curvature are used. This does not remain true for moderate or low frequencies. Therefore, other criteria have to be satisfied to distinguish between complete radiation conditions of the same order. This can be accomplished either by numerical experiments or by examining particular examples. In our opinion, a theoretical treatment of this problem is a harder task.

4.8. *Surface Radiation Conditions for an Ellipsoidal Scatterer*

This part is devoted to the explicit expression of the previous surface radiation conditions for the special case of an ellipsoid centered at the origin. We intend to compute the OSRC solution to a scattering problem with a Neumann boundary condition: $\partial_{\mathbf{n}} u = g$ on Γ. To this end, we use the weak OSRC formulation [1, 2]: find $u \in H^1(\Gamma)$ such that

$$\int_\Gamma A_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma v + \beta uv \, d\Gamma = \int_\Gamma gv \, d\Gamma. \tag{30}$$

The test functions v are defined in the $H^1(\Gamma)$ space. The operator A_Γ is a zeroth-order operator of the tangent plane, and β is a complex valued function of the surface Γ. They are both fixed by the chosen surface radiation condition.

The different geometrical parameters involved in such a problem are the following. The chosen coordinate system is

$$\Psi : (\theta, \varphi) \in \,]0; 2\pi[\, \times \,]0; \pi[\, \to (a \cos \theta \sin \varphi, b \sin \varphi \sin \theta, c \cos \varphi) \in \Gamma.$$

We designate by a, b, and c ($\neq 0$), respectively, the semiaxis of the obstacle along the axes (Ox_1), (Ox_2), and (Ox_3) of the Cartesian coordinate system. We obtain the basis vectors $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ of the tangent plane as

$$\begin{aligned}
\boldsymbol{\tau}_1 &= (-a \sin \varphi \sin \theta, b \sin \varphi \cos \theta, 0), \\
\boldsymbol{\tau}_2 &= (a \cos \varphi \cos \theta, b \cos \varphi \sin \theta, -c \sin \varphi).
\end{aligned} \tag{31}$$

If the function α is set as

$$\alpha = \left(a^2 b^2 \cos^2 \varphi + c^2 \sin^2 \varphi (a^2 \sin^2 \theta + b^2 \cos^2 \theta)\right)^{1/2},$$

it follows that the mean and Gauss curvatures are easily computed as

$$\mathscr{H} = \frac{abc}{2\alpha^3} \left[a^2 (\cos^2 \theta \cos^2 \varphi + \sin^2 \theta) \right.$$
$$\left. + b^2 (\sin^2 \theta \cos^2 \varphi + \cos^2 \theta) + c^2 \sin^2 \varphi \right],$$

$$\mathscr{K} = \frac{a^2 b^2 c^2}{\alpha^4}.$$

Let us introduce $\mathscr{G}$ as the metric tensor operating from the cotangent plane onto the tangent plane and defined by

$$\mathscr{G} = (\boldsymbol{\tau}_i \cdot \boldsymbol{\tau}_j) = \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

It can be shown from (31) that the first fundamental form coefficients E, F, and G are given by

$$E = (a^2 \sin^2 \theta + b^2 \cos^2 \theta) \sin^2 \varphi, \quad F = (b^2 - a^2) \cos \varphi \sin \varphi \cos \theta \sin \theta,$$
$$G = (a^2 \cos^2 \theta + b^2 \sin^2 \theta) \cos^2 \varphi + c^2 \sin^2 \varphi.$$

Since the vector bases $(\boldsymbol{\tau}^1, \boldsymbol{\tau}^2)$ of the cotangent plane are related to $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ by

$$(\boldsymbol{\tau}^1, \boldsymbol{\tau}^2) = \mathscr{G}^{-1}(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2),$$

we obtain

$$\boldsymbol{\nabla}_\Gamma u = \partial_\theta u \, \boldsymbol{\tau}^1 + \partial_\varphi u \, \boldsymbol{\tau}^2,$$

with

$$\boldsymbol{\tau}^1 = \frac{1}{\alpha^2 \sin^2 \varphi} (G \boldsymbol{\tau}_1 - F \boldsymbol{\tau}_2) \quad \text{and} \quad \boldsymbol{\tau}^2 = \frac{1}{\alpha^2 \sin^2 \varphi} (-F \boldsymbol{\tau}_1 + E \boldsymbol{\tau}_2).$$

The Laplace—Beltrami operator applied to the mean curvature can be computed using

$$\Delta_\Gamma \mathscr{H} = \frac{\alpha^2}{abc} (\partial_\theta, \partial_\varphi) \left(\frac{abc}{\alpha^2} \mathscr{G} (\partial_\theta \mathscr{H}, \partial_\varphi \mathscr{H}) \right).$$

Finally, the approximation of the solution to (30) is based upon the use of a surface finite-element method [1, 2].

4.9. *Some Numerical Results*

We present some radar cross section (RCS) calculations in the OSRC context. We consider the scattering problem for a Neumann boundary condition. The problem of the approximation is treated in [1, 2]. The reference solution is given by a boundary integral equation method. The first example consists of the two-dimensional scattering problem with an incident plane wave illuminating an elliptical cylinder. The obstacle is characterized by its semiaxis a and b along the axis $(0x_1)$ and $(0x_2)$. As can be seen in Fig. 1, the use of the Bayliss—Turkel-like second-order radiation condition improves the accuracy of the method compared with the Engquist—Majda-like second-order radiation condition. A second example (cf. Fig. 2) is the three-dimensional scattering problem of a plane
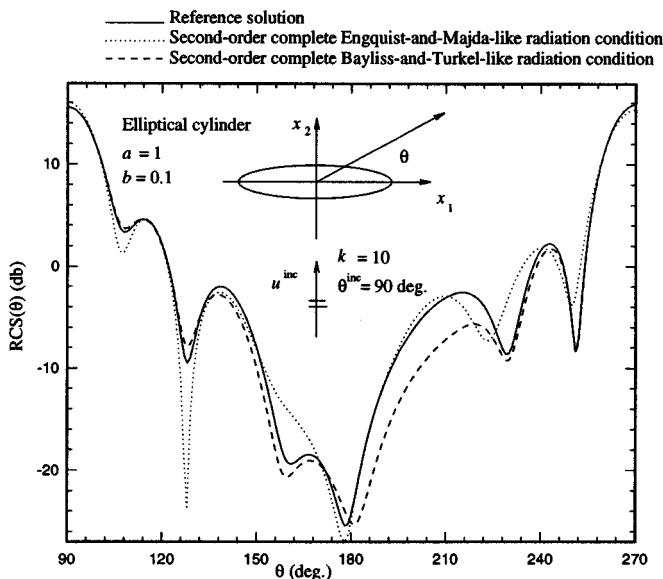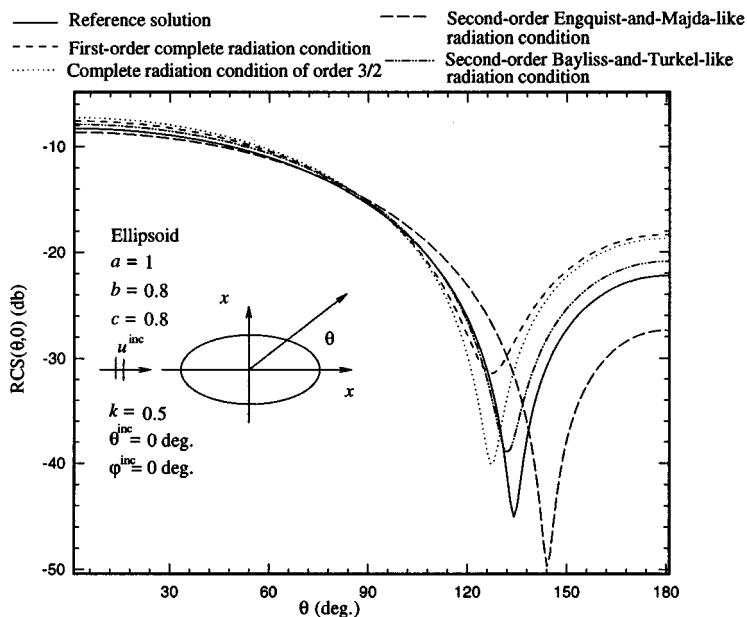

FIG. 1. Bistatic RCS of an elliptical cylinder.

FIG. 2. Bistatic RCS of an ellipsoidal scatterer: complete radiation conditions of order 1, 3/2, and 2.

wave by an ellipsoidal scatterer. If an improvement is obtained by using the complete radiation condition of order $3/2$ (22) compared with the first-order condition (23), we can conclude that the Bayliss–Turkel-like radiation condition (25) is the most accurate second-order condition.

ACKNOWLEDGMENTS

REFERENCES

1. X. Antoine, "Conditions de radiation sur le bord," Ph.D. thesis, UFR Sciences, Université de Pau et des Pays de l'Adour, Pau, France, 1997.

2. X. Antoine, Fast Computation of time-harmonic scattered field using the on-surface radiation condition method, submitted to *J. Comput. Phys.* and available as preprint 98/14, Analyse Numérique des EDP, Laboratoire de Mathématiques Appliquées pour l'Industrie and UPRES-A 5033, Université de Pau et des Pays de l'Adour, France.

3. X. Antoine, H. Barucq, and A. Bendali, Formulations symétriques dans le calcul de diffraction d'ondes par des conditions de radiation sur le bord, preprint 95/11, Analyse Numérique des EDP, Laboratoire de Mathématiques Appliquées pour l'Industrie and UPRES-A 5033, Univerisité de Pau et des Pays de l'Adour, France.

4. X. Antoine, H. Barucq, and A. Bendali, New absorbing boundary conditions including derivatives of the curvature for the two-dimensional Helmholtz equation, preprint 96/07, Analyse Numérique des EDP, Laboratoire de Mathématiques Appliquées pour l'Industrie and UPRES-A 5033, Université de Pau et des Pays de l'Adour, France.

5. A. Bayliss, M. Gunzburger, and E. Turkel, Boundary conditions for the numerical solution of elliptic equations in exterior regions, *SIAM J. Appl. Math.* **42** (1982), 430−451.

6. J. Chazarain and A. Piriou, ''Introduction to the Theory of Linear Partial Differential Equations,'' North-Holland, Amsterdam/New York, 1982.

7. G. Chen and J. Zhou, ''Boundary Element Methods,'' Academic Press, Harcourt Brace Jovanovitch, San Diego, 1992.

8. M. P. Do Carmo, ''Differential Geometry of Curves and Surfaces,'' Prentice Hall International, Englewood Cliffs, NJ, 1976.

9. B. Engquist and A. Majda, Absorbing boundary conditions for the numerical simulation of waves, *Math. Comp.* **31** (1977), 629−651.

10. L. Halpern and L. N. Trefethen, Wide-angle one-way wave equation, numerical analysis report 86-5, Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA, July 1986.

11. B. Hanouzet and M. Sesquès, Absorbing boundary conditions for Maxwell's equations, *in* ''Non-Linear Hyperbolic Problems: Theoretical, Applied and Computational Aspects,'' (A. Donato and F. Olivieri, Eds.), *Notes Numer. Fluid Dynamics* **43** (1992), 315−322.

12. J. Jin, ''The Finite Element Method in Electromagnetics,'' Wiley, New York, 1993.

13. D. S. Jones, Surface radiation conditions, *IMA J. Appl. Math.* **41** (1988), 21−30.

14. D. S. Jones, An approximate boundary condition in acoustics, *J. Sound Vibration* **121**(1) (1988), 37−45.

15. J. J. Kohn and L. Nirenberg, An algebra of pseudodifferential operators, *Comm. Pure Appl. Math.* **18** (1965), 269−305.

16. G. A. Kriegsmann, A. Taflove, and K. R. Umashankar, A new formulation of electromagnetic wave scattering using the on-surface radiation condition approach, *IEEE Trans. Antennas and Propagation* **35** (1987), 153−161.

17. A. Majda and S. Osher, Reflection of singularities at the boundary, *Comm. Pure Appl. Math.* **28** (1975), 479−499.

18. D. B. Meade, A. F. Peterson, and C. Piellusch-Castle, Derivation and comparison of radiation boundary conditions for the two-dimensional Helmholtz equation with non-circular artificial boundaries, *in* ''Proceedings of the Third International Conference on Mathematical and Numerical Aspects of Wave Propagation, Mandelieu-La-Napoule, 1995'' (Gary Cohen, Ed.), 506−514.

19. R. Mittra and O. Ramahi, Absorbing boundary conditions for the direct solution of partial differential equations arising in electromagnetic scattering problems, *in* ''Progress in Electromagnetics Research: Finite Element and Finite Difference Methods in Electromagnetic Scattering'' (M. A. Morgan, Ed.), Elsevier, Newark, NJ, 1990.

20. G. Mur, Absorbing boundary conditions for the finite-difference approximation of time-domain electromagnetic field equations, *IEEE Trans. Electromagnet. Compat.* **23** (1981), 377−382.

21. L. Nirenberg, Pseudodifferential operators and some applications, Lectures on Linear Partial Differential Equations, *CBMS Regional Conf. Ser. in Math.* **17** (1973), 19−58.

22. B. Stupfel, Absorbing boundary conditions on arbitrary boundaries for the scalar and vector wave equations, *IEEE Trans. Antennas and Propagation* **42**(6) (1994), 773−780.

23. M. E. Taylor, ''Pseudodifferential Operators,'' Princeton Univ. Press, Princeton, NJ, 1981.