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Periodic solutions for a kind of Liénard equation with a deviating argument ☆

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Abstract

By means of continuation theorem of coincidence degree theory, we study a kind of Liénard equation with a deviating argument as follows

$$x''(t) + f\big(x(t)\big)x'(t) + g\big(x\big(t - \tau\big(t, x(t)\big)\big)\big) = e(t).$$

Some new results on the existence of periodic solutions are obtained.
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Keywords: Periodic solution; Continuation theorem; Deviating argument

1. Introduction

The problems of periodic solutions for second order ordinary differential equation have been extensively studied [1–6]. In recent years, some results on the existence of periodic solutions of delay differential equation have appeared by applying continuation theorem, see Refs. [7–11]. In [7], Huang and Xiang studied the following type of Duffing equation with a single constant deviating argument

$$x''(t) + g\big(x(t - \tau)\big) = p(t). \tag{1}$$

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Under a one-sided boundedness condition imposed on $g(x)$ and $xg(x) > 0,\ |x| > M$, the authors obtained a periodic solution for Eq. (1). In [9], Ma studied a kind of delay Duffing equation of the type

$$x''(t) + m^2 x(t) + g\big(x(t-\tau)\big) = p(t). \tag{2}$$

He established several criteria to guarantee the existence of periodic solutions of Eq. (2) by assuming $M = \sup_{x\in R} |g(x)| < \infty$. Lu [11] discussed the existence of periodic solutions for a kind of second order differential equation with two deviating arguments

$$x''(t) = f\big(t, x\big(t-\tau_0(t)\big)\big)x'(t) + \beta(t)g\big(x\big(t-\tau(t)\big)\big) + p(t). \tag{3}$$

However, the domain of function $g(x)$ in [7–11] is the space R of real numbers.

The purpose of this article is to investigate the existence of periodic solutions for nonautonomous equation with a deviating argument

$$x''(t) + f\big(x(t)\big)x'(t) + g\big(x\big(t-\tau\big(t,x(t)\big)\big)\big) = e(t), \tag{4}$$

where $f(x), g(x)$ are continuous on a given bounded interval, $e(t)$ is a continuous periodic function with period $T > 0$, $\tau(t,x) \equiv \tau(t+T,x), \forall x \in R$. As far as we know, there are few papers to deal with such a problem. By using continuation theorem of coincidence degree theory and some new analysis techniques, we obtain some new results on the existence of periodic solutions of Eq. (4). The conditions imposed on function $g(x)$ and the methods to estimate a priori bounds of periodic solution are different from those of [7–11]. The domain of definition of function $g(x)$ and $f(x)$ in this paper is only a bounded interval, and our results are related to the deviating argument $\tau(t,x(t))$.

2. Main lemmas

Let $C_T = \{x \mid x \in C(R,R),\ x(t+T) \equiv x(t)\}$ with the norm $|\varphi|_0 = \max_{t\in[0,T]} |\varphi(t)|$, $\forall \varphi \in C_T$, and $C_T^1 = \{x \mid x \in C^1(R,R),\ x(t+T) \equiv x(t)\}$ with the norm $\|\varphi\| = \max\{|\varphi|_0, |\varphi'|_0\}$. Clearly, C_T and C_T^1 are two Banach spaces. We also define the operator L as follows.

$$L : D(L) \subset C_T^1 \to C_T, \quad Lx = x'',$$

where $D(L) = \{x \mid x \in C^2(R,R),\ x(t+T) \equiv x(t)\}$. It is easy to see that $\operatorname{Ker} L = R$ and $\operatorname{Im} L = \{x \mid x \in C_T,\ \int_0^T x(s)\,ds = 0\}$. Thus, L is a Fredholm operator with index zero. Let

$$P : C_T^1 \to \operatorname{Ker} L, \qquad Q : C_T \to C_T/\operatorname{Im} L$$

be defined respectively by

$$Px = x(0), \qquad Qx = \frac{1}{T}\int_0^T x(s)\,ds$$

and let

$$L_P = L|_{C_T \cap \operatorname{Ker} P} : C_T^1 \cap \operatorname{Ker} P \to \operatorname{Im} L.$$

Then L_P has a unique continuous pseudo-inverse L_P^{-1} on $\operatorname{Im} L$ defined by

$$\left(L_P^{-1}y\right)(t) = \int_0^T G(t,s)y(s)\,ds, \tag{5}$$

where

$$G(t,s) = \begin{cases} s(T-t)/T, & 0 \leqslant s < t, \\ t(T-s)/T, & t \leqslant s \leqslant T. \end{cases}$$

Lemma 1. *Let $0 \leqslant \alpha \leqslant T$ be a constant, $s \in C_T$ with $|s|_0 \leqslant \alpha$. Then for $\forall x \in C_T^1$, we have*

$$\int_0^T \left|x(t) - x\left(t - s(t)\right)\right|^2 dt \leqslant 2\alpha^2 \int_0^T \left|x'(t)\right|^2 dt. \tag{6}$$

Proof. Let $\Delta_1 = \{t\colon\ t \in [0,T],\ s(t) \geqslant 0\}$ and $\Delta_2 = \{t\colon\ t \in [0,T],\ s(t) < 0\}$. Then for $\forall t \in \Delta_1$,

$$\begin{aligned} \left|\int_{t-s(t)}^t x'(\sigma)\,d\sigma\right|^2 &\leqslant \left(\int_{t-s(t)}^t \left|x'(\sigma)\right| d\sigma\right)^2 \leqslant \left|s(t)\right| \int_{t-s(t)}^t \left|x'(\sigma)\right|^2 d\sigma \\ &\leqslant \alpha \int_{t-\alpha}^t \left|x'(\sigma)\right|^2 d\sigma \end{aligned} \tag{7}$$

and for $\forall t \in \Delta_2$,

$$\begin{aligned} \left|\int_{t-s(t)}^t x'(\sigma)\,d\sigma\right|^2 &\leqslant \left(\int_t^{t-s(t)} \left|x'(\sigma)\right| d\sigma\right)^2 \leqslant \left|s(t)\right| \int_t^{t-s(t)} \left|x'(\sigma)\right|^2 d\sigma \\ &\leqslant \alpha \int_t^{t+\alpha} \left|x'(\sigma)\right|^2 d\sigma. \end{aligned} \tag{8}$$

From (7) and (8), we find

$$\begin{aligned} \int_0^T \left|x(t) - x\left(t - s(t)\right)\right|^2 dt &\leqslant \int_{\Delta_1} \left|\int_{t-s(t)}^t x'(\sigma)\,d\sigma\right|^2 dt + \int_{\Delta_2} \left|\int_{t-s(t)}^t x'(\sigma)\,d\sigma\right|^2 dt \\ &\leqslant \alpha \int_{\Delta_1} \int_{t-\alpha}^t \left|x'(\sigma)\right|^2 d\sigma\,dt + \alpha \int_{\Delta_2} \int_t^{t+\alpha} \left|x'(\sigma)\right|^2 d\sigma\,dt \\ &\leqslant \alpha \int_0^T \int_{t-\alpha}^{t+\alpha} \left|x'(\sigma)\right|^2 d\sigma\,dt. \end{aligned} \tag{9}$$

Case 1. If $\alpha \in [0, T/2]$, then we have

$$\begin{aligned}\int_0^T \int_{t-\alpha}^{t+\alpha} |x'(\sigma)|^2 \, d\sigma \, dt &= \int_{-\alpha}^{\alpha} \int_0^{\sigma+\alpha} |x'(\sigma)|^2 \, dt \, d\sigma + \int_{\alpha}^{T-\alpha} \int_{\sigma-\alpha}^{\sigma+\alpha} |x'(\sigma)|^2 \, dt \, d\sigma \\ &\quad + \int_{T-\alpha}^{T+\alpha} \int_{\sigma-\alpha}^{T} |x'(\sigma)|^2 \, dt \, d\sigma \\ &= \int_{-\alpha}^{\alpha} |x'(\sigma)|^2 (\sigma+\alpha) \, d\sigma + 2\alpha \int_{\alpha}^{T-\alpha} |x'(\sigma)|^2 \, d\sigma \\ &\quad + \int_{T-\alpha}^{T+\alpha} |x'(\sigma)|^2 (T-\sigma+\alpha) \, d\sigma \\ &= 2\alpha \int_{-\alpha}^{\alpha} |x'(\sigma)|^2 d\sigma + 2\alpha \int_{\alpha}^{T-\alpha} |x'(\sigma)|^2 \, d\sigma \\ &= 2\alpha \int_{-\alpha}^{T-\alpha} |x'(\sigma)|^2 \, d\sigma = 2\alpha \int_0^T |x'(\sigma)|^2 \, d\sigma.\end{aligned}$$

So it follows from (9) that

$$\int_0^T \left|x(t) - x\big(t - s(t)\big)\right|^2 dt \leqslant 2\alpha^2 \int_0^T |x'(\sigma)|^2 \, d\sigma. \tag{10}$$

Case 2. If $\alpha \in (T/2, T]$, then

$$\begin{aligned}\int_0^T \int_{t-\alpha}^{t+\alpha} |x'(\sigma)|^2 \, d\sigma \, dt &= \int_{-\alpha}^{T-\alpha} \int_0^{\sigma+\alpha} |x'(\sigma)|^2 \, dt \, d\sigma + \int_{T-\alpha}^{\alpha} \int_0^{T} |x'(\sigma)|^2 \, dt \, d\sigma \\ &\quad + \int_{\alpha}^{T+\alpha} \int_{\sigma-\alpha}^{T} |x'(\sigma)|^2 \, dt \, d\sigma \\ &= \int_{-\alpha}^{T-\alpha} |x'(\sigma)|^2 (\sigma+\alpha) \, d\sigma + T \int_{T-\alpha}^{\alpha} |x'(\sigma)|^2 \, d\sigma \\ &\quad + \int_{\alpha}^{T+\alpha} |x'(\sigma)|^2 (T-\sigma+\alpha) \, d\sigma.\end{aligned} \tag{11}$$

As

$$\int_{\alpha}^{T+\alpha} \left|x'(\sigma)\right|^2 (T-\sigma+\alpha)\,d\sigma = \int_{\alpha}^{-\alpha} \left|x'(\sigma)\right|^2 (T-\sigma+\alpha)\,d\sigma + \int_{-\alpha}^{T-\alpha} \left|x'(\sigma)\right|^2 (T-\sigma+\alpha)\,d\sigma + \int_{T-\alpha}^{T+\alpha} \left|x'(\sigma)\right|^2 (T-\sigma+\alpha)\,d\sigma,$$

it follows from (11) that

$$\begin{aligned}
\int_0^T \int_{t-\alpha}^{t+\alpha} \left|x'(\sigma)\right|^2 d\sigma\,dt &= (T+2\alpha) \int_{-\alpha}^{T-\alpha} \left|x'(\sigma)\right|^2 d\sigma + T \int_{T-\alpha}^{\alpha} \left|x'(\sigma)\right|^2 d\sigma \\
&\quad + \int_{\alpha}^{-\alpha} \left|x'(\sigma)\right|^2 (T-\sigma+\alpha)\,d\sigma + \int_{T-\alpha}^{T+\alpha} \left|x'(\sigma)\right|^2 (T-\sigma+\alpha)\,d\sigma \\
&= (T+2\alpha) \int_{-\alpha}^{T-\alpha} \left|x'(\sigma)\right|^2 d\sigma + T \int_{T-\alpha}^{\alpha} \left|x'(\sigma)\right|^2 d\sigma \\
&\quad + \int_{\alpha}^{-\alpha} \left|x'(\sigma)\right|^2 (T-\sigma+\alpha)\,d\sigma + \int_{-\alpha}^{\alpha} \left|x'(\sigma)\right|^2 (-\sigma+\alpha)\,d\sigma \\
&= (T+2\alpha) \int_{-\alpha}^{T-\alpha} \left|x'(\sigma)\right|^2 d\sigma - T \int_{\alpha}^{T-\alpha} \left|x'(\sigma)\right|^2 d\sigma - T \int_{-\alpha}^{\alpha} \left|x'(\sigma)\right|^2 d\sigma \\
&= (T+2\alpha) \int_{-\alpha}^{T-\alpha} \left|x'(\sigma)\right|^2 d\sigma - T \int_{-\alpha}^{T-\alpha} \left|x'(\sigma)\right|^2 d\sigma = 2\alpha \int_0^T \left|x'(\sigma)\right|^2 d\sigma.
\end{aligned}$$

Substituting it into (9), we have

$$\int_0^T \left|x(t) - x\big(t-s(t)\big)\right|^2 dt \leqslant 2\alpha^2 \int_0^T \left|x'(\sigma)\right|^2 d\sigma. \tag{12}$$

The conclusion of Lemma 1 immediately follows from (10) and (12). □

Throughout this paper, we denote that $|h|_p := (\int_0^T |h(s)|^p\,ds)^{1/p}$, $p \geqslant 1$, $h \in C_T$, $X = C_T^1$, $Y = C_T$ and $E(t) = \int_0^t e(s)\,ds$. Further, we assume that $\int_0^T e(s)\,ds = 0$ and there is an integer m such that $\Lambda := \{\tau(t,x)\colon\ t \in [0,T],\ x \in R\} \subset [(m-1)T, (m+1)T]$. Let

$$l := \sup_{x\in R} \left|\tau(t,x) - mT\right|_0. \tag{13}$$

Clearly, $l \leqslant T$. So by applying Lemma 1, for $\forall x \in C_T^1$ we have

$$\int_0^T \left|x(t) - x\big(t - \tau\big(t, x(t)\big)\big)\right|^2 dt$$
$$= \int_0^T \left|x(t) - x\big[t - \big(\tau\big(t, x(t)\big) - mT\big)\big]\right|^2 dt = 2l^2 \int_0^T \left|x'(t)\right|^2 dt. \tag{14}$$

Lemma 2 [12]. *Let X and Y be two Banach spaces, $L : D(L) \subset X \to Y$ a Fredholm operator with index zero. $\Omega \subset X$ is an open bounded set, and $N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$. If all the following conditions hold*:

(1) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap D(L)$, $\lambda \in (0, 1)$;
(2) $Nx \notin \operatorname{Im} L$, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$;
(3) $\deg\{QN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$,

then equation $Lx = Nx$ has a solution on $\overline{\Omega} \cap D(L)$.

3. Main results

Theorem 1. *Let $M > 0$, $d \geqslant 0$ be two constants with $M > d + T^{1/2}(|E|_2 + \sqrt{T d g_d})$ and*

$$\sigma = \frac{(M-d)T^{-1/2} - |E|_2 - \sqrt{d g_d T}}{M},$$

where $g_d = \max_{|x| \leqslant d} |g(x)|$.Further, we assume that the following conditions hold.

(1) *$f(x)$, $g(x)$ are continuous on S_M and there exists a constant $L > 0$ such that*

$$\big|g(x_1) - g(x_2)\big| \leqslant L|x_1 - x_2|, \quad \forall x_1, x_2 \in S_M,$$

where $S_M = \{x\colon\ x \in R,\ |x| \leqslant M\}$.
(2) *$xg(x) < 0$, whenever $d < |x| \leqslant M$.*

Then Eq. (4) *has a T-periodic solution for*

$$0 \leqslant l < \frac{\sigma}{\sqrt{2}\, L T^{1/2}}. \tag{15}$$

Proof. Consider the auxiliary equation as follows

$$x''(t) + f_1\big(x(t)\big)x'(t) + g_1\big(x\big(t - \tau\big(t, x(t)\big)\big)\big) = e(t), \tag{16}$$

where

$$f_1(x) = \begin{cases} f(x), & |x| \leqslant M, \\ f(M), & x \in (M, +\infty), \\ f(-M), & x \in (-\infty, -M); \end{cases}$$

$$g_1(x) = \begin{cases} g(x), & |x| \leqslant M, \\ g(M), & x \in (M, +\infty), \\ g(-M), & x \in (-\infty, -M). \end{cases}$$

Obviously, $g_1(x)$ satisfies condition (1), and

$$xg_1(x) < 0, \quad |x| > d. \tag{17}$$

It is easy to see that Eq. (16) is equivalent to following operator equation.

$$Lx = Nx,$$

where

$$Nx = -f_1\big(x(t)\big)x'(t) - g_1\big(x\big(t - \tau\big(t, x(t)\big)\big)\big) + e(t). \tag{18}$$

Let

$$\Omega_1 = \big\{x \mid x \in X \cap D(L),\ Lx = \lambda Nx,\ \lambda \in (0,1)\big\},$$

$\forall x \in \Omega_1$, x must satisfy

$$x''(t) + \lambda f_1\big(x(t)\big)x'(t) + \lambda g_1\big(x\big(t - \tau\big(t, x(t)\big)\big)\big) = \lambda e(t). \tag{19}$$

Integrating two sides of Eq. (19) on the interval $[0, T]$, we have

$$\int_0^T g_1\big(x\big(t - \tau\big(t, x(t)\big)\big)\big)\,dt = 0. \tag{20}$$

By the integral mean value theorem, we know that there is a constant $\xi \in [0, T]$ such that

$$g_1\big(x\big(\xi - \tau\big(\xi, x(\xi)\big)\big)\big) = 0.$$

It follows from (17) that

$$\big|x\big(\xi - \tau\big(\xi, x(\xi)\big)\big)\big| \leqslant d.$$

Let k be a constant and $t_0 \in [0, T]$ such that $\xi - \tau(\xi, x(\xi)) = kT + t_0$. Thus,

$$|x|_0 \leqslant \big|x(t_0)\big| + \int_{t_0}^{T+t_0} \big|x'(t)\big|\,dt \leqslant d + \int_0^T \big|x'(t)\big|\,dt. \tag{21}$$

On the other hand, multiplying two sides of Eq. (19) with $x(t)$ and integrating them over $[0, T]$, together with $E(0) = E(T) = 0$, we find

$$\begin{aligned} \int_0^T \big|x'(t)\big|^2\,dt &= \int_0^T x(t)g_1\big(x\big(t - \tau\big(t, x(t)\big)\big)\big)\,dt - \int_0^T x(t)e(t)\,dt \\ &= \int_0^T x(t)g_1\big(x(t)\big)\,dt + \int_0^T x'(t)E(t)\,dt \\ &\quad + \int_0^T x(t)\big[g_1\big(x\big(t - \tau\big(t, x(t)\big)\big)\big) - g_1\big(x(t)\big)\big]\,dt. \end{aligned} \tag{22}$$

Let $E_1 = \{t\colon\ t \in [0,T],\ |x(t)| \leqslant d\}$, $E_2 = \{t\colon\ t \in [0,T],\ |x(t)| > d\}$, then from (17) we have

$$\begin{aligned}\int_0^T x(t)g_1\big(x(t)\big)\,dt &= \int_{E_1} x(t)g_1\big(x(t)\big)\,dt + \int_{E_2} x(t)g_1\big(x(t)\big)\,dt \\ &\leqslant Td\max_{|x|\leqslant d}\big|g_1(x)\big| = dg_dT. \end{aligned} \tag{23}$$

By applying Lemma 1 and (14), we obtain

$$\begin{aligned}&\int_0^T x(t)\big[g_1\big(x\big(t-\tau\big(t,x(t)\big)\big)\big) - g_1\big(x(t)\big)\big]\,dt \\ &\quad\leqslant |x|_0 L\int_0^T \big|x(t) - x\big(t-\tau\big(t,x(t)\big)\big)\big|\,dt \\ &\quad\leqslant |x|_0 LT^{1/2}\left(\int_0^T \big|x(t) - x\big(t-\tau\big(t,x(t)\big)\big)\big|^2\,dt\right)^{1/2} \\ &\quad= |x|_0 LT^{1/2}\left(\int_0^T \big|x(t) - x\big(t-\tau\big(t,x(t)\big) - mT\big)\big|^2\,dt\right)^{1/2} \\ &\quad\leqslant \left(d + \int_0^T \big|x'(t)\big|\,dt\right)\sqrt{2}\,LT^{1/2}l\left(\int_0^T \big|x'(t)\big|^2\,dt\right)^{1/2} \\ &\quad= \sqrt{2}\,dLT^{1/2}l\left(\int_0^T \big|x'(t)\big|^2\,dt\right)^{1/2} + \sqrt{2}\,LTl\int_0^T \big|x'(t)\big|^2\,dt, \end{aligned} \tag{24}$$

and Hölder inequality yields

$$\int_0^T x'(t)E(t)\,dt \leqslant |E|_2|x'|_2. \tag{25}$$

Substituting (22)–(25) into (21), we have

$$|x'|_2^2 \leqslant Tdg_d + \big[\sqrt{2}\,dLT^{1/2}l + |E|_2\big]|x'|_2 + \sqrt{2}\,LTl|x'|_2^2.$$

As $\sigma < T^{-1/2}$, it follows from (15) that $\sqrt{2}\,LTl < 1$. Hence,

$$|x'|_2 < \frac{\sqrt{2}\,dLT^{1/2}l + |E|_2 + \sqrt{(\sqrt{2}\,dLT^{1/2}l + |E|_2)^2 + 4Tdg_d}}{2(1-\sqrt{2}\,LTl)}.$$

According to the definition of σ, δ, we find

$$
\begin{aligned}
|x|_0 &\leqslant d + |x'|_1 \leqslant d + T^{1/2}|x'|_2 \\
&\leqslant d + \frac{d\sigma + |E|_2 + \sqrt{(d\sigma + |E|_2)^2 + 4dg_dT}}{2(1 - T^{1/2}\sigma)} T^{1/2} \\
&\leqslant d + \frac{d\sigma + |E|_2 + \sqrt{dg_dT}}{1 - T^{1/2}\sigma} T^{1/2} = M. \qquad (26)
\end{aligned}
$$

Again by (19), we have

$$
\begin{aligned}
|x''|_1 &\leqslant f_{1M}|x'|_1 + g_{1M}T + |e|_1 \leqslant f_{1M}T^{1/2}|x'|_2 + g_{1M}T + |e|_1 \\
&\leqslant f_M T^{1/2} \frac{\sqrt{2}\,dLT^{1/2}l + |E|_2 + \sqrt{(dLT^{1/2}l + |E|_2)^2 + 4Tdg_d}}{2(1 - \sqrt{2}\,LTl)} \\
&\quad + g_M T + |e|_1 := M_1, \qquad (27)
\end{aligned}
$$

where f_{1M}, g_{1M} and f_M are defined similarly as g_d. Noting that $x(0) = x(T)$, we find that there is a $\eta \in [0, T]$ such that $x'(\eta) = 0$. So

$$|x'|_0 \leqslant \big|x'(\eta)\big| + |x''|_1 \leqslant M_1.$$

If we set $\Omega = \{x\colon\ x \in X,\ |x|_0 < M,\ |x'|_0 < M_1\}$, then $\Omega_1 \subset \Omega$. Let $\Omega_2 = \{x \mid x \in \ker L,\ Nx \in \operatorname{Im} L\}$. Clearly $\forall x \in \Omega_2$, $x(t) \equiv C$ is a constant function and $g_1(C) = 0$, i.e., $|C| \leqslant d$.

So $\Omega_2 \subset \Omega$. Take

$$H(x, \mu)\colon \Omega \times [0, T] \to X\colon\ H(x, \mu) = \mu x - \frac{1 - \mu}{T} \int_0^T g_1\big(x\big(t - \tau\big(t, x(t)\big)\big)\big)\, dt,$$

it is easy to see that

$$H(x, \mu) \neq 0, \quad \forall (x, \mu) \in (\partial\Omega \cap \ker L) \times [0, 1].$$

Hence

$$
\begin{aligned}
\deg\{QN, \Omega \cap \ker L, 0\} &= \deg\big\{H(x, 0), \Omega \cap \ker L, 0\big\} \\
&= \deg\big\{H(x, 1), \Omega \cap \ker L, 0\big\} \neq 0.
\end{aligned}
$$

From (5) and (18), we can verify that N is L-compact on $\overline{\Omega}$. By applying Lemma 2, Eq. (16) has a T-periodic solution $x = x^*(t) \in \overline{\Omega}$. Taking into account the definition of $g_1(x)$ and $|x^*|_0 \leqslant M$, we can easily find $x = x^*(t)$ is a T-periodic solution of Eq. (4). □

Corollary 1. *Suppose that the following conditions hold.*

(1′) $f(x) \in C(R, R)$, $g(x) \in C(R, R)$ *and there is a constant* $L > 0$ *such that*

$$\big|g(x_1) - g(x_2)\big| \leqslant L|x_1 - x_2|, \quad \forall x_1, x_2 \in R.$$

(2′) *There is a constant* $d > 0$ *such that*

$$xg(x) < 0, \quad x \in R \text{ with } |x| > d.$$

Then Eq. (4) *has a* T*-periodic solution for* $l < 1/(\sqrt{2}\,LT)$.

Proof. Let

$$\varepsilon = \left(\frac{1}{\sqrt{2}LT} - l\right)\sqrt{2}LT^{1/2} \quad \text{and} \quad \sigma(y) = \frac{(y-d)T^{-1/2} - |E|_2 - \sqrt{dg_dT}}{y}.$$

As $\lim_{y\to+\infty}\sigma(y) = T^{-1/2}$, it follows that there is a constant $M > 0$ such that

$$\begin{aligned}\sigma(M) &= \frac{(M-d)T^{-1/2} - |E|_2 - \sqrt{dg_dT}}{M} > T^{-1/2} - \varepsilon \\ &= T^{-1/2} - \left(\frac{1}{\sqrt{2}LT} - l\right)\sqrt{2}LT^{1/2} = \sqrt{2}LlT^{1/2},\end{aligned}$$

which yields that

$$l < \frac{\sigma(M)}{\sqrt{2}LT^{1/2}}.$$

Meanwhile, it is easy to see that if the condition (1′) and (2′) hold, then the condition (1) and (2) of Theorem 1 is also satisfied. Thus, the conclusion of Corollary 1 immediately follows from Theorem 1. □

Theorem 2. *Assume there are three constants $a > 0, M > 0$ and $d > 0$ with $M > d + |e|_2/aT^{1/2}$. Let $\delta = a - |e|_2/(M-d)T^{1/2}$. Further, assume that the following conditions hold.*

(1) *$f(x), g(x)$ are continuous on S_M, $|f(x)| \geqslant a$, $\forall x \in S_M$ and there is a constant $L > 0$ such that*

$$\big|g(x_1) - g(x_2)\big| \leqslant L|x_1 - x_2|, \quad \forall x_1, x_2 \in S_M,$$

where $S_M = \{x\colon x \in R, |x| \leqslant M\}$.

(2) *$xg(x) < 0$ or $xg(x) > 0$, whenever $d < |x| \leqslant M$.*

Then Eq. (4) *has a T-periodic solution for $l < \delta/(\sqrt{2}L)$.*

Proof. Considering the following equation

$$x''(t) + \lambda f_1\big(x(t)\big)x'(t) + \lambda g_1\big(x\big(t - \tau\big(t, x(t)\big)\big)\big) = \lambda e(t) \tag{28}$$

where $f_1(x)$ and $g_1(x)$ are defined in the same way as in the proof of Theorem 1. Let $x(t)$ be an arbitrary T-periodic solution of Eq. (28), then from the proof of Theorem 1, we know that

$$|x|_0 \leqslant d + \int_0^T \big|x'(t)\big|\,dt. \tag{29}$$

Multiplying the two sides of Eq. (28) with $x'(t)$ and integrating them over $[0, T]$, we have

$$\int_0^T f_1\big(x(t)\big)\big|x'(t)\big|^2\,dt + \int_0^T g_1\big(x\big(t - \tau\big(t, x(t)\big)\big)\big)x'(t)\,dt = \int_0^T e(t)x'(t)\,dt. \tag{30}$$

From the condition of $|f(x)| > a,\ x \in S_M$, it is easy to see that $|f_1(x)| > a, \forall x \in R$. Thus, by (30)

$$\begin{aligned} a\int_0^T |x'(t)|^2\,dt &\leqslant \left|\int_0^T g_1\big(x\big(t-\tau\big(t,x(t)\big)\big)\big)x'(t)\,dt\right| + \left|\int_0^T e(t)x'(t)\,dt\right| \\ &\leqslant \int_0^T \big|g_1\big(x(t)\big) - g_1\big(x\big(t-\tau\big(t,x(t)\big)\big)\big)\big|\big|x'(t)\big|\,dt \\ &\quad + \left|\int_0^T g_1\big(x(t)\big)x'(t)\,dt\right| + \int_0^T \big|e(t)\big|\big|x'(t)\big|\,dt \\ &\leqslant \int_0^T \big|g_1\big(x(t)\big) - g_1\big(x\big(t-\tau\big(t,x(t)\big)\big)\big)\big|\big|x'(t)\big|\,dt \\ &\quad + \int_0^T \big|e(t)\big|\big|x'(t)\big|dt. \end{aligned} \tag{31}$$

Using Lemma 1 and (14), we find

$$\begin{aligned} &\int_0^T \big|g_1\big(x(t)\big) - g_1\big(x\big(t-\tau\big(t,x(t)\big)\big)\big)\big|\big|x'(t)\big|\,dt \\ &\quad\leqslant \left(\int_0^T \big|g_1\big(x(t)\big) - g_1\big(x\big(t-\tau\big(t,x(t)\big)\big)\big)\big|^2\,dt\right)^{1/2}\left(\int_0^T |x'(t)|^2\,dt\right)^{1/2} \\ &\quad\leqslant L\left(\int_0^T \big|x(t) - x\big(t-\tau\big(t,x(t)\big)\big)\big|^2\,dt\right)^{1/2}\left(\int_0^T |x'(t)|^2\,dt\right)^{1/2} \\ &\quad\leqslant \sqrt{2}lL\int_0^T |x'(t)|^2\,dt. \end{aligned} \tag{32}$$

Substituting (32) into (31), we have

$$\begin{aligned} a\int_0^T |x'(t)|^2\,dt &\leqslant \sqrt{2}lL\int_0^T |x'(t)|^2\,dt + \int_0^T \big|e(t)\big|\big|x'(t)\big|\,dt \\ &\leqslant \sqrt{2}lL\int_0^T |x'(t)|^2\,dt + |e|_2|x'|_2. \end{aligned}$$

That is

$$|x'|_2 \leqslant \frac{|e|_2}{a-\sqrt{2}lL}.$$

Thus

$$\begin{aligned}|x|_0 &\leqslant d + |x'|_1 \leqslant d + T^{1/2}|x'|_2 \\ &\leqslant d + \frac{|e|_2}{a-\sqrt{2}lL}T^{1/2} \leqslant d + \frac{|e|_2}{a-\delta}T^{1/2} = M.\end{aligned}$$

The remainder of the proof is the same as in the proof of Theorem 1. □

Corollary 2. *Suppose that the following conditions hold.*

(1′) $f(x) \in C(R, R)$, $g(x) \in C(R, R)$ *and there are two constants* $L > 0$ *and* $a > 0$ *such that* $|f(x)| \geqslant a$, $\forall x \in R$ *and*

$$\big|g(x_1) - g(x_2)\big| \leqslant L|x_1 - x_2|, \quad \forall x_1, x_2 \in R.$$

(2′) *There is a constant* $d > 0$ *such that*

$$xg(x) < 0, \quad x \in R \text{ with } |x| > d.$$

Then Eq. (4) *has a* T*-periodic solution for* $l < a/(\sqrt{2}L)$.

As applications, we list the following examples

Example 1. Consider

$$x''(t) + f\big(x(t)\big)x'(t) + g\left(x\left(t - \frac{7/10\sin^3 2\pi t}{1+x^2(t)}\right)\right) = \frac{\pi}{10\sqrt{2}}\cos 2\pi t, \tag{33}$$

where $f \in C([-1/2, 1/2], R)$, $g(x) = -x^3$, $x \in [-1/2, 1/2]$.

Clearly,

$$l = \sup_{x\in R}\big|\tau(t,x)\big|_0 = \sup_{x\in R}\left|\frac{7/10\sin^3 2\pi t}{1+x^2}\right|_0 \leqslant 7/10.$$

$|E|_2 = 1/40$, $T = 1$. From the definition of $g(x)$, we can take $d = 0$, $M = 1/2$. So

$$\sigma = \frac{(M-d)T^{-1/2} - |E|_2 - \sqrt{dg_dT}}{M} = \frac{19}{20}, \qquad L = \max_{|x|\leqslant 1/2}\big|g'(x)\big| = 3/4$$

and

$$\frac{\sigma}{\sqrt{2}LT^{1/2}} = \frac{4\sigma}{3\sqrt{2}} = \frac{19}{30}\sqrt{2}.$$

It follows that

$$l < \frac{\sigma}{\sqrt{2}LT^{1/2}}.$$

Using Theorem 1, we know that Eq. (33) has a 1-periodic solution.

Example 2. Consider the following equation.

$$x''(t) + f\big(x(t)\big)x'(t) + g\left(x\left(t - \frac{(\sqrt{6}-2)\pi}{1+x^2(t)}\sin t\right)\right) = \cos t, \tag{34}$$

where $f(x) = 3\sqrt{3}\pi/(1+x)$, $x \in [-1/2, 1/2]$, $g(x) = x^3$, $x \in [-1/2, 1/2]$.

As

$$\min_{x\in[-1/2,1/2]}\big|f(x)\big| = \min_{x\in[-1/2,1/2]} f(x) = 2\sqrt{3}\pi, \qquad |e|_2 = \sqrt{\pi}, \qquad d = 0,$$

$$\frac{|e|_2}{2\sqrt{3}\pi}T^{1/2} = \frac{|e|_2}{2\sqrt{3}}(2\pi)^{1/2} = \frac{\sqrt{2}\pi}{2\sqrt{3}\pi} < 1/2.$$

Thus, M can be chosen as $1/2$, $a = 2\sqrt{3}\pi$. So

$$\delta = a - \frac{|e|_2}{M-d}(2\pi)^{1/2} = 2(\sqrt{3}-\sqrt{2})\pi, \qquad L = 3/4,$$

and

$$\frac{\delta}{\sqrt{2}L} = \frac{4(\sqrt{6}-2)}{3}\pi > \big(\sqrt{6}-2\big)\pi = \max_{x\in R, t\in[0,T]}\left|\frac{(\sqrt{6}-2)\pi\sin t}{1+x^2(t)}\right|.$$

By using Theorem 2, we obtain that Eq. (34) has a 2π-periodic solution.

Remark. It is easy to see from Eqs. (33) and (34) that the functions $g(x)$ and $f(x)$ are only defined on bounded interval $[-1/2, 1/2]$ and the results are related to the deviating argument, which is different from those of papers [7–11].

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