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Discrete approximations to optimization of neutral functional differential inclusions

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Abstract

The paper deals with the convergence of discrete approximations to optimization problems governed by neutral functional differential inclusions. The discrete approximations through Euler finite difference are constructed and the convergence of discrete approximations is proved.

Keywords: Discrete approximations; Neutral functional differential inclusions; Euler finite difference; Convergence

1. Introduction

Let $F:\mathbb{R}^{2n}\times[a,b]\rightrightarrows\mathbb{R}^n$ be a set-valued mapping, and let $f:\mathbb{R}^{4n}\times[a,b]\to\mathbb{R}^n$ be a nonlinear function. Consider the following optimization problem:

$$\text{minimize} \quad J[x]=\varphi\big(x(a),x(b)\big)+\int_a^b f\big(x(t),x(t-\tau),\dot{x}(t),\dot{x}(t-\tau),t\big)\,dt \tag{1.1}$$

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over a set of functions $x : [a-\tau, b] \to \mathbb{R}^n$, which are absolutely continuous on $[a-\tau, a]$ and $[a, b]$ with a possible jump at $t = a$ and satisfy the following neutral functional differential inclusion:

$$\begin{cases} \dot{x}(t) - A(t)\dot{x}(t-\tau) \in F(x(t), x(t-\tau), t), & \text{a.e. } t \in [a, b], \\ x(t) = c(t), & t \in [a-\tau, a), \end{cases} \tag{1.2}$$

with the endpoint constraint

$$\big(x(a), x(b)\big) \in \Omega \subset \mathbb{R}^{2n}, \tag{1.3}$$

where Ω is a closed set, $\tau > 0$ is a delay, and $A(t)$ is a $n \times n$ continuous matrix on $[a, b]$. The above optimization problem is labelled (P). To solve and analyze (P) numerically, discrete approximation through Euler finite difference turns out to be a useful tool. Such an approach to optimization problems goes back to Euler, who used it to prove the classical Euler–Lagrange equation in the calculus of variations. In this paper we construct discrete approximations (P_N) to (P) and prove the convergence of discrete approximations in the sense described in the following.

In [6], Mordukhovich and Wang discussed the convergence of discrete approximations and optimality conditions of (1.1) with the special case $f = f(x(t), x(t-\tau), t)$ subject to the following neutral functional differential inclusion in the so-called Hale form:

$$\begin{cases} d/dt[x(t) - Ax(t-\tau)] \in F(x(t), x(t-\tau), t), & \text{a.e. } t \in [a, b], \\ x(t) = c(t), & t \in [a-\tau, a), \end{cases}$$

where A is a constant matrix. Note that in this case the trajectories $x(t)$ are not necessarily absolutely continuous on $[a, b]$, such are combinations $x(t) - Ax(t-\tau)$.

The general framework of this paper is based on Mordukhovich [4] for ordinary differential inclusions, Mordukhovich and Wang [5] for delayed differential inclusions, and Mordukhovich and Wang [6] for neutral functional differential inclusions in the Hale form.

Let $\bar{x}(t)$ be a trajectory of (1.2). We assume that $F(x, y, t)$ is locally bounded, locally Lipschitzian continuous around $\bar{x}(t)$, and Hausdorff continuous a.e. on $[a, b]$. More precisely, the following assumptions are imposed throughout the paper:

(A1) There are an open set $U \subset \mathbb{R}^n$ and two positive numbers ℓ_F, m_F such that $\bar{x}(t) \in U$ for any $t \in [a-\tau, b]$. Also, the sets $F(x, y, t)$ are closed and

$$F(x, y, t) \subset m_F \mathbb{B},$$

$$F(x_1, y_1, t) \subset F(x_2, y_2, t) + \ell_F\big(|x_1 - x_2| + |y_1 - y_2|\big)\mathbb{B}$$

for all (x, y), (x_1, y_1), $(x_2, y_2) \in U \times U$ and $t \in [a, b]$.

(A2) There is a null set $\mathcal{A}$ such that $F(x, y, \cdot)$ is Hausdorff continuous on $[a, b] \setminus \mathcal{A}$ uniformly in $U \times U$.

(A3) $c(\cdot)$ is continuous differentiable on $[a-\tau, a]$.

Consider the averaged modulus of continuity for the set-valued mapping $F(x, y, t)$ defined by

$$\tau(F; h) := \int_a^b \sigma(F; t, h)\, dt,$$

where

$$\sigma(F; t, h) := \sup\{\omega(F; x, y, t, h) \mid (x, y) \in U \times U\},$$

with

$$\omega(F; x, y, t, h) \\ := \sup\{\mathrm{haus}(F(x, y, t'), F(x, y, t'')) \mid (t', t'') \in [t - h/2, t + h/2] \cap [a, b]\},$$

where $\mathrm{haus}(F(x, y, t'), F(x, y, t''))$ is the Hausdorff distance between the sets $F(x, y, t')$ and $F(x, y, t'')$. It is proved in [2] that $\tau(F; h) \to 0$ as $h \to 0$ under assumption (A2).

Let us construct discrete approximations to differential inclusion (1.2) (not (P)). Replacing the derivatives in (1.2) by the Euler finite differences

$$\dot{x}(t) \approx [x(t + h) - x(t)]/h, \qquad \dot{x}(t - \tau) \approx [x(t + h - \tau) - x(t - \tau)]/h$$

implies the following discrete approximations to (1.2):

$$\begin{cases} x_N(t_{j+1}) - A(t_{j+1})x_N(t_{j+1} - \tau) \\ \quad \in x_N(t_j) - A(t_j)x_N(t_j - \tau) + h_N F(x_N(t_j), x_N(t_j - \tau), t_j) \quad \text{for } j = 0, \ldots, k, \\ x_N(t_j) = c(t_j) \quad \text{for } j = -N, \ldots, -1. \end{cases} \tag{1.4}$$

Here, $N \in \mathbb{N}$, the set of all natural numbers, $h_N := \tau/N$, $t_j := a + jh_N$ for $j = -N, \ldots, k$, where k is a natural number determined by $a + kh_N \leqslant b < a + (k + 1)h_N$, and $t_{k+1} := b$.

A collection of vectors $\{x_N(t_j) \mid j = -N, \ldots, k + 1\}$ satisfying (1.4) is called a discrete trajectory, and the collection $\{[x_N(t_{j+1}) - x_N(t_j)]/h_N \mid j = -N, \ldots, k\}$ is called a discrete velocity.

The extended discrete velocities on $[a - \tau, b]$, denoted by $\dot{x}_N(t)$, are defined as piecewise constant extensions of discrete velocities on $[a - \tau, a)$ and $[a, b]$ respectively by

$$\dot{x}_N(t) := [x_N(t_{j+1}) - x_N(t_j)]/h_N, \quad t \in [t_j, t_{j+1}),\ j = -N, \ldots, k.$$

The extended discrete trajectories on $[a - \tau, b]$, denoted by $x_N(t)$, are defined as piecewise linear extensions of discrete trajectories on $[a - \tau, b]$. Clearly,

$$x_N(t) = \bar{x}(a) + \int_a^t \dot{x}_N(s)\, ds, \quad t \in [a, b].$$

2. Convergence of discrete approximations

In this section, we consider the convergence of discrete approximations constructed in Section 1. We prove that the extended discrete trajectories converge to $\bar{x}(t)$ uniformly on $[a - \tau, b]$, and the extended discrete velocities converge to $\dot{\bar{x}}(t)$ in L^2-norm on $[a - \tau, b]$.

Theorem 2.1. *Assume that $\bar{x}(t)$ is a trajectory of* (1.2) *under hypotheses* (A1)–(A3). *Then there is a sequence of solutions to* (1.4), $z_N(t_j)$, $j = -N, \dots, k+1$, *such that* $z_N(t_0) = \bar{x}(a)$, *the extended discrete trajectories* $z_N(t)$ *converge to* $\bar{x}(t)$ *uniformly on* $[a-\tau, b]$, *and the extended discrete velocities* $\dot{z}_N(t)$ *converge to* $\dot{\bar{x}}(t)$ *in* L^2*-norm on* $[a-\tau, b]$ *as* $N \to \infty$.

Proof. Let $\wp_N(t)$ be an arbitrary sequence of functions on $[a,b]$ such that $\wp_N(t)$ is constant on $[t_j, t_{j+1})$ for every $j = 0, \dots, k$, and $\wp_N(t)$ converges to $\dot{\bar{x}}(t) - A(t)\dot{\bar{x}}(t-\tau)$ as $N \to \infty$ in the norm of $L^1[a,b]$; such a sequence exists because of the density of step-functions in $L^1[a,b]$. It follows from (A1) that for $t \in [a,b]$,

$$\left|\wp_N(t)\right| \leqslant \left|\dot{\bar{x}}(t) - A(t)\dot{\bar{x}}(t-\tau) - \wp_N(t)\right| + \left|\dot{\bar{x}}(t) - A(t)\dot{\bar{x}}(t-\tau)\right| \leqslant 1 + m_F.$$

Let

$$\xi_N := \int_a^b \left|\dot{\bar{x}}(t) - A(t)\dot{\bar{x}}(t-\tau) - \wp_N(t)\right| dt.$$

Then $\xi_N \to 0$ follows from the construction of $\wp_N(t)$. Define discrete functions $u_N(t_j)$, $j = -N, \dots, k+1$, as follows:

$$\begin{cases} u_N(t_{j+1}) - A(t_{j+1})u_N(t_{j+1} - \tau) & \\ \quad := u_N(t_j) - A(t_j)u_N(t_j - \tau) + h_N \wp_N(t_j), & j = 0, \dots, k, \\ u_N(t_j) := \bar{x}(t_j), & j = -N, \dots, 0. \end{cases}$$

The extended discrete functions are

$$\begin{cases} u_N(t) - A(t)u_N(t-\tau) & \\ \quad = \bar{x}(a) - A(a)\bar{x}(a-\tau) + \int_a^t \wp_N(s)\,ds, & t \in [a,b], \\ u_N(t) = \bar{x}(t_j) + \frac{\bar{x}(t_{j+1}) - \bar{x}(t_j)}{h_N}(t - t_j), & t \in [t_j, t_{j+1}), \\ & j = -N, \dots, -2, \\ u_N(t) = \bar{x}(t_{-1}) + \frac{\bar{x}(t_0-) - \bar{x}(t_{-1})}{h_N}(t - t_{-1}), & t \in [t_{-1}, t_0). \end{cases} \tag{2.1}$$

Let $\mathcal{E}_N(t) := u_N(t) - \bar{x}(t)$. We are to show that $|\mathcal{E}_N(t)| \to 0$ uniformly on $[a-\tau, b]$ as $N \to \infty$. To this end, define $y_N(t) := |\mathcal{E}_N(t) - A(t)\mathcal{E}_N(t-\tau)|$. For any $t \in [a,b]$,

$$\begin{aligned} y_N(t) &= \left|u_N(t) - A(t)u_N(t-\tau) - \left[\bar{x}(t) - A(t)\bar{x}(t-\tau)\right]\right| \\ &\leqslant \int_a^t \left|\dot{\bar{x}}(s) - A(t)\dot{\bar{x}}(s-\tau) - \wp_N(s)\right| ds \leqslant \xi_N, \end{aligned}$$

then

$$\begin{aligned} \left|\mathcal{E}_N(t)\right| &\leqslant y_N(t) + M\left|\mathcal{E}_N(t-\tau)\right| \leqslant y_N(t) + My_N(t-\tau) + M^2\left|\mathcal{E}_N(t-2\tau)\right| \\ &\leqslant \cdots \leqslant y_N(t) + My_N(t-\tau) + M^2\left|y_N(t-2\tau)\right| \\ &\quad + \cdots + M^{m+1}\left|\mathcal{E}_N\big(t - (m+1)\tau\big)\right|, \end{aligned}$$

where $M := \max_{t \in [a,b]} |A(t)|$.

Choose a sequence $\beta_N \to 0$ as $N \to \infty$. (A3) implies the uniform continuity of $c(\cdot)$ and $\dot{c}(\cdot)$ on $[a-\tau, a]$, then $|c(t') - c(t'')| \leqslant \beta_N$ and $|\dot{c}(t') - \dot{c}(t'')| \leqslant \beta_N$ for any sufficiently large N and any $t', t'' \in [t_j, t_{j+1}]$ for $j = -N, \ldots, -1$. Similarly, the uniform continuity of $A(t)$ on $[a, b]$ implies that

$$\left|A(t') - A(t'')\right| \leqslant \beta_N$$

for any sufficiently large N and any $t', t'' \in [t_j, t_{j+1}]$ for $j = 0, \ldots, k$.

Let m be an integer with $a - \tau \leqslant b - (m+1)\tau < a$. Then $t - (m+1)\tau \in [t_j, t_{j+1}]$ for some $j \in \{-N, \ldots, -1\}$, and

$$\begin{aligned}\left|\mathcal{E}_N\big(t-(m+1)\tau\big)\right| &\leqslant \left|c(t_j) + \frac{c(t_{j+1}) - c(t_j)}{h_N}(t - t_j) - c\big(t-(m+1)\tau\big)\right| \\ &\leqslant \left|c(t_j) - c\big(t-(m+1)\tau\big)\right| + \left|c(t_{j+1}) - c(t_j)\right| \leqslant 2\beta_N.\end{aligned}$$

Since m does not depend on N, one has that

$$\left|\mathcal{E}_N(t)\right| \leqslant \xi_N\big(1 + M + \cdots + M^m\big) + 2M^{m+1}\beta_N \to 0 \quad \text{as } N \to \infty. \tag{2.2}$$

Define

$$\zeta_N := h_N \sum_{j=0}^{k} \operatorname{dist}\big(\wp_N(t_j); F\big(u_N(t_j), u_N(t_j - \tau), t_j\big)\big).$$

We have

$$\begin{aligned}\zeta_N &= \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \operatorname{dist}\big(\wp_N(t_j); F\big(u_N(t_j), u_N(t_j-\tau), t_j\big)\big)\,dt \\ &= \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \operatorname{dist}\big(\wp_N(t_j); F\big(u_N(t_j), u_N(t_j-\tau), t\big)\big)\,dt \\ &\quad + \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \big[\operatorname{dist}\big(\wp_N(t_j); F\big(u_N(t_j), u_N(t_j-\tau), t_j\big)\big)\,dt \\ &\quad - \operatorname{dist}\big(\wp_N(t_j); F\big(u_N(t_j), u_N(t_j-\tau), t\big)\big)\big]\,dt \\ &\leqslant \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \operatorname{dist}\big(\wp_N(t_j); F\big(u_N(t_j), u_N(t_j-\tau), t\big)\big)\,dt + \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \sigma(F; t, h_N)\,dt \\ &\leqslant \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \operatorname{dist}\big(\wp_N(t_j); F\big(u_N(t_j), u_N(t_j-\tau), t\big)\big)\,dt + \tau(F; h_N).\end{aligned}$$

(A1) implies that for any $t \in [t_j, t_{j+1})$ $(j = 0, \ldots, k)$,

$$\begin{aligned}&\operatorname{dist}\bigl(\wp_N(t_j); F\bigl(u_N(t_j), u_N(t_j-\tau), t\bigr)\bigr) - \operatorname{dist}\bigl(\wp_N(t_j); F\bigl(u_N(t), u_N(t-\tau), t\bigr)\bigr)\\&\quad\leqslant \operatorname{dist}\bigl(F\bigl(u_N(t_j), u_N(t_j-\tau), t\bigr), F\bigl(u_N(t), u_N(t-\tau), t\bigr)\bigr)\\&\quad\leqslant \ell_F\bigl(\bigl|u_N(t_j)-u_N(t)\bigr| + \bigl|u_N(t_j-\tau)-u_N(t-\tau)\bigr|\bigr)\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}&\bigl|\bigl[u_N(t_j)-A(t_j)u_N(t_j-\tau)\bigr]-\bigl[u_N(t)-A(t)u_N(t-\tau)\bigr]\bigr|\\&\quad=\left|\int_{t_j}^{t}\wp_N(s)\,ds\right| \leqslant (1+m_F)(t_{j+1}-t_j) = (1+m_F)h_N := a_N.\end{aligned}$$

It follows from (2.1) that

$$\begin{aligned}\bigl|u_N(t)\bigr| &\leqslant \bigl|A(t)u_N(t-\tau)\bigr| + \bigl|\bar{x}(a)-A(a)\bar{x}(a-\tau)\bigr| + (b-a)(1+m_F)\\&\leqslant M\bigl|u_N(t-\tau)\bigr| + M_1\\&\leqslant M_1\bigl(1+M+\cdots+M^m\bigr) + M^{m+1}\bigl|c\bigl(t-(m+1)\tau\bigr)\bigr|\\&\leqslant M_1\bigl(1+M+\cdots+M^m\bigr) + M^{m+1}M_2 := K,\end{aligned}$$

where

$$M_1 = \bigl|\bar{x}(a)-A(a)\bar{x}(a-\tau)\bigr| + (b-a)(1+m_F), \qquad M_2 = \max_{t\in[a-\tau,a]}\bigl|c(t)\bigr|.$$

Therefore,

$$\begin{aligned}&\bigl|u_N(t)-u_N(t_j)\bigr|\\&\quad\leqslant \bigl|\bigl[u_N(t_j)-A(t_j)u_N(t_j-\tau)\bigr]-\bigl[u_N(t)-A(t)u_N(t-\tau)\bigr]\bigr|\\&\qquad+\bigl|A(t_j)\bigl(u_N(t_j-\tau)-u_N(t-\tau)\bigr)\bigr| + \bigl|A(t_j)-A(t)\bigr|\bigl|u_N(t-\tau)\bigr|\\&\quad\leqslant a_N + M\bigl|u_N(t-\tau)-u_N(t_j-\tau)\bigr| + \beta_N\bigl|u_N(t-\tau)\bigr|\\&\quad\leqslant (a_N+K\beta_N) + M\bigl|u_N(t-\tau)-u_N(t_j-\tau)\bigr| \leqslant \cdots\\&\quad\leqslant (a_N+K\beta_N)\bigl(1+M+\cdots+M^m\bigr)\\&\qquad+M^{m+1}\bigl|u_N\bigl(t-(m+1)\tau\bigr)-u_N\bigl(t_j-(m+1)\tau\bigr)\bigr|\\&\quad\leqslant (a_N+K\beta_N)\bigl(1+M+\cdots+M^m\bigr) + M^{m+1}\beta_N := b_N.\end{aligned}\tag{2.4}$$

Note that (2.3) and (2.4) imply

$$\begin{aligned}&\operatorname{dist}\bigl(\wp_N(t_j); F\bigl(u_N(t_j), u_N(t_j-\tau), t\bigr)\bigr)\\&\quad-\operatorname{dist}\bigl(\wp_N(t_j); F\bigl(u_N(t), u_N(t-\tau), t\bigr)\bigr) \leqslant 2\ell_F b_N.\end{aligned}\tag{2.5}$$

By (A1) and (2.2), for any $t\in[t_j,t_{j+1})$, $j=0,\ldots,k$, we have

$$\begin{aligned}&\operatorname{dist}\bigl(\wp_N(t_j); F\bigl(u_N(t), u_N(t-\tau), t\bigr)\bigr) - \operatorname{dist}\bigl(\wp_N(t); F\bigl(\bar{x}(t), \bar{x}(t-\tau), t\bigr)\bigr)\\&\quad\leqslant \operatorname{dist}\bigl(F\bigl(u_N(t), u_N(t-\tau), t\bigr),\ F\bigl(\bar{x}(t), \bar{x}(t-\tau), t\bigr)\bigr)\\&\quad\leqslant \ell_F\bigl(\bigl|u_N(t)-\bar{x}(t)\bigr| + \bigl|u_N(t-\tau)-\bar{x}(t-\tau)\bigr|\bigr)\\&\quad\leqslant 2\ell_F\xi_N\bigl(1+M+\cdots+M^m\bigr) + 2\ell_F M^{m+1}\beta_N.\end{aligned}\tag{2.6}$$

Taking into account (2.3)–(2.6), we get

$$\begin{aligned}
&\operatorname{dist}\big(\wp_N(t_j); F\big(u_N(t_j), u_N(t_j-\tau), t\big)\big)\\
&\quad\leqslant 2\ell_F b_N + \operatorname{dist}\big(\wp_N(t_j); F\big(u_N(t), u_N(t-\tau), t\big)\big)\\
&\quad\leqslant 2\ell_F b_N + 2\ell_F \xi_N\big(1+M+\cdots+M^m\big) + 2\ell_F M^{m+1}\beta_N\\
&\qquad + \operatorname{dist}\big(\wp_N(t_j); F\big(\bar{x}(t), \bar{x}(t-\tau), t\big)\big)\\
&\quad\leqslant \ell_F \mu_N + \big|\dot{\bar{x}}(t) - A(t)\dot{\bar{x}}(t-\tau) - \wp_N(t_j)\big|,
\end{aligned}$$

where

$$\mu_N := 2b_N + 2\xi_N\big(1+M+\cdots+M^m\big) + 2M^{m+1}\beta_N.$$

Therefore,

$$\begin{aligned}
\zeta_N &\leqslant \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \big(\big|\dot{\bar{x}}(t) - A(t)\dot{\bar{x}}(t-\tau) - \wp_N(t_j)\big| + \ell_F\mu_N\big)\,dt + \tau(F; h_N)\\
&= \xi_N + \ell_F\mu_N(b-a) + \tau(F; h_N) := \gamma_N \to 0 \quad (N\to\infty).
\end{aligned}$$

Since the functions $u_N(t_j)$, $j=-N,\ldots,k+1$, may not be trajectories of (1.4). In the following we construct trajectories of (1.4) which satisfy the required convergence properties in Theorem 2.1. Define the trajectories $z_N(t_j)$, $j=-N,\ldots,k+1$ as follows:

$$\begin{cases}
z_N(t_0) = \bar{x}(a), & \\
z_N(t_j) = c(t_j), & j=-N,\ldots,-1,\\
z_N(t_{j+1}) - A(t_{j+1})z_N(t_{j+1}-\tau) & \\
\quad = z_N(t_j) - A(t_j)z_N(t_j-\tau) + h_N v_N(t_j), & j=0,\ldots,k,\\
v_N(t_j) \in F(z_N(t_j), z_N(t_j-\tau), t_j), & \\
|v_N(t_j) - \wp_N(t_j)| & \\
\quad = \operatorname{dist}(\wp_N(t_j);\ F(z_N(t_j), z_N(t_j-\tau), t_j)), & j=1,\ldots,k+1.
\end{cases}$$

Clearly, $z_N(t_j)$ are trajectories of (1.4). Note that

$$\big|z_N(t) - \bar{x}(t)\big| = \left|c(t_j) + \frac{c(t_{j+1}) - c(t_j)}{h_N}(t-t_j) - c(t)\right| < 2\beta_N$$

for $t\in[t_j, t_{j+1})$, $j=-N,\ldots,-1$, then $z_N(t)$ converge to $\bar{x}(t)$ uniformly on $[a-\tau, a)$.

Applying the mean value theorem for $\dot{c}(t)$ on $[t_j, t_{j+1}]$, we have

$$\begin{aligned}
\int_{a-\tau}^{a} \big|\dot{z}_N(t) - \dot{\bar{x}}(t)\big|^2 &= \sum_{j=-N}^{-1} \int_{t_j}^{t_{j+1}} \left|\frac{c(t_{j+1}) - c(t_j)}{h_N} - \dot{c}(t)\right|^2\\
&= \sum_{j=-N}^{-1} \int_{t_j}^{t_{j+1}} \big|\dot{c}\big(t_j^*\big) - \dot{c}(t)\big|^2 \leqslant \tau\beta_N.
\end{aligned} \tag{2.7}$$

Therefore, $\dot{z}_N(t)$ converge to $\dot{\bar{x}}(t)$ in L^2-norm on $[a-\tau, a]$.

In the following, we are to prove that $\dot{z}_N(t)$ converge to $\dot{\bar{x}}(t)$ in L^2-norm on $[a, b]$. To prove that, firstly, we need to show $z_N(t_j) \in U$ for $j = 0, \dots, k+1$ through mathematical induction. The case $j = 0$ is obvious due to $|z_N(t_0) - \bar{x}(t_0)| = 0$. Assume $z_N(t_l) \in U$ for $l = 0, \dots, i$. Note that

$$\begin{aligned}
&\bigl|z_N(t_{l+1}) - u_N(t_{l+1})\bigr| \\
&\quad = \bigl|A(t_{l+1}) z_N(t_{l+1} - \tau) + z_N(t_l) - A(t_l) z_N(t_l - \tau) + h_N v_{N_l} \\
&\qquad - \bigl(A(t_{l+1}) u_N(t_{l+1} - \tau) + u_N(t_l) - A(t_l) u_N(t_l - \tau) + h_N \wp_{N_l}\bigr)\bigr| \\
&\quad \leqslant M\bigl|z_N(t_{l+1} - \tau) - u_N(t_{l+1} - \tau)\bigr| + M\bigl|z_N(t_l - \tau) - u_N(t_l - \tau)\bigr| \\
&\qquad + \bigl|z_N(t_l) - u_N(t_l)\bigr| + h_N \operatorname{dist}\bigl(\wp_{N_l}; F\bigl(z_N(t_l), z_N(t_l - \tau), t_l\bigr)\bigr), \\
&\bigl|z_N(t_l) - u_N(t_l)\bigr| \\
&\quad \leqslant \bigl|z_N(t_{l-1}) - u_N(t_{l-1})\bigr| + M\bigl|z_N(t_{l-1-N}) - u_N(t_{l-1-N})\bigr| \\
&\qquad + M\bigl|z_N(t_{l-N}) - u_N(t_{l-N})\bigr| \\
&\qquad + h_N \operatorname{dist}\bigl(\wp_{N_{l-1}}; F\bigl(z_N(t_{l-1}), z_N(t_{l-1} - \Delta)\bigr), t_{l-1}\bigr),
\end{aligned}$$

as well as $|z_N(t_l) - u_N(t_l)| = 0$ for $l \leqslant 0$, there exists a real number $H > 0$ such that

$$\begin{aligned}
&\bigl|z_N(t_{l+1}) - u_N(t_{l+1})\bigr| \\
&\quad \leqslant H h_N \sum_{j=0}^{l} \operatorname{dist}\bigl(\wp_N(t_j); F\bigl(u_N(t_j), u_N(t_j - \tau), t_j\bigr)\bigr) \leqslant H \gamma_N.
\end{aligned} \tag{2.8}$$

It follows from (2.2) that $|z_N(t_{l+1}) - \bar{x}(t_{l+1})| \leqslant \xi_N + H\gamma_N$. Hence, $z_N(t_{l+1}) \in U$.

Employing (2.8),

$$\begin{aligned}
\sum_{j=0}^{k+1} \bigl|z_N(t_j) - u_N(t_j)\bigr| &\leqslant \sum_{j=0}^{k+1} H \sum_{l=0}^{j-1} h_N \operatorname{dist}\bigl(\wp_N(t_l); F\bigl(u_N(t_l), u_N(t_l - \tau), t_l\bigr)\bigr) \\
&\leqslant (b-a) H \sum_{j=0}^{k} \operatorname{dist}\bigl(\wp_N(t_j); F\bigl(u_N(t_j), u_N(t_j - \tau), t_j\bigr)\bigr).
\end{aligned} \tag{2.9}$$

From the construction of $v_N(t_j)$ and (2.9), we have

$$\int_a^b \bigl|\dot{z}_N(t) - A(t)\dot{z}_N(t-\tau) - \wp_N(t)\bigr|\, dt$$

$$
\begin{aligned}
&= \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \left|\dot{z}_N(t) - A(t)\dot{z}_N(t-\tau) - \wp_N(t_j)\right| dt \\
&= \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \left|v_N(t_j) - \wp_N(t_j)\right| dt \\
&= \sum_{j=0}^{k} h_N \operatorname{dist}\bigl(\wp_N(t_j); F\bigl(z_N(t_j), z_N(t_j-\tau), t_j\bigr)\bigr) \\
&= \sum_{j=0}^{k} h_N \operatorname{dist}\bigl(\wp_N(t_j); F\bigl(u_N(t_j), u_N(t_j-\tau), t_j\bigr)\bigr) \\
&\quad + \sum_{j=0}^{k} h_N \bigl[\operatorname{dist}\bigl(\wp_N(t_j); F\bigl(z_N(t_j), z_N(t_j-\tau), t_j\bigr)\bigr) \\
&\quad - \operatorname{dist}\bigl(\wp_N(t_j); F\bigl(u_N(t_j), u_N(t_j-\tau), t_j\bigr)\bigr)\bigr] \\
&\leqslant \gamma_N + \sum_{j=0}^{k} \ell_F h_N \bigl[\left|z_N(t_j) - u_N(t_j)\right| + \left|z_N(t_j-\tau) - u_N(t_j-\tau)\right|\bigr] \\
&\leqslant \gamma_N + 2(b-a)H\ell_F\gamma_N := \sigma_N.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\int_a^b \left|\dot{z}_N(t) - A(t)\dot{z}_N(t-\tau) - \bigl[\dot{\bar{x}}(t) - A(t)\dot{\bar{x}}(t-\tau)\bigr]\right| dt \\
&\quad \leqslant \int_a^b \left|\dot{z}_N(t) - A(t)\dot{z}_N(t-\tau) - \wp_N(t)\right| dt + \int_a^b \left|\dot{\bar{x}}(t) - A(t)\dot{\bar{x}}(t-\tau) - \wp_N(t)\right| dt \\
&\quad \leqslant \sigma_N + \xi_N.
\end{aligned}
$$

Note that $\bar{x}(t) \in U$, $z_N(t) \in U$, and (A1), we have

$$
\left|\dot{z}_N(t) - A(t)\dot{z}_N(t-\tau)\right| \leqslant m_F, \qquad \left|\dot{\bar{x}}(t) - A(t)\dot{\bar{x}}(t-\tau)\right| \leqslant m_F.
$$

Hence

$$
\begin{aligned}
&\int_a^b \left|\dot{z}_N(t) - A(t)\dot{z}_N(t-\tau) - \bigl[\dot{\bar{x}}(t) - A(t)\dot{\bar{x}}(t-\tau)\bigr]\right|^2 dt \\
&\quad = \int_a^b \left|\dot{z}_N(t) - A(t)\dot{z}_N(t-\tau) - \bigl[\dot{\bar{x}}(t) - A(t)\dot{\bar{x}}(t-\tau)\bigr]\right| \\
&\qquad \times \left|\dot{z}_N(t) - A(t)\dot{z}_N(t-\tau) + \dot{\bar{x}}(t) - A(t)\dot{\bar{x}}(t-\tau)\right| dt \\
&\quad \leqslant 2m_F(\sigma_N + \xi_N). \qquad (2.10)
\end{aligned}
$$

Taking into account (2.10) and (2.7), we obtain

$$\begin{aligned}
&\int_a^b \left|\dot{z}_N(t)-\dot{\bar{x}}(t)\right|^2 dt \\
&\quad\leqslant 2\int_a^b \left|\dot{z}_N(t)-A(t)\dot{z}_N(t-\tau)-\left[\dot{\bar{x}}(t)-A(t)\dot{\bar{x}}(t-\tau)\right]\right|^2 dt \\
&\qquad + 2M\int_a^b \left|\dot{z}_N(t-\tau)-\dot{\bar{x}}(t-\tau)\right|^2 dt \\
&\quad\leqslant 2\int_a^b \left|\dot{z}_N(t)-A(t)\dot{z}_N(t-\tau)-\left[\dot{\bar{x}}(t)-A(t)\dot{\bar{x}}(t-\tau)\right]\right|^2 dt \\
&\qquad + 4M\int_a^b \left|\dot{z}_N(t-\tau)-A(t-\tau)\dot{z}_N(t-2\tau)\right. \\
&\qquad \left.-\left[\dot{\bar{x}}(t-\tau)-A(t-\tau)\dot{\bar{x}}(t-2\tau)\right]\right|^2 dt \\
&\qquad + 4M^2\int_a^b \left|\dot{z}_N(t-2\tau)-\dot{\bar{x}}(t-2\tau)\right|^2 dt \\
&\quad\leqslant \cdots \leqslant 4m_F\left(1+2M+\cdots+2^m M^m\right)(\sigma_N+\xi_N)+(2M)^{m+1}\tau\beta_N \to 0,
\end{aligned}$$

where m is an integer such that $a-\tau \leqslant b-m\tau < a$. Consequently, the extended discrete velocities $z_N(t)$ converge to $\dot{\bar{x}}(t)$ in L^2-norm on $[a,b]$ as $N\to\infty$. To complete the proof of the theorem, we need to show the uniform convergence of $z_N(t)$ to $\bar{x}(\cdot)$ on $[a,b]$, this exactly follows from

$$\max_{t\in[a,b]}\left|z_N(t)-\bar{x}(t)\right| \leqslant \sqrt{b-a}\left(\int_a^b \left|\dot{z}_N(t)-\dot{\bar{x}}(t)\right|^2 dt\right)^{1/2}. \qquad \square$$

3. Discrete approximations to optimization problem

In this section, we construct a sequence of discrete optimization problems (P_N) to (P) and consider the convergence of optimal solutions of (P_N). Hereafter, let $\bar{x}(t)$ be an optimal solution to (P). Consider the following optimization problem (P_N) for each $N\in\mathbb{N}$:

$$\text{minimize}\quad J_N[x_N] := \varphi\big(x_N(a), x_N(b)\big) + \left|x_N(a)-\bar{x}(a)\right|^2$$

$$+ h_N \sum_{j=0}^{k} f\left(x_N(t_j), x_N(t_{j-N}), \frac{x_N(t_{j+1}) - x_N(t_j)}{h_N}, \frac{x_N(t_{j+1-N}) - x_N(t_{j-N})}{h_N}, t_j\right)$$

$$+ \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \left| \frac{x_N(t_{j+1}) - x_N(t_j)}{h_N} - \dot{\bar{x}}(t) \right|^2 dt, \tag{3.1}$$

subject to the following constraints:

$$x_N(t_{j+1}) - A(t_{j+1})x_N(t_{j+1-N}) \in x_N(t_j) - A(t_j)x_N(t_{j-N}) + h_N F\big(x_N(t_j), x_N(t_{j-N}), t_j\big), \quad j = 0, \dots, k, \tag{3.2}$$

$$x_N(t_0) = \bar{x}(a), \tag{3.3}$$

$$x_N(t_j) = c(t_j), \quad j = -N, \dots, -1, \tag{3.4}$$

$$\big(x_N(a), x_N(b)\big) \in \Omega_N := \Omega + \eta_N B, \tag{3.5}$$

$$\big|x_N(t_j) - \bar{x}(t_j)\big| \leqslant \delta, \quad j = 0, \dots, k+1, \tag{3.6}$$

where $\delta > 0$ is a fixed number and η_N is any sequence such that $\eta_N \to 0$ as $N \to \infty$.

Clearly, $z_N(t_j)$, $j = -N, \dots, k+1$, is a feasible solution to (P_N) from the proof in Theorem 2.1. Select δ in (3.6) such that $\bar{x}(t) + \delta\mathbb{B} \subset U$ for all $t \in [a - \tau, b]$ and $\eta_N < \delta$. By (3.2)–(3.6) both the set $\{x_N(t_j) \mid j = -N, \dots, k+1\}$ and the set $\{[x_N(t_{j+1}) - x_N(t_j)]/h_N \mid j = 0, \dots, k+1\}$ are bounded. The classical Weierstrass theorem implies the existence of an optimal solution, let it be $\bar{x}_N(t)$.

To deal with the convergence of optimal solutions for (P_N), we need an additional assumption called the relaxation stability for (P). Consider the following convexified neutral functional differential inclusion:

$$\begin{cases} \dot{x}(t) - A(t)\dot{x}(t-\tau) \in \operatorname{co} F(x(t), x(t-\tau), t), & \text{a.e. } t \in [a, b], \\ x(t) = c(t), & t \in [a - \tau, a), \end{cases} \tag{3.7}$$

where "co" stands for the convex hull of the set. The corresponding relaxed problem (R) to (P) is the problem in which we minimize the following cost functional:

$$\hat{J}[x] := \varphi\big(x(a), x(b)\big) + \int_a^b \hat{f}_F\big(x(t), x(t-\tau), \dot{x}(t), \dot{x}(t-\tau), \dot{x}(t) - A(t)\dot{x}(t-\tau), t\big)\, dt \tag{3.8}$$

over all trajectories of (3.7) with the endpoint constraint (1.3). Here, the function $\hat{f}_F(x, y, z, w, v, t)$ is the convexification of the function

$$f_F(x, y, z, w, v, t) := f(x, y, z, w, t) + \delta\big(v; F(x, y, t)\big),$$

in the v variable, and $\delta(v; F)$ is the indicator function of the set $F(x, y, t)$.

Definition 3.1. The problem (P) is said to be stable with respect to relaxation if

$$\inf(P) = \inf(R).$$

Obviously, (P) is stable with respect to relaxation if the sets $F(x, y, t)$ are convex. More detailed discussion can be found in [1,3,4,6–8].

4. Convergence of optimal solutions

This section deals with the convergence of discrete optimization problems (P_N). In addition to previous assumptions, we also need the following additional hypotheses:

(A4) $\varphi(x, y)$ is continuous on $(x, y) \in U \times U$, $f(x, y, z, w, t)$ is continuous for a.e. $t \in [a, b]$ uniformly in $(x, y, z, w) \in U \times U \times \mathbb{R}^{2n}$, and continuous on $(x, y, z, w) \in U \times U \times \mathbb{R}^{2n}$ uniformly in $t \in [a, b]$.

(A5) $f(x, y, z, w, t)$ is convex in (z, w) for a.e. $t \in [a, b]$.

Theorem 4.1. *Let $\bar{x}(t)$ be an optimal solution to (P),* (A1)–(A5) *be fulfilled, and (P) be stable with respect to relaxation. Then for any sequence of optimal solutions $\bar{x}_N(t_j)$, $j = -N, \dots, k+1$, to (P_N), the extended trajectories $\bar{x}_N(t)$ converge uniformly to $\bar{x}(t)$ on $[a-\tau, b]$, the extended velocities $\dot{\bar{x}}_N(t)$ converge to $\dot{\bar{x}}(t)$ in the L^2-norm on $[a-\tau, b]$ as $N \to \infty$.*

Proof. Note that

$$\begin{aligned} J_N[z_N] = {} & \varphi\big(z_N(a), z_N(b)\big) \\ & + h_N \sum_{j=0}^{k} f\left(z_N(t_j), z_N(t_{j-N}), \frac{z_N(t_{j+1}) - z_N(t_j)}{h_N}, \right. \\ & \qquad \left. \frac{z_N(t_{j+1-N}) - z_N(t_{j-N})}{h_N}, t_j\right) \\ & + \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \left| \frac{z_N(t_{j+1}) - z_N(t_j)}{h_N} - \dot{\bar{x}}(t) \right|^2 dt \\ := {} & I_1 + I_2 + I_3. \end{aligned}$$

It follows from (A4) and Theorem 2.1 that

$$\begin{aligned} I_1 & \to \varphi\big(\bar{x}(a), \bar{x}(b)\big), \\ I_2 & = h_N \sum_{j=0}^{k} f\left(z_N(t_j), z_N(t_{j-N}), \frac{z_N(t_{j+1}) - z_N(t_j)}{h_N}, \frac{z_N(t_{j+1-N}) - z_N(t_{j-N})}{h_N}, t_j\right) \\ & = \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} f\big(z_N(t_j), z_N(t_j - \tau), \dot{z}_N(t), \dot{z}_N(t-\tau), t\big)\, dt \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}\Big[f\big(z_N(t_j), z_N(t_j-\tau), \dot z_N(t), \dot z_N(t-\tau), t_j\big) \\
&\quad - f\big(z_N(t_j), z_N(t_j-\tau), \dot z_N(t), \dot z_N(t-\tau), t\big)\Big]\,dt \\
&= \sum_{j=0}^{k}\int_{t_j}^{t_{j+1}} f\big(z_N(t_j), z_N(t_j-\tau), \dot z_N(t), \dot z_N(t-\tau), t\big)\,dt + \tau(f; h_N) \\
&\to \int_a^b f\big(\bar x(t), \bar x(t-\tau), \dot{\bar x}(t), \dot{\bar x}(t-\tau), t\big)\,dt,
\end{aligned}
$$

and

$$
I_3 = \sum_{j=0}^{k}\int_{t_j}^{t_{j+1}} \big|\dot z_N(t) - \dot{\bar x}(t)\big|^2\,dt = \int_a^b \big|\dot z_N(t) - \dot{\bar x}(t)\big|^2\,dt \to 0.
$$

Therefore,

$$
J_N[z_N] \to J[\bar x] \quad \text{as } N \to \infty. \tag{4.1}
$$

Since $\bar x_N(t_j)$ is an optimal solution to (P_N) and $z_N(t_j)$ is a feasible solution to (P_N), then

$$
J_N[\bar x_N] \leqslant J_N[z_N]. \tag{4.2}
$$

Hence,

$$
\limsup_{N\to\infty} J_N[\bar x_N] \leqslant J[\bar x]. \tag{4.3}
$$

Let

$$
\rho_N = \int_a^b \big|\dot{\bar x}_N(t) - \dot{\bar x}(t)\big|^2\,dt.
$$

Due to (2.7) and

$$
\max_{t\in[a,b]}\big|\bar x_N(t) - \bar x(t)\big| \leqslant \sqrt{b-a}\left(\int_a^b \big|\dot{\bar x}_N(t) - \dot{\bar x}(t)\big|^2\,dt\right)^{1/2},
$$

the proof will be complete if we show that $\rho_N \to 0$. Suppose that it is not the case, we can find a constant $c > 0$ and a subsequence N_k such that $\rho_{N_k} \to c$. Without loss of generality, we may assume $c = \lim \rho_N$ as $N \to \infty$. From (3.2), (3.4), and (3.6), both $\bar x_N(t)$ and $\dot{\bar x}_N(t)$ are uniformly bounded on $[a-\tau, b]$. Therefore, there exists an absolutely continuous function $\tilde x : [a-\tau, b] \to \mathbb{R}^n$ with

$$
\begin{aligned}
&\bar x_N(t) \to \tilde x(t) \quad \text{uniformly on } [a-\tau, b], \\
&\dot{\bar x}_N(t) \to \dot{\tilde x}(t) \quad \text{weakly in } L^2[a-\tau, b].
\end{aligned} \tag{4.4}
$$

Thus, $\dot{\bar{x}}_N(t) - A(t)\dot{\bar{x}}_N(t-\tau)$ converge to $\dot{\tilde{x}}(t) - A(t)\dot{\tilde{x}}(t-\tau)$ weakly in $L^2[a,b]$. The classical Mazur's theorem implies that $\dot{\tilde{x}}(t) - A(t)\dot{\tilde{x}}(t-\tau) \in \operatorname{co} F(\tilde{x}(t), \tilde{x}(t-\tau), t)$. Consequently, $\tilde{x}(t)$ is a feasible solution to the relaxed problem (R). From the definition of $\hat{f}_F$, we have

$$\begin{aligned}
&\int_a^b \hat{f}_F\big(\tilde{x}(t), \tilde{x}(t-\tau), \dot{\tilde{x}}(t), \dot{\tilde{x}}(t-\tau), \dot{\tilde{x}}(t) - A(t)\dot{\tilde{x}}(t-\tau), t\big)\, dt \\
&\quad \leqslant \int_a^b f\big(\tilde{x}(t), \tilde{x}(t-\tau), \dot{\tilde{x}}(t), \dot{\tilde{x}}(t-\tau), t\big)\, dt \\
&\quad \leqslant \liminf_{N\to\infty} h_N \sum_{j=0}^{k} f\big(\bar{x}_N(t_j), \bar{x}_N(t_j-\tau), \dot{\bar{x}}_N(t), \dot{\bar{x}}_N(t-\tau), t_j\big).
\end{aligned}$$

Since the integral functional

$$I[v] := \int_a^b \big|v(t) - \dot{\tilde{x}}(t)\big|^2\, dt$$

is lower semicontinuous in the weak topology of $L^2[a,b]$ due to the convexity of the integrand in v. Then

$$\int_a^b \big|\dot{\tilde{x}}(t) - \dot{\bar{x}}(t)\big|^2\, dt \leqslant \liminf_{N\to\infty} \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \left|\frac{\bar{x}_N(t_{j+1}) - \bar{x}_N(t_j)}{h_N} - \dot{\bar{x}}(t)\right|^2 dt.$$

Taking into account (4.3) and (4.4), we have

$$\begin{aligned}
\hat{J}[\tilde{x}] &= \varphi\big(\tilde{x}(a), \tilde{x}(b)\big) + \int_a^b f\big(\tilde{x}(t), \tilde{x}(t-\tau), \dot{\tilde{x}}(t), \dot{\tilde{x}}(t-\tau), t\big)\, dt \\
&\leqslant \varphi\big(\tilde{x}(a), \tilde{x}(b)\big) + \liminf_{N\to\infty} h_N \sum_{j=0}^{k} f\big(\bar{x}_N(t_j), \bar{x}_N(t_j-\tau), \dot{\bar{x}}_N(t), \dot{\bar{x}}_N(t-\tau), t_j\big) \\
&\leqslant \varphi\big(\tilde{x}(a), \tilde{x}(b)\big) + \liminf_{N\to\infty} \Bigg[J_N[\bar{x}_N] - \varphi\big(\bar{x}_N(a), \bar{x}_N(b)\big) \\
&\qquad - \int_a^b \big|\dot{\bar{x}}_N(t) - \dot{\bar{x}}(t)\big|^2\, dt \Bigg] \\
&\leqslant J[\bar{x}] - c.
\end{aligned}$$

The assumption of relaxation stability implies $c = 0$ and the proof is complete. □

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