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On an integrable modified (2 + 1)-dimensional Lotka–Volterra equation

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Abstract

In this paper, an integrable (2 + 1)-dimensional modified Lotka–Volterra equation is considered. This is a semi-discrete system, having discrete space and continuous time which has a known (1 + 1)-dimensional self-dual network equation as a reduction. Its bilinear Bäcklund transformation and Lax pair are also presented and explicit solutions including soliton solutions expressed in terms of pfaffians are obtained. Finally, the reduction of these solutions to (1 + 1) dimensions is considered.
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Keywords: Modified (2 + 1)-dimensional Lotka–Volterra equation; Self-dual network equation; Bäcklund transformation; Pfaffian solutions; Soliton solutions

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1. Introduction

There has been a good deal of interest for some time in $(2+1)$-dimensional integrable systems. In general, these systems may be reduced to the known $(1+1)$-dimensional integrable equations. Two contrasting examples are the Kadomtsev–Petviashvili equation (KP) [1] which reduces to the Korteweg–de Vries equation and the Davey–Stewartson (DS) equations [2,3] which reduce to the nonlinear Schrödinger equation. In the KP equation, the two space variables appear in an asymmetric way with only of these variables having a clear link with the space variable in the KdV equation, and the other having a lesser role; solutions of the KdV equation are obtained by seeking solutions independent of the lesser variable. On the other hand, in the DS equations the two spatial variables appear on an equal footing with each linked with the space variable in the nonlinear Schrödinger equation and hence allow reduction to be made by identifying the two space variables. For these reasons, one refers to KP and DS as weak and strong $(2+1)$-dimensional generalisations, respectively. Other interesting example include the Nizhnik–Novikov–Veselov [4,5] and Loewner–Konopelchenko–Rogers [6] equations which strongly generalise the KdV and sine-Gordon equations, respectively.

Until quite recently, such strong generalisations were only known for *continuous* integrable systems. However, strong $(2+1)$-dimensional generalisation for differential–difference equations, the Lotka–Volterra equation [7], hereafter referred to as LV2, and of the Toda lattice equation [8], have recently been discovered. In this paper we consider a third example, a symmetric $(2+1)$-dimensional modified Lotka–Volterra equation, which we call mLV2. It turns out that this system is a strong $(2+1)$-dimensional generalisation of a potential self-dual network equation [9–11].

This new system is obtained from a Bäcklund transformation of the LV2 equation in the same way as the mKdV equation is obtained from the Bäcklund transformation of the KdV equation. A Bäcklund transformation and Lax pair are found for the mLV2 equation and pfaffian solutions expressed in terms of eigenfunctions are also obtained.

The paper is organised as follows. Section 2 summarises the key results related to the LV2 equation, including its Bäcklund transformation, and then presents the mLV2 equation. A Bäcklund transformation for the mLV2 equation is given in Section 3 and from this a Lax pair is constructed. Sections 4 and 5 discuss pfaffian solutions of this Bäcklund transformation. First how it relates solutions of LV2 and then how it has solutions of the mLV2 equation. In Section 6, a special case of the pfaffian solutions of mLV2 is shown to give multisoliton solutions. Finally, in Section 7 we discuss the reduction of the pfaffian solutions to solutions of the self-dual network equation.

2. Lotka–Volterra and modified Lotka–Volterra equations

One of the most familiar semi-discrete $(1+1)$-dimensional integrable systems is the Lotka–Volterra (LV) equation

$$u_t + e^{u+u_n} - e^{u+u_{\bar{n}}} = 0, \tag{1}$$

in which $u \equiv u(n,t)$ where n is a discrete space variable and t is a continuous time variable. The subscript t denotes partial derivative as usual and the subscript involving n denote shifts; $u_n \equiv u(n+1,t)$ and $u_{\tilde{n}} \equiv u(n-1,t)$. In [7], an integrable symmetric $(2+1)$-dimensional generalisation of LV,

$$\begin{aligned} &2u_t + e^{\Delta_n^2\phi+u_n} + e^{\Delta_m^2\phi+u_m} - e^{u+\Delta_n^2\phi_{\tilde{n}}} - e^{u+\Delta_m^2\phi_{\tilde{m}}} \\ &\quad = \Delta_n\big(e^{\Delta_n^2\phi_{\tilde{n}}-u_{\tilde{n}}}\big) + \Delta_m\big(e^{\Delta_m^2\phi_{\tilde{m}}-u_{\tilde{m}}}\big), \quad u = \Delta_m\Delta_n\phi, \end{aligned} \tag{2}$$

was presented. In addition to the subscript notation to express shifts in discrete variables, we use the standard difference operator notation $\Delta_m F \equiv F(m+1,n) - F(m,n)$, and similarly for Δ_n. Hereafter we refer to this system as LV2.

This system describes evolution with respect to continuous time in two-dimensional discrete space in which the space variables appear on equal footing. Indeed, in the special case $m = n$, we have $\Delta_m^2\phi = \Delta_n^2\phi = u$ and so (2) becomes (1) and the choice of the name LV2 is justified.

By making the dependent variable transformation $\phi = \log f$ and $u = \Delta_m\Delta_n \log f$, (2) may be written in multilinear form:

$$\begin{aligned} &\sinh\big(\tfrac{1}{2}D_n\big)\big[\big(D_t e^{\frac{1}{2}D_m} - e^{D_n-\frac{1}{2}D_m} + e^{D_n+\frac{1}{2}D_m}\big)f\cdot f\big]\cdot\big(e^{\frac{1}{2}D_m}f\cdot f\big) \\ &\quad + \sinh\big(\tfrac{1}{2}D_m\big)\big[\big(D_t e^{\frac{1}{2}D_n} - e^{D_m-\frac{1}{2}D_n} + e^{D_m+\frac{1}{2}D_n}\big)f\cdot f\big]\cdot\big(e^{\frac{1}{2}D_n}f\cdot f\big) = 0, \end{aligned} \tag{3}$$

where the bilinear operators D_x, D_y, and D_n [12] are defined by

$$\begin{aligned} &D_x^p D_y^q a\cdot b \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^p \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^q a(x,y)b(x',y')\Big|_{x'=x,\,y'=y}, \\ &\exp(\delta D_n)a\cdot b \equiv a(n+\delta)b(n-\delta), \end{aligned}$$

respectively. Introducing auxiliary independent variables x and y such that $\partial_x + \partial_y = 2\partial_t$, this may be decoupled to obtain the Hirota bilinear form of (3):

$$\big(D_x e^{\frac{1}{2}D_m} - e^{D_n-\frac{1}{2}D_m} + e^{D_n+\frac{1}{2}D_m}\big)f\cdot f = 0, \tag{4}$$

$$\big(D_y e^{\frac{1}{2}D_n} - e^{D_m-\frac{1}{2}D_n} + e^{D_m+\frac{1}{2}D_n}\big)f\cdot f = 0. \tag{5}$$

One considers solutions of this system depending on the four variables m, n, x and y and then reduces to solutions of LV2 by setting $x = y = t$.

It was also shown in [7] that a bilinear Bäcklund transformation (BT) for (4), (5) is given by

$$\sinh\big(\tfrac{1}{2}D_m\big)\sinh\big(\tfrac{1}{2}D_n\big)f\cdot f' = 0, \tag{6}$$

$$\big(D_t + \sinh(D_m) + \sinh(D_n)\big)f\cdot f' = 0. \tag{7}$$

This means that (6), (7) relate particular solutions f and f' of (4), (5). By introducing the "eigenfunction" $\theta = f'/f$, the BT may be converted into a Lax pair for LV2,

$$\theta_{mn} + \theta = \frac{f_m f_n}{f_{mn} f}(\theta_m + \theta_n), \tag{8}$$

$$2\theta_t + \frac{f_m f_{\tilde{m}}}{f^2}(\theta_m - \theta_{\tilde{m}}) + \frac{f_n f_{\tilde{n}}}{f^2}(\theta_n - \theta_{\tilde{n}}) = 0. \tag{9}$$

In [7] it was shown how a class of pfaffian solutions defined in terms of an arbitrary number of eigenfunctions of its Lax pair of (2) could be found. In the next section we will show that the BT can indeed be satisfied by two such solutions.

As is well known [12], one may often reinterpret a BT as a new integrable system. To this end, we rewrite $f' = g$ in (6), (7) to obtain the system

$$\sinh\left(\tfrac{1}{2}D_m\right)\sinh\left(\tfrac{1}{2}D_n\right) f \cdot g = 0, \tag{10}$$

$$\left(D_t + \sinh(D_m) + \sinh(D_n)\right) f \cdot g = 0. \tag{11}$$

The prototype of this process is the well-known result that the BT of the Korteweg–de Vries equation is the modified Korteweg–de Vries equation and for this reason we call (10)–(11) the symmetric $(2+1)$-dimension *modified* Lotka–Volterra equation (mLV2). Through the dependent variable transformation $v = \log(f/g)$ and $w = \log(gf)$, we obtain its nonlinear form

$$\exp\left(\tfrac{1}{2}\Delta_m\Delta_n w\right) = \frac{\cosh\frac{1}{2}(v_m - v_n)}{\cosh\frac{1}{2}(v_{mn} - v)}, \tag{12}$$

$$v_t + \exp\left(\tfrac{1}{2}\Delta_m^2 w_{\bar{m}}\right)\sinh\tfrac{1}{2}(v_m - v_{\bar{m}}) + \exp\left(\tfrac{1}{2}\Delta_n^2 w_{\bar{n}}\right)\sinh\tfrac{1}{2}(v_n - v_{\bar{n}}) = 0. \tag{13}$$

From (12) we can derive a Miura transformation between mLV2 and LV2. The solutions of the two systems are related by

$$u = \log f = \tfrac{1}{2}(v + w).$$

It then follows, using (12), that

$$\frac{e^{-u_m}e^{-u_n}}{e^{-u}e^{-u_{mn}}} = \frac{e^{-v_m} + e^{-v_n}}{e^{-v} + e^{-v_{mn}}}. \tag{14}$$

In the reduction $m = n$, (12)–(13) becomes the $(1+1)$-dimensional equation,

$$v_t + 2\tanh\tfrac{1}{2}(v_n - v_{\bar{n}}) = 0, \tag{15}$$

having bilinear form

$$\sinh^2\left(\tfrac{1}{2}D_n\right) f \cdot g = 0, \tag{16}$$

$$\left(D_t + 2\sinh(D_n)\right) f \cdot g = 0, \tag{17}$$

where $v = \log(f/g)$. This is the potential form of the self-dual nonlinear network equation, which arises from discretisation of the mKdV equation [9–11].

It is noteworthy that (14) reduces to the classical Miura transformation after reducing to one dimension and taking the obvious continuum limit. In one dimension, we have the Miura transformation

$$\frac{e^{-2u_n}}{e^{-u}e^{-u_{nn}}} = \frac{2e^{-v_n}}{e^{-v} + e^{-v_{nn}}}, \tag{18}$$

linking (15) with (1). Then, letting $u(n+1,t) = u(x+\epsilon,t)$ and $v(n+1,t) = v(x+\epsilon,t)$ and using Taylor's theorem, one obtains at leading order

$$2u_{xx} = v_{xx} - v_x^2,$$

which is the classical Miura transformation between (potential) mKdV and (potential) KdV.

3. Bäcklund transformation and Lax pair

In this section we will show that the bilinear equations (10) and (11) have bilinear Bäcklund transformation:

$$\left(e^{\frac{1}{2}D_m}+\mu e^{-\frac{1}{2}D_m}\right)g\bullet f'+\left(\lambda e^{\frac{1}{2}D_m}+\lambda\mu e^{-\frac{1}{2}D_m}\right)f\bullet g'=0, \tag{19}$$

$$\left(e^{\frac{1}{2}D_n}+\gamma e^{-\frac{1}{2}D_n}\right)g\bullet f'-\left(\lambda e^{\frac{1}{2}D_n}+\lambda\gamma e^{-\frac{1}{2}D_n}\right)f\bullet g'=0, \tag{20}$$

$$\left(2D_t+\mu e^{-D_m}-\frac{1}{\mu}e^{D_m}+\gamma e^{-D_n}-\frac{1}{\gamma}e^{D_n}\right)f\bullet f'=0, \tag{21}$$

$$\left(2D_t+\mu e^{-D_m}-\frac{1}{\mu}e^{D_m}+\gamma e^{-D_n}-\frac{1}{\gamma}e^{D_n}\right)g\bullet g'=0, \tag{22}$$

where λ, μ and γ are arbitrary constants.

To verify this result, we let f, g satisfy (10)–(11) and suppose that f', g' are given in terms of f, g by (19)–(22). Now consider the relations

$$\begin{aligned}
P_1\equiv{}&\left[\left(e^{\frac{1}{2}D_m+\frac{1}{2}D_n}+e^{-\frac{1}{2}D_m-\frac{1}{2}D_n}-e^{\frac{1}{2}D_m-\frac{1}{2}D_n}-e^{-\frac{1}{2}D_m+\frac{1}{2}D_n}\right)f\bullet g\right]\\
&\times\left[\left(e^{\frac{1}{2}D_m-\frac{1}{2}D_n}+e^{-\frac{1}{2}D_m+\frac{1}{2}D_n}\right)f'\bullet g'\right]\\
&-\left[\left(e^{\frac{1}{2}D_m-\frac{1}{2}D_n}+e^{-\frac{1}{2}D_m+\frac{1}{2}D_n}\right)f\bullet g\right]\\
&\times\left[\left(e^{\frac{1}{2}D_m+\frac{1}{2}D_n}+e^{-\frac{1}{2}D_m-\frac{1}{2}D_n}-e^{\frac{1}{2}D_m-\frac{1}{2}D_n}-e^{-\frac{1}{2}D_m+\frac{1}{2}D_n}\right)f'\bullet g'\right]=0,\\
P_2\equiv{}&\left[\left(2D_t+e^{D_m}-e^{-D_m}+e^{D_n}-e^{-D_n}\right)f\bullet g\right]f'g'\\
&-fg\left[\left(2D_t+e^{D_m}-e^{-D_m}+e^{D_n}-e^{-D_n}\right)f'\bullet g'\right]=0.
\end{aligned}$$

If we can show that $P_1=P_2=0$, then f', g' must also satisfy (10)–(11) and hence (19)–(22) form a BT for this system. In fact, by using bilinear operator identities given in Appendix A and (19)–(22), we can show that

$$\begin{aligned}
P_1={}&\left[\left(e^{\frac{1}{2}D_m+\frac{1}{2}D_n}f\bullet g\right)\left(e^{\frac{1}{2}D_m-\frac{1}{2}D_n}f'\bullet g'\right)-\left(e^{\frac{1}{2}D_m+\frac{1}{2}D_n}f'\bullet g'\right)\left(e^{\frac{1}{2}D_m-\frac{1}{2}D_n}f\bullet g\right)\right]\\
&+\left[\left(e^{-\frac{1}{2}D_m-\frac{1}{2}D_n}f\bullet g\right)\left(e^{\frac{1}{2}D_m-\frac{1}{2}D_n}f'\bullet g'\right)\right.\\
&\left.-\left(e^{-\frac{1}{2}D_m-\frac{1}{2}D_n}f'\bullet g'\right)\left(e^{\frac{1}{2}D_m-\frac{1}{2}D_n}f\bullet g\right)\right]\\
&+\left[\left(e^{\frac{1}{2}D_m+\frac{1}{2}D_n}f\bullet g\right)\left(e^{-\frac{1}{2}D_m+\frac{1}{2}D_n}f'\bullet g'\right)\right.\\
&\left.-\left(e^{\frac{1}{2}D_m+\frac{1}{2}D_n}f'\bullet g'\right)\left(e^{-\frac{1}{2}D_m+\frac{1}{2}D_n}f\bullet g\right)\right]\\
&+\left[\left(e^{-\frac{1}{2}D_m-\frac{1}{2}D_n}f\bullet g\right)\left(e^{-\frac{1}{2}D_m+\frac{1}{2}D_n}f'\bullet g'\right)\right.\\
&\left.-\left(e^{-\frac{1}{2}D_m-\frac{1}{2}D_n}f'\bullet g'\right)\left(e^{-\frac{1}{2}D_m+\frac{1}{2}D_n}f\bullet g\right)\right]\\
={}&2\sinh\left(\tfrac{1}{2}D_n\right)\left(e^{\frac{1}{2}D_m}f\bullet g'\right)\bullet\left(e^{-\frac{1}{2}D_m}g\bullet f'\right)\\
&+2\sinh\left(\tfrac{1}{2}D_m\right)\left(e^{\frac{1}{2}D_n}g\bullet f'\right)\bullet\left(e^{-\frac{1}{2}D_n}f\bullet g'\right)\\
&+2\sinh\left(\tfrac{1}{2}D_m\right)\left(e^{\frac{1}{2}D_n}f\bullet g'\right)\bullet\left(e^{-\frac{1}{2}D_n}g\bullet f'\right)
\end{aligned}$$

$$
\begin{aligned}
&+2\sinh\left(\tfrac{1}{2}D_n\right)\left(e^{\frac{1}{2}D_m}g\bullet f'\right)\bullet\left(e^{-\frac{1}{2}D_m}f\bullet g'\right)\\
&=2\sinh\left(\tfrac{1}{2}D_n\right)\left(e^{\frac{1}{2}D_m}f\bullet g'\right)\bullet\left(-\lambda e^{-\frac{1}{2}D_m}f\bullet g'-\frac{1}{\mu}e^{\frac{1}{2}D_m}g\bullet f'\right)\\
&\quad+2\sinh\left(\tfrac{1}{2}D_m\right)\left(e^{\frac{1}{2}D_n}g\bullet f'\right)\bullet\left(\frac{1}{\lambda}e^{-\frac{1}{2}D_n}g\bullet f'-\frac{1}{\gamma}e^{\frac{1}{2}D_n}f\bullet g'\right)\\
&\quad+2\sinh\left(\tfrac{1}{2}D_m\right)\left(e^{\frac{1}{2}D_n}f\bullet g'\right)\bullet\left(\lambda e^{-\frac{1}{2}D_n}f\bullet g'-\frac{1}{\gamma}e^{\frac{1}{2}D_n}g\bullet f'\right)\\
&\quad+2\sinh\left(\tfrac{1}{2}D_n\right)\left(e^{\frac{1}{2}D_m}g\bullet f'\right)\bullet\left(-\frac{1}{\lambda}e^{-\frac{1}{2}D_m}g\bullet f'-\frac{1}{\mu}e^{\frac{1}{2}D_m}f\bullet g'\right)=0,\\
P_2&=2\left[(D_tf\bullet f')gg'-ff'D_tg\bullet g'\right]+2\sinh\left(\frac{D_m}{2}\right)\left(e^{\frac{1}{2}D_m}f\bullet g'\right)\bullet\left(e^{-\frac{1}{2}D_m}g\bullet f'\right)\\
&\quad+2\sinh\left(\frac{D_m}{2}\right)\left(e^{-\frac{1}{2}D_m}f\bullet g'\right)\bullet\left(e^{\frac{1}{2}D_m}g\bullet f'\right)\\
&\quad+2\sinh\left(\frac{D_n}{2}\right)\left(e^{\frac{1}{2}D_n}f\bullet g'\right)\bullet\left(e^{-\frac{1}{2}D_n}g\bullet f'\right)\\
&\quad+2\sinh\left(\frac{D_n}{2}\right)\left(e^{-\frac{1}{2}D_n}f\bullet g'\right)\bullet\left(e^{\frac{1}{2}D_n}g\bullet f'\right)\\
&=2\left[(D_tf\bullet f')gg'-ff'(D_tg\bullet g')\right]\\
&\quad+2\sinh\left(\tfrac{1}{2}D_m\right)\left(e^{\frac{1}{2}D_m}f\bullet g'\right)\left[-\lambda e^{-\frac{1}{2}D_m}f\bullet g'-\frac{1}{\mu}e^{\frac{1}{2}D_m}g\bullet f'\right]\\
&\quad+2\sinh\left(\tfrac{1}{2}D_m\right)\left(e^{-\frac{1}{2}D_m}f\bullet g'\right)\bullet\left[-\lambda e^{\frac{1}{2}D_m}f\bullet g'-\mu e^{-\frac{1}{2}D_m}g\bullet f'\right]\\
&\quad+2\sinh\left(\tfrac{1}{2}D_n\right)\left(e^{\frac{1}{2}D_n}f\bullet g'\right)\bullet\left[\lambda e^{-\frac{1}{2}D_n}f\bullet g'-\frac{1}{\gamma}e^{\frac{1}{2}D_n}g\bullet f'\right]\\
&\quad+2\sinh\left(\tfrac{1}{2}D_n\right)\left(e^{-\frac{1}{2}D_n}f\bullet g'\right)\bullet\left[\lambda e^{\frac{1}{2}D_n}f\bullet g'-\gamma e^{-\frac{1}{2}D_n}g\bullet f'\right]\\
&=2(D_tf\bullet f')gg'-2ff'(D_tg\bullet g')-\frac{1}{\mu}\left(e^{D_m}f\bullet f'\right)gg'+\frac{1}{\mu}ff'e^{D_m}g\bullet g'\\
&\quad+\mu\left(e^{-D_m}f\bullet f'\right)gg'-\mu ff'e^{-D_m}g\bullet g'-\frac{1}{\gamma}\left(e^{D_n}f\bullet f'\right)gg'+\frac{1}{\gamma}ff'e^{D_n}g\bullet g'\\
&\quad+\gamma\left(e^{-D_n}f\bullet f'\right)gg'-\gamma ff'e^{-D_n}gg'=0.
\end{aligned}
$$

Finally, we must show that this BT is consistent and this is done by using it to obtain a Lax pair for mLV2.

This Lax pair is derived from this BT in the standard way. We introduce eigenfunctions $\phi=f'/f$ and $\psi=g'/g$ and then rewrite (19)–(22) in terms of these variables to obtain:

$$(\phi+\lambda\mu\psi_m)g_mf+(\mu\phi_m+\lambda\psi)f_mg=0, \tag{23}$$

$$(\phi-\lambda\gamma\psi_n)g_nf+(\gamma\phi_n-\lambda\psi)f_ng=0, \tag{24}$$

$$-2\phi_t+\frac{f_mf_{\bar{m}}}{f^2}\left(\mu\phi_m-\frac{1}{\mu}\phi_{\bar{m}}\right)+\frac{f_nf_{\bar{n}}}{f^2}\left(\gamma\phi_n-\frac{1}{\gamma}\phi_{\bar{n}}\right)=0, \tag{25}$$

$$-2\psi_t + \frac{g_m g_{\bar{m}}}{g^2}\left(\mu\psi_m - \frac{1}{\mu}\psi_{\bar{m}}\right) + \frac{g_n g_{\bar{n}}}{g^2}\left(\gamma\psi_n - \frac{1}{\gamma}\psi_{\bar{n}}\right) = 0. \tag{26}$$

It may be verified in a straightforward way that the compatibility conditions of this Lax pair yields the mLV2 equation.

By using the gauge transformation

$$\psi \to (-\mu)^{-m}(-\gamma)^{-n}\psi, \qquad \phi \to \lambda(-\mu)^{-m}(-\gamma)^{-n}\phi, \tag{27}$$

we may transform (23)–(26) into

$$(\phi - \psi_m)g_m f - (\phi_m - \psi) f_m g = 0, \tag{28}$$

$$(\phi + \psi_n)g_n f - (\phi_n + \psi) f_n g = 0, \tag{29}$$

$$2\phi_t + \frac{f_m f_{\bar{m}}}{f^2}(\phi_m - \phi_{\bar{m}}) + \frac{f_n f_{\bar{n}}}{f^2}(\phi_n - \phi_{\bar{n}}) = 0, \tag{30}$$

$$2\psi_t + \frac{g_m g_{\bar{m}}}{g^2}(\psi_m - \psi_{\bar{m}}) + \frac{g_n g_{\bar{n}}}{g^2}(\psi_n - \psi_{\bar{n}}) = 0, \tag{31}$$

effectively choosing $\lambda = -\mu = -\gamma = 1$. From (28) and (29) and making use of (10), we may show that

$$\phi_{mn} + \phi = \frac{f_m f_n}{f_{mn} f}(\phi_m + \phi_n), \tag{32}$$

$$\psi_{mn} + \psi = \frac{g_m g_n}{g_{mn} g}(\psi_m + \psi_n). \tag{33}$$

By comparing the Lax pairs (8)–(9) of LV2 and (30)–(33) of mLV2, we see that the mLV2 eigenfunction ϕ and ψ are LV2 eigenfunctions for two solutions f and g of LV2, respectively. To have this property, we made the choice of gauge (27).

4. Solutions related by the Bäcklund transformation

In [7], it was shown that the symmetric $(2+1)$-dimensional LV equation has solutions expressed in terms of pfaffians. In this section, we will describe the relationship between two such solutions f and f' which are related by the Bäcklund transformation (6)–(7).

Suppose that τ is any solution of LV2 (4)–(5) and let θ_i, $i = 1, \ldots, 2N+1$, be the corresponding eigenfunctions, that is, solutions of (8)–(9) with $f = \tau$. The pfaffian element (i, j) is defined in terms of these quantities by the compatible relations

$$(i, j)_m = (i, j) - \theta_{i,m}\theta_j + \theta_i\theta_{j,m}, \tag{34}$$

$$(i, j)_n = (i, j) + \theta_{i,n}\theta_j - \theta_i\theta_{j,n}, \tag{35}$$

$$2(i, j)_t = \frac{\tau_n \tau_{\bar{n}}}{\tau^2}(\theta_{i,\bar{n}}\theta_{j,n} - \theta_{i,n}\theta_{j,\bar{n}}) + \frac{\tau_m \tau_{\bar{m}}}{\tau^2}(\theta_{i,m}\theta_{j,\bar{m}} - \theta_{i,\bar{m}}\theta_{j,m}), \tag{36}$$

where, as described above, the subscript t denotes the t-derivative, and m and n denote increments with respect to these discrete variables. We also use the index c_i^j defined by

$$\left(k, c_i^j\right) = \theta_k(m+i, n+j), \qquad \left(c_i^j, c_k^l\right) = 0. \tag{37}$$

We will now show that the BT relates

$$f = \tau(1, 2, \ldots, 2N) \equiv \tau(), \tag{38}$$

$$f' = \tau\big(1, 2, \ldots, 2N+1, c_0^0\big) \equiv \tau\big(2N+1, c_0^0\big), \tag{39}$$

where, for a more compact notation, we have omitted the indices $1, 2, \ldots, 2N$. The derivatives and shifts of f and f' are given by

$$f_m = \tau_m() + \tau_m\big(c_1^0, c_0^0\big), \qquad f_{\bar{m}} = \tau_{\bar{m}}() + \tau_{\bar{m}}\big(c_{-1}^0, c_0^0\big), \tag{40}$$

$$f_n = \tau_n() + \tau_n\big(c_0^0, c_0^1\big), \qquad f_{\bar{n}} = \tau_{\bar{n}}() + \tau_{\bar{n}}\big(c_0^0, c_0^{-1}\big), \tag{41}$$

$$f_{mn} = \tau_{mn}() + \frac{\tau_m\tau_n}{\tau}\big(c_1^0, c_0^1\big),$$

$$2f_t = 2\tau_t() + \frac{\tau_m\tau_{\bar{m}}}{\tau}\big(c_{-1}^0, c_1^0\big) + \frac{\tau_n\tau_{\bar{n}}}{\tau}\big(c_0^1, c_0^{-1}\big), \tag{42}$$

$$f'_m = \tau_m\big(2N+1, c_1^0\big), \qquad f'_{\bar{m}} = \tau_{\bar{m}}\big(2N+1, c_{-1}^0\big), \tag{43}$$

$$f'_n = \tau_n\big(2N+1, c_0^1\big), \qquad f'_{\bar{n}} = \tau_{\bar{n}}\big(2N+1, c_0^{-1}\big), \tag{44}$$

and

$$f'_{mn} = \frac{\tau_m\tau_n}{\tau}\big(2N+1, c_1^0\big) + \frac{\tau_m\tau_n}{\tau}\big(2N+1, c_0^1\big) - \tau_{mn}\big(2N+1, c_0^0\big) + \frac{\tau_m\tau_n}{\tau}\big(2N+1, c_1^0, c_0^0, c_0^1\big), \tag{45}$$

$$2f'_t = 2\tau_t\big(2N+1, c_0^0\big) + \frac{\tau_n\tau_{\bar{n}}}{\tau}\big(2N+1, c_0^{-1} - c_0^1\big) + \frac{\tau_m\tau_{\bar{m}}}{\tau}\big(2N+1, c_{-1}^0 - c_1^0\big) + \frac{\tau_n\tau_{\bar{n}}}{\tau}\big(2N+1, c_0^0, c_0^1, c_0^{-1}\big) + \frac{\tau_m\tau_{\bar{m}}}{\tau}\big(2N+1, c_0^0, c_{-1}^0, c_1^0\big). \tag{46}$$

With these expressions, (6) and (7) become the pfaffian identities:

$$\big(2N+1, c_1^0, c_0^0, c_0^1\big)() - \big(c_1^0, c_0^0\big)\big(2N+1, c_0^1\big) + \big(c_1^0, c_0^1\big)\big(2N+1, c_0^0\big) - \big(2N+1, c_1^0\big)\big(c_0^0, c_0^1\big) = 0, \tag{47}$$

$$\begin{aligned}&\tau_n\tau_{\bar{n}}\Big(\big(2N+1, c_0^0, c_0^1, c_0^{-1}\big)() - \big(c_0^0, c_0^1\big)\big(2N+1, c_0^{-1}\big) + \big(c_0^0, c_0^{-1}\big)\big(2N+1, c_0^1\big) - \big(2N+1, c_0^0\big)\big(c_0^1, c_0^{-1}\big)\Big)\\ &+ \tau_m\tau_{\bar{m}}\Big(\big(2N+1, c_0^0, c_{-1}^0, c_1^0\big)() - \big(c_0^0, c_{-1}^0\big)\big(2N+1, c_1^0\big) + \big(c_0^0, c_1^0\big)\big(2N+1, c_{-1}^0\big) - \big(2N+1, c_0^0\big)\big(c_{-1}^0, c_1^0\big)\Big) = 0.\end{aligned} \tag{48}$$

This confirms that the solutions f and f' of (4)–(5) given by (38) and (39) are related by the BT (6)–(7).

5. Pfaffian solutions of the modified equation

In this section, we will present pfaffian solutions of the modified equation expressed in terms of eigenfunctions of its Lax pair. Let ϕ_i, ψ_i, $i = 1, \ldots, 2N$, be pairs of eigenfunc-

tions, that is, solutions of (28)–(31), corresponding to a given solution $f = \tau$, $g = \sigma$ of (10)–(11). Since each of ϕ_i and ψ_i are also solutions of (8)–(9), we may define *two* pfaffian elements; (i, j) defined by (34)–(36) with $\theta_i = \phi_i$ and $(i, j)'$ defined by (34)–(36) with $\theta_i = \psi_i$. Then, in a natural way, the expressions for the solution f, g, which we will later verify, are

$$f = \tau(1, 2, \ldots, 2N), \tag{49}$$

$$g = \sigma(1, 2, \ldots, 2N)'. \tag{50}$$

The proof that these satisfy (10)–(11), however, will require to express f and g in terms of the *same* pfaffian element. Indeed, we may show that

$$(i, j)' = (i, j) + \phi_i\psi_j - \phi_j\psi_i = (i, j) - \big(i, j, c_0^0, d_0^0\big), \tag{51}$$

where indices c_i^j and d_i^j are defined by

$$\big(k, c_i^j\big) = \phi_k(m + i, n + j), \qquad \big(k, d_i^j\big) = \psi_k(m + i, n + j). \tag{52}$$

The proof of (51) is obtained by showing that the m- and n-differences and t-derivatives of LHS and RHS all agree. This is sufficient since the pfaffian element is only defined up to an arbitrary constant.

From (51), it follow by [12, p. 99] that $(1, 2, \ldots, 2N)' = (1, 2, \ldots, 2N) - (1, 2, \ldots, 2N, c_0^0, d_0^0)$ and so we have pfaffian expressions for f and g in terms of the same pfaffian element (i, j):

$$f = \tau(), \qquad g = \sigma() - \sigma\big(c_0^0, d_0^0\big), \tag{53}$$

in which, once again, we omit the indices $1, 2, \ldots, 2N$ for brevity.

To verify these solutions, we must calculate the m- and n-shifts and t-derivatives of f and g. The required expressions for f are given in the last section and for g we have:

$$g_m = \sigma_m() - \frac{\tau_m\sigma}{\tau}\big(c_1^0, d_0^0\big), \qquad g_{\bar m} = \sigma_{\bar m}() - \frac{\tau_{\bar m}\sigma}{\tau}\big(c_{-1}^0, d_0^0\big), \tag{54}$$

$$g_n = \sigma_n() - \frac{\tau_n\sigma}{\tau}\big(c_0^1, d_0^0\big), \qquad g_{\bar n} = \sigma_{\bar n}() - \frac{\tau_{\bar n}\sigma}{\tau}\big(c_0^{-1}, d_0^0\big), \tag{55}$$

$$\begin{aligned} g_{mn} = {} & \sigma_{mn}() + \frac{\tau_{mn}\sigma}{\tau}\big(c_0^0, d_0^0\big) - \frac{\tau_m\sigma_n}{\tau}\big(c_0^0, c_1^0\big) - \frac{\tau_n\sigma_m}{\tau}\big(c_0^1, c_0^0\big) \\ & + \frac{\sigma\tau_m\tau_n}{\tau^2}\big(\big(c_0^1, c_1^0\big) + \big(d_0^0, c_1^0\big) + \big(d_0^0, c_0^1\big) - \big(d_0^0, c_0^1, c_0^0, c_1^0\big)\big), \end{aligned} \tag{56}$$

$$\begin{aligned} 2g_t = {} & 2\sigma_t() - 2\sigma_t\big(c_0^0, d_0^0\big) + \frac{\sigma_m\tau_{\bar m}}{\tau}\big(\big(c_0^0, c_{-1}^0\big) + \big(d_0^0, c_0^0\big)\big) \\ & + \frac{\tau_m\sigma_{\bar m}}{\tau}\big(\big(c_1^0, c_0^0\big) + \big(c_0^0, d_0^0\big)\big) \\ & + \frac{\tau_{\bar n}\sigma_n}{\tau}\big(\big(c_0^{-1}, c_0^0\big) + \big(d_0^0, c_0^0\big)\big) + \frac{\tau_n\sigma_{\bar n}}{\tau}\big(\big(c_0^0, c_0^1\big) + \big(c_0^0, d_0^0\big)\big) \\ & + \frac{\sigma\tau_n\tau_{\bar n}}{\tau^2}\big(-\big(c_0^0, d_0^0, c_0^1, c_0^{-1}\big) + \big(c_0^1, c_0^{-1}\big) + \big(d_0^0, c_0^{-1}\big) - \big(d_0^0, c_0^1\big)\big) \\ & + \frac{\sigma\tau_m\tau_{\bar m}}{\tau^2}\big(-\big(c_0^0, d_0^0, c_{-1}^0, c_1^0\big) + \big(c_{-1}^0, c_1^0\big) + \big(d_0^0, c_{-1}^0\big) - \big(d_0^0, c_1^0\big)\big). \end{aligned} \tag{57}$$

Substituting (53) into (10)–(11) and using the fact that τ, σ satisfy (10)–(11) lead to the two pfaffian identities:

$$\begin{aligned}
&\big(d_0^0, c_0^1, c_0^0, c_1^0\big)() - \big(d_0^0, c_0^1\big)\big(c_0^0, c_1^0\big) + \big(d_0^0, c_0^0\big)\big(c_0^1, c_1^0\big) - \big(d_0^0, c_1^0\big)\big(c_0^1, c_0^0\big) = 0, \quad (58)\\
&\frac{\sigma\tau_n\tau_{\bar{n}}}{\tau}\Big(\big(c_0^0, d_0^0, c_0^1, c_0^{-1}\big)() - \big(c_0^0, d_0^0\big)\big(c_0^1, c_0^{-1}\big) + \big(c_0^0, c_0^1\big)\big(d_0^0, c_0^{-1}\big)\\
&\quad - \big(c_0^0, c_0^{-1}\big)\big(d_0^0, c_0^1\big)\Big)\\
&\quad + \frac{\sigma\tau_m\tau_{\bar{m}}}{\tau}\Big(\big(c_0^0, d_0^0, c_{-1}^0, c_1^0\big)() - \big(c_0^0, d_0^0\big)\big(c_{-1}^0, c_1^0\big) + \big(c_0^0, c_{-1}^0\big)\big(d_0^0, c_1^0\big)\\
&\quad - \big(c_0^0, c_1^0\big)\big(d_0^0, c_{-1}^0\big)\Big) = 0. \quad (59)
\end{aligned}$$

This completes the proof of the pfaffian solutions of the modified equation.

6. Soliton solutions

In this section, we will derive explicit soliton solutions as a special case of the pfaffian solutions found in the previous section. Let $\tau = \sigma = 1$, then (53) becomes

$$f = (1, 2, \ldots, 2N), \qquad g = (1, 2, \ldots, 2N) - \big(1, 2, \ldots, 2N, c_0^0, d_0^0\big). \quad (60)$$

For this choice of solutions τ, σ, (28)–(29) and (32)–(33) give

$$\Delta_m(\phi + \psi) = 0, \qquad \Delta_n(\phi - \psi) = 0, \quad \text{and} \quad (61)$$

$$\Delta_m\Delta_n\phi = 0, \qquad \Delta_m\Delta_n\psi = 0, \quad (62)$$

which imply that eigenfunctions ϕ and ψ can be decomposed as

$$\phi(m, n, t) = \Phi(n, t) + \Psi(m, t), \qquad \psi(m, n, t) = \Phi(n, t) - \Psi(m, t). \quad (63)$$

Further, by considering (30)–(31), we see that the only condition on Φ and Ψ is that they satisfy

$$2\Phi_t = \Phi_{\bar{n}} - \Phi_n, \quad (64)$$

$$2\Psi_t = \Psi_{\bar{m}} - \Psi_m. \quad (65)$$

The simplest choice of eigenfunctions is hence $\phi_i = \Phi_i + \Psi_i$, $\psi_i = \Phi_i - \Psi_i$ where

$$\Phi_i = \alpha_i\left(\frac{1 - p_i}{1 + p_i}\right)^{-n} \exp\left(\frac{-2p_i t}{1 - p_i^2}\right), \quad (66)$$

$$\Psi_i = \beta_i\left(\frac{1 - q_i}{1 + q_i}\right)^{-m} \exp\left(\frac{-2q_i t}{1 - q_i^2}\right), \quad (67)$$

in which α_i, β_i, p_i, and q_i are arbitrary constants.

Although in the previous section, we were only able to define the pfaffian element (i, j) implicitly through relations (34)–(36), for the above eigenfunctions, we may calculate it explicitly. From these relations, we get

$$(i,j)=\frac{p_i-p_j}{p_i+p_j}\Phi_i\Phi_j-\frac{q_i-q_j}{q_i+q_j}\Psi_i\Psi_j+\Phi_i\Psi_j-\Psi_i\Phi_j+C_{i,j}, \tag{68}$$

$$\left(i,j,c_0^0,d_0^0\right)=2\Phi_j\Psi_i-2\Phi_i\Psi_j, \tag{69}$$

where $C_{i,j}=-C_{j,i}$.

First, let us consider the case $N=1$ and choose $C_{1,2}=0$. Then

$$f=(1,2)=\frac{p_1-p_2}{p_1+p_2}\Phi_1\Phi_2-\frac{q_1-q_2}{q_1+q_2}\Psi_1\Psi_2+\Phi_1\Psi_2-\Psi_1\Phi_2,$$

$$g=(1,2)-2\Phi_1\Psi_2+2\Phi_2\Psi_1=\frac{p_1-p_2}{p_1+p_2}\Phi_1\Phi_2-\frac{q_1-q_2}{q_1+q_2}\Psi_1\Psi_2-\Phi_1\Psi_2+\Psi_1\Phi_2.$$

In order to compare these expressions with the more usual form of soliton solution, we will scale each of f and g by the same factor $(q_1+q_2)/((q_1-q_2)\Psi_1\Psi_2)$ which has the form am^be^ct, where a,b,c are given constants. It is easy to check that the bilinear form (10)–(11) is invariant under such a scaling. Thus we have the solution

$$f=1+\frac{\Phi_1}{\Psi_1}+\frac{\Phi_2}{\Psi_2}+\frac{(p_1-p_2)(q_1-q_2)}{(p_1+p_2)(q_1+q_2)}\frac{\Phi_1}{\Psi_1}\frac{\Phi_2}{\Psi_2}, \tag{70}$$

$$g=1-\frac{\Phi_1}{\Psi_1}-\frac{\Phi_2}{\Psi_2}+\frac{(p_1-p_2)(q_1-q_2)}{(p_1+p_2)(q_1+q_2)}\frac{\Phi_1}{\Psi_1}\frac{\Phi_2}{\Psi_2}, \tag{71}$$

with Φ_i and Ψ_i given by (66)–(67), which has the form of the two-soliton solution as described in [15]. Notice also that when we set $p_2=q_2=0$, we obtain (up to an irrelevant factor)

$$f=1+\frac{\Phi_1}{\Psi_1},\qquad g=1-\frac{\Phi_1}{\Psi_1}, \tag{72}$$

which is a one-soliton solution.

Similarly, we can obtain the $2N$- and then the $(2N-1)$-soliton solutions for any N. For example, the 3-soliton solutions are given by

$$\begin{aligned}f=1&+\frac{\Phi_1}{\Psi_1}+\frac{\Phi_2}{\Psi_2}+\frac{\Phi_3}{\Psi_3}+\frac{(p_1-p_2)(q_1-q_2)}{(p_1+p_2)(q_1+q_2)}\frac{\Phi_1}{\Psi_1}\frac{\Phi_2}{\Psi_2}\\&+\frac{(p_1-p_3)(q_1-q_3)}{(p_1+p_3)(q_1+q_3)}\frac{\Phi_1}{\Psi_1}\frac{\Phi_3}{\Psi_3}+\frac{(p_2-p_3)(q_2-q_3)}{(p_2+p_3)(q_2+q_3)}\frac{\Phi_2}{\Psi_2}\frac{\Phi_3}{\Psi_3}\\&+\frac{(p_1-p_2)(q_1-q_2)(p_1-p_3)(q_1-q_3)(p_2-p_3)(q_2-q_3)}{(p_1+p_2)(q_1+q_2)(p_1+p_3)(q_1+q_3)(p_2+p_3)(q_2+q_3)}\frac{\Phi_1\Phi_2\Phi_3}{\Psi_1\Psi_2\Psi_3},\\g=1&-\frac{\Phi_1}{\Psi_1}-\frac{\Phi_2}{\Psi_2}-\frac{\Phi_3}{\Psi_3}+\frac{(p_1-p_2)(q_1-q_2)}{(p_1+p_2)(q_1+q_2)}\frac{\Phi_1}{\Psi_1}\frac{\Phi_2}{\Psi_2}\\&+\frac{(p_1-p_3)(q_1-q_3)}{(p_1+p_3)(q_1+q_3)}\frac{\Phi_1}{\Psi_1}\frac{\Phi_3}{\Psi_3}+\frac{(p_2-p_3)(q_2-q_3)}{(p_2+p_3)(q_2+q_3)}\frac{\Phi_2}{\Psi_2}\frac{\Phi_3}{\Psi_3}\\&-\frac{(p_1-p_2)(q_1-q_2)(p_1-p_3)(q_1-q_3)(p_2-p_3)(q_2-q_3)}{(p_1+p_2)(q_1+q_2)(p_1+p_3)(q_1+q_3)(p_2+p_3)(q_2+q_3)}\frac{\Phi_1\Phi_2\Phi_3}{\Psi_1\Psi_2\Psi_3}.\end{aligned}$$

Here we would like to point out that in [13–15], 3-soliton solutions of continuous mKdV-type bilinear equations were considered. In the above, we obtained 3-soliton solutions for

the discrete mKdV-type equation, that is, mLV2 equation, from pfaffian solutions which can be also derived similarly as in [13–15].

7. Reduction to one discrete variable

In Section 2 we mentioned the fact that when $m = n$, mLV2 becomes the $(1 + 1)$-dimensional self-dual nonlinear network equation (15). In this section, we will give details of how this reduction comes through, showing how one obtains solutions of mLV2 depending only on $m + n$, that permit the reduction $m = n$. We will thereby obtain pfaffian and soliton solutions of (15). We remark that this procedure also gives the same types of solutions of the Lotka–Volterra equation as a reduction of LV2, but we will give full details for the modified equation only.

Consider eigenfunctions of the Lax pair (28)–(31) of the form $\phi(m, n, t) = \lambda^k \tilde{\phi}(\ell, t)$, $\psi(m, n, t) = \lambda^k \tilde{\psi}(\ell, t)$, where $k = m - n$, $\ell = m + n$, and λ is a constant that will become the eigenvalue in the reduced Lax pair. By making this substitution in (28)–(31) and (32)–(33) and assuming $f = f(\ell)$ and $g = g(\ell)$, we obtain

$$(\phi - \lambda\psi_\ell) g_\ell f - (\lambda\phi_\ell - \psi) f_\ell g = 0, \tag{73}$$

$$\big(\phi + \lambda^{-1}\psi_\ell\big) g_\ell f - \big(\lambda^{-1}\phi_\ell + \psi\big) f_\ell g = 0, \tag{74}$$

$$2\phi_t + \big(\lambda + \lambda^{-1}\big)\frac{f_\ell f_{\bar{\ell}}}{f^2}(\phi_\ell - \phi_{\bar{\ell}}) = 0, \tag{75}$$

$$2\psi_t + \big(\lambda + \lambda^{-1}\big)\frac{g_\ell g_{\bar{\ell}}}{g^2}(\psi_\ell - \psi_{\bar{\ell}}) = 0, \tag{76}$$

and

$$\phi_\ell + \phi_{\bar{\ell}} = \big(\lambda + \lambda^{-1}\big)\frac{f^2}{f_\ell f_{\bar{\ell}}}\phi, \tag{77}$$

$$\psi_\ell + \psi_{\bar{\ell}} = \big(\lambda + \lambda^{-1}\big)\frac{g^2}{g_\ell g_{\bar{\ell}}}\psi, \tag{78}$$

where, for a simpler notation, we have omitted $\tilde{}$ above ϕ and ψ.

In a similar way, the pfaffian element (i, j) defined by (34)–(36) can be written as $(\lambda_i\lambda_j)^k \widetilde{(i, j)}(\ell, t)$, and the defining relations become

$$\lambda_i\lambda_j(i, j)_\ell = (i, j) - \lambda_i\phi_{i\,\ell}\phi_j + \lambda_j\phi_i\phi_{j\,\ell}, \tag{79}$$

$$\lambda_i^{-1}\lambda_j^{-1}(i, j)_\ell = (i, j) + \lambda_i^{-1}\phi_{i\,\ell}\phi_j - \lambda_j^{-1}\phi_i\phi_{j\,\ell}, \tag{80}$$

$$(i, j)_t = 2\big(\lambda_i\lambda_j^{-1} - \lambda_i^{-1}\lambda_j\big)\frac{f_\ell f_{\bar{\ell}}}{f^2}(\phi_{i\,\ell}\phi_{j\,\bar{\ell}} + \phi_{i\,\ell}\phi_{j\,\bar{\ell}}). \tag{81}$$

Now we may solve (79)–(80) to obtain an explicit expression for the reduced pfaffian element:

$$(i, j) = \big(\lambda_i\lambda_j - \lambda_i^{-1}\lambda_j^{-1}\big)^{-1}\Big(\big(\lambda_i + \lambda_i^{-1}\big)\phi_i\phi_{j\,\ell} - \big(\lambda_j + \lambda_j^{-1}\big)\phi_{i\,\ell}\phi_j\Big). \tag{82}$$

As would be expected, it is straightforward to check that this expressions is compatible with all of the defining relations (79)–(81). Finally, we use (77) to obtain

$$(i,j) = \left(\lambda_i\lambda_j - \lambda_i^{-1}\lambda_j^{-1}\right)^{-1}\frac{f_\ell f_{\bar{\ell}}}{f^2}\left(\phi_{i\,\bar{\ell}}\phi_{j\,\ell} - \phi_{i\,\ell}\phi_{j\,\bar{\ell}}\right). \tag{83}$$

In the same way, we obtain

$$(i,j)' = \left(\lambda_i\lambda_j - \lambda_i^{-1}\lambda_j^{-1}\right)^{-1}\frac{g_\ell g_{\bar{\ell}}}{g^2}(\psi_{i\,\bar{\ell}}\psi_{j\,\ell} - \psi_{i\,\ell}\psi_{j\,\bar{\ell}}) \tag{84}$$

$$= (i,j) + \phi_i\psi_j - \phi_j\psi_i = (i,j) - (i,j,c_0,d_0), \tag{85}$$

where $(k,c_j) = \phi_k(\ell + j)$ and $(k,d_j) = \psi_k(\ell + j)$. It then follows that the pfaffian solutions for the potential self-dual network equation are given by

$$f = \tau(1,2,\ldots,2N), \qquad g = \sigma(1,2,\ldots,2N)',$$

where τ and σ are arbitrary solutions of (16)–(17), (i,j) and $(i,j)'$ are defined by (83)–(85) and ϕ and ψ satisfy the Lax pair (73)–(76).

To illustrate this procedure, we obtain the soliton solutions of the potential self-dual network equation and take $\tau = \sigma = 1$. We choose eigenfunctions ϕ_i and ψ_i as

$$\phi_i = \alpha_i\lambda_i^{-\ell}\exp\left(\tfrac{1}{2}\left(\lambda_i^2 - \lambda_i^{-2}\right)t\right) + \beta_i\lambda_i^{\ell}\exp\left(-\tfrac{1}{2}\left(\lambda_i^2 - \lambda_i^{-2}\right)t\right), \tag{86}$$

$$\psi_i = \alpha_i\lambda_i^{-\ell}\exp\left(\tfrac{1}{2}\left(\lambda_i^2 - \lambda_i^{-2}\right)t\right) - \beta_i\lambda_i^{\ell}\exp\left(-\tfrac{1}{2}\left(\lambda_i^2 - \lambda_i^{-2}\right)t\right). \tag{87}$$

This result can be achieved either by using the reduction described using the soliton eigenfunctions (66)–(67) or directly by seeking solutions of the reduced Lax pair (73)–(76). In the first method, one must choose

$$\lambda_i = \sqrt{\frac{1-p_i}{1+p_i}} = \sqrt{\frac{1+q_i}{1-q_i}}$$

and it is also seen that the reduction of the soliton eigenfunction requires $q_i = -p_i$ for each i.

8. Conclusion

In this paper, we have discussed a symmetric $(2+1)$-dimensional modified Lotka–Volterra equation (mLV2) which arises as a Bäcklund transformation for the $(2+1)$-dimensional symmetric Lotka–Volterra equation (LV2). It is reduced to a known $(1+1)$-dimensional differential–difference equation which is an integrable discretisation of the mKdV equation. We also presented its bilinear Bäcklund transformation and from this derived a Lax pair.

We proved that there are two kinds of pfaffian solutions of the LV2 Bäcklund transformation. First, we showed that it relates two solutions of LV2; it is "soliton adding" in the sense that one solution is defined in terms of N eigenfunctions of the LV2 Lax pair and the other in terms of the same N eigenfunctions together with an extra one. Then, regarding the Bäcklund transformation as a new system mLV2, we obtained pfaffian solutions of mLV2 defined in terms of eigenfunctions of the mLV2 Lax pair.

By making a suitable choice of eigenfunctions, we show that the class of pfaffian solutions we found contain the multisoliton solutions. Finally, we considered the reduction of the these solution to the case $m = n$ and thereby obtained solutions of the potential self-dual network equation in a novel pfaffian form.

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Appendix A. Hirota's bilinear operator identities

The following bilinear operator identities hold for arbitrary functions a, b, c, and d. They are used in establishing BT (19)–(22).

$$\sinh\left(\tfrac{1}{2}D_m\right)a\bullet a = 0, \tag{A.1}$$

$$\begin{aligned}&\left(e^{D_1}a\bullet b\right)\left(e^{D_2}c\bullet d\right) - \left(e^{D_2}a\bullet b\right)\left(e^{D_1}c\bullet d\right)\\ &\quad = 2\sinh\tfrac{1}{2}(D_1 - D_2)\left(e^{\frac{1}{2}(D_1+D_2)}a\bullet d\right)\bullet\left(e^{-\frac{1}{2}(D_1+D_2)}b\bullet c\right),\end{aligned} \tag{A.2}$$

$$\left(e^{D_m}a\bullet b\right)cd - ab\left(e^{D_m}c\bullet d\right) = 2\sinh\left(\tfrac{1}{2}D_m\right)\left(e^{\frac{1}{2}D_m}a\bullet d\right)\bullet\left(e^{-\frac{1}{2}D_m}b\bullet c\right), \tag{A.3}$$

$$\left(e^{-D_m}a\bullet b\right)cd - ab\left(e^{-D_m}c\bullet d\right) = -2\sinh\left(\tfrac{1}{2}D_m\right)\left(e^{-\frac{1}{2}D_m}a\bullet d\right)\bullet\left(e^{\frac{1}{2}D_m}b\bullet c\right), \tag{A.4}$$

$$\begin{aligned}&\sinh\left(\tfrac{1}{2}D_n\right)\left(e^{\frac{1}{2}D_m}a\bullet b\right)\bullet\left(e^{-\frac{1}{2}D_m}a\bullet b\right)\\ &\quad = \sinh\left(\tfrac{1}{2}D_m\right)\left(e^{\frac{1}{2}D_n}a\bullet b\right)\bullet\left(e^{-\frac{1}{2}D_n}a\bullet b\right),\end{aligned} \tag{A.5}$$

$$(D_t a\bullet b)cd - abD_t c\bullet d = (D_t a\bullet c)bd - acD_t b\bullet d. \tag{A.6}$$

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