

# On quadratic integral equation of fractional orders

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## Abstract

We present an existence theorem for a nonlinear quadratic integral equations of fractional orders, arising in the queuing theory and biology, in the Banach space of real functions defined and continuous on a bounded and closed interval. The concept of measure of noncompactness and a fixed point theorem due to Darbo are the main tool in carrying out our proof.

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## 1. Introduction

Quadratic integral equations are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in the traffic theory. Especially, the so-called quadratic integral equation of Chandrasekher type can be very often encountered in many applications (cf. [1,3,5,9]). On the other hand, the first serious attempt to give a logical definition of a fractional derivative is due to Liouville, see [8] and references

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therein. Now, the fractional calculus topic is enjoying growing interest among scientists and engineers, see [8,10,11].

Some problems in the queuing theory and biology lead to the following nonlinear integral equation (cf. [6]):

$$y(t) = f(t) + \frac{y(t)}{\Gamma(\alpha)} \int_0^t \frac{u(\tau, y(\tau))}{(t-\tau)^{1-\alpha}} d\tau, \quad t \in [0, T]. \quad (1.1)$$

This equation creates an example of the so-called quadratic integral equation [5]. Throughout we have  $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  and  $f : [0, T] \rightarrow \mathbb{R}$  are functions satisfies special assumptions, see Section 3.

**Definition 1.** Let  $g \in L_1(a, b)$ ,  $0 \leq a < b < \infty$ , and let  $\alpha > 0$  be a real number. The (Riemann–Liouville) fractional integral of order  $\alpha$  is defined by (see [10,11])

$$I^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s)}{(t-s)^{1-\alpha}} ds.$$

Rewrite Eq. (1.1) in the form

$$y(t) = f(t) + y(t)I^\alpha u(t, y(t)), \quad t \in [0, T], \quad (1.2)$$

where  $I^\alpha$  is the standard Riemann–Liouville fractional integral.

Using the technique associated with measures of noncompactness, we show that Eq. (1.2) has solutions belong to  $C([0, T])$  and is nondecreasing on the interval  $[0, T]$ .

In fact, our results in this paper are motivated by the extensions of the work of Banaś and Martinon [1] based on the a measure of noncompactness and fixed point theorem due to Darbo.

## 2. Auxiliary facts and results

This section is devoted to collect some definitions and results which will be needed further on. Assume  $(E, \|\cdot\|)$  is an infinitely dimensional Banach space with zero element  $\theta$ . Let  $B(x, r)$  denotes the closed ball centered at  $x$  and with radius  $r$ . The symbol  $B_r$  stands for the ball  $B(\theta, r)$ .

If  $X$  is a subset of  $E$ , then  $\bar{X}$  and  $\text{Conv } X$  denote the closure and convex closure of  $X$ , respectively. Moreover, we denote by  $\mathcal{M}_E$  the family of all nonempty and bounded subsets of  $E$  and  $\mathcal{N}_E$  its subfamily consisting of all relatively compact subsets.

Next we give the concept of a measure of noncompactness [4]:

**Definition 2.** A mapping  $\mu : \mathcal{M}_E \rightarrow [0, +\infty)$  is said to be a measure of noncompactness in  $E$  if it satisfies the following conditions:

- (1) The family  $\text{Ker } \mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$  is nonempty and  $\text{Ker } \mu \subset \mathcal{N}_E$ .

- (2)  $X \subseteq Y \Rightarrow \mu(X) \leq \mu(Y)$ .  
 (3)  $\mu(\bar{X}) = \mu(\text{Conv } X) = \mu(X)$ .  
 (4)  $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$  for  $0 \leq \lambda \leq 1$ .  
 (5) If  $X_n \in \mathcal{M}_E$ ,  $X_n = \bar{X}_n$ ,  $X_{n+1} \subset X_n$  for  $n = 1, 2, 3, \dots$  and  $\lim_{n \rightarrow \infty} \mu(X_n) = 0$ , then  $\bigcap_{n=1}^{\infty} X_n \neq \emptyset$ .

In what follows we will work in the Banach space  $C[0, T]$  consisting of all real functions defined and continuous on  $[0, T]$ . The space  $C([0, T])$  is equipped with the standard norm

$$\|x\| = \max\{|x(t)| : t \geq 0\}.$$

Now, we recollect the construction of the measure of noncompactness which will be used in the next section (see [1,2]).

Let us fix a nonempty and bounded subset  $X$  of  $C([0, T])$ . For  $x \in X$  and  $\varepsilon \geq 0$  denoted by  $\omega(x, \varepsilon)$ , the modulus of continuity of the function  $x$ , i.e.,

$$\omega(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, T], |t - s| \leq \varepsilon\}.$$

Further, let us put

$$\omega(X, \varepsilon) = \sup\{\omega(x, \varepsilon) : x \in X\}, \quad \omega_0(X) = \lim_{\varepsilon \rightarrow 0} \omega(X, \varepsilon).$$

Define

$$\begin{aligned} d(x) &= \sup\{|x(s) - x(t)| - [x(s) - x(t)] : t, s \in [0, T], t \leq s\}, \\ d(X) &= \sup\{d(x) : x \in X\}, \\ i(x) &= \sup\{|x(t) - x(s)| - [x(t) - x(s)] : t, s \in [0, T], t \leq s\}, \\ i(X) &= \sup\{i(x) : x \in X\}. \end{aligned}$$

All functions belonging to  $X$  are nondecreasing on  $[0, T]$  if and only if  $d(X) = 0$ . Similarly, we can characterize the set  $X$  with  $i(X) = 0$ .

Now, let us define the function  $\mu$  on the family  $\mathcal{M}_{C([0, T])}$  by the formula

$$\mu(X) = \omega_0(X) + d(X).$$

The function  $\mu$  is a measure of noncompactness in the space  $C([0, T])$  [2].

**Remark 1.** In a similar way, we can define the measure of noncompactness associated with the set  $i(X)$ , but we omit the details concerning that measure.

Finally, the fixed point theorem due to Darbo will be recalled [7]:

**Theorem 1.** Let  $Q$  be a nonempty, bounded, closed and convex subset of the space  $E$  and let

$$H : Q \rightarrow Q$$

be a continuous transformation which is a contraction with respect to the measure of noncompactness  $\mu$ , i.e., there exists a constant  $0 \leq k < 1$  such that  $\mu(HX) \leq k\mu(X)$  for any nonempty subset  $X$  of  $Q$ .

Then  $H$  has a fixed point in the set  $Q$ .

### 3. Main theorem

In this section, we will study Eq. (1.2) assuming that the following assumptions are satisfied:

- (a<sub>1</sub>)  $f : [0, T] \rightarrow \mathbb{R}$  is a continuous, nondecreasing and nonnegative function on  $[0, T]$ .
- (a<sub>2</sub>)  $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function such that  $u : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and for arbitrary  $y \in \mathbb{R}_+$  the function  $t \rightarrow u(t, y)$  is nondecreasing on  $[0, T]$ .
- (a<sub>3</sub>) There exists a nondecreasing function  $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that

$$|u(t, y)| \leq \phi(|y|)$$

for all  $t \in [0, T]$  and  $y \in \mathbb{R}$ .

- (a<sub>4</sub>) The inequality

$$\Gamma(\alpha + 1)\|f\| + rT^\alpha\phi(r) \leq r\Gamma(\alpha + 1)$$

has a positive solution  $r_0$  such that  $T^\alpha\phi(r_0) < \Gamma(\alpha + 1)$ .

Now, we are in a position to state and prove our main result in papers.

**Theorem 2.** Let  $\|f\| \neq 0$ . Let the assumptions (a<sub>1</sub>)–(a<sub>4</sub>) be satisfied. Then Eq. (1.2) has at least one solution  $y \in C([0, T])$  being nondecreasing on the interval  $[0, T]$ .

**Proof.** Denote by  $U$  the operator associated with the right-hand side of Eq. (1.2), i.e., Eq. (1.2) takes the form

$$y = Uy, \tag{3.1}$$

where

$$(Uy)(t) = f(t) + y(t)I^\alpha u(t, y(t)), \quad t \in [0, T]. \tag{3.2}$$

Solving Eq. (1.2) is equivalent to finding a fixed point of the operator  $U$  defined on the space  $C([0, T])$ .

First, observe that for a given  $y \in C([0, T])$ , we have  $Uy \in C([0, T])$ , thanks (a<sub>1</sub>) and (a<sub>2</sub>), i.e., the operator  $U$  maps  $C([0, T])$  into itself. Moreover, in virtue of (a<sub>1</sub>)–(a<sub>3</sub>) we have

$$\begin{aligned} |(Uy)(t)| &\leq |f(t)| + |y(t)| |I^\alpha u(t, y(t))| \\ &\leq \|f\| + \frac{1}{\Gamma(\alpha)} \|y\| \int_0^t \frac{\phi(\|y\|)}{(t-s)^{1-\alpha}} ds \\ &\leq \|f\| + \frac{1}{\Gamma(\alpha+1)} \|y\| \phi(\|y\|) T^\alpha. \end{aligned}$$

Hence

$$\|Uy\| \leq \|f\| + \frac{1}{\Gamma(\alpha+1)} \|y\| \phi(\|y\|) T^\alpha,$$

which means that the operator  $U$  transforms the ball  $B_{r_0}$  into itself, where  $r_0 = \frac{\Gamma(\alpha+1)\|f\|}{\Gamma(\alpha+1)-T^\alpha\phi(r_0)}$ .

Further, let us consider the operator  $U$  on the subset  $Q$  of  $B_{r_0}$  defined in the following way:

$$Q = \{y \in B_{r_0} : y(t) \geq 0, \text{ for } t \in [0, T]\}.$$

Then the set  $Q$  is nonempty, bounded, closed and convex in  $C([0, T])$ . In view of these facts and assumptions  $(a_1)$  and  $(a_2)$  we conclude that  $U$  maps the set  $Q$  into itself.

**We claim** that the operator  $U$  is continuous. To establish this claim, let us fix  $\varepsilon > 0$  and take arbitrary  $x, y \in Q$  such that  $\|y - x\| \leq \varepsilon$ . Then, for  $t \in [0, T]$ , we have

$$\begin{aligned} & |(Uy)(t) - (Ux)(t)| \\ & \leq |y(t)I^\alpha u(t, y(t)) - x(t)I^\alpha u(t, x(t))| \\ & \leq |y(t)I^\alpha u(t, y(t)) - x(t)I^\alpha u(t, y(t))| + |x(t)I^\alpha u(t, y(t)) - x(t)I^\alpha u(t, x(t))| \\ & \leq |y(t) - x(t)| |I^\alpha u(t, y(t))| + |x(t)| |I^\alpha u(t, y(t)) - I^\alpha u(t, x(t))| \\ & \leq \frac{1}{\Gamma(\alpha)} \|y - x\| \int_0^t \frac{\phi(\|y\|)}{(t-s)^{1-\alpha}} ds + \frac{1}{\Gamma(\alpha)} \|x\| \int_0^t \frac{\beta_{r_0}(\varepsilon)}{(t-s)^{1-\alpha}} ds \\ & \leq \frac{\varepsilon}{\Gamma(\alpha+1)} \phi(r_0) T^\alpha + \frac{r_0}{\Gamma(\alpha+1)} T^\alpha \beta_{r_0}(\varepsilon), \end{aligned}$$

where

$$\beta_{r_0}(\varepsilon) = \sup\{|u(t, y) - u(t, x)| : t \in [0, T], x, y \in [0, r], \|y - x\| \leq \varepsilon\}.$$

By uniform continuity of the function  $U$  on the set  $[0, T] \times [0, r_0]$ , it is easy to see that  $\beta_{r_0}(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . From the above estimate, we have

$$\|Uy - Ux\| \leq \frac{\varepsilon}{\Gamma(\alpha+1)} \phi(r_0) T^\alpha + \frac{r_0}{\Gamma(\alpha+1)} T^\alpha \beta_{r_0}(\varepsilon),$$

which implies the continuity of the operator  $U$  on the set  $Q$ .

Now, let us take a nonempty set  $X \subset Q$ . Fix arbitrarily a number  $\varepsilon > 0$  and choose  $y \in X$  and  $s, t \in [0, T]$  such that  $|s - t| \leq \varepsilon$ . Then, in view of our assumptions, we obtain

$$\begin{aligned} |(Uy)(s) - (Uy)(t)| & \leq |f(s) - f(t)| + |y(s)I^\alpha u(s, y(s)) - y(t)I^\alpha u(t, y(t))| \\ & \leq \omega(f, \varepsilon) + |y(s)I^\alpha u(s, y(s)) - y(t)I^\alpha u(s, y(s))| \\ & \quad + |y(t)I^\alpha u(s, y(s)) - y(t)I^\alpha u(t, y(t))| \\ & \leq \omega(y, \varepsilon) + |y(s) - y(t)| |I^\alpha u(s, y(s))| \\ & \quad + \frac{|y(t)|}{\Gamma(\alpha)} \left| \int_0^s \frac{u(\tau, y(\tau))}{(s-\tau)^{1-\alpha}} d\tau - \int_0^t \frac{u(\tau, y(\tau))}{(s-\tau)^{1-\alpha}} d\tau \right| \\ & \quad + \frac{|y(t)|}{\Gamma(\alpha)} \left| \int_0^t \frac{u(\tau, y(\tau))}{(s-\tau)^{1-\alpha}} d\tau - \int_0^t \frac{u(\tau, y(\tau))}{(t-\tau)^{1-\alpha}} d\tau \right| \end{aligned}$$

$$\begin{aligned}
&\leq \omega(f, \varepsilon) + \frac{\omega(y, \varepsilon)}{\Gamma(\alpha)} \int_0^s \frac{\phi(\|y\|)}{(s-\tau)^{1-\alpha}} d\tau \\
&\quad + \frac{\|y\|}{\Gamma(\alpha)} \int_t^s \frac{|u(\tau, y(\tau))|}{(s-\tau)^{1-\alpha}} d\tau \\
&\quad + \frac{\|y\|}{\Gamma(\alpha)} \int_0^t |(s-\tau)^{\alpha-1} - (t-\tau)^{\alpha-1}| |u(\tau, y(\tau))| d\tau \\
&\leq \omega(f, \varepsilon) + \frac{\omega(y, \varepsilon)}{\Gamma(\alpha+1)} \phi(r_0) T^\alpha + \frac{r_0}{\Gamma(\alpha+1)} \phi(r_0) |s-t|^\alpha \\
&\quad + \frac{r_0}{\Gamma(\alpha+1)} \phi(r_0) \{|s-t|^\alpha + (s^\alpha - t^\alpha)\} \\
&\leq \omega(f, \varepsilon) + \frac{\omega(y, \varepsilon)}{\Gamma(\alpha+1)} \phi(r_0) T^\alpha \\
&\quad + \frac{r_0}{\Gamma(\alpha+1)} \phi(r_0) \{2|s-t|^\alpha + (s^\alpha - t^\alpha)\}.
\end{aligned}$$

Then we have

$$\omega_0(UX) \leq \frac{\phi(r_0)}{\Gamma(\alpha+1)} T^\alpha \omega_0(X). \quad (3.3)$$

In what follows, fix arbitrary  $x \in X$  and  $t, s \in [0, T]$  such that  $s \geq t$ . Then we have

$$\begin{aligned}
&|(Uy)(s) - (Uy)(t)| - [(Uy)(s) - (Uy)(t)] \\
&= |f(s) + y(s)I^\alpha u(s, y(s)) - f(t) - y(t)I^\alpha u(t, y(t))| \\
&\quad - [f(s) + y(s)I^\alpha u(s, y(s)) - f(t) - y(t)I^\alpha u(t, y(t))] \\
&\leq \{|f(s) - f(t)| - [f(s) - f(t)]\} + |y(s)I^\alpha u(s, y(s)) - y(t)I^\alpha u(t, y(t))| \\
&\quad - [y(s)I^\alpha u(s, y(s)) - y(t)I^\alpha u(t, y(t))] \\
&\leq |y(s)I^\alpha u(s, y(s)) - y(t)I^\alpha u(s, y(s))| + |y(t)I^\alpha u(s, y(s)) - y(t)I^\alpha u(t, y(t))| \\
&\quad - [y(s)I^\alpha u(s, y(s)) - y(t)I^\alpha u(s, y(s))] - [y(t)I^\alpha u(s, y(s)) - y(t)I^\alpha u(t, y(t))] \\
&\leq |y(s) - y(t)| |I^\alpha u(s, y(s))| + |y(t)| |I^\alpha u(s, y(s)) - I^\alpha u(t, y(t))| \\
&\quad - [y(s) - y(t)] I^\alpha u(s, y(s)) - y(t) [I^\alpha u(s, y(s)) - I^\alpha u(t, y(t))] \\
&\leq \{|y(s) - y(t)| - [y(s) - y(t)]\} |I^\alpha u(s, y(s))| \\
&\quad + \frac{|y(t)|}{\Gamma(\alpha)} \left| \int_0^t \frac{u(\tau, y(\tau))}{(s-\tau)^{1-\alpha}} d\tau + \int_t^s \frac{u(\tau, y(\tau))}{(s-\tau)^{1-\alpha}} d\tau - \int_0^t \frac{u(\tau, y(\tau))}{(t-\tau)^{1-\alpha}} d\tau \right| \\
&\quad - \frac{y(t)}{\Gamma(\alpha)} \left[ \int_0^s \frac{u(\tau, y(\tau))}{(s-\tau)^{1-\alpha}} d\tau - \int_0^t \frac{u(\tau, y(\tau))}{(t-\tau)^{1-\alpha}} d\tau \right]
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha)} \{ |y(s) - y(t)| - [y(s) - y(t)] \} \int_0^s \frac{\phi(r_0) d\tau}{(s - \tau)^{1-\alpha}} \\
&\quad + \frac{|y(t)|}{\Gamma(\alpha)} \left\{ \int_0^t |(s - \tau)^{\alpha-1} - (t - \tau)^{\alpha-1}| u(\tau, y(\tau)) d\tau \right. \\
&\quad \left. + \left| \int_t^s \frac{u(\tau, y(\tau)) d\tau}{(s - \tau)^{1-\alpha}} \right| \right\} - \frac{y(t)}{\Gamma(\alpha)} \left[ \int_0^s \frac{u(\tau, y(\tau)) d\tau}{(s - \tau)^{1-\alpha}} - \int_0^t \frac{u(\tau, y(\tau)) d\tau}{(t - \tau)^{1-\alpha}} \right] \\
&\leq \frac{\phi(r_0)}{\Gamma(\alpha + 1)} T^\alpha \{ |y(s) - y(t)| - [y(s) - y(t)] \} \\
&\quad + \frac{|y(t)|}{\Gamma(\alpha)} \left[ \int_0^t \frac{u(\tau, y(\tau)) d\tau}{(s - \tau)^{1-\alpha}} + \int_t^s \frac{u(\tau, y(\tau)) d\tau}{(s - \tau)^{1-\alpha}} - \int_0^t \frac{u(\tau, y(\tau)) d\tau}{(t - \tau)^{1-\alpha}} \right] \\
&\quad - \frac{y(t)}{\Gamma(\alpha)} \left[ \int_0^s \frac{u(\tau, y(\tau)) d\tau}{(s - \tau)^{1-\alpha}} - \int_0^t \frac{u(\tau, y(\tau)) d\tau}{(t - \tau)^{1-\alpha}} \right] \\
&\leq \frac{\phi(r_0)}{\Gamma(\alpha + 1)} T^\alpha \{ |y(s) - y(t)| - [y(s) - y(t)] \} \\
&= \frac{\phi(r_0)}{\Gamma(\alpha + 1)} T^\alpha d(y).
\end{aligned}$$

Hence

$$d(Uy) \leq \frac{\phi(r_0)}{\Gamma(\alpha + 1)} T^\alpha d(y)$$

and consequently

$$d(UX) \leq \frac{\phi(r_0)}{\Gamma(\alpha + 1)} T^\alpha d(X). \quad (3.4)$$

Finally, from (3.3) and (3.4) and the definition of the measure of noncompactness  $\mu$ , we obtain

$$\mu(Uy) \leq \frac{\phi(r_0)}{\Gamma(\alpha + 1)} T^\alpha \mu(X).$$

Now, the above obtained inequality together with the fact that  $T^\alpha \phi(r_0) < \Gamma(\alpha + 1)$  enable us to apply Theorem 1. This complete the proof.  $\square$

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