ELSEVIER

On the existence of periodic solutions for a kind of high-order neutral functional differential equation ☆

Kai Wang [*], Shiping Lu

Department of Mathematics, Anhui Normal University, Wuhu 241000, PR China

Received 2 January 2006

Available online 2 May 2006

Submitted by J.S.W. Wong

Abstract

By using the coincidence degree theory of Mawhin, we study a kind of high-order neutral functional differential equation with distributed delay as follows:

$$\left(x(t) - cx(t-\sigma)\right)^{(n)} + f\left(x(t)\right)x'(t) + g\left(\int_{-r}^{0} x(t+s)\,d\alpha(s)\right) = p(t).$$

Some new results on the existence of periodic solutions are obtained.
© 2006 Elsevier Inc. All rights reserved.

Keywords: Periodic solution; High-order; Neutral functional differential equation; Coincidence degree theory

1. Introduction

There are many results on the existence of periodic solutions for second-order ordinary differential equations. In recent years, by applying the coincidence degree theory of Mawhin, some

people have studied second-order delay functional differential equations, see papers [1–7]. These papers were devoted mainly to studying the following several types of equations:

$$x''(t) + g\bigl(x(t - \tau)\bigr) = p(t),$$
$$x''(t) + ax'(t) + bx(t) + g\bigl(x(t - \tau)\bigr) = p(t),$$
$$\frac{d^2}{dt^2}\bigl(u(t) - ku(t - \tau)\bigr) = f\bigl(u(t)\bigr)u'(t) + \alpha(t)g\bigl(u(t)\bigr) + \sum_{j=1}^{n}\beta_j(t)g\bigl(u(t) - \gamma_j(t)\bigr) + p(t),$$

and

$$\frac{d^2}{dt^2}\left(u(t) - \sum_{j=1}^{n}c_ju(t - r_j)\right)$$
$$= f\bigl(u(t)\bigr)u'(t) + \alpha(t)g\bigl(u(t)\bigr) + \sum_{j=1}^{n}\beta_j(t)g\bigl(u(t) - \gamma_j(t)\bigr) + p(t).$$

The delays of these equations are discrete. It is well known that distributed delay has played a very important role in the circle of biologies in nature. But the work to get the existence of periodic solutions of neutral distributed delay functional differential equations—especially high-order neutral distributed functional differential equations—rarely appeared.

In this paper, we discuss the existence of periodic solutions to a kind of high-order neutral functional differential equation with distributed delay as follows:

$$\bigl(x(t) - cx(t - \sigma)\bigr)^{(n)} + f\bigl(x(t)\bigr)x'(t) + g\left(\int_{-r}^{0} x(t + s)\,d\alpha(s)\right) = p(t), \qquad (1.1)$$

where $f, g : R \to R$ are continuous functions, p is a continuous periodic function defined on R with period $T > 0$, $r > 0$, n is a positive integer, $|c| \neq 1$, $\sigma \in R$, $\alpha : [-r, 0] \to R$ is a bounded variation function.

By employing the coincidence degree theory of Mawhin, we obtain some new results on the existence of periodic solutions of Eq. (1.1). The significance is that we generalize the results of papers [1–7]. Even if for $n = 2$, the conditions imposed on function $g(x)$ are weaker than the corresponding ones of papers [6,7].

2. Main lemmas

We set the following notations: $\bigvee_{-r}^{0}(\alpha) = 1$, where $\bigvee_{-r}^{0}(\alpha)$ is the total variation of $\alpha(s)$ over $[-r, 0]$. Z^+ is a set of positive integers, $\bar{p} = \frac{1}{T}\int_{0}^{T}p(t)\,dt$. $X = \{x : x \in C(R, R),\ x(t + T) \equiv x(t)\}$, with the norm $|x|_0 = \max_{t \in [0, T]}|x(t)|$. $Y = \{x : x \in C^1(R, R),\ x(t + T) \equiv x(t)\}$, with the norm $\|x\| = \max\{|x|_0, |x'|_0\}$. Clearly, X and Y are two *Banach* spaces. We also define operators A and L in the following form:

$$A : X \to X, \quad (Ax)(t) = x(t) - cx(t - \sigma);$$
$$L : \mathrm{Dom}(L) \subset Y \to X, \quad [Lx] = (Ax)^{(n)},$$

where $\mathrm{Dom}(L) = \{x \in C^n(R, R) : x(t + T) \equiv x(t)\}$.

Lemma 2.1. [7] *If* $|c| \neq 1$, *then* A *has continuous bounded inverse on* X, *and*

[H$_1$] $\|A^{-1}x\| \leqslant \frac{\|x\|}{||c|-1|}, \ \forall x \in X$;

[H$_2$] $\int_0^T |(A^{-1}f)(t)| \, dt \leqslant \frac{1}{|1-|c||} \int_0^T |f(s)| \, ds, \ \forall f \in X.$

By Hale's terminology [8], a solution $x(t)$ of Eq. (1.1) is that $x(t) \in C^1(R, R)$ such that $Ax \in C^n(R, R)$ and Eq. (1.1) is satisfied on R. In general, $x(t)$ does not belong to $C^n(R, R)$. But under the condition $|c| \neq 1$, we can see easily from Lemma 2.1 that $(Ax)'(t) = Ax'(t)$, $(Ax)''(t) = Ax''(t), \ldots, (Ax)^{(n)}(t) = Ax^{(n)}(t)$. So a solution $x(t)$ of Eq. (1.1) must belong to $C^{(n)}(R, R)$. According to the first part of Lemma 2.1, we can easily obtain that $\ker L = R$, $\operatorname{Im} L = \{x: \ x \in X, \ \int_0^T x(s) \, ds = 0\}$. So L is a *Fredholm* operator with index zero. Let project operators P and Q as follows:

$$P: Y \to \ker L, \quad Px = (Ax)(0); \qquad Q: X \to X/\operatorname{Im} L, \quad Qy = \frac{1}{T} \int_0^T y(s) \, ds.$$

Then $\operatorname{Im} P = \ker L$, $\ker Q = \operatorname{Im} L$. Set $L_p = L|_{\operatorname{Dom} L \cap \ker P}: \operatorname{Dom} L \cap \ker P \to \operatorname{Im} L$ and let $L_p^{-1}: \operatorname{Im} L \to \operatorname{Dom} L \cap \ker P$ denote the inverse of L_p, then

$$\left[L_p^{-1}y\right](t) = A^{-1}\left(\sum_{i=1}^{n-1} \frac{1}{i!}(Ax)^{(i)}(0)t^i + \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1}y(s) \, ds\right), \tag{2.1}$$

where $(Ax)^{(i)}(0) \ (i = 1, 2, \ldots, n-1)$ are defined by the equation

$$AX = B,$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ c_1 & 1 & 0 & \cdots & 0 & 0 \\ c_2 & c_1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-3} & c_{n-4} & c_{n-5} & \cdots & 1 & 0 \\ c_{n-2} & c_{n-3} & c_{n-4} & \cdots & c_1 & 1 \end{pmatrix}_{(n-1)\times(n-1)}$$

$$X' = \left((Ax)^{(n-1)}(0), \ldots, (Ax)''(0), (Ax)'(0)\right),$$

$$B' = (b_1, b_2, \ldots, b_{n-1}),$$

$$b_i = -\frac{1}{i!T} \int_0^T (T-s)^i y(s) \, ds,$$

and

$$c_j = \frac{T^j}{(j+1)!}, \quad j = 1, 2, \ldots, n-2.$$

Lemma 2.2. [9] *Let* X *and* Y *be two* Banach *spaces*, $L: \operatorname{Dom}(L) \subset X \to Y$ *be a* Fredholm *operator with index zero,* $\Omega \subset X$ *be an open bounded set, and* $N: \bar{\Omega} \to Y$ *be* L-*compact on* $\bar{\Omega}$. *If all the following conditions hold:*

[A$_1$] $Lx \neq \lambda Nx$, $\forall x \in \partial \Omega \cap \text{Dom}(L)$, $\forall \lambda \in (0, 1)$;

[A$_2$] $Nx \notin \text{Im } L$, $\forall x \in \partial \Omega \cap \text{Ker } L$;

[A$_3$] $\deg\{JQN, \Omega \cap \text{Ker } L, 0\} \neq 0$, *where* $J : \text{Im } Q \to \ker L$ *is an isomorphism*,

then equation $Lx = Nx$ *has at least one solution on* $\bar{\Omega} \cap \text{Dom}(L)$.

3. Main results

Theorem 3.1. *Suppose n is an even integer and $|c| < 1$, if there exist constants $a \geqslant 0$ and $M > 0$ such that*

[B$_1$] $x(g(x) - \bar{p}) > 0$, $|x| > M$; *or* $x(g(x) - \bar{p}) < 0$, $|x| > M$;

[B$_2$] $\lim_{x \to +\infty} \sup \frac{|g(x) - \bar{p}|}{|x|} \leqslant a$,

then Eq. (1.1) has at least one T-periodic solution, if $\frac{2aT^n}{1 - |c|} < 1$.

Proof. It is easy to see that Eq. (1.1) has a T-periodic solution if and only if the following operator equation

$$Lx = Nx, \tag{3.1}$$

has a T-periodic solution, where

$$N : Y \to X, \qquad (Nx)(t) = -f(x(t))x'(t) - g\left(\int_{-r}^{0} x(t + s) \, d\alpha(s)\right) + p(t).$$

From (2.1), we see that N is L-compact on $\bar{\Omega}$, where Ω is any open and bounded subset of Y. Take

$$\Omega_1 = \{x : x \in \text{Dom}(L), \ Lx = \lambda Nx, \ \lambda \in (0, 1)\}$$

$\forall x \in \Omega_1$, then x must satisfy

$$\left(x(t) - cx(t - \sigma)\right)^{(n)} + \lambda f(x(t))x'(t) + \lambda g\left(\int_{-r}^{0} x(t + s) \, d\alpha(s)\right) = \lambda p(t). \tag{3.2}$$

Without loss of generality, we may assume that $x(g(x) - \bar{p}) > 0$, $\forall t \in R$, $|x| > M$. By integrating the two sides of Eq. (3.2) over $[0, T]$, we have

$$\int_{0}^{T} \left(g\left(\int_{-r}^{0} x(t + s) \, d\alpha(s)\right) - \bar{p}\right) dt = 0. \tag{3.3}$$

By integral mean value theorem, there is a constant $\xi \in (0, T)$ such that $g(\int_{-r}^{0} x(\xi + s) \, d\alpha(s)) - \bar{p} = 0$. So from assumption [B$_1$] we get $|\int_{-r}^{0} x(\xi + s) \, d\alpha(s)| \leqslant M$. By the properties of Riemann–Stieltjes integral, there is a constant $\zeta \in (-r, 0)$ such that $|x(\xi + \zeta)| \leqslant M$. Be-

cause $\xi + \zeta \in R$, there must exist an integer k_0 such that $\xi + \zeta = k_0 T + t^*$, $t^* \in [0, T)$, then $|x(t^*)| \leqslant M$. Hence we have

$$\big|x(t)\big| \leqslant M + \int_{t^*}^{t} \big|x'(s)\big| \, ds \leqslant M + \int_{0}^{T} \big|x'(s)\big| \, ds, \quad \forall t \in [0, T],$$

i.e.,

$$|x|_0 \leqslant M + \int_{0}^{T} \big|x'(t)\big| \, dt, \quad \forall t \in [0, T]. \tag{3.4}$$

On the other hand, because n is even, there is an integer k such that $n = 2k$, $k \in Z^+$, then multiplying the two sides of Eq. (3.2) by $x(t)$ and integrating them over $[0, T]$, we get

$$(-1)^k \int_{0}^{T} \big|x^{(k)}(t)\big|^2 \, dt = (-1)^k c \int_{0}^{T} x^{(k)}(t - \sigma) x^{(k)}(t) \, dt$$

$$- \lambda \int_{0}^{T} x(t) \left(g \left(\int_{-r}^{0} x(t + s) \, d\alpha(s) \right) - \bar{p} \right) dt.$$

By Hölder inequality, we have

$$\int_{0}^{T} \big|x^{(k)}(t)\big|^2 \, dt \leqslant |c| \left(\int_{0}^{T} \big|x^{(k)}(t)\big|^2 \, dt \right)^{1/2} \left(\int_{0}^{T} \big|x^{(k)}(t - \sigma)\big|^2 \, dt \right)^{1/2}$$

$$+ \int_{0}^{T} \big|x(t)\big| \left| g \left(\int_{-r}^{0} x(t + s) \, d\alpha(s) \right) - \bar{p} \right| dt$$

$$= |c| \left(\int_{0}^{T} \big|x^{(k)}(t)\big|^2 \, dt \right)^{1/2} \left(\int_{-\sigma}^{T - \sigma} \big|x^{(k)}(t)\big|^2 \, dt \right)^{1/2}$$

$$+ \int_{0}^{T} \big|x(t)\big| \left| g \left(\int_{-r}^{0} x(t + s) \, d\alpha(s) \right) - \bar{p} \right| dt$$

$$\leqslant \frac{1}{1 - |c|} |x|_0 \int_{0}^{T} \left| g \left(\int_{-r}^{0} x(t + s) \, d\alpha(s) \right) - \bar{p} \right| dt. \tag{3.5}$$

In view of $\frac{2aT^n}{1 - |c|} < 1$, so there exists a constant $\varepsilon > 0$ such that $\frac{2(a + \varepsilon)T^n}{1 - |c|} < 1$. From condition $[B_2]$, and by the properties of bounded variation function, we get that there exists a constant $\rho > M$ such that

$$\left| g \left(\int_{-r}^{0} x(t + s) \, d\alpha(s) \right) - \bar{p} \right| \leqslant (a + \varepsilon) \left| \int_{-r}^{0} x(t + s) \, d\alpha(s) \right| \leqslant (a + \varepsilon)|x|_0,$$

$$\forall t \in R, \quad \int_{-r}^{0} x(t+s)\,d\alpha(s) > \rho. \tag{3.6}$$

Let $X(t) = \int_{-r}^{0} x(t+s)\,d\alpha(s)$, we set

$$E_1 = \{t \in [0, T]:\ X(t) > \rho\}, \qquad E_2 = \{t \in [0, T]:\ |X(t)| \leqslant \rho\},$$
$$E_3 = \{t \in [0, T]:\ X(t) < -\rho\}.$$

By (3.3) it is easy to see that

$$\left(\int_{E_1} + \int_{E_2} + \int_{E_3} \right) \left(g\left(\int_{-r}^{0} x(t+s)\,d\alpha(s) \right) - \bar{p} \right) dt = 0. \tag{3.7}$$

Hence

$$\int_{E_3} \left| g\left(\int_{-r}^{0} x(t+s)\,d\alpha(s) \right) - \bar{p} \right| dt = -\int_{E_3} \left(g\left(\int_{-r}^{0} x(t+s)\,d\alpha(s) \right) - \bar{p} \right) dt$$

$$= \left(\int_{E_1} + \int_{E_2} \right) \left(g\left(\int_{-r}^{0} x(t+s)\,d\alpha(s) \right) - \bar{p} \right) dt$$

$$\leqslant \left(\int_{E_1} + \int_{E_2} \right) \left| g\left(\int_{-r}^{0} x(t+s)\,d\alpha(s) \right) - \bar{p} \right| dt$$

$$\leqslant (a + \varepsilon)T|x|_0 + \tilde{g}_\rho T. \tag{3.8}$$

Therefore by (3.6) and (3.8), we get

$$\int_{0}^{T} \left| g\left(\int_{-r}^{0} x(t+s)\,d\alpha(s) \right) - \bar{p} \right| dt = \left(\int_{E_1} + \int_{E_2} + \int_{E_3} \right) \left| g\left(\int_{-r}^{0} x(t+s)\,d\alpha(s) \right) - \bar{p} \right| dt$$

$$\leqslant 2\left(\int_{E_1} + \int_{E_2} \right) \left| g\left(\int_{-r}^{0} x(t+s)\,d\alpha(s) \right) - \bar{p} \right| dt$$

$$\leqslant 2(a + \varepsilon)T|x|_0 + 2\tilde{g}_\rho T, \tag{3.9}$$

where $\tilde{g}_\rho = \max_{t \in E_2} |g(\int_{-r}^{0} x(t+s)\,d\alpha(s)) - \bar{p}|$. From (3.5), (3.6) and (3.9), we have

$$\int_{0}^{T} \left| x^{(k)}(t) \right|^2 dt \leqslant \frac{2T(a + \varepsilon)}{1 - |c|} |x|_0^2 + \frac{2T}{1 - |c|} \tilde{g}_\rho |x|_0. \tag{3.10}$$

From $x(0) = x(T)$, $x'(0) = x'(T)$, ..., $x^{(n-1)}(0) = x^{(n-1)}(T)$, we know that there exist $\xi_i \in (0, T)$, $i = 1, 2, \ldots, n$, such that $x'(\xi_1) = x''(\xi_2) = \cdots = x^{(n)}(\xi_n) = 0$. Hence we get

$$\int_{0}^{T} \left| x'(t) \right| dt \leqslant T \int_{0}^{T} \left| x''(t) \right| dt \leqslant T^2 \int_{0}^{T} \left| x'''(t) \right| dt \leqslant \cdots \leqslant T^{n-1} \int_{0}^{T} \left| x^{(n)}(t) \right| dt. \tag{3.11}$$

Following (3.4) and (3.11), we have

$$|x|_0 \leqslant M + \int_0^T |x'(t)| \, dt \leqslant M + T^{k-1} \int_0^T |x^{(k)}(t)| \, dt \tag{3.12}$$

and

$$|x'|_0 \leqslant \int_0^T |x''(t)| \, dt \leqslant T \int_0^T |x'''(t)| \, dt \leqslant \cdots \leqslant T^{k-2} \int_0^T |x^{(k)}(t)| \, dt. \tag{3.13}$$

So it follows from (3.10) and (3.12), we have the following inequality:

$$
\begin{aligned}
\int_0^T |x^{(k)}(t)|^2 \, dt &\leqslant \frac{2T(a+\varepsilon)}{1-|c|} |x|_0^2 + \frac{2T}{1-|c|} \tilde{g}_\rho |x|_0 \\
&\leqslant \frac{2T(a+\varepsilon)}{1-|c|} \left(M + T^{k-1} \int_0^T |x^{(k)}(t)| \, dt \right)^2 \\
&\quad + \frac{2T}{1-|c|} \tilde{g}_\rho \left(M + T^{k-1} \int_0^T |x^{(k)}(t)| \, dt \right) \\
&\leqslant \frac{2T^{2k-1}(a+\varepsilon)}{1-|c|} \left(\int_0^T |x^{(k)}(t)| \, dt \right)^2 + d_1 \int_0^T |x^{(k)}(t)| \, dt + d_2 \\
&\leqslant \frac{2T^n(a+\varepsilon)}{1-|c|} \int_0^T |x^{(k)}(t)|^2 \, dt + d_1 \int_0^T |x^{(k)}(t)| \, dt + d_2,
\end{aligned}
\tag{3.14}
$$

where $d_1 = \frac{2T^k}{1-|c|}(2(a+\varepsilon)+\tilde{g}_\rho)$, $d_2 = \frac{2TM}{1-|c|}((a+\varepsilon)M+\tilde{g}_\rho)$. As $\frac{2(a+\varepsilon)T^n}{1-|c|} < 1$, there is a constant $M_2 > 0$ such that

$$\int_0^T |x^{(k)}(t)|^2 \, dt < M_2. \tag{3.15}$$

From (3.12), (3.13) and (3.15), there exist two constants M_0 and M_1 independent of λ and x such that

$$|x|_0 < M_0, \qquad |x'|_0 < M_1.$$

Let $\widetilde{M} = \max\{M_0, M_1\}$, $\Omega = \{x: \|x\| \leqslant \widetilde{M}\}$ and $\Omega_2 = \{x \in \partial\Omega: x \in \ker L\}$, then

$$QNx = -\frac{1}{T} \int_0^T \left(g\left(\int_{-r}^0 x(t+s) \, d\alpha(s) \right) - \bar{p} \right) dt.$$

If $x = \widetilde{M}$ or $-\widetilde{M}$, then $|x| > M$, we know $g(\int_{-r}^0 x(t+s) \, d\alpha(s)) - \bar{p} > 0$, we have $QNx \neq 0$, which yields a contradiction, i.e., $\forall x \in \Omega$, $x \notin \operatorname{Im} L$. So conditions [A$_1$] and [A$_2$] of Lemma 2.2

are both satisfied. Next we show that condition [A_3] of Lemma 2.2 is also satisfied. Define the isomorphism $J : \operatorname{Im} Q \to \ker L$ as follows: $J(x) = -x$.

Let

$$H(x, \mu) = \mu x + \frac{1 - \mu}{T} J Q N x, \quad (x, \mu) \in \Omega \times [0, 1].$$

Then we have

$$H(x, \mu) = \mu x + \frac{1 - \mu}{T} \left(\frac{1}{T} \int_0^T \left(g \left(\int_{-r}^0 x(t + s) \, d\alpha(s) \right) - \bar{p} \right) dt \right),$$

$$\forall (x, \mu) \in (\partial \Omega \cap \ker L) \times [0, 1].$$

Similar to the above proof we can see that $H(x, \mu) \neq 0$. Hence

$$\deg\{J Q N, \Omega \cap \ker L, 0\}$$
$$= \deg\{H(x, 0), \Omega \cap \ker L, 0\} = \deg\{H(x, 1), \Omega \cap \ker L, 0\} \neq 0.$$

So condition [A_3] of Lemma 2.2 is satisfied. By applying Lemma 2.2, we conclude that Eq. (3.1) has at least one solution $x(t)$ on $\bar{\Omega} \cap D(L)$, i.e., Eq. (1.1) has at least one T-periodic solution $x(t)$. □

Corollary. *Suppose n is an even integer and $|c| < 1$. If there exist constants $a \geqslant 0$ and $M > 0$ such that*

[B_1] $x(g(x) - \bar{p}) > 0, |x| > M$; *or* $x(g(x) - \bar{p}) < 0, |x| > M$;
[B_2] $\lim_{x \to -\infty} \sup \frac{|g(x) - \bar{p}|}{|x|} \leqslant a$,

then Eq. (1.1) has at least one T-periodic solution, if $\frac{2aT^n}{1 - |c|} < 1$.

Theorem 3.2. *Suppose n is an odd integer and $|c| < 1$. If there exist constants $a \geqslant 0$ and $M > 0$ such that*

[B_1] $x(g(x) - \bar{p}) > 0, |x| > M$; *or* $x(g(x) - \bar{p}) < 0, |x| > M$;
[B_2] $\lim_{x \to +\infty} \sup \frac{|g(x) - \bar{p}|}{|x|} \leqslant a$;
[B_3] $f(y) \geqslant 0, \forall y \in R$,

then Eq. (1.1) has at least one T-periodic solution, if $\frac{2aT^n}{1 - |c|} < 1$.

Proof. Without loss of generality, we assume that $x(g(x) - \bar{p}) > 0, \forall t \in R, |x| > M$, and $x(t)$ is an arbitrary T-periodic solution of Eq. (3.2). Notice that n is an odd number, so there is a constant k such that $n = 2k - 1, k \in Z^+$, multiplying the two sides of Eq. (3.2) by $x'(t)$ and integrating them on the interval $[0, T]$, we have

$$(-1)^{k-1} \int_0^T \left|x^{(k)}(t)\right|^2 dt = (-1)^{k-1} c \int_0^T x^{(k)}(t - \sigma) x^{(k)}(t) \, dt - \lambda \int_0^T f(x(t)) [x'(t)]^2 \, dt$$

$$-\lambda \int_0^T x'(t)\left(g\left(\int_{-r}^0 x(t+s)\,d\alpha(s)\right)-\bar{p}\right)dt$$

$$\leqslant (-1)^{k-1}c\int_0^T x^{(k)}(t-\sigma)x^{(k)}(t)\,dt$$

$$-\lambda \int_0^T x'(t)\left(g\left(\int_{-r}^0 x(t+s)\,d\alpha(s)\right)-\bar{p}\right)dt. \tag{3.16}$$

In view of $|x|_0 \leqslant M + \int_0^T |x'(t)|\,dt \leqslant M + T|x'|_0$, by Hölder inequality, we have

$$\int_0^T \left|x^{(k)}(t)\right|^2 dt \leqslant |c|\left(\int_0^T \left|x^{(k)}(t)\right|^2 dt\right)^{1/2}\left(\int_{-\sigma}^{T-\sigma}\left|x^{(k)}(t)\right|^2 dt\right)^{1/2}$$

$$+\int_0^T |x'(t)|\left|g\left(\int_{-r}^0 x(t+s)\,d\alpha(s)\right)-\bar{p}\right|dt$$

$$\leqslant \frac{1}{1-|c|}|x'|_0\int_0^T\left|g\left(\int_{-r}^0 x(t+s)\,d\alpha(s)\right)-\bar{p}\right|dt$$

$$\leqslant \frac{2T(a+\varepsilon)}{1-|c|}|x|_0|x'|_0 + \frac{2T}{1-|c|}\tilde{g}_\rho|x|_0$$

$$\leqslant \frac{2T^2(a+\varepsilon)}{1-|c|}|x'|_0^2 + \frac{2T}{1-|c|}\left(M(a+\varepsilon)+T\tilde{g}_\rho\right)|x'|_0 + \frac{2T}{1-|c|}\tilde{g}_\rho M. \tag{3.17}$$

So from (3.13) and (3.17), we get

$$\int_0^T \left|x^{(k)}(t)\right|^2 dt \leqslant \frac{2T^{2k-2}(a+\varepsilon)}{1-|c|}\left(\int_0^T \left|x^{(k)}(t)\right|dt\right)^2 + d_3\int_0^T \left|x^{(k)}(t)\right|dt + d_4$$

$$\leqslant \frac{2T^n(a+\varepsilon)}{1-|c|}\int_0^T \left|x^{(k)}(t)\right|^2 dt + d_3\int_0^T \left|x^{(k)}(t)\right|dt + d_4, \tag{3.18}$$

where $d_3 = \frac{2T^{k-1}}{1-|c|}(M(a+\varepsilon)+T\tilde{g}_\rho)$, $d_4 = \frac{2T}{1-|c|}\tilde{g}_\rho M$. From assumption $\frac{2aT^n}{1-|c|} < 1$, there exists a constant ε such that $\frac{2(a+\varepsilon)T^n}{1-|c|} < 1$. Hence from (3.18), there is a constant M_2 independent of λ and x such that

$$\int_0^T \left|x^{(k)}(t)\right|^2 dt \leqslant M_2. \tag{3.19}$$

The remainder can be proved in the same way as that in Theorem 3.1. $\quad\square$

Corollary. *Suppose n is an odd integer and $|c| < 1$. If there exist constants $a \geqslant 0$ and $M > 0$ such that*

[B$_1$] $x(g(x) - \bar{p}) > 0$, $|x| > M$; *or* $x(g(x) - \bar{p}) < 0$, $|x| > M$;
[B$_2$] $\lim_{x \to -\infty} \sup \frac{|g(x) - \bar{p}|}{|x|} \leqslant a$;
[B$_3$] $f(y) \geqslant 0$, $\forall y \in R$,

then Eq. (1.1) has at least one T-periodic solution, if $\frac{2aT^n}{1-|c|} < 1$.

Theorem 3.3. *If there exist constants $a \geqslant 0$, $b > 0$ and $M > 0$ such that*

[B$_1$] $x(g(x) - \bar{p}) > 0$, $|x| > M$; *or* $x(g(x) - \bar{p}) < 0$, $|x| > M$;
[B$_2$] $\lim_{x \to +\infty} \sup \frac{|g(x) - \bar{p}|}{|x|} \leqslant a$;
[B$_3$] $|f(y)| \leqslant b$, $\forall y \in R$,

then Eq. (1.1) has at least one T-periodic solution, if $\frac{T^{n-1}(b+2aT)}{|1-|c||} < 1$.

Proof. Without loss of generality, we assume that $x(g(x) - \bar{p}) > 0$, $\forall t \in R$, $|x| > M$, and $x(t)$ is an arbitrary T-periodic solution of Eq. (3.2). So we get

$$\left|Ax^{(n)}(t)\right| \leqslant \lambda\left|f\left(x(t)\right)x'(t)\right| + \lambda\left|g\left(\int_{-r}^{0} x(t+s)\,d\alpha(s)\right)\right| + \lambda\left|p(t)\right|. \tag{3.20}$$

Integrating both sides of (3.20) on the interval $[0, T]$, then

$$\int_0^T \left|Ax^{(n)}(t)\right|dt \leqslant \lambda \int_0^T \left|f\left(x(t)\right)x'(t)\right|dt + \lambda \int_0^T \left|g\left(\int_{-r}^{0} x(t+s)\,d\alpha(s)\right) - \bar{p}\right|dt$$

$$+ \lambda \int_0^T \left|p(t)\right|dt + \lambda \int_0^T |\bar{p}|\,dt$$

$$\leqslant T\left(b|x'|_0 + 2(a + \varepsilon)|x|_0 + 2\tilde{g}_\rho + |p|_0 + |\bar{p}|\right), \tag{3.21}$$

where $|p|_0 = \max_{t \in [0,T]} |p(t)|$. From (3.21) and the second part of Lemma 2.1, we have

$$\int_0^T \left|x^{(n)}(t)\right|dt = \int_0^T \left|A^{-1}Ax^{(n)}(t)\right|dt$$

$$\leqslant \frac{1}{|1-|c||} \int_0^T \left|Ax^{(n)}(t)\right|dt$$

$$\leqslant \frac{T}{|1-|c||}\left(b|x'|_0 + 2(a+\varepsilon)|x|_0 + 2\tilde{g}_\rho + |p|_0 + |\bar{p}|\right). \tag{3.22}$$

Hence from (3.13) and (3.22), we get

$$|x'|_0 \leqslant T^{n-2} \int_0^T |x^{(n)}(t)| \, dt$$

$$\leqslant \frac{T^{n-1}}{|1-|c||} \left(b|x'|_0 + 2(a+\varepsilon)|x|_0 + 2\tilde{g}_\rho + |p|_0 + |\bar{p}| \right)$$

$$\leqslant \frac{T^{n-1}}{|1-|c||} \left((b + 2(a+\varepsilon)T)|x'|_0 + 2(a+\varepsilon)M + 2\tilde{g}_\rho + |p|_0 + |\bar{p}| \right). \qquad (3.23)$$

From assumption $\frac{T^{n-1}(b+2aT)}{|1-|c||} < 1$, there exists a small constant $\varepsilon > 0$ such that $\frac{T^{n-1}(b+2(a+\varepsilon)T)}{|1-|c||} < 1$. So following (3.23), there must exist a constant M_1 independent of λ and x such that

$$|x'|_0 < M_1. \qquad (3.24)$$

The remainder is similar to that in Theorem 3.1. $\quad \square$

Corollary. *If there exist constants $a \geqslant 0$, $b > 0$ and $M > 0$ such that*

[B$_1$] $x(g(x) - \bar{p}) > 0$, $|x| > M$; *or* $x(g(x) - \bar{p}) < 0$, $|x| > M$;
[B$_2$] $\lim_{x \to -\infty} \sup \frac{|g(x) - \bar{p}|}{|x|} \leqslant a$;
[B$_3$] $|f(y)| \leqslant b$, $\forall y \in R$,

then Eq. (1.1) has at least one T-periodic solution, if $\frac{T^{n-1}(b+2aT)}{|1-|c||} < 1$.

Remark 3.4. If $f(x) = 0$, then Eq. (1.1) is a high-order Duffing equation. Furthermore, the conditions in Theorems 3.1–3.3 are all satisfied. If $c = 0$, then Eq. (1.1) is a high-order Liénard equation, and all conditions in the above theorems are still satisfied.

4. Examples

As an application, we list the following examples.

Example 1.

$$\left(x(t) - \frac{1}{2}x(t-\tau) \right)^{(6)} + 2xe^{4(x^2(t)-1)}x'(t)$$

$$+ \int_{-\pi}^0 x(t+s)\,ds\, e^{-(\int_{-\pi}^0 x(t+s)\,ds)^2} = -\frac{1}{2}\cos t. \qquad (4.1)$$

By Theorem 3.1 we let

$$g(x) = xe^{-x^2}, \qquad p(t) = -\frac{1}{2}\cos t, \qquad r = -\pi, \qquad T = 2\pi, \qquad n = 6.$$

Then we get

$$c = \frac{1}{2}, \qquad k = 3, \qquad T = 2\pi, \qquad a = \lim_{x \to +\infty} \left| \frac{g(x) - \bar{p}}{x} \right| = 0, \quad \text{and}$$

$$\frac{2aT^n}{1 - |c|} = 0 < 1.$$

So by applying Theorem 3.1, Eq. (4.1) has at least one T-periodic solution.

Example 2.

$$\left(x(t) - \frac{1}{2}x(t - \tau) \right)^{(5)} + 2e^{4(x^2(t)-1)}x'(t) + \int_{-\pi}^{0} x(t+s)\,ds\, e^{-\left(\int_{-\pi}^{0} x(t+s)\,ds \right)^2} = -\frac{1}{2}\sin t. \tag{4.2}$$

By Theorem 3.2 we let

$$f(x) = 2e^{4(x^2(t)-1)}, \qquad g(x) = xe^{-x^2}, \qquad p(t) = -\frac{1}{2}\sin t, \qquad r = -\pi,$$

$$T = 2\pi, \qquad n = 5.$$

Then we get

$$c = \frac{1}{2}, \qquad k = 3, \qquad T = 2\pi, \qquad a = \lim_{x \to +\infty} \left| \frac{g(x) - \bar{p}}{x} \right| = 0,$$

$$f(x) = 2e^{4(x^2(t)-1)} > 0,$$

and $\frac{2aT^n}{1-|c|} = 0 < 1$. So by employing Theorem 3.2, Eq. (4.2) has at least one T-periodic solution.

Example 3.

$$\left(x(t) - 2x(t - \tau) \right)^{(5)} + \frac{1}{16\pi^3(1 + 2\pi)}e^{-4(x^2(t)+1)}x'(t)$$

$$+ \int_{-\pi}^{0} x(t+s)\,ds\, e^{-\left(\int_{-\pi}^{0} x(t+s)\,ds \right)^2} = \cos t. \tag{4.3}$$

By Theorem 3.3 we let

$$f(x) = \frac{1}{16\pi^3(1 + 2\pi)}e^{-4(x^2(t)+1)}, \qquad g(x) = xe^{-x^2}, \qquad p(t) = \cos t,$$

$$r = -\pi, \qquad T = 2\pi.$$

Then we get

$$c = 2, \qquad n = 5, \qquad T = 2\pi, \qquad a = \lim_{x \to +\infty} \left| \frac{g(x) - \bar{p}}{x} \right| = 0,$$

$$\left| f(x) \right| = \frac{1}{16\pi^3(1 + 2\pi)}e^{-4(x^2(t)+1)} < \frac{1}{16\pi^3(1 + 2\pi)}.$$

Obviously all conditions of Theorem 3.3 are satisfied, and $\frac{(2aT+b)T^{n-1}}{||1-|c|||} = \frac{\pi}{2(1+2\pi)} < 1$. So by using Theorem 3.3, Eq. (4.3) has at least one T-periodic solution.

References

[1] X.K. Huang, Z.G. Xiang, On the existence of 2π-periodic solutions of Duffing type equation $x''(t) + g(t, x(t - \tau)) = p(t)$, Chinese Sci. Bull. 39 (3) (1994) 201–203.
[2] W. Layton, Periodic solutions of a nonlinear delay equations, J. Math. Anal. Appl. 77 (2) (1980) 198–204.
[3] S.W. Ma, Z.C. Wang, J.S. Yu, An abstract theorem at resonance and its applications, J. Differential Equations 145 (2) (1998) 274–294.
[4] S.P. Lu, W.G. Ge, Z.X. Zheng, Periodic solutions for a kind of Rayleigh equation with a deviating argument, Appl. Math. Lett. 17 (2004) 443–449.
[5] S.P. Lu, On the existence of positive solutions for neutral functional differential equation with multiple deviating arguments, J. Math. Anal. Appl. 280 (2) (2003) 321–333.
[6] S.P. Lu, W.G. Ge, On the existence of periodic solutions for a kind of second order neutral functional differential equation, Acta. Math. Sinica 21 (2) (2005) 381–392.
[7] S.P. Lu, W.G. Ge, Existence of periodic solutions for a kind of second order neutral functional differential equation, Appl. Math. Comput. 157 (2004) 433–448.
[8] J.K. Hale, Theory of Functional Differential Equations, Springer, New York, 1977.
[9] R.E. Gaines, J.L. Mawhin, Coincidence Degree and Nonlinear Differential Equations [M], Springer, Berlin, 1977, pp. 40–41.