Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 326 (2007) 1268–1290
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Structure of Stieltjes classes of moment-equivalent probability laws

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Received 5 August 2005
Available online 2 May 2006
Submitted by J. Rosinski

Abstract

A Stieltjes class is a one-parameter family of moment-equivalent distribution functions constructed by modulation of a given indeterminate distribution function F, called the center of the class. Members of a Stieltjes class are mutually absolutely continuous, and conversely, any pair of moment-equivalent and mutually absolutely continuous distribution functions can be joined by a Stieltjes class. The center of a Stieltjes class is an equally weighted mixture of its extreme members, and this places restrictions on which distributions can belong to a Stieltjes class with a given center. The lognormal law provides interesting illustrations of the general ideas. In particular, it is possible for two moment equivalent infinitely divisible distributions to be joined by a Stieltjes class, and random scaling can be used to construct new Stieltjes classes from a given Stieltjes class.

Keywords: Indeterminate distributions; Stieltjes class; Lognormal distribution; Infinite divisibility

1. Introduction

Stoyanov [21] formalized a construction first used by Stieltjes [20, §56] to exhibit some probability distributions which are moment equivalent (denoted M-equivalent), meaning that they possess the same moment sequence. Specifically, let $F(x)$ be a distribution function with support

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.03.035

supp(F) $\subset [0,\infty)$ and for a given measurable function $h(x)$ let $\mathcal{S}(F,h) := \{F_\epsilon\colon -1 \leqslant \epsilon \leqslant 1\}$ where

$$dF_\epsilon(x) = \bigl(1 + \epsilon h(x)\bigr)\,dF(x). \tag{1.1}$$

We assume throughout that F has finite moments of all orders, $m_n = \int x^n\,dF(x)$ for $n = 1, 2, \ldots$, where $\int(\cdot)\,dF(x)$ denotes integration over supp(F). If $h(x)$ satisfies $-1 \leqslant h(x) \leqslant 1$ and $\int x^n h(x)\,dF(x) \equiv 0$, then each member of $\mathcal{S}(F,h)$ is a distribution function having the same moments as F. We then say that $\mathcal{S}(F,h)$ is a Stieltjes class with center F and extreme elements $F_{\pm 1}$, and that the distinct elements are *joined* by the Stieltjes class. If F is uniquely determined by its moments then $\mathcal{S}(F,h)$ is a Stieltjes class iff $h(x) \equiv 0$, a trivial situation we will normally exclude.

Stoyanov's [21] definition is for the case of absolutely continuous F. He defines a dissimilarity index for a Stieltjes class as the total variation distance between F_{-1} and F_1, and values are given for some known Stieltjes classes centered on the lognormal law and powers of normal and exponential laws. The latter are explored more fully by Stoyanov and Tolmatz [23], and powers of inverse Gaussian distributions are treated by Ostrovska and Stoyanov [16]. Shifting, dilation and linear combination of functions can be used to construct new Stieltjes classes from old, a theme pursued by Stoyanov and Tolmatz [22] in connection with the lognormal distribution, and powers of the logistic and inverse Gaussian. These papers exhibit some new Stieltjes classes.

The present paper enlarges on themes mentioned by Stoyanov [21]. In Section 2 we explore simple consequences of the definition (1.1). We will see that members of a Stieltjes class are mutually absolutely continuous, and an example shows that there exist many pairs of M-equivalent density functions which cannot be joined by a Stieltjes class. Any pair of M-equivalent and mutually absolutely distribution functions can be the extreme members of a Stieltjes class, but boundedness relations between members impose constraints on distribution functions which can be joined to a putative center; see Theorem 2.1 and Corollaries 2.1–2.3. A distribution function is a center if it is not an extreme point of the convex set of M-equivalent distribution functions; see Theorem 2.2. Further examples are explored in Section 3 for the lognormal law. A well-known distributional identity for this law extends to certain Stieltjes classes; see Theorem 3.1. A general construction discussed by Pakes [17] provides a vehicle for examples and counter-examples. In particular, an M-equivalent family due to Berg [5] is exhibited and discussed further in Section 4 as an infinitely divisible (abbreviated to infdiv) example which, subject to parameter constraints, can be joined to the lognormal acting as a center. The topic of Section 5 is showing that new Stieltjes classes can be constructed from a given Stieltjes class by random scaling, and we illustrate this using a connection between the lognormal and the q-gamma laws.

2. Structural consequences

Let F be indeterminate (i.e., not determined by its moments) and $\mathcal{M}(F)$ comprise the convex set of all distribution functions which are M-equivalent to F. A Stieltjes class is a one-parameter subset of $\mathcal{M}(F)$, and hence it can be regarded as a kind of fiber element in the cushion $\mathcal{M}(F)$. This comment is made precise in Theorems 2.1 and 2.2. Our first result shows that it need not be the case that any two points in $\mathcal{M}(F)$ are joined by a Stieltjes class. If F and G are distribution functions, write $F \ll G$ if F is absolutely continuous with respect to G, and $F \asymp G$ if $F \ll G$ and $G \ll F$.

Theorem 2.1. *If A and B are elements of the Stieltjes class $\mathcal{S}(F,h)$ then $A \asymp B$, and hence* $\operatorname{supp}(A) = \operatorname{supp}(B)$. *Conversely, if A and B are M-equivalent and $A \asymp B$ then there is a Stieltjes class $\mathcal{S}(F,h)$ with $A = F_{-1}$ and $B = F_1$ which is specified by*

$$F(x) = \frac{1}{2}\big(A(x) + B(x)\big) \quad \textit{and} \quad h(x) = \frac{1-\lambda(x)}{1+\lambda(x)},$$

where $\lambda(x) = dA(x)/dB(x)$.

Proof. The direct assertion is obvious from (1.1). The converse follows by expressing the M-equivalent distribution functions $K_p(x) = pA(x) + (1-p)B(x)$ $(0 \leqslant p \leqslant 1)$ in terms of the class index $\epsilon = 1 - 2p$ and $F_\epsilon(x) = K_p(x)$,

$$dF_\epsilon(x) = \frac{1}{2}\big(dA(x) + dB(x)\big) + (\epsilon/2)\big(dB(x) - dA(x)\big)$$

and $F(x) = K_0(x)$. □

The following example builds on one due to Simon [19, p. 87], and it demonstrates the existence of many mutually singular but M-equivalent density functions.

Example 2.1. Let $b(u)$ be a non-trivial $C^\infty([0,1])$ function, and

$$\hat{b}(x) = (2\pi)^{-1/2} \int_0^\infty e^{ixu} b(u)\, du$$

be its Fourier transform. Differentiating the inverse transform yields

$$\int_{-\infty}^{\infty} x^n \hat{b}(x)\, dx = \sqrt{2\pi}(-i)^n b^{(n)}(0).$$

Choose b such that $b^{(n)}(0+) = 0$ for all n, for example, $b(u) = \exp(-u^{-1})$. Next, let

$$g_1(x) = K_1\big(\Re\hat{b}(x)\big)^+ \quad \text{and} \quad g_2(x) = K_2\big(\Re\hat{b}(x)\big)^-,$$

where K_i $(i = 1, 2)$ is a normalization constant. Then g_1 and g_2 are density functions whose supports have disjoint interiors and which share the same set of moments. The distributions corresponding to g_1 and g_2 are indeterminate in the Hamburger sense. Since $\Re\hat{b}(x)$ is an even function, the density functions $f_i(x) = x^{-1/2} g_i(\sqrt{x})$ are M-equivalent in the Stieltjes sense, but not joined by a Stieltjes class.

The following corollaries show that members of a Stieltjes class share some boundedness properties.

Corollary 2.1. *Suppose F and G are M-equivalent, $F \asymp G$, and $dG(x)/dF(x) \leqslant 2$ for all $x \in \operatorname{supp}(F)$. Then $G \in \mathcal{S}(F,h)$ where $h(x) = (dG(x)/dF(x)) - 1$, $G = F_1$ and*

$$\frac{1}{2}G(x) \leqslant F(x) \leqslant \frac{1}{2}\big(1 + G(x)\big).$$

Observe that if the boundedness condition is omitted then F and G determine a *one-sided* Stieltjes class defined as at (1.1) but with $0 \leqslant \epsilon \leqslant 1$. This is just the set of mixtures $F_\epsilon(x) = (1-\epsilon)F(x) + \epsilon G(x)$.

It follows from Theorem 2.1 that if F and G are M-equivalent and $F \asymp G$, then F and G are the end points of some Stieltjes class, but if the bound condition in Corollary 2.1 is violated, then it need not be the case that there is a Stieltjes class centered on F and with $G = F_{\pm 1}$. In Theorem 2.1, if A and B have density functions (with respect to Lebesgue measure) $a(x)$ and $b(x)$, respectively, then $\lambda(x) = [b(x) - a(x)]/[b(x) + a(x)]$, and in the corollary, if $f(x) = F'(x)$ and $g(x) = G'(x)$, then the additional requirement is that $g(x) \leqslant 2f(x)$, and then $h(x) = (g(x)/f(x)) - 1$. The following result adds a little more.

Corollary 2.2. *If the center F of $\mathcal{S}(F,h)$ has a density function f then the member F_ϵ has a density function*

$$f_\epsilon(x) = f(x)\big(1 + \epsilon h(x)\big) \quad (-1 \leqslant \epsilon \leqslant 1).$$

In addition, if f is bounded then f_ϵ is bounded for each ϵ. Conversely, if f is unbounded, then f_ϵ is unbounded for each ϵ with one possible exception.

Proof. If f is unbounded there exists a sequence of positive numbers $x_n \to x_\infty$ such that $f(x_n) \to \infty$. If there exists $\bar\epsilon \neq 0$ such that $\lim_{n\to\infty} f_{\bar\epsilon}(x_n) < \infty$ then $h(x_n) \to -1/\bar\epsilon$, and hence $f_\epsilon(x_n) \to \infty$ for all $\epsilon \neq \bar\epsilon$. $\square$

The following result supplements Corollary 2.1.

Corollary 2.3. *If $b = \sup_{x>0} dG(x)/dF(x) < \infty$ then $\mathcal{S}(F, h_b)$ is a Stieltjes class where*

$$h_b(x) = \frac{dG(x)/dF(x) - 1}{1 \vee (b-1)},$$

and if $b > 2$, then

$$F_1(x) = \frac{(b-2)F(x) + G(x)}{b-1}, \qquad F_{-1}(x) = \frac{bF(x) - G(x)}{b-1},$$

and

$$G(x)/b \leqslant F(x) \leqslant 1 - b^{-1} + b^{-1}G(x).$$

If $b = \infty$ then there is no Stieltjes class $\mathcal{S}(F,h)$ containing G.

Which members of $\mathcal{M}(F)$ can be centers of a Stieltjes class? A theorem of Naimark asserts that G is an extreme point of $\mathcal{M}(F)$ iff the polynomials are dense in $L_1(G)$. The functional analytic proof given in Akhiezer [1, p. 47] is due to Gelfand, and Berg [7] mentions that it is equivalent to the following assertion which we prove by elementary means.

Theorem 2.2. *Suppose F is indeterminate. Then $G \in \mathcal{M}(F)$ is the center of a Stieltjes class iff it is not an extreme point of $\mathcal{M}(F)$. In particular, no N-extremal member of $\mathcal{M}(F)$ is the center of a Stieltjes class.*

Proof. Theorem 2.1 implies that if $G \in \mathcal{M}(F)$ is a center then it is not an extreme point. Conversely, if it is not an extreme point then there exist distribution functions $G_i \in \mathcal{M}(F)$ $(i = 1, 2)$

and a number $0 < \alpha < 1$ such that $G = \alpha G_1 + (1-\alpha)G_2$. Clearly $\operatorname{supp}(G_i) \subset \operatorname{supp}(G) = \operatorname{supp}(G_1) \cup \operatorname{supp}(G_2)$. Hence the distribution functions $A = (1-(1-\alpha)^2)G_1 + (1-\alpha)^2 G_2$ and $B = \alpha^2 G_1 + (1-\alpha)^2 G_2$ satisfy the conditions of the converse part of Theorem 2.1, and hence $G = \frac{1}{2}(A+B)$ is a center. $\square$

Theorem 2.1 shows that the center of a Stieltjes class is an equally weighted mixture of its extreme elements, $F_{\pm 1}$. Is this the only such representation of the center? We generalize with reference to (1.1) by defining $\tilde{\epsilon} = \inf\{-1 \leqslant \epsilon \leqslant 0\colon\ F_\epsilon$ is a distribution function$\}$. Thus $\tilde{\epsilon} = -1$ for a Stieltjes class, $\tilde{\epsilon} = 0$ for the one-sided version, and if $b > 2$ in Corollary 2.3 then $\tilde{\epsilon} = -2/b$. Since $F_\epsilon(0) \geqslant 0$ for any ϵ, it is clear by choosing a sequence $\epsilon_n \downarrow \tilde{\epsilon}$ that $F_{\tilde{\epsilon}}$ is a distribution function. Let $\bar{\mathcal{S}}(F,h) = \{F_\epsilon\colon\ \tilde{\epsilon} \leqslant \epsilon \leqslant 1\}$. The following theorem, proved by simple manipulation, exhibits the mixture closure properties of a Stieltjes class.

Theorem 2.3. *The center F of $\bar{\mathcal{S}}(F,h)$ is a mixture of two other distinct members iff $\tilde{\epsilon} < 0$. More generally, if $-1 \leqslant \tilde{\epsilon} \leqslant \epsilon' < \epsilon'' \leqslant 1$, then*

$$F_\epsilon = \frac{(\epsilon'' - \epsilon)F_{\epsilon'} + (\epsilon - \epsilon')F_{\epsilon''}}{\epsilon'' - \epsilon'}$$

is a two-component mixture iff $\epsilon' < \epsilon < \epsilon''$.

Suppose X is a random variable having the distribution function F. The following result gives two transformations which map a Stieltjes class into a second one. We need the following notation for weighted distribution functions. Let $w(x) \geqslant 0$ $(x \geqslant 0)$ be a weight function satisfying $m_w := E[w(X)] < \infty$, and let $\hat{X}_w$ denote a random variable having the weighted distribution function $\hat{F}_w(x) = m_w^{-1} \int_0^x w(y)\, dF(y)$. Recall that $m_n = E(X^n)$.

Theorem 2.4. *Suppose $\mathcal{S}(F,h)$ is a Stieltjes class.* (a) *Let $w(x)$ be a weight function which, for all $x \geqslant 0$, has a power series expansion $w(x) = \sum_{j\geqslant 0} a_j x^j$ such that $\sum_{j \geqslant 0} |a_j| m_{j+n} < \infty$ for $n = 0, 1, \ldots.$ Then $\mathcal{S}(\hat{F}_w, h)$ is a Stieltjes class.*

(b) *Suppose that $\tau(x)$ is a strictly increasing polynomial with inverse $\eta(y)$. Then $\mathcal{S}(G,\gamma)$ is a Stieltjes class, where $G(y) = F(\eta(y))$ and $\gamma(y) = h(\eta(y))$.*

Proof. (a) Since members of $\mathcal{S}(\hat{F}_w, h)$ have the form $d\hat{F}_{w,\epsilon}(x) = (1+\epsilon h(x))w(x)\, dF(x)$, it suffices to observe that

$$\int x^n h(x) w(x)\, dF(x) = \sum_{j \geqslant 0} a_j \int x^{j+n} h(x)\, dF(x) \equiv 0.$$

Interchanging summation and integration is justified by the summability assumption and Fubini's theorem.

(b) The distribution function of $Y = \tau(X)$ is $G(y) = F(\eta(y))$ and a change of variable yields

$$\int y^n h\bigl(\eta(y)\bigr)\, dG(y) = \int \tau^n(x) h(x)\, dF(x), \tag{2.1}$$

and since $\tau^n(x)$ is a polynomial it follows from the hypothesis that the right-hand side is zero for all $n = 0, 1, \ldots.$ $\square$

The assumptions for (a) are satisfied if $w(x)$ is a polynomial, thus covering almost any example of size biasing. Part (b) is a little unexpected in view of Stoyanov's [21] counter-example based on the lognormal law that there exist transformations $Y=\tau(X)$ having a determinate law even though F is indeterminate, in which case $\mathcal{S}(G,\gamma)$ is not a Stieltjes class. The assertion (b) is valid under the weaker condition that $\tau(x)$ is increasing and can be extended as an entire function and that the integral on the right-hand side of (2.1) can be evaluated term by term from the power series expansion of τ^n. Stoyanov's discussion is based on the transformation $\tau(x)=\log x$ which does not have a MacLaurin series expansion.

3. The lognormal law

The lognormal law is a fruitful source of interesting examples. The general lognormal law $LN(\mu,\sigma^2)$ has the density function

$$f_L(x;\mu,\sigma)=\frac{1}{x}\phi\left(\frac{\log x-\mu}{\sigma}\right)\quad (x>0), \tag{3.1}$$

where ϕ is the standard normal density function. The standard lognormal law is defined by $\mu=0$ and $\sigma=1$. The moment function of $X\sim LN(\mu,\sigma^2)$ is

$$M_L(t):=E\big(X^t\big)=e^{\mu t+\frac{1}{2}\sigma^2t^2}.$$

Stieltjes' [20] oft-mentioned example of laws which are M-equivalent to the lognormal is based on the function $x^{-\log x}$ which is proportional to $f_L(x;-\frac{1}{2},\frac{1}{2})$. He remarked that equivalent laws are obtained by taking $h(x)=\omega(\log x)$ where ω is an odd and periodic function satisfying $-1\leqslant\omega(z)\leqslant 1$ and $\omega(z+\frac{1}{2})=\pm\omega(z)$. His specific example is $\omega(z)=\sin(2\pi z)$, and this was later exhibited by Heyde [13] in a statistical context, and for an even more general parametrization than we will use.

Following Stieltjes, let $h(x)=\omega(\log x-\mu)$ where ω is odd with $|\omega(z)|\leqslant 1$ and $\omega(z+\sigma^2)=\pm\omega(z)$, and let Z have a standard normal law. Then

$$\begin{aligned}E\big[X^th(X)\big]&=E\big[e^{(\mu+\sigma Z)t}\omega(\sigma Z)\big]=(2\pi)^{-\frac{1}{2}}e^{\mu t+\frac{1}{2}\sigma^2t^2}\int e^{-\frac{1}{2}(z-\sigma t)^2}\omega(\sigma z)\,dz\\&=M_L(t)E\big[\omega\big(\sigma Z+t\sigma^2\big)\big].\end{aligned}$$

If $n=0,1,\ldots$ then $E[\omega(\sigma Z+n\sigma^2)]=\pm E\omega(\sigma Z)=0$. It follows that $f_L(x;\mu,\sigma)$ and

$$f_\epsilon(x)=f_L(x;\mu,\sigma)\big(1+\epsilon h(x)\big)$$

are M-equivalent. A more analytical derivation of this conclusion is given by White [24], but our simple argument is easier than that usually associated with the sinusoidal case simply because in this general context there is no temptation to explicit evaluation of the final expectation. This construction generalizes in a minor way by recognizing that center densities can be created by mutiplying $\phi((z-\mu)/\sigma)$ by a function $\rho(z-\mu)$ where $\rho(v)$ is even and non-negative, $\rho(z+\sigma^2)\equiv\rho(z)$, and $\int\rho(\sigma z)\phi(z)e^{\theta z}\,dz<\infty$ for all real θ, a condition which is satisfied if $\rho(v)$ is bounded. The special features making this proof work do not seem to be available for other common examples of indeterminate laws.

An interesting corollary of Stieltjes' construction relates to the fact that if $X\sim LN(\mu,\sigma^2)$ then $X^{-1}\stackrel{L}{=}e^{2\mu}X$. Does this extend to other laws in a Stieltjes class? We show now that this property is a special case of a more general relation.

Theorem 3.1. *Suppose $\mathcal{L}(X_\epsilon)$ comprises a Stieltjes class centered on the $LN(\mu,\sigma^2)$ law with $h(x)=\omega(\log x-\mu)$ where $\omega(z)$ is odd and periodic with period σ^2. Then*

$$X_\epsilon^{-1} \stackrel{L}{=} e^{2\mu}X_{-\epsilon} \quad (-1\leqslant\epsilon\leqslant 1).$$

Proof. Observe that $h(1/x)=\omega(-\log x-\mu)=-\omega(\log x+\mu)=-h(xe^{2\mu})$. So if $f_\epsilon(x)$ is the density function of X_ϵ, then X_ϵ^{-1} has the density function

$$\begin{aligned} x^{-2}f_\epsilon\big(x^{-1}\big) &= x^{-2}f_L\big(x^{-1};\mu,\sigma\big)\big(1+\epsilon h\big(x^{-1}\big)\big)=e^{2\mu}f_L\big(xe^{2\mu};\mu,\sigma\big)\big(1-\epsilon h\big(xe^{2\mu}\big)\big)\\ &= e^{2\mu}f_{-\epsilon}\big(xe^{2\mu}\big), \end{aligned}$$

which is the density function of $e^{2\mu}X_{-\epsilon}$. □

Suppose X has the standard lognormal law, $\omega(z)=\sin(2\pi z)$, and $Z_\epsilon=\log X_\epsilon$. Then $Z_0\sim N(0,1)$ is moment-determinate and hence the law of Z_ϵ is moment-determinate for all ϵ. Stoyanov [21] observes that the density of Z_ϵ is

$$\phi_\epsilon(z)=\phi(z)\big(1+\epsilon\sin(2\pi z)\big),$$

where $\phi(z)$ is the standard normal density. He observes too that if $M_\epsilon(t)=E(Z_\epsilon^t)$, then the even order moments are independent of ϵ, $M_\epsilon(2n)=E(Z^{2n})$, and that the odd order moments $M_\epsilon(2n+1)$ are not all zero if $\epsilon\neq 0$. This constancy in ϵ holds more generally. If the pdf of X is $f_L(x;\mu,\sigma)$ and that of X_ϵ is $f_L(x)(1+\epsilon h(x))$, where $h(x)=\omega(\log x-\mu)$ and ω is odd, then it is easily checked that

$$E\big[(\log X_\epsilon-\mu)^{2n}\big]=\sigma^{2n}E\big(Z^n\big).$$

The corresponding odd-order moments are $\epsilon\sigma^{2n+1}E[Z^{2n+1}\omega(\sigma Z)]$. This could in principle be evaluated if ω can be expressed as a Fourier sine series. Indeed, in Stoyanov's case we have

$$\begin{aligned} M_\epsilon(2n+1) &= \epsilon E\big[Z^{2n+1}\sin(2\pi Z)\big]=i^{-1}\epsilon E\big[Z^{2n+1}e^{2\pi iZ}\big]\\ &= i^{-1}\epsilon E\big[Z^{2n+1}e^{i\theta Z}\big]\big|_{\theta=2\pi}=i^{-2n-2}\epsilon\frac{d^{2n+1}}{d\theta^{2n+1}}E\big[e^{i\theta Z}\big]\big|_{\theta=2\pi}\\ &= (-1)^{-n-1}\epsilon\frac{d^{2n+1}}{d\theta^{2n+1}}e^{-\frac{1}{2}\theta^2}\big|_{\theta=2\pi}\\ &= (-1)^n\epsilon e^{-2\pi^2}2^{-n-1/2}H_{2n+1}(\sqrt{2}\pi) \quad (n=0,1,\ldots), \end{aligned}$$

where H_n is a Hermite polynomial and we have used Rodrigues' formula in the form

$$\frac{d^n}{d\theta^n}e^{-\frac{1}{2}\theta^2}=(-1)^n e^{-\frac{1}{2}\theta^2}H_n(\theta/\sqrt{2}).$$

(See Willink [25] for a similar approach to moment calculations for normal laws.) These moments are proportional to ϵ, and the first order moment $M_\epsilon(1)/\epsilon=2\pi e^{-2\pi^2}=1.680933\times 10^{-8}$. In addition

$$\big|M_\epsilon(2n+1)\big|\sim\epsilon\sqrt{2/e}\cdot e^{-\pi^2}(n/e)^{n+1/2}S_n \quad\text{and}\quad M_\epsilon(2n)\sim\sqrt{e/2}\cdot 2^n(n/e)^{n+1/2},$$

where $S_n=|\sin[\sqrt{8n+6}\pi]|$. Thus the even-order moments increase much faster than the odd-order moments, in fact, $|M_\epsilon(2n+1)|=O(2^{-n}M_\epsilon(2n))$.

We now extend discussion of a general construction methodology [17] for laws which are M-equivalent to the lognormal. The moment function $M_L(t)$ solves the functional equation

$$M(t+1) = m\ell^t M(t), \tag{3.2}$$

where $m = M_L(1) = e^{\mu+\frac{1}{2}\sigma^2}$ and $\ell = e^{\sigma^2} > 1$. For any non-negative random variable X with distribution function $F(x)$ and $E(X^r) < \infty$ for some $r > 0$, we define its order-r length-biased version to be $\hat{X}(r) := \hat{X}_w$ in the case that $w(x) = x^r$, and its distribution function is denoted by $\hat{F}_r(x)$. The argument r is omitted in the case $r = 1$. So if $X \sim LN(\mu, \sigma^2)$ and $q = \ell^{-1}$, then (3.2) can be expressed as

$$X \stackrel{L}{=} q\hat{X}, \tag{3.3}$$

i.e., the law of X is recovered be rescaling the stochastically larger $\hat{X}$. We call this the length-bias scaling property, abbreviated to LBS-property. The right-hand side represents an operator $\mathcal{T}$ acting on distributions, and hence (3.3) asserts that distribution functions having the LBS-property are fixed points of $\mathcal{T}$. Christiansen [11, §3] obtains interesting results about $\mathcal{T}$ in relation to N-extremal and canonical solutions of the lognormal moment problem.

On the other hand, (3.2) has uncountably many moment function solutions and the corresponding laws satisfy (3.3). It is easily seen that all such solutions are M-equivalent to the $LN(\mu, \sigma^2)$ law. In addition, the quotient function $M(t)/M_L(t)$ is periodic with unit period; see Pakes [17] for the explicit construction of these solutions. Almost all known explicit examples of laws equivalent to $LN(\mu, \sigma^2)$ are particular cases of this construction. Indeed, solutions are in 1-1 correspondence with the set of finite measures on $(q, 1]$, and it follows that solution laws can be absolutely or singular continuous with respect to Lebesgue measure, be discrete, or be a mixture of any of these. On the other hand, although the solution set of (3.3) is convex, its extreme points are not N-extremal solutions of the lognormal moment problem [17, p. 836] and hence this construction gives only a proper subset of the full set of M-equivalent laws.

Denote the distribution function of $\hat{X}$ by $\hat{F}(x)$. The distribution function version of (3.3) is

$$d\hat{F}(x) = (x/m)\,dF(x) = dF(qx). \tag{3.4}$$

If $\mathcal{S}(F, h)$ is a Stieltjes class, then Theorem 2.4(a) says that $\mathcal{S}(\hat{F}, h)$ also is a Stieltjes class, and we may ask whether $F_\epsilon(x)$ has the LBS-property. The following result gives an answer.

Theorem 3.2. *Let $F(x)$ have the LBS-property and let $\mathcal{S}(F, h)$ be a Stieltjes class. If $h(x) = \pm h(qx)$ and the conditions of Theorem* 2.4(a) *hold, then members of $\mathcal{S}(\hat{F}_w, h)$ have the form*

$$\hat{F}_{w,\epsilon}(x) = m_w^{-1} \sum_{n\geqslant 0} a_n m_n F_{(\pm 1)^n \epsilon}\big(q^n x\big). \tag{3.5}$$

Proof. Observe first that iterating (3.3) yields $\hat{X}(n) \stackrel{L}{=} q^{-n}X$, whence $\hat{F}_n(x) = F(q^n x)$. Since $\hat{F}_w(x) = m_w^{-1} \sum_{n\geqslant 0} a_n m_n \hat{F}_n(x)$, the distribution function of $\hat{X}_{w,\epsilon}$ is

$$\begin{aligned} d\hat{F}_{w,\epsilon}(x) &= m_w^{-1}\big(1+\epsilon h(x)\big)\,d\hat{F}_w(x) \\ &= m_w^{-1}\sum_{n\geqslant 0} a_n m_n \big(1+\epsilon h(x)\big)\,d\hat{F}_n(x) \\ &= m_w^{-1}\sum_{n\geqslant 0} a_n m_n \big(1+(\pm 1)^n \epsilon h\big(q^n x\big)\big)\,dF\big(q^n x\big), \end{aligned}$$

and this is the right-hand side of (3.5). □

If $X^{(n)}$ $(n = 1, 2, \dots)$ are independent copies of X then a random variable version of (3.5) can be expressed as

$$\hat{X}_{w,\epsilon} \stackrel{L}{=} m_w^{-1} \sum_{n=0}^{\infty} a_n m_n q^{-n} X^{(n)}_{(\pm 1)^n \epsilon}. \tag{3.6}$$

In particular, if $a_n = \delta_{1n}$ then

$$q\hat{X}_\epsilon \stackrel{L}{=} X_{\pm\epsilon},$$

where the sign corresponds to $h(x) = \pm h(qx)$. Almost all explicit Stieltjes class solutions of the lognormal moment problem have center satisfying the LBS-property and a perturbation function satisfying $h(x) = h(qx)$, in which case all members of the Stieltjes class have the LBS-property. Christiansen [11, (2.9)] exhibits a Stieltjes class of discrete laws M-equivalent to the $LN(\frac{1}{2}\sigma^2, \sigma^2)$ law. Its center allocates mass proportional to $q^{\binom{n+1}{2}}$ to the point q^n and $h(q^n) = (-1)^n$, where $n = 0, \pm 1, \pm 2, \dots$, and hence members with $\epsilon \neq 0$ do not have the LBS-property. However, taking $a_n = \delta_{2n}$ in (3.6) gives the following corollary of Theorem 3.2, noted by Christiansen [11] in relation to his Stieltjes class.

Corollary 3.1. *Any member of a Stieltjes class constructed as in Theorem* 3.2 *with* $w(x) = x$ *is a fixed point of* $\mathcal{T}^2$.

Let $f(x)$ be the density function of a law $\mathcal{L}(X)$ with $E(X) = m$ and satisfying (3.3). The expression of this relation in terms of f is

$$mf(x) = \ell^2 x f(\ell x), \quad \text{equivalently,} \quad xf(x) = mqf(qx). \tag{3.7}$$

The second relation yields

$$m_n = \int_0^\infty x^n f(x)\,dx = mq^{-(n-1)} m_{n-1} = m^n q^{-\frac{1}{2}n(n-1)} \quad (n \geqslant 1).$$

Moreover, if $f(x) > 0$ for $q < x \leqslant 1$ then $f(x) > 0$ for all $x > 0$. If $f(x)$ is such a solution and $g(x)$ is another density satisfying (3.7) then

$$\frac{g(x/q)}{f(x/q)} = \frac{g(x)}{f(x)} \quad (x > 0). \tag{3.8}$$

The function $h(x) := (g(x)/f(x)) - 1$ is defined in $(0, \infty)$ and $h(x) \equiv h(x/q) \equiv h(qx)$, i.e., $\omega(z) := h(e^z)$ has period σ^2. Since $\int_0^\infty x^n f(x)h(x)\,dx = \int_0^\infty x^n g(x)\,dx - \int_0^\infty x^n f(x)\,dx = 0$, $(n = 0, 1, \dots)$, and since $h(x) \geqslant -1$, we conclude that

$$f_\epsilon(x) = f(x)\big(1 + \epsilon h(x)\big) \quad (0 \leqslant \epsilon \leqslant 1) \tag{3.9}$$

specifies a one-sided Stieltjes class which is moment equivalent to the $LN(\mu, \sigma^2)$ law. In particular $f_L(x; \mu, \sigma)$ is joined to *any* other solution of (3.7) in this way. A one-sided Stieltjes class can be extended to $\epsilon \in [-1, 0)$ iff $h(x) \leqslant 1$, a condition which needs to be checked only in the base interval $(q, 1]$. This boundedness condition can be satisfied by choosing $g(x)$ to be a small perturbation of $f(x)$. On the other hand, g can be chosen unbounded in $(q, 1]$, and hence unbounded in $(\ell^{n-1}, \ell^n]$ for all integers n. In particular, if f is bounded and g is unbounded then so is h, and hence the one-sided Stieltjes class (3.9) cannot be extended to negative values of ϵ.

Summarizing, there are uncountably many absolutely continuous laws joined to the $LN(\mu,\sigma^2)$ law as center of a Stieltjes class, and uncountably many M-equivalent absolutely continuous laws which cannot be so joined to $LN(\mu,\sigma^2)$.

The following result exhibits solutions of (3.7).

Theorem 3.3. *If $f(x)\geqslant 0$ $(q<x\leqslant 1)$ is specified and satisfies $0<\int_q^1 f(x)\,dx<\infty$, then defining*

$$f\left(\ell^n x\right)=q^{\frac{1}{2}(n^2+3n)}(m/x)^n f(x)\quad (q<x\leqslant 1;\ n=0,\pm 1,\pm 2,\dots),$$

equivalently,

$$f(x)=\ell^{\frac{1}{2}(n^2-3n)}(m/x)^n f\left(q^n x\right)\quad \left(\ell^{n-1}<x\leqslant \ell^n;\ n=0,\pm 1,\pm 2,\dots\right), \tag{3.10}$$

yields a solution of (3.7). *If $\nu(t):=\int_q^1 x^t f(x)\,dx$ then*

$$M_f(t):=\int_0^\infty x^t f(x)\,dx=\sum_{n=-\infty}^{\infty} m^{-n}q^{\frac{1}{2}(n^2-n)+nt}\nu(t+n). \tag{3.11}$$

In particular, the normalized function $f(x)/M_f(0)$ is a density function solution of (3.7).

Proof. The integral defining $M_f(t)$ is evaluated by writing it as a sum of integrals $\int_{\ell^{n-1}}^{\ell^n} x^t f(x)\,dx$, substituting (3.10), evaluating and then changing the sign of n. $\square$

If $f(x)$ is continuous in $(q,1]$ then it is continuous everywhere except perhaps where $x=\ell^n$, and at these points (3.10) implies that it is continuous from the left. Letting $x\downarrow 1$ in the second member of (3.7) yields $f(1+)=mqf(q+)$. It follows that $f(x)$ is continuous at $x=1$ iff

$$mqf(q+)=f(1). \tag{3.12}$$

Lemma 3.1. *The condition* (3.12) *is necessary and sufficient for continuity of $f(x)$ in* $(0,\infty)$.

Obviously lognormal density functions satisfy this condition. Another example is the scaled version of density functions described by Berg [5]. Let $e^\mu=m\sqrt{q}$ and define the continuous density functions

$$f_B(x;c,\mu)=\frac{x^{c-1}}{\mathcal{N}(c,\mu)L(xe^{-\mu}q^{-c})}\quad (x>0), \tag{3.13}$$

where

$$L(x)=\sum_{n=-\infty}^{\infty} x^n q^{\frac{1}{2}n^2},$$

c is real, and $\mathcal{N}(c,\mu)$ is the normalization constant. Berg has $\mu=0$, and he notes that $L(x)=\sqrt{q}xL(qx)$. This identity implies that $f_B(x;c,\mu)$ satisfies (3.7).

The Jacobi triple product formula yields the symmetric identification

$$L(x)=(-\sqrt{q}x;q)(-\sqrt{q}/x;q)(q;q), \tag{3.14}$$

where we write $(a;q)=\prod_{n\geqslant 0}(1-aq^n)$, slightly abbreviating the conventional notation for this product. If X has the density (3.13) and X_0 has the density $f(x;c,0)$ then X and $e^\mu X_0$ have the same distribution whence, as in Berg [5], $f(x; c+1, \mu) = f(x; c, \mu)$, so we can restrict c to $[0, 1]$. It follows too that $\mathcal{N}(c, \mu) = e^{\mu}\mathcal{N}(c, 0)$.

Calculations about X are facilitated by observing that the density function of $X_A = q^{-c+1/2}X_0$ has the simpler form introduced by Askey [4, p. 315]. For example, Askey's normalization constant leads to the evaluation: If $0 < c < 1$ then

$$\mathcal{N}(c, \mu) = \frac{\pi}{\sin(\pi c)}\left[e^{\mu}q^{c-1/2}\right]^{c}\frac{(q^c; q)(q^{1-c}; q)}{(q; q)^2}. \tag{3.15}$$

Observe that as $c \to 0+$ we have $(q^c; q) \sim (1 - e^{-c\sigma^2})(q; q) \sim c\sigma^2(q; q)$, and hence $\mathcal{N}(0, \mu) = \sigma^2$. In addition, the evaluation of the moment function of X_A in [17] (which follows immediately from the known evaluation of the q-beta integral (4.12) in [4]) yields: If $0 < c < 1$ then

$$M_B(t; c, \mu) = m^t q^{ct}\frac{\sin(\pi c)}{\sin(\pi(c+t))}\cdot\frac{(q^{c+t}; q)(q^{1-c-t}; q)}{(q^c; q)(q^{1-c}; q)} \tag{3.16}$$

and

$$M_B(t; 0, \mu) = m^t\frac{\pi}{\sigma^2\sin(\pi t)}\cdot\frac{(q^t; q)(q^{1-t}; q)}{(q; q)^2}.$$

Berg's [5, Remark 2.3] independent evaluation of the normalization constant for $c \neq 0$ also starts from Askey's integral.

Continuous solutions of (3.7) probably are the exception. We exhibit a discontinuous solution of (3.7) as follows. Rewrite (3.10) as

$$f(x) = C(x)f\left(q^n x\right) \quad \left(\ell^{n-1} < x \leqslant \ell^n;\ n = 0, \pm 1, \pm 2, \ldots\right),$$

where $C(0) = 0$ and

$$C(x) = m^n q^{-\frac{1}{2}(n^2-3n)}x^{-n} \quad \left(\ell^{n-1} < x \leqslant \ell^n;\ n = 0, \pm 1, \pm 2, \ldots\right).$$

It is easy to check that

$$C\left(\ell^n+\right) = m^{n+1}q^{\frac{1}{2}(n^2+3n+2)} = mqC\left(\ell^n\right),$$

and hence $C(x)$ is continuous in $(0, \infty)$ iff $mq = 1$, i.e., $\mu = \frac{1}{2}\sigma^2$. This explains the condition (3.12) for continuity of $f(x)$.

The graph of $C(x)$ comprises left-continuous convex increasing arcs proportional to x^n in $(q^{n+1}, q^n]$ $(n \geqslant 1)$, $C(x) \equiv 1$ if $q < x \leqslant 1$, and concavely decreasing arcs proportional to x^{-n} in $(\ell^{n-1}, \ell^n]$ $(n \geqslant 1)$. Also, $C(x) \to 0$ as $x \to 0$ and $x \to \infty$. Discontinuities at ℓ^n (all n) are upward (respectively downward) jumps if $mq > 1$ (respectively $mq < 1$). So if $mq > 1$ then the graph of $C(x)$ is strictly increasing in $(0, q]$ and it has local maxima in $[1, \infty)$ at $x = \ell^n$. This pattern is reversed in an obvious way if $mq < 1$. The quotient

$$C\left(\ell^n\right)/C\left(\ell^{n-1}\right) = mq^{n+1} \quad (n = 0, \pm 1, \pm 2, \ldots),$$

can exceed unity for small positive n if $mq > 1$, but clearly it tends to 0 as $n \to \infty$, and it tends to ∞ as $n \to -\infty$. Thus relative values of $C(x)$ at successive discontinuities are very large near the origin, and very small in the neighborhood of ∞.

Finally, if $\ell^{n-1} < x \leqslant \ell^n$ then $n = \sigma^{-2}\log x + r(x)$ where $0 < r(x) \leqslant 1$. Substituting and simplifying yields $C(x) = f_L(x; \mu, \sigma)R(x)$ where

$$R(x) = \sqrt{2\pi\sigma^2}\exp\left[\mu^2/2\sigma^2 + \left(\mu - \sigma^2\right)r(x) + \frac{1}{2}\sigma^2 r^2(x)\right].$$

Dividing $C(x)$ by $\int_0^\infty C(v)\,dv$ yields a density function M-equivalent to $LN(\mu,\sigma^2)$ and whose graph has the multimodal properties described above.

This construction obviously generalizes by setting $f(x) \propto x^a$ $(q < x \leqslant 1)$, where a is real, giving via (3.10) a density function equivalent to $LN(\mu,\sigma^2)$. Its moment function is given by (3.11) as

$$M(t;a) = K_a^{-1} \sum_{n=-\infty}^{\infty} m^{-n} q^{\binom{n}{2}+nt} \frac{1-q^{n+a+t+1}}{n+a+t+1},$$

provided $a+t$ is non-integral. Writing $(n+A)^{-1} = \int_0^\infty e^{-(n+A)y}\,dy$ and using the triple product formula, we find for real but non-integral A that

$$\sum_{n=-\infty}^{\infty} q^{\binom{n}{2}} \frac{x^n}{n+A} = (q;q) \int_0^\infty e^{-Ay} \bigl(-xe^{-y};q\bigr)\bigl(-qx^{-1}e^{y};q\bigr)\,dy = (q;q)\beta(x;A),$$

where

$$\beta(x;A) = \int_0^1 v^{A-1}(-xv;q)(-q/xv;q)\,dv.$$

It follows that

$$M(t;a) = \frac{\beta(m^{-1}q^t;a+t+1) - q^{a+t+1}\beta(m^{-1}q^{1+t};a+t+1)}{\beta(m^{-1};a+1) - q^{a+1}\beta(m^{-1}q;a+1)}.$$

In particular

$$\int_0^\infty x^t C(x)\,dx = (q;q)\bigl[\beta\bigl(m^{-1}q^t;t+1\bigr) - q^{t+1}\beta\bigl(m^{-1}q^{t+1};t+1\bigr)\bigr],$$

and setting $t=0$ gives the normalization constant required to produce a density function from $C(x)$.

Chihara [10] and Leipnik [14] independently constructed discrete laws M-equivalent to the $LN(-\frac{1}{2}\sigma^2,\sigma^2)$ law. By virtue of Theorem 2.1, discrete laws M-equivalent to $LN(\mu,\sigma^2)$ do not belong to any Stieltjes class centered on this lognormal law. However, Theorem 2.1 shows that M-equivalent discrete laws can be joined if they have a common support. To see this more explicitly, let Ω_i $(i=0,1)$ be discrete measures having the same countable support in $(q,1]$, and define distribution functions

$$F_i(x) = K_i \sum_{n=-\infty}^{\infty} c(n) m^{-n} \int_0^{\ell^n x} v^n\,\Omega_i(dv),$$

where $c(n) = q^{\frac{1}{2}n(n-1)}$ and K_i is a normalization constant. Then $F_i(x)$ is M-equivalent to the $LN(\mu,\sigma^2)$ law [17]. If $x \in (q,1]$ is a support point of Ω_i, then $x_n = q^n x$ $(n = 0,\pm1,\pm2,\ldots)$ is a support point of $F_i(x)$. The proof of Theorem 3.1 in [17] implies the identity

$$\frac{F_1(\{x_n\})}{F_0(\{x_n\})} = \frac{\Omega_0(\{x\})/K_0}{\Omega_1(\{x\})/K_1}.$$

It follows that if the right-hand side lies in $[0, 2]$ for all supporting $x \in (q, 1]$ then F_1 and F_0 are joined in the Stieltjes class $\mathcal{S}(F_0, h)$, where $1 + h(x)$ is taken as the right-hand side of the above identity. The boundedness condition can be achieved if, for example, the generating measures have a finite support at the atoms of which they attribute almost equal masses.

4. Infinite divisibility

In this section we consider whether two M-equivalent and infinitely divisible (infdiv) laws can belong to the same Stieltjes class. This question was raised by Stoyanov [21]. A density function $g(x)$ belongs to the class of hyperbolically completely monotone (HCM) densities if, for all $u > 0$, the function $g(uv)g(u/v)$ is a completely monotone function of $w = v + v^{-1}$. Any HJM density is self-decomposable, and hence infdiv. See Bondesson [8] for a thorough account of this concept.

In this section we concentrate on the $LN(\mu, \sigma^2)$ law which is known to be HJM from the following argument due in essence to Bondesson [8, p. 59]. It is easily checked that

$$f_L(uv; \mu, \sigma) f_L(u/v; \mu, \sigma) = \chi(u) \exp\bigl(-(\log v)^2/2\sigma^2\bigr),$$

where $\chi(u)$ is functionally independent of v. The representation

$$\log v = \int_0^\infty \left(\frac{1}{1+y} - \frac{1}{y+v}\right) dy$$

and the identity $2\log v = \log v - \log v^{-1}$ combine to yield

$$\frac{d}{dw}(\log v)^2 = \frac{2\log v}{v}\frac{dv}{dw} = \frac{2\log v}{v}\bigl(1 - v^{-2}\bigr)^{-1} = \int_0^\infty \frac{dy}{y^2 + yw + 1},$$

which is completely monotone in w. Hence $\exp(-(\log v)^2/2\sigma^2)$ is completely monotone. See [9] for numerical investigation of the Lévy measure (and the Thorin measure) of the $LN(0, 1)$ law.

It follows from (3.14) that if $a > 0$ is a constant, then

$$\begin{aligned} &L(auv)L(au/v) \\ &\quad = \prod_{n\geqslant 0} \bigl[1 + auwq^{n+1/2} + a^2u^2q^{2n+1}\bigr]\bigl[1 + (w/au)q^{n+1/2} + q^{2n+1}/a^2u^2\bigr], \end{aligned}$$

and the reciprocal of this expression is completely monotone in w. Hence $f_B(x; c, \mu)$ at (3.13) is an infdiv density function. This has been observed by Berg [6]; see the proof of his Theorem 2.7.

The class of HJM density functions coincides with the Bondesson class $\mathcal{B}$ whose members have, by definition, a representation

$$g(x) = Kx^{\beta-1}h_1(x)h_2\bigl(x^{-1}\bigr) \quad (x > 0),$$

where β is a real constant,

$$h_j(x) = \exp\left[-b_jx + \int_0^\infty \log\frac{y+1}{y+x}\Gamma_j(dy)\right] \quad (j = 1, 2),$$

with $b_j \geqslant 0$, and Γ_j is a measure on $(0, \infty)$ satisfying $\int_0^\infty (1+y)^{-1}\,\Gamma_j(dy) < \infty$. The b_j and the Γ_j are not unique, but can be made so by requiring that Γ_1 and Γ_2 are concentrated in

$(1,\infty)$ and $[1,\infty)$, respectively. Bondesson [8] calls the result of this convention the canonical representation, but he remarks that it is not always the neatest one. He shows [8, p. 74] that $\beta=\mu/\sigma^2$, $b_j=0$, and $\Gamma_j(dy)=\sigma^{-2}y^{-1}\,dy$ $(y>1)$ gives the canonical representation for the $LN(\mu,\sigma^2)$ law.

A $\mathcal{B}$-class non-canonical representation of $f_B(x;c,\mu)$ emanates from the identity

$$\frac{1+\gamma}{1+\gamma x}=\exp\int\log\frac{y+1}{y+x}\delta_{\gamma^{-1}}(dy),$$

where $\gamma>0$ and $\delta_a(dy)$ assigns unit mass to $y=a$. It is easy to check that

$$h(x;b):=(-b;q)/(-bx;q)=\exp\int_0^\infty\log\frac{y+1}{y+x}\Gamma(dy;b),$$

where $b>0$ and $\Gamma(dy;b)=\sum_{n\geqslant 0}\delta_{\ell^n/b}(dy)$. Setting $\beta=c$, $b_j=0$ and $h_1(x)=h(x;m^{-1}q^{-c})$, $h_2(x)=h(x;mq^{1+c})$ and $K=\mathcal{N}(c,\mu)L(e^{-\mu}q^{-c})$ gives a representation. Because both the log-normal law and the Berg laws are infdiv, it follows that their Lévy measures are indeterminate and that there are infinitely many other infdiv laws having the same moment sequence; see [6].

These results show that the infdiv laws $LN(\mu,\sigma^2)$ and any Berg law comprise the extreme members of a one-sided Stieltjes class. For investigating the two-sided case we can assume without loss of generality that $\mu=0$ and $0\leqslant c<1$, and seek conditions ensuring that $f_B(x;c,0)/f_L(x;0,\sigma)\leqslant 2$ for $q\leqslant x\leqslant 1$. Manipulation with (3.1), (3.13), (3.14) and (3.15) represents this condition as

$$\begin{aligned}\lambda_1(x;\sigma)+\lambda_2(x;\sigma)+\left(c-\frac{1}{2}\right)c\sigma^2-\log\sigma+\log\bigl(q^{1+c};q\bigr)\\+\log\bigl(q^{1-c};q\bigr)-\log(q;q)+\delta(c,\sigma)\geqslant-\frac{1}{2}\log(2\pi),\end{aligned}\tag{4.1}$$

where

$$\lambda_1(x;\sigma)=\sum_{n\geqslant 0}\log\bigl(1+xq^{n+1/2-c}\bigr)+\sum_{n\geqslant 0}\log\bigl(1+x^{-1}q^{n+1/2+c}\bigr),\tag{4.2}$$

$$\lambda_2(x;\sigma)=-c\log x-\bigl(2\sigma^2\bigr)^{-1}(\log x)^2=\bigl[c-\bigl(\log x^{-1}\bigr)/2\sigma^2\bigr]\log x^{-1},\tag{4.3}$$

and

$$\delta(c,\sigma)=\begin{cases}-\log\sin(\pi c)+\log(1-q^c), & \text{if } 0<c<1,\\ 2\log\sigma-\log\pi, & \text{if } c=0.\end{cases}$$

We have used the identity $(q^c;q)=(1-q^c)(q^{1+c};q)$ to obtain this form of $\delta(c,\sigma)$ and clearly it is continuous at $c=0$ for any positive σ.

The inequality (4.1) is difficult to check in general, but we can gain some understanding of it by looking at two extreme cases. For the first of these we consider the case of large σ.

Theorem 4.1. *If $1/2<c<1$ then there exists $\sigma(c)>0$ such that the density functions*

$$f_\epsilon(x)=(1-\epsilon)f_L(x;0,\sigma)+\epsilon f_B(x;c,0)\quad(-1\leqslant\epsilon\leqslant 1),$$

comprises a Stieltjes class for all $\sigma\geqslant\sigma(c)$. This is not true if $0\leqslant c\leqslant 1/2$; if $-1\leqslant\epsilon<0$ there exists $x>0$ and $\sigma(c)$ such that $f_\epsilon(x)<0$ if $\sigma\geqslant\sigma(c)$.

Proof. Observe that as $\sigma \to \infty$, the base interval $\mathcal{B}_q := (q, 1] \uparrow (0, 1]$ and if $a > 0$ then terms of the form $\log(q^a; q)$ tend to zero. In addition, if $x \in \bar{\mathcal{B}}_q$ we can write $x = q^\zeta$ where $0 \leqslant \zeta \leqslant 1$, and then $\lambda_2(x) = (c - \frac{1}{2}\zeta)\zeta\sigma^2$. So if $c > 1/2$ then $\inf_{x \in \bar{\mathcal{B}}_q} \lambda(x) \geqslant 0$ and $(c - 1/2)c\sigma^2 - \log\sigma \to \infty$. The first assertion follows.

Let $c = 1/2$ and $\alpha \leqslant x \leqslant 1$ where $0 < \alpha < 1$ is fixed. Then $[\alpha, 1] \subset \mathcal{B}_q$ if σ^2 is large enough, and the left-hand side of (4.1) equals $\lambda(x) - \log\sigma + o(1) \to -\infty$ as $\sigma^2 \to \infty$. Hence (4.1) eventually fails for such x.

Now let $0 \leqslant c < 1/2$. Then, uniformly in $\mathcal{B}_q$, the first sum at (4.2) is bounded above by $q^{1/2-c}/(1-q) \to 0$ and the second sum taken over $n \geqslant 1$ is similarly bounded by $q^{1/2+c}/(1-q) \to 0$. It follows that $\lambda_1(x) = \log(1 + x^{-1}q^{1/2+c}) + \bar{\lambda}(x)$ where $0 < \sup_{x \in \mathcal{B}_q} \bar{\lambda}(x) \to 0$ as $\sigma \to 0$, and hence the significant portion on the left-hand side of (4.1) is

$$T(x, \sigma) := \log\bigl(1 + q^{1/2+c-\zeta}\bigr) + Q_1(\zeta, c)\sigma^2 - \log\sigma + \delta(c, \sigma),$$

where

$$Q_1(\zeta, c) = \left(c - \frac{1}{2}\zeta\right)\zeta + \left(c - \frac{1}{2}\right)c.$$

Suppose that $c > 0$. If $\zeta < c + 1/2$ then $\log(1 + q^{1/2+c-\zeta})$ is negligible. Algebra will show that $Q_1(0, c) < 0$ and that $Q_1(\zeta, c) = 0$ iff $1/3 \leqslant c < 1/2$ and

$$\zeta = \zeta_u(c) := c + \sqrt{3c^2 - c} \quad \text{or} \quad \zeta = \zeta_l(c) := c - \sqrt{3c^2 - c}.$$

Observe that $\zeta_u(1/3) = \zeta_l(1/3) = 1/3$. The graph of $\zeta_u(c)$ increases to unity in $[1/3, 1/2]$ and it lies beneath the line $\zeta = c + 1/2$. The graph of $\zeta_l(c)$ decreases to zero in this interval. Hence, as $\sigma \to \infty$, $T(x, \sigma) \to \infty$ if $1/3 < c < 1/2$ and $\zeta_l(c) < \zeta < \zeta_u(c)$, and $T(x, \zeta) \to -\infty$ if $\zeta > \zeta_u(c)$ or $\zeta < \zeta_l(c)$, or if $c < 1/3$. If $c + 1/2 < \zeta \leqslant 1$ then $T(x, \sigma) \sim Q_2(\zeta, c)\sigma^2$ where

$$Q_2(\zeta, c) = \zeta - c - \frac{1}{2} + Q_1(\zeta, c) = -\frac{1}{2}(1 - \zeta)^2 - (3/2 - \zeta - c) < 0.$$

We conclude that $T(x, \zeta) \to -\infty$ for all $x \in \mathcal{B}_q$ if $0 < c < 1/3$.

Finally, if $c = 0$, then taking account of $\delta(0, \sigma)$ shows the dominant part of the left-hand side of (4.1) now is $\lambda(x) + \log\sigma \to \infty$ if $0 < \alpha \leqslant x \leqslant 1$. But if $0 < \zeta < 1$ then $Q_2(\zeta, 0) + \log\sigma = -\frac{1}{2}(1 - \zeta)^2\sigma^2 + \log\sigma \to -\infty$.

These cases show that if $0 \leqslant c \leqslant 1/2$ then (4.1) is violated for some values of x in $\mathcal{B}_q$ and sufficiently large σ. □

We now suppose that $\sigma \to 0$, in which case $q \to 1$ and for any $a > 0$ the product terms $(\pm q^a; q) \to 0$ or $\to \infty$, respectively. We can derive an asymptotic estimate of $\log(\pm q^a; q)$ by using Euler's summation formula. If $f(x)$ is positive and differentiable in $[0, \infty)$, and decreasing to zero then the usual expression of the summation formula gives

$$\sum_{n=0}^{\infty} f(n) = \int_0^\infty f(x)\,dx + \frac{1}{2}f(0) + \int_0^\infty P(x)f'(x)\,dx,$$

where $P(x) = x - [x] - 1/2$, and we assume that the left-hand side is finite. We will need the slight rearrangement obtained by observing that $\int_0^1 P(x) f'(x)\,dx = f(0) + \frac{1}{2} f(1) - \int_0^1 f(x)\,dx$, giving

$$\sum_{n=0}^{\infty} f(n) = \int_1^{\infty} f(x)\,dx + f(0) + \frac{1}{2} f(1) + R, \tag{4.4}$$

where

$$R = \int_1^{\infty} P(x) f'(x)\,dx = \sum_{n\geqslant 1} \int_0^{1/2} u\left[f'\left(n + \frac{1}{2} + u\right) - f'\left(n + \frac{1}{2} - u\right)\right] du. \tag{4.5}$$

The following lemmas have interest in their own right.

Lemma 4.1. *If $a > 0$ and $q = e^{-V}$ then, as $V \to 0+$,*

$$-\log\bigl(q^a; q\bigr) = \frac{\pi^2}{6V} + \left(a - \frac{1}{2}\right) \log V + L(a) + Z(a) + o(1),$$

where

$$L(a) = \left(a + \frac{1}{2}\right) \log(1 + a) - \log a - 1 - a,$$

$$Z(a) = \frac{1}{2} \sum_{j=2}^{\infty} (-1)^j \frac{j-1}{j(j+1)} \zeta(j, a+1) \tag{4.6}$$

and, for $A > 0$ and $s > 1$, $\zeta(s, A) = \sum_{n\geqslant 0} (A+n)^{-s}$ is the Hurwitz zeta function.

Proof. Let $f(x) = -\log(1 - q^{a+x})$ in (4.4). Setting $A = 1 + a$, the first integral is

$$I(V, a) = \sum_{j=1}^{\infty} \bigl(q^{aj}/j\bigr) \int_1^{\infty} e^{-Vjx}\,dx = V^{-1}\left[\sum_{j=1}^{\infty} j^{-2} - \sum_{j=1}^{\infty} \bigl(1 - j^{-2} q^{Aj}\bigr)\right]. \tag{4.7}$$

The first sum equals $\zeta(2) = \pi^2/6$. Since $j^{-2} = \int_0^{\infty} z e^{-jz}\,dz$, the second sum reduces to

$$\bigl(1 - q^A\bigr) \int_0^{\infty} \frac{z e^{-z}}{(1 - e^{-z})(1 - q^A e^{-z})}\,dz$$

$$= \bigl(1 - q^A\bigr)\left[\int_0^{\infty} \frac{z e^{-z}}{(1 - q^A e^{-z})}\,dz + \int_0^1 \frac{-u^{-1}\log(1-u) - 1}{1 - q^A(1-u)}\,du\right],$$

where we have used the substitution $u = 1 - e^{-z}$ for the second integral. The first integral equals $-q^A \log(1 - q^A)$. Letting $q \to 1$ and expanding the log term as a power series will show that the second integral equals $\sum_{j\geqslant 1} j^{-1} \int_0^1 u^{j-2}\,du + o(1) = 1 + o(1)$. Hence the second sum at (4.7) equals

$$-\bigl(e^{AV} - 1\bigr) \log\bigl(1 - e^{-AV}\bigr) + \bigl(1 - e^{-AV}\bigr) + o(V) = AV[-\log V - \log A + 1] + o(V).$$

We thus have the expansion

$$I(V,a)=\frac{\pi^2}{6V}+A\log V+A\log A-A+o(1).$$

Some algebra yields $f(0)+\frac{1}{2}f(1)=-(3/2)\log V-\log a-\frac{1}{2}\log A+o(1)$. Next, we evaluate the remainder R as follows. Differentiation yields $f'(n+1/2+u)-f'(n+1/2-u)=2Vq^a\sinh(Vu)T(n,u)$ where, as $V\to 0$,

$$T(n,u):=\frac{q^{n+1/2}}{(1-q^{a+n+1/2+u})(1-q^{a+n+1/2-u})}\sim\frac{2u}{(a+n+1/2+u)(a+n+1/2-u)}.$$

In particular we can interchange integration and summation in R. If $0<B,C<1$ and $0\leqslant y\leqslant 1$ then the function $g(y)=y/(1-By)(1-Cy)$ is increasing in y. It follows that $T(n,u)$ is decreasing in n and hence the terms of the series in R (after interchanging) can be bounded above by the corresponding integral plus a constant term. It can be shown that the dominated convergence theorem is applicable to this bound and hence, using Pratt's lemma [18, p. 232], we conclude that the remainder

$$\begin{aligned}R&=\sum_{n\geqslant 1}\int_0^{1/2}\frac{2u^2}{(a+n+1/2+u)(a+n+1/2-u)}\,du+o(1)\\&=\sum_{n\geqslant 1}\left[\left(a+n+\frac{1}{2}\right)\log\frac{a+n+1}{a+n}-1\right]+o(1).\end{aligned}$$

The form given in the assertion arises by expanding the log term in inverse powers of $a+n$. □

Lemma 4.2. *If $a>0$ and $q=e^{-V}$ then as $V\to 0$,*

$$\log\bigl(-q^a;q\bigr)=\frac{\pi^2}{12V}-\left(a-\frac{1}{2}\right)\log 2+o(1).$$

Proof. The identity $(1+q^a)(1-q^a)=1-q^{2a}$ implies that

$$\log\bigl(-q^a;q\bigr)=-\log\bigl(q^a;q\bigr)+\log\bigl(q^{2a};q^2\bigr)$$

and the assertion follows directly from Lemma 4.1 because the expansion of the second term on the right follows from the mere replacement of V with $2V$. □

Our final result shows that the lognormal law and Berg's laws determine a Stieltjes class if σ is small.

Theorem 4.2. *If $0\leqslant c<1$ there exists $\sigma(c)>0$ such that* (4.1) *is satisfied for $x\in\mathcal{B}_q$ if $0<\sigma\leqslant\sigma(c)$.*

Proof. Let $V:=\sigma^2$. Observe that if $x=q^\zeta$ then $\lambda_2(x)=(c-\frac{1}{2}\zeta)\zeta\sigma^2\to 0$ uniformly for $x\in\mathcal{B}_q$. In addition, by combining corresponding summands in (4.2) we see that $\lambda_1(x)$ is an increasing function of $xq^{-c}+x^{-1}q^c$, and this takes its least value of 2 at $x=q^c$. Hence $\lambda_1(x)\geqslant 2\log(-q^{1/2};q)$. Applying Lemma 4.2 to this bound and applying Lemma 1 to the three

(q^a, q) terms at (4.1) will show that the contributions from terms in the asymptotic expansions proportional to V^{-1} cancel. Similarly, observing that

$$\delta(c,\sigma) = \log V + \log(c/\sin\pi c) + o(1),$$

the terms proportional to $\log V$ also vanish. Thus the left-hand side of (4.1) is bounded below by $B_1(c) + B_2(c) + o(1)$ where

$$\begin{aligned} B_1(c) &= -L(1+c) - L(1-c) + L(1) + \log\bigl(c/\sin(\pi c)\bigr) \\ &= -(3/2+c)\log(2+c) - (3/2-c)\log(2-c) + \log\bigl(1-c^2\bigr) \\ &\quad + 2 + (3/2)\log 2 + \log\bigl(c/\sin(\pi c)\bigr), \\ B_2(c) &= -Z(1+c) - Z(1-c) + Z(1), \end{aligned}$$

and the $o(1)$ term holds uniformly for $x \in \mathcal{B}_q$.

Careful numerical computation shows that $B_1(c)$ decreases from -0.18445 at $c = 0$ to -0.15839 at $c = 1$. This last value is $B_1(1-)$, obtained using the limit relation $\log(1-c^2) + \log(c/\sin(\pi c)) \to \log 2 - \log\pi$ as $c \uparrow 1$.

The representation (4.6) exhibits $Z(a)$ as an alternating series whose unsigned terms decrease as j increases. Consequently $Z(a)$ is bounded above and below by successive partial sums. It follows that

$$\begin{aligned} B_2(c) &> -\frac{1}{12}\bigl[\zeta(2,2+c) + \zeta(2,2-c) - \zeta(2,2) + \zeta(3,2)\bigr] \\ &> -\frac{1}{12}\bigl[\zeta(2,3) + \zeta(2,1) - \zeta(2,1) + \zeta(3,2)\bigr], \end{aligned}$$

since $\zeta(s,2+c) + \zeta(s,2-c)$ is increasing in c. But $\zeta(s,1) = \zeta(s)$, the Riemann zeta function, and $\zeta(s,i) = \zeta(s) - 1 - \cdots - (i-1)^{-s}$, so we find for all $0 \leqslant c \leqslant 1$ that

$$B_2(c) > -\frac{1}{12}\bigl[\zeta(2) + \zeta(3) - 1 - 1/4\bigr] = -0.13310.$$

We conclude that if $0 \leqslant c < 1$ then the left-hand side of (4.1) is uniformly bounded below by $-0.1845 - 0.1331 + o(1)$ which exceeds the right-hand side, -0.91894, if σ is small enough. □

5. Construction by random scaling

Let V be a random variable with distribution function $K(x)$ and finite moments ν_n, and which is independent of X. If $Y = VX$ then $E(Y^n) = \nu_n m_n$, and its distribution function $G(x)$ is indeterminate if $F(x)$ is so. This simple construction by random scaling allows endless examples of M-equivalent laws to be constructed from a single pair of M-equivalent laws. Our next result shows this possibility extends to Stieltjes classes. In this section the factors in any product of random variables are assumed to be independent.

Let X_ϵ have the distribution function $F_\epsilon(x)$ at (1.1), assumed to comprise a Stieltjes class. The distribution function of $Y_\epsilon := VX_\epsilon$ is

$$G_\epsilon(x) = \int F_\epsilon(x/v)\,dK(v), \tag{5.1}$$

so $G_0(x) = G(x)$.

Theorem 5.1. *If $\mathcal{S}(F,h)$ is a Stieltjes class, then so is $\mathcal{S}(G,\kappa)=\{G_\epsilon(x)\colon -1\leqslant\epsilon\leqslant 1\}$ where $\kappa(x)$ is given implicitly by*

$$\int_0^x \kappa(y)\,dG(y)=\int_0^\infty\int_0^{x/v} h(z)\,dF(z)\,dK(v). \tag{5.2}$$

If $b_1<b_2$ are real constants such that $b_1\leqslant h(x)\leqslant b_2$ for almost every x with respect to $F(x)$ then $b_1\leqslant\kappa(x)\leqslant b_2$ for almost every x with respect to $G(x)$.

Suppose $P(V>0)=1$ and $F(x)$ has a density function $f(x)$. Then $G(x)$ has a density function

$$g(x)=\int v^{-1}f(x/v)\,dK(v) \tag{5.3}$$

and

$$\kappa(x)g(x)=\int v^{-1}h(x/v)f(x/v)\,dK(v). \tag{5.4}$$

Proof. Substituting (1.1) into (5.1) yields

$$G_\epsilon(x)=G(x)+\epsilon\int_0^x \kappa(y)\,dG(y),$$

where $\kappa(x)$ satisfies (5.2), and it can be calculated by inverting the Mellin transform

$$\int x^t\kappa(x)\,dG(x)=\int v^t\left(\int x^t h(x)\,dF(x)\right)dK(v).$$

The right-hand side vanishes for $t=0,1,\ldots$. If $h(x)$ satisfies the asserted boundedness condition then for all $0\leqslant x'<x''$, $b_1(G(x'')-G(x'))\leqslant\int_{x'}^{x''}\kappa(y)\,dG(y)\leqslant b_2(G(x'')-G(x'))$, and this implies the boundedness assertion for $\kappa(x)$. By choosing $b_1=-1$ and $b_2=1$, it follows that $\mathcal{S}(G,\kappa)$ is a Stieltjes class. Finally, if $F(x)$ is absolutely continuous and $V>0$ then $G(x)$ is absolutely continuous with a density function (5.3), and (5.4) follows from (5.2). □

The following example may be seen as an alternative rendition of portions of Sections 2 and 3 in [5]. We need a little notation from [2, Chapter 10]. Denote the q-factorials by $n!_q:=(1-q)^{-1}(q;q)_n$ where $(a;q)_n=(1-a)\times\cdots\times(1-aq^{n-1})$ $(n\geqslant 1)$, and $(a;q)_0=1$. One version of a q-exponential function is

$$e_q(x)=\sum_{n\geqslant 0}x^n/n!_q=1/\big((1-q)x;q\big),$$

where the second equality is an identity of Euler. For any fixed $a>0$ [3, §4] defines a weight function which, after normalization, is the density function of a random variable $C_q(a)$,

$$f_A(x;a)=x^{a-1}e_q(-x)/A_q(a)\quad (x>0), \tag{5.5}$$

where

$$A_q(a)=\frac{\Gamma(a)\Gamma(1-a)}{\Gamma_q(1-a)}\quad\text{and}\quad \Gamma_q(a):=(1-q)^{1-a}\frac{(q;q)}{(q^a;q)};$$

see [17, (6.21)]. This density function defines a continuous q-gamma distribution in the sense that $f_A(x;a) \to x^{a-1}e^{-x}/\Gamma(a)$ as $q \uparrow 1$. This limit distribution is determinate, but each of those defined by (5.5) is indeterminate. Its moment sequence is given by

$$M_A(n) = (1-q)^{-n}\big(q^a;q\big)_n q^{-an-\binom{n}{2}}, \tag{5.6}$$

and Askey [3, (4.4) and (4.6)] exhibits a discrete distribution having the same moments. Berg [6] calls this the q-Laguerre moment problem because (5.5) is an orthogonality measure for q-Laguerre polynomials; see [15].

The factor $q^{-an-\binom{n}{2}}$ is the moment sequence of laws M-equivalent to the $LN(\mu(a),\sigma^2)$ law, where $\mu(a) := (a-1/2)\sigma^2$. The factor $(1-q)^{-n}(q^a;q)_n$ are the moments of the discrete q-gamma random variable $\gamma_q(a)$,

$$P\big(\gamma_q(a) = (1-q)^{-1}q^j\big) = \big(q^a;q\big)q^{aj}/(q;q)_j \quad (j = 0,1,\ldots); \tag{5.7}$$

see [17, §5]. This name is appropriate because the moment function for (5.7),

$$M_{dg}(t;a) := E\big(\gamma_q^t(a)\big) = \frac{\Gamma_q(a+t)}{\Gamma_q(a)}, \tag{5.8}$$

converges as $q \uparrow 1$ to $\Gamma(a+t)/\Gamma(a)$, the moment function of the gamma law.

It is clear that solutions to the q-Laguerre moment problem can be obtained as the distribution of $Y = \gamma_q(a)X$ where X has the moments of the $LN(\mu(a),\sigma^2)$ law. For example, if $0 < c < 1$ then the moment function (3.16) can be expressed in terms of gamma functions as

$$M_B(t;c,\mu) = m^t q^{ct}\frac{\Gamma_q(c)\Gamma_q(1-c)}{\Gamma(c)\Gamma(1-c)} \cdot \frac{\Gamma(c+t)\Gamma(1-c-t)}{\Gamma_q(c+t)\Gamma_q(1-c-t)}. \tag{5.9}$$

If $B(c,\mu)$ is a random variable having this moment function, then that of the product $\gamma_q(a) \times B(a,\mu(a))$ is

$$M_B\big(t;a,\mu(a)\big)M_{dg}(t;a) = \frac{\Gamma_q(1-a)}{\Gamma(a)\Gamma(1-a)} \cdot \frac{\Gamma(a+t)\Gamma(1-a-t)}{\Gamma_q(1-a-t)},$$

which is the moment function of $C_q(a)$. This multiplicative representation of $C_q(a)$ contradicts one of our results; see Theorem 6.2(a) in [17]. In fact, the log-convexity assertion is not correct. This false assertion is mentioned in [17, §7], but it is not relevant to the discussion there.

To make this assertion correct, let $\Lambda(a)$ have the $LN(\mu(a),\sigma^2)$ law. Then the moment function of $W_a = \gamma_q(a)\Lambda(a)$ is $q^{at+\binom{t}{2}}M_{dg}(t;a)$. Its density function is a special case of the following lemma.

Lemma 5.1. *If X has the $LN(\mu,\sigma^2)$ law then the density function of $\gamma_q(a)X$ is*

$$g(x;a,\mu,\sigma) = \big(q^a;q\big)\big(-(1-q)^{-1}x^{-1}q^{a+1/2}e^{\mu};q\big)f_L\big((1-q)x;\mu,\sigma\big).$$

Proof. Evaluation of (5.3) gives

$$\begin{aligned}
&g(x;a,\mu,\sigma) \\
&\quad = E\big[\gamma_q^{-1}(a)f_L\big(x/\gamma_q(a);\mu,\sigma\big)\big] \\
&\quad = \frac{(q^a;q)}{\sqrt{2\pi\sigma^2}}\sum_{j=0}^{\infty}\frac{q^{aj}}{(q;q)_j}\exp\big[-\big(2\sigma^2\big)^{-1}\big(\log x - \mu + j\sigma^2 + \log(1-q)\big)^2\big]
\end{aligned}$$

$$= (q^a;q) f_L\big(x;\mu-\log(1-q),\sigma\big) \sum_{j=0}^{\infty} \big[(1-q)^{-1}x^{-1}q^{a+1/2}e^{\mu}\big]^j q^{\binom{j}{2}} \big/ (q;q)_j.$$

Another Euler identity asserts that the sum equals the q-product factor in the assertion. □

Remark. Observe that $q^{a+1/2}e^{\mu(a)} = q$, and $g(x;a,\mu(a),\sigma)$ has the same moments as $C_q(a)$.

Since the moments of $B(c,\mu(a))$ do not depend on c, the product $Y(a,c) := \gamma_q(a)B(c,\mu(a))$ has the same moments as $C_q(a)$. It follows from Theorem 5.1 that the density function of $Y(a,c)$ can be joined in a Stieltjes class to the density function of $C_q(a)$ as center under the stipulations of Theorems 4.1 and 4.2. The density function of $Y(a,c)$ can be evaluated in two ways.

The first evaluation begins as for Lemma 5.1 but putting the sum into hypergeometric form using the identities $1/(-zq^j;q) = (-z;q)_j/(-z;q)$ and

$$1/\big(-zq^{-j};q\big) = z^{-j}q^{\frac{1}{2}j(j+1)}/(-z;q)(q/z;q)_j.$$

This gives the density function

$$b(x;a,c) = \frac{(1-q)^c x^{c-1}}{(-\beta;q)(-q/\beta;q)\mathcal{N}(c,\mu(a))} {}_1\phi_1\big(-\beta;\beta;q,-q/(1-q)x\big),$$

where $\beta = q^{c-a+1}/(1-q)x$ and ${}_1\phi_1$ is a basic hypergeometric function [2, §10.9].

The second evaluation is based on a limiting form of the Askey–Roy q-beta integral [2, p. 514] which yields

$$K(b,c) := \int_0^{\infty} y^{c-1}\frac{(-q^{b+1}/y;q)}{(-y;q)(-q/y;q)}\,dy = \frac{\Gamma(c)\Gamma(1-c)}{\Gamma_q(c)\Gamma_q(1-c)}\Gamma_q(b+c)(1-q)^{b+c},$$

valid for $b > -c$ and $0 < c < 1$, and a limit is taken when $c = 0$. Denoting the integrand by $\psi(y;b,c)$, we see that $f_Y(y) := \psi(y;a-c,c)/K(a-c,c)$ is the density function of Y, say, and its moment function is

$$\int_0^{\infty} y^t f_Y(y)\,dy = \frac{K(a-c,c+t)}{K(a-c,c)}.$$

Multiplication by $(1-q)^{-t}q^{(c-a)t}$ gives the product $M_B(t;c,\mu(a))M_{dg}(t;a)$ (see (5.8) and (5.9)), i.e., the moment function of $Y(a,c)$. It follows that the density function of $Y(a,c)$ is

$$\begin{aligned} b(x;a,c) &= (1-q)q^{a-c} f_Y\big((1-q)q^{a-c}x\big) \\ &= \frac{(1-q)^{c-a}q^{(a-c)c}}{\mathcal{N}(a,c)} \cdot \frac{x^{c-1}(-q/(1-q)x;q)}{(-(1-q)q^{a-c}x;q)(-q^{c-a+1}/(1-q)x;q)}. \end{aligned}$$

The distribution of $Y(a,c)$ is a two-parameter continuous q-gamma law in the sense that its moment function converges to $\Gamma(a+t)/\Gamma(a)$. The two expressions of $b(x;a,c)$ give an evaluation of the above q-hypergeometric function. Formulae for the case $c = 0$ can be derived from the above results by taking a limit.

The above solutions of the q-Laguerre moment problem satisfy a weight-scaling relation dual to (3.7). Theorem 5.1 in [17] asserts that $V\hat{\gamma}_q(a) \stackrel{L}{=} \gamma_q(a)$ where $P(V = q^j) = (1-q^a)q^{aj}$ for

$j = 0, 1, \ldots$. Let X satisfy (3.3) and $Y = \gamma_q(a)X$. Then since $\hat{Y} \stackrel{L}{=} \hat{\gamma}_q(a)\hat{X}$, we obtain $V\hat{Y} \stackrel{L}{=} q^{-1}Y$. So, if $N(t) = E(Y^t)$, then it follows from the relation $E(V^t) = (1-q^a)/(1-q^{a+t})$ that

$$\frac{1-q^a}{1-q^{a+t}} \cdot \frac{N(t+1)}{N(1)} = q^{-t}N(t),$$

and $N(1) = (1-q^a)m/(1-q)$. A rearrangement of this functional equation yields

$$q^a N(t) + \left(1-q^a\right)\frac{N(t+1)}{N(1)} = q^{-t}N(t),$$

which is expressed in terms of the distribution function $G(x)$ of Y as

$$\left[q^a + \left(1-q^a\right)x/N(1)\right]dG(x) = dG(qx). \tag{5.10}$$

The steps leading to this identity can be reversed, thus establishing a one-to-one correspondence between solutions of (3.7) and (5.10). Christiansen [12] explores implications of (5.10) in the case $\mu = q^{-a}$, for which the coefficient of $dG(x)$ simplifies to $q^a(1+x)$. This correspondence shows that N-extremal solutions of the lognormal and the q-Laguerre moment problems are not related through multiplication by $\gamma_q(a)$, a proposition which follows also from the fact that the support of the distribution of such a product has zero as a limit point.

Acknowledgments

I thank Professor Christian Berg for alerting me to the significance of Gelfand's proof of Naimark's theorem. My thanks too for the referee's careful reading of the manuscript.

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