

Attracting and invariant sets for a class of impulsive functional differential equations [☆]

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Received 28 June 2005

Available online 8 August 2006

Submitted by K. Gopalsamy

Abstract

In this article, a class of nonlinear and nonautonomous functional differential systems with impulsive effects is considered. By developing a delay differential inequality, we obtain the attracting set and invariant set of the impulsive system. An example is given to illustrate the theory.

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Keywords: Attracting set; Invariant set; Stability; Impulsive differential equation; Differential inequality

1. Introduction

Impulsive differential equations have attracted increasing interest both in theoretical research and applications in the past 20 years. In particular, the stability of the zero solution of impulsive differential equations has recently been widely studied by many authors (see [1–10]). However, under impulsive perturbation, the equilibrium point sometimes does not exist in many real physical systems, especially in nonlinear and nonautonomous dynamical systems. Therefore, an interesting subject is to discuss the attracting set and the invariant set of impulsive systems. Some significant progress has been made in the techniques and methods of determining the invariant set and attracting set for the continuous differential systems including ordinary differ-

[☆] The work is supported by National Natural Science Foundation of China under Grant 10371083.

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ential equations, partial differential equations and delay differential equations and so on [11–18]. Unfortunately, the corresponding problems for impulsive functional differential equations have not been considered prior to this work.

Motivated by the above discussions, our objective in this paper is to determine the invariant set and the global attracting set for a class of nonlinear nonautonomous functional differential systems with impulsive effects. Our method is based on a differential inequality with the impulsive initial conditions. An example is given to illustrate our results.

2. Preliminaries

Let N be the set of all positive integers, R^n be the space of n -dimensional real column vectors and $R^{m \times n}$ be the set of $m \times n$ real matrices. E denotes an $n \times n$ unit matrix. For $A, B \in R^{m \times n}$ or $A, B \in R^n$, $A \geq B$ ($A \leq B$, $A > B$, $A < B$) means that each pair of corresponding elements of A and B satisfies the inequality “ \geq (\leq , $>$, $<$)”. Especially, A is called a nonnegative matrix if $A \geq 0$, and z is called a positive vector if $z > 0$.

For $x(t) = (x_1(t), \dots, x_n(t))^T : R \rightarrow R^n$, we define

$$\begin{aligned} D^+x(t) &= \limsup_{s \rightarrow 0^+} \frac{x(t+s) - x(t)}{s}, \\ x(t^+) &= \lim_{s \rightarrow 0^+} x(t+s), \quad x(t^-) = \lim_{s \rightarrow 0^-} x(t+s), \\ [x(t)]^+ &= (|x_1(t)|, \dots, |x_n(t)|)^T, \quad [x_i(t)]_\tau = \sup_{-\tau \leq s \leq 0} \{x_i(t+s)\}, \\ [x(t)]_\tau &= ([x_1(t)]_\tau, \dots, [x_n(t)]_\tau)^T \quad \text{and} \quad [x(t)]_\tau^+ = [[x(t)]^+]_\tau. \end{aligned}$$

Let $\tau > 0$ and $t_0 < t_1 < t_2 < \dots$ be the fixed points with $\lim_{k \rightarrow \infty} t_k = \infty$ (called impulsive moments).

$C[X, Y]$ denotes the space of continuous mappings from the topological space X to the topological space Y . Especially, let $C \triangleq C[[-\tau, 0], R^n]$.

$PC = \{\phi : [-\tau, 0] \rightarrow R^n \mid \phi(t^+) = \phi(t) \text{ for } t \in [-\tau, 0), \phi(t^-) \text{ exists for } t \in (-\tau, 0], \phi(t^-) = \phi(t) \text{ for all but at most a finite number of points } t \in (-\tau, 0]\}$. PC is a space of piecewise right-hand continuous functions with the norm $\|\phi\| = \sup_{-\tau \leq s \leq 0} |\phi(s)|$, $\phi \in PC$, where $|\cdot|$ is a norm in R^n .

$PC[[t_0, \infty), R^{m \times n}] \triangleq \{\psi : [t_0, \infty) \rightarrow R^{m \times n} \mid \psi(t) \text{ is continuous at } t \neq t_k, \psi(t_k^+) \text{ and } \psi(t_k^-) \text{ exist, } \psi(t_k) = \psi(t_k^+), \text{ for } k \in N\}$.

In this paper, we shall consider an impulsive functional differential equation

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t, x_t), & t \neq t_k, t \geq t_0, \\ \Delta x = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-)), & k \in N, \end{cases} \quad (1)$$

where $A(t) \in PC[[t_0, \infty), R^{n \times n}]$, $f \in C[[t_{k-1}, t_k) \times PC, R^n]$ and the limit

$$\lim_{(t, \phi) \rightarrow (t_k^-, \phi)} f(t, \phi) = f(t_k^-, \phi)$$

exists, $I_k \in C[R^n, R^n]$, $x_t \in PC$ is defined by $x_t(s) = x(t+s)$, $s \in [-\tau, 0]$, $\dot{x}(t)$ denotes the right-hand derivative of $x(t)$.

Definition 1. A function $x(t) : [t_0 - \tau, \infty) \rightarrow R^n$ is said to be a solution of (1) through (t_0, ϕ) , if $x(t) \in PC[[t_0, \infty), R^n]$ as $t \geq t_0$, and satisfies (1) with the initial condition

$$x(t_0 + s) = \phi(s), \quad s \in [-\tau, 0], \quad \phi \in PC.$$

Throughout the paper, we always assume that for any $\phi \in PC$, system (1) has at least one solution through (t_0, ϕ) , denoted by $x(t, t_0, \phi)$ or $x_t(t_0, \phi)$ (simply $x(t)$ and x_t if no confusion should occur), where $x_t(t_0, \phi) = x(t + s, t_0, \phi) \in PC$, $s \in [-\tau, 0]$.

Definition 2. The set $S \subset PC$ is called a positive invariant set of (1), if for any initial value $\phi \in S$, we have the solution $x_t(t_0, \phi) \in S$ for $t \geq t_0$.

Definition 3. The set $S \subset PC$ is called a global attracting set of (1), if for any initial value $\phi \in PC$, the solution $x_t(t_0, \phi)$ converges to S as $t \rightarrow +\infty$. That is,

$$\text{dist}(x_t, S) \rightarrow 0 \quad \text{as } t \rightarrow +\infty,$$

where $\text{dist}(\varphi, S) = \inf_{\psi \in S} \text{dist}(\varphi, \psi)$, $\text{dist}(\varphi, \psi) = \sup_{s \in [-\tau, 0]} |\varphi(s) - \psi(s)|$, for $\varphi \in PC$.

Definition 4. The zero solution of (1) is said to be globally exponentially stable if for any solution $x(t, t_0, \phi)$, there exist constants $\lambda > 0$ and $\kappa \geq 1$ such that

$$|x(t, t_0, \phi)| \leq \kappa \|\phi\| e^{-\lambda(t-t_0)}, \quad t \geq t_0.$$

Definition 5. [21,22] Let the matrix $D = (d_{ij})_{n \times n}$ with $d_{ii} > 0$ and $d_{ij} \leq 0$, $i \neq j$, $i, j = 1, 2, \dots, n$. Then each of the following conditions is equivalent to the statement “ D is a nonsingular M -matrix”:

- (i) All the leading principle minors of D are positive.
- (ii) D^{-1} exists and $D^{-1} \geq 0$.
- (iii) There exists a positive vector d such that $Dd > 0$ or $D^T d > 0$.
- (iv) $D = C - M$ and $\rho(C^{-1}M) < 1$, where $M \geq 0$, $C = \text{diag}\{c_1, \dots, c_n\}$ and $\rho(\cdot)$ is the spectral radius of the matrix.

Based on Halanay inequality [19] and its extension [10,20], we develop the following differential inequality with the impulsive initial condition.

Lemma 1. Let $\sigma < b \leq +\infty$ and $v(t) \in C[[\sigma, b), \mathbb{R}^n]$ satisfies

$$\begin{cases} D^+ v(t) \leq P v(t) + Q[v(t)]_\tau + J, & t \in [\sigma, b), \\ v(\sigma + s) \in PC, & s \in [-\tau, 0], \end{cases} \quad (2)$$

where $P = (p_{ij})_{n \times n}$, $p_{ij} \geq 0$ for $i \neq j$, $Q = (q_{ij})_{n \times n} \geq 0$ and $J = (J_1, \dots, J_n)^T \geq 0$, $i, j = 1, 2, \dots, n$. Suppose that there exist a scalar $\lambda > 0$ and a vector $z = (z_1, z_2, \dots, z_n)^T > 0$ such that

$$[\lambda E + P + Q e^{\lambda \tau}] z < 0. \quad (3)$$

If the initial condition satisfies

$$v(t) \leq \kappa z e^{-\lambda(t-\sigma)} - (P + Q)^{-1} J, \quad \kappa \geq 0, \quad t \in [\sigma - \tau, \sigma], \quad (4)$$

then $v(t) \leq \kappa z e^{-\lambda(t-\sigma)} - (P + Q)^{-1} J$ for $t \in [\sigma, b)$.

Proof. From (3), we have $(P + Q)z < 0$. Together with Definition 5 and the negativeness of nondiagonal entries of $P + Q$, this implies that $-(P + Q)^{-1}$ exists and $-(P + Q)^{-1} \geq 0$. Denote

$$u(t) = (u_1(t), \dots, u_n(t))^T = v(t) + (P + Q)^{-1} J, \quad t \in [\sigma - \tau, b).$$

Then, by (2) and (4),

$$\begin{aligned} D^+u(t) &\leq Pv(t) + Q[v(t)]_\tau + J \\ &\leq P[u(t) - (P + Q)^{-1}J] + Q[u(t) - (P + Q)^{-1}J]_\tau + J \\ &= Pu(t) + Q[u(t)]_\tau, \quad t \in [\sigma, b), \end{aligned} \quad (5)$$

and

$$u(t) \leq \kappa z e^{-\lambda(t-\sigma)}, \quad \kappa \geq 0, \quad t \in [\sigma - \tau, \sigma]. \quad (6)$$

In the following, we shall prove that for any positive constant ϵ

$$u_i(t) \leq (\kappa + \epsilon)z_i e^{-\lambda(t-\sigma)} \triangleq y_i(t), \quad t \in [\sigma, b), \quad i = 1, \dots, n. \quad (7)$$

If this is not true, from (6) and the continuity of $u(t)$ as $t \in [\sigma, b)$, then there must be a constant $t^* > \sigma$ and some integer m such that

$$u_m(t^*) = y_m(t^*), \quad D^+u_m(t^*) \geq \dot{y}_m(t^*), \quad (8)$$

$$u_i(t) \leq y_i(t), \quad t \in [\sigma - \tau, t^*], \quad i = 1, \dots, n. \quad (9)$$

Using (5), (7)–(9), $p_{ij} \geq 0$ ($i \neq j$) and $Q \geq 0$, we obtain that

$$\begin{aligned} D^+u_m(t^*) &\leq \sum_{j=1}^n [p_{mj}u_j(t^*) + q_{mj}[u_j(t^*)]_\tau] \\ &\leq \sum_{j=1}^n [p_{mj}(\kappa + \epsilon)z_j e^{-\lambda(t^*-\sigma)} + q_{mj}(\kappa + \epsilon)z_j e^{-\lambda(t^*-\tau-\sigma)}] \\ &= \sum_{j=1}^n [p_{mj} + q_{mj}e^{\lambda\tau}]z_j(\kappa + \epsilon)e^{-\lambda(t^*-\sigma)}. \end{aligned} \quad (10)$$

From (3), we have $\sum_{j=1}^n [p_{mj} + q_{mj}e^{\lambda\tau}]z_j < -\lambda z_m$. Then (10) becomes

$$D^+u_m(t^*) < -\lambda z_m(\kappa + \epsilon)e^{-\lambda(t^*-\sigma)} = \dot{y}_m(t^*).$$

This contradicts the inequality in (8), and so (7) holds. Letting $\epsilon \rightarrow 0^+$ in (7), we have

$$u(t) = v(t) + (P + Q)^{-1}J \leq \kappa z e^{-\lambda(t-\sigma)}, \quad t \in [\sigma, b).$$

The proof is complete. \square

3. Main results

In this paper, we always suppose the following.

(A1) $[f(t, \varphi)]^+ \leq B[\varphi]_\tau^+ + J$ for $t \geq t_0$ and $\varphi \in PC$, where $B = (b_{ij})_{n \times n} \geq 0$ and $J = (J_1, J_2, \dots, J_n)^T \geq 0$.

(A2) There exist a scalar $\lambda > 0$ and a vector $z = (z_1, z_2, \dots, z_n)^T > 0$ such that

$$[\lambda E + \bar{A} + B e^{\lambda\tau}]z < 0,$$

where $\bar{A} = (\bar{a}_{ij})_{n \times n}$ satisfies $a_{ii}(t) \leq \bar{a}_{ii} < 0$ and $|a_{ij}(t)| \leq \bar{a}_{ij}$ for $i \neq j, i, j = 1, 2, \dots, n$.

(A3) $[x + I_k(x)]^+ \leq \Gamma_k[x]^+, k \in N$, for any $x \in R^n$, where $\Gamma_k = (\gamma_{ij}^{(k)})_{n \times n} \geq 0$.

Theorem 1. Assume that (A1)–(A3) hold. If

$$\ln \mu_k \leq \lambda(t_k - t_{k-1}) \quad \text{and} \quad v = \sum_{k=1}^{\infty} \ln v_k < \infty, \quad k \in N, \quad (11)$$

where $\mu_k, v_k \geq 1$ satisfy

$$\Gamma_k z \leq \mu_k z \quad \text{and} \quad \Gamma_k(-\bar{A} - B)^{-1} J \leq v_k(-\bar{A} - B)^{-1} J, \quad (12)$$

then $S = \{\phi \in PC \mid [\phi]_{\tau}^+ \leq e^v(-\bar{A} - B)^{-1} J\}$ is a global attracting set of (1).

Proof. From (A1) and the definition of \bar{A} , we calculate the upper right derivative along the solutions of (1),

$$\begin{aligned} D^+ |x_i(t)| &= \operatorname{sgn}(x_i(t)) \dot{x}_i(t) \\ &\leq \operatorname{sgn}(x_i(t)) \left[\sum_{j=1}^n a_{ij}(t)x_j(t) + f_i(t, x_t) \right] \\ &\leq a_{ii}(t)|x_i(t)| + \sum_{j \neq i} |a_{ij}(t)||x_j(t)| + \sum_{j=1}^n b_{ij}[x_j(t)]_{\tau}^+ + J_i \\ &\leq \bar{a}_{ii}|x_i(t)| + \sum_{j \neq i} \bar{a}_{ij}|x_j(t)| + \sum_{j=1}^n b_{ij}[x_j(t)]_{\tau}^+ + J_i, \quad i = 1, 2, \dots, n, \end{aligned}$$

where $\operatorname{sgn}(\cdot)$ is the sign function. That is,

$$D^+[x(t)]^+ \leq \bar{A}[x(t)]^+ + B[x(t)]_{\tau}^+ + J, \quad t \in [t_{k-1}, t_k), \quad k \in N. \quad (13)$$

From (A2) and Definition 5, we have $(\bar{A} + B)z < 0$ and $-(\bar{A} + B)$ is an M -matrix. Then $-(\bar{A} + B)^{-1} \geq 0$, and so $w = -(\bar{A} + B)^{-1} J \geq 0$. Furthermore, we can find an enough small $\epsilon > 0$ such that

$$[(\lambda + \epsilon)E + \bar{A} + B e^{(\lambda + \epsilon)\tau}]z < 0. \quad (14)$$

For the initial conditions $x(t_0 + s) = \phi(s)$, $s \in [-\tau, 0]$, where $\phi \in PC$, we have

$$[x(t)]^+ \leq \kappa_0 z, \quad \kappa_0 = \frac{\|\phi\|}{\min_{1 \leq i \leq n} \{z_i\}}, \quad t_0 - \tau \leq t \leq t_0,$$

and so

$$[x(t)]^+ \leq \kappa_0 z e^{-(\lambda + \epsilon)(t - t_0)} + w, \quad t_0 - \tau \leq t \leq t_0. \quad (15)$$

By (13)–(15) and Lemma 1,

$$[x(t)]^+ \leq \kappa_0 z e^{-(\lambda + \epsilon)(t - t_0)} + w, \quad t_0 \leq t < t_1. \quad (16)$$

Suppose that for all $m = 1, \dots, k$ the inequalities

$$[x(t)]^+ \leq \mu_0 \cdots \mu_{m-1} \kappa_0 z e^{-(\lambda + \epsilon)(t - t_0)} + v_0 \cdots v_{m-1} w, \quad t_{m-1} \leq t < t_m, \quad (17)$$

hold, where $\mu_0 = v_0 = 1$. Then, from (A3), (12) and (17),

$$\begin{aligned}
[x(t_k)]^+ &= [x(t_k^-) + I_k(x(t_k^-))]^+ \\
&\leq \Gamma_k [\mu_0 \cdots \mu_{k-1} \kappa_0 z e^{-(\lambda+\epsilon)(t_k-t_0)} + v_0 \cdots v_{k-1} w] \\
&\leq \mu_0 \cdots \mu_{k-1} \mu_k \kappa_0 z e^{-(\lambda+\epsilon)(t_k-t_0)} + v_0 \cdots v_{k-1} v_k w.
\end{aligned} \tag{18}$$

This, together with (17) and $\mu_k, v_k \geq 1$, lead to

$$[x(t)]^+ \leq \mu_0 \cdots \mu_{k-1} \mu_k \kappa_0 z e^{-(\lambda+\epsilon)(t-t_0)} + v_0 \cdots v_{k-1} v_k w \quad \text{for } t \in [t_k - \tau, t_k]. \tag{19}$$

On the other hand,

$$D^+[x(t)]^+ \leq \bar{A}[x(t)]^+ + B[x(t)]_\tau^+ + v_0 v_1 \cdots v_k J, \quad t \neq t_k. \tag{20}$$

It follows from (14), (19), (20) and Lemma 1 that

$$[x(t)]^+ \leq \mu_0 \cdots \mu_{k-1} \mu_k \kappa_0 z e^{-(\lambda+\epsilon)(t-t_0)} + v_0 \cdots v_{k-1} v_k w \quad \text{for } t \in [t_k, t_{k+1}). \tag{21}$$

By the induction, we can conclude that

$$[x(t)]^+ \leq \mu_0 \cdots \mu_{k-1} \kappa_0 z e^{-(\lambda+\epsilon)(t-t_0)} + v_0 \cdots v_{k-1} w, \quad t_{k-1} \leq t < t_k, \quad k \in N. \tag{22}$$

From (11),

$$\mu_k \leq e^{\lambda(t_k-t_{k-1})}, \quad v_0 \cdots v_{k-1} \leq e^v,$$

we can use (22) to conclude that

$$\begin{aligned}
[x(t)]^+ &\leq e^{\lambda(t_1-t_0)} \cdots e^{\lambda(t_{k-1}-t_{k-2})} \kappa_0 z e^{-(\lambda+\epsilon)(t-t_0)} + v_0 \cdots v_{k-1} w \\
&\leq \kappa_0 z e^{\lambda(t-t_0)} e^{-(\lambda+\epsilon)(t-t_0)} + e^v w \\
&= \kappa_0 z e^{-\epsilon(t-t_0)} + e^v w \quad \text{for all } t \in [t_0, t_k), \quad k \in N.
\end{aligned}$$

This implies that the conclusion holds and the proof is complete. \square

Remark 1. In condition (A2), λ and z are easily found if $-(\bar{A} + B)$ is an M -matrix. In fact, from (iii) in Definition 5, there exists a positive vector z such that $-(\bar{A} + B)z > 0$. Then, by using continuity, there is a λ satisfying (A2).

By using Lemma 1 with $\kappa_0 = 0$, we can obtain a positive invariant set of (1).

Theorem 2. Assume that (A1)–(A3) with $\Gamma_k = E$ hold. Then $S = \{\phi \in PC \mid [\phi]_\tau^+ \leq (-\bar{A} - B)^{-1} J\}$ is a positive invariant set and also a global attracting set of (1).

Proof. Similarly, the inequality (14) holds by (A1). For the initial condition $x(t_0 + s) = \phi(s)$, $s \in [-\tau, 0]$, where $\phi \in S$, we have

$$[x(t)]^+ \leq (-\bar{A} - B)^{-1} J, \quad t_0 - \tau \leq t \leq t_0. \tag{23}$$

By (A2), (14), (23) and Lemma 1 with $\kappa = 0$,

$$[x(t)]^+ \leq (-\bar{A} - B)^{-1} J, \quad t_0 \leq t < t_1.$$

Also,

$$[x(t_1^+)]^+ = [x(t_1^-) + I_k(x(t_1^-))]^+ \leq [x(t_1^-)]^+ \leq (-\bar{A} - B)^{-1} J.$$

Thus,

$$[x(t)]^+ \leq (-\bar{A} - B)^{-1}J, \quad t_1 - \tau \leq t \leq t_1.$$

Using Lemma 1 again, we obtain

$$[x(t)]^+ \leq (-\bar{A} - B)^{-1}J, \quad t_1 \leq t < t_2.$$

By an induction, we have

$$[x(t)]^+ \leq (-\bar{A} - B)^{-1}J, \quad t_{k-1} \leq t < t_k, \quad k \in N.$$

Therefore, $S = \{\phi \in PC \mid [\phi]_\tau^+ \leq (-\bar{A} - B)^{-1}J\}$ is a positive invariant set. Since $\Gamma_k = E$, a direct calculation shows that $\mu_k = \nu_k = 1$ and $\nu = 1$ in Theorem 1. It follows from Theorem 1 that the set S is also a global attracting set of (1). The proof is complete. \square

For the case $J = 0$, we easily observe $x(t) = 0$ is a solution of (1) from (A1) and (A3). In the following, we give the attractivity of the zero solution and the proof is similar to that of Theorem 1.

Theorem 3. Assume that (A1)–(A3) with $J = 0$ hold. If

$$\ln \mu_k \leq \lambda(t_k - t_{k-1}), \quad \text{where } \mu_k \geq 1 \text{ satisfy } \Gamma_k z \leq \mu_k z, \quad k \in N,$$

then the zero solution of (1) is globally exponentially stable.

Remark 2. According to the properties of M -matrix given in Definition 5, one can see that the above theorems are extension and improvement of the results on continuous dynamical systems in [17,18].

4. Illustrative example

Example 1. Consider a 2-dimensional impulsive delay system

$$\begin{cases} \dot{x}_1(t) = -4x_1(t) + \sin(t)x_2(t) + \sin(x_1(t-1)) + x_2(t-1) + J_1(t), & t \geq 0, \\ \dot{x}_2(t) = \cos(t)x_1(t) - 4x_2(t) - x_1(t-1) + \sin(x_2(t-1)) + J_2(t), & t \neq t_k, \\ \Delta x_1 = x_1(t_k^+) - x_1(t_k^-) = I_1(x(t_k^-)), & t_k = k, \\ \Delta x_2 = x_2(t_k^+) - x_2(t_k^-) = I_2(x(t_k^-)), & k \in N, \end{cases} \quad (24)$$

where $J(t) = (J_1(t), J_2(t))^T$ with $|J_1(t)| \leq J_1$ and $|J_2(t)| \leq J_2$, $I_k(x) = (\beta_{1k}x_1, \beta_{2k}x_2)^T$. Taking $\tau = 1$, $\lambda = 0.3$, $z = (1, 1)^T$, we easily verify the conditions (A1)–(A3) with

$$\begin{aligned} \bar{A} &= \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}, \\ \Gamma_k &= \begin{pmatrix} |1 + \beta_{1k}| & 0 \\ 0 & |1 + \beta_{2k}| \end{pmatrix}, \\ (\lambda E + \bar{A} + e^{\lambda\tau}B)z &\approx \begin{pmatrix} -2.3501 & 2.3499 \\ 2.3499 & -2.3501 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.0002 \\ -0.0002 \end{pmatrix} < 0. \end{aligned}$$

Now, we discuss the asymptotical behavior of the system (24) as follows:

(i) If $J(t) = (\sin(t), \cos(t))^T$ and $-e^{\frac{1}{4k}} - 1 \leq \beta_{ik} \leq e^{\frac{1}{4k}} - 1$, $i = 1, 2$, $k \in N$, then $\Gamma_k = e^{\frac{1}{4k}} E$ and $J = (1, 1)^T$. Thus, $\mu_k = \nu_k = e^{\frac{1}{4k}}$ and $v = \frac{1}{3}$, which implies that the conditions (11) and (12) hold. Therefore, by Theorem 1, $S = \{\phi \in PC \mid [\phi]_{\tau}^+ \leq e^{\frac{1}{3}}(-\bar{A} - B)^{-1}J = (e^{\frac{1}{3}}, e^{\frac{1}{3}})^T\}$ is a global attracting set of (24).

(ii) If $J(t) = (4 \cos(t), 5 \sin(t))^T$ and $-2 \leq \beta_{ik} \leq 0$, $i = 1, 2$, then $\Gamma_k = E$ and $J = (4, 5)^T$. According to Theorem 2, $S = \{\phi \in PC \mid [\phi]_{\tau}^+ \leq w = (-\bar{A} - B)^{-1}J = (4.4, 4.6)^T\}$ is a positive invariant set and also a global attracting set of (24).

(iii) If $J(t) = (0, 0)^T$ and $-2.3 \leq \beta_{ik} \leq 0.3$, $i = 1, 2$, then $\Gamma_k = 1.3E$ and $x = (0, 0)^T$ is the solution of (24). Taking $\mu_k = 1.3$, it follows from Theorem 3 that the zero solution of (24) is globally exponentially stable.

Acknowledgments

The authors are thankful to the reviewers for their encouragements and helpful suggestions as well as detailed annotations.

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