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# Inversion of the Bessel potential operator in weighted variable Lebesgue spaces ☆

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**Abstract**

We study the inversion problem of the Bessel potential operator within the frameworks of the weighted Lebesgue spaces with variable exponent. The inverse operator is constructed by using approximative inverse operators. This generalizes some classical results to the variable exponent setting.
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## 1. Introduction

The Bessel potentials $\mathcal{B}^\alpha\varphi = G_\alpha * \varphi$, where $G_\alpha$ is the well-known Bessel kernel of order $\alpha > 0$ (see Section 2.4 below), were firstly investigated within the frameworks of variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^d)$ by the first author and Samko [3]. Several results were obtained in [3], such as the boundedness of the Bessel potential operator $\mathcal{B}^\alpha$ in $L^{p(\cdot)}(\mathbb{R}^d)$, the characterization of the Bessel potentials in terms of the convergence of hypersingular integrals and the coincidence of the Bessel potential space $\mathcal{B}^k[L^{p(\cdot)}(\mathbb{R}^d)]$ with the Sobolev space $W^{k,p(\cdot)}(\mathbb{R}^d)$ (when $k$ is a natural number). We note that the relationship between Bessel potential spaces with variable exponent and Sobolev spaces was independently studied in [13].

In this paper we consider the problem of the inversion of the Bessel potential operator in the weighted spaces $L^{p(\cdot)}(\mathbb{R}^d, \varrho)$. The continuity of the operator $\mathcal{B}^\alpha$ is also proved over these spaces.

One possible way to deal with the inversion problem is by using the so-called *approximative inverse operator* (AIO) method, where the inverse operator is constructed as the limit of a certain sequence of convolution operators with integrable kernels (see [28, Chapter 11], for details and references). For constant exponents $p$, the integrability of the kernel is sufficient for the continuity of such operators in classical $L^p$-spaces. But for variable exponents $p(\cdot)$, this is not enough, since Young's inequality for convolutions does not hold in general. Nevertheless, the continuity of

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the convolution operators may be obtained if they are dominated by the maximal operator (see Proposition 5). This tool will play an important role below.

We refer to [28] for the inversion of the Bessel operator in the non-weighted case and constant exponent $p$, and for further references. As far as the inversion of other potential operators in variable Lebesgue spaces is concerned, we mention [23] where the AIO method was also used to get the inversion of the Riesz potential operator in variable Lebesgue spaces, and [2] where the same question was considered but using a different approach based on hypersingular integrals. We note that the inversion problem of the Riesz potential operator, acting on classical weighted $L^p$-spaces, was studied in [21] via AIO method.

We notice that Lebesgue spaces with variable exponent $L^{p(\cdot)}(\Omega)$ have been intensively studied during the last decade. Apparently, the research was strongly motivated by their applications to differential equations with non-standard growth conditions and by modeling problems of fluid mechanics (see, for instance, [1,24]). A significant progress has also been made concerning the study of classical integral operators in the context of the $L^{p(\cdot)}$-spaces. We refer to the papers [18,27] for the basics properties of these spaces. For details on the recent development of operator theory in such spaces see the surveying papers [9,15,29] and other references therein. We also mention the paper [5] where the boundedness of several classical operators was obtained through extrapolation tools.

The outline of the paper is as follows: in Section 4 we observe the boundedness of the Bessel potential operator on variable Lebesgue spaces and establish our main inversion results. Sections 2 and 3 provide some necessary preliminaries.

## 2. Preliminaries

By $|E|$ we denote the (Lebesgue) measure of a measurable subset $E$ of the Euclidean space $\mathbb{R}^d$. We write $B(x,r)$ for the open ball centered at $x \in \mathbb{R}^d$ and radius $r > 0$. The symbol $\mathcal{S}(\mathbb{R}^d)$ stands for the Schwartz class of all infinitely differentiable and rapidly decreasing functions in $\mathbb{R}^d$. The Fourier transform of an integrable function $\varphi$ is denoted by $\mathcal{F}\varphi$ (or $\hat{\varphi}$),

$$(\mathcal{F}\varphi)(\xi) := \int_{\mathbb{R}^d} e^{ix\cdot\xi}\varphi(x)\,dx, \quad \xi \in \mathbb{R}^d,$$

whereas $\mathcal{F}^{-1}\varphi$ denotes its inverse Fourier transform.

By $C$ we denote generic positive constants, which may have different values even in the same line. Although the exact value of the constant be irrelevant for our purposes, sometimes we emphasize its dependence on certain parameters (e.g. $C(\alpha)$ means that $C$ depends on $\alpha$, etc.). The notation is standard or it will be properly introduced throughout the text.

### 2.1. On Lebesgue spaces with variable exponent

The basics on variable Lebesgue spaces may be found in [18] and [27], but we recall here some necessary definitions. Let $\Omega \subset \mathbb{R}^d$ be a non-empty open set. For a measurable function $p : \Omega \to [1,\infty)$, we put

$$p^+ = p^+(\Omega) := \operatorname*{ess\,sup}_{x\in\Omega} p(x) \quad \text{and} \quad p^- = p^-(\Omega) := \operatorname*{ess\,inf}_{x\in\Omega} p(x).$$

In the sequel we use the notation

$$\mathcal{P}(\Omega) := \left\{p \in L^\infty(\Omega):\ 1 < p^- \leqslant p(x) \leqslant p^+ < \infty\right\} \tag{1}$$

and

$$\mathcal{P}_1(\Omega) := \left\{p \in L^\infty(\Omega):\ 1 \leqslant p^- \leqslant p(x) \leqslant p^+ < \infty\right\}. \tag{2}$$

The generalized Lebesgue space $L^{p(\cdot)}(\Omega)$ with variable exponent $p \in \mathcal{P}_1(\Omega)$ is introduced as the set of all functions $\varphi$ on $\Omega$ for which

$$I_{p(\cdot)}(\varphi) := \int_\Omega \left|\varphi(x)\right|^{p(x)} dx < \infty.$$

This is a Banach space equipped with the norm

$$\|\varphi\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0\colon\ I_{p(\cdot)}\left(\frac{\varphi}{\lambda}\right) \leqslant 1\right\}.$$

Given a weight $\varrho$ on $\Omega$ (i.e. a measurable non-negative and a.e. finite function on $\Omega$), the weighted variable Lebesgue space $L^{p(\cdot)}(\Omega,\varrho)$ is defined as the set of all measurable functions $\varphi$ on $\Omega$ for which $\varrho\varphi \in L^{p(\cdot)}(\Omega)$, normed by

$$\|\varphi\|_{L^{p(\cdot)}(\Omega,\varrho)} := \|\varrho\varphi\|_{L^{p(\cdot)}(\Omega)}.$$

The notation $\|\varphi\|_{L^{p(\cdot)}(\Omega,\varrho)}$ and $\|\varphi\|_{p(\cdot),\varrho}$ will be used interchangeably, when no ambiguity arises.

By $w$-$Lip(\Omega)$ we denote the class of all exponents $p \in L^{\infty}(\Omega)$ satisfying the (local) logarithmic condition

$$\big|p(x) - p(y)\big| \leqslant \frac{C}{-\ln|x-y|}, \quad |x-y| \leqslant \frac{1}{2},\ x, y \in \Omega. \tag{3}$$

In case of unbounded domains, we also refer to the decay condition

$$\big|p(x) - p_{\infty}\big| \leqslant \frac{C}{\ln(e+|x|)}, \quad x \in \Omega, \tag{4}$$

where $p_{\infty} := \lim_{|x|\to\infty} p(x)$. For simplicity, $w_{\infty}$-$Lip(\Omega)$ will stand for the class of all exponents $p(\cdot)$ that belong to $w$-$Lip(\Omega)$ and at the same time satisfy the decay condition (4).

By $p'(\cdot)$ we denote the conjugate exponent, given by $\frac{1}{p(x)} + \frac{1}{p'(x)} \equiv 1$.

In [16] the following theorem was proved.

**Theorem 1.** *If $p \in \mathcal{P}_1(\mathbb{R}^d)$, the class $C_0^{\infty}(\mathbb{R}^d)$ is dense in the space $L^{p(\cdot)}(\mathbb{R}^d,\varrho)$ if the following conditions are satisfied*:

$$\big|\big\{x \in \mathbb{R}^d\colon\ \varrho(x) = 0\big\}\big| = 0 \quad \textit{and} \quad \big[\varrho(x)\big]^{p(x)} \in L^1_{\mathrm{loc}}\big(\mathbb{R}^d\big). \tag{5}$$

### *2.2. The maximal operator in $L^{p(\cdot)}$-spaces*

Let

$$(\mathcal{M}\varphi)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)\cap\Omega} \big|\varphi(y)\big|\,dy \tag{6}$$

be the Hardy–Littlewood maximal operator. The following theorem for the power weights

$$\varrho(x) = \big(1+|x|\big)^{\gamma} \prod_{k=1}^{m} |x - x_k|^{\beta_k}, \quad x_k \in \Omega,\ m \in \mathbb{N}, \tag{7}$$

was proved in [17] for bounded domains $\Omega$, and in [14], in the unbounded case.

**Theorem 2.** *Let $p \in \mathcal{P}(\Omega) \cap w$-$Lip(\Omega)$ and $\varrho$ be weight* (7).

(I) *When $\Omega$ is bounded, the maximal operator is bounded in $L^{p(\cdot)}(\Omega,\varrho)$ if and only if*

$$-\frac{d}{p(x_k)} < \beta_k < \frac{d}{p'(x_k)}, \quad k = 1, 2, \ldots, m. \tag{8}$$

(II) *Let $\Omega$ be unbounded and $p$ be constant outside some ball of large radius $R > 0$: $p(x) \equiv p_{\infty} > 1$, if $|x| > R$. Then the maximal operator is bounded in $L^{p(\cdot)}(\Omega,\varrho)$ if and only if condition* (8) *and the condition*

$$-\frac{d}{p_{\infty}} < \gamma + \sum_{k=1}^{m} \beta_k < \frac{d}{p'_{\infty}} \tag{9}$$

*are satisfied.*

**Remark 3.** The boundedness of the maximal operator on variable Lebesgue spaces $L^{p(\cdot)}(\Omega)$ remained an open problem for a long time. It was first proved by Diening [7] over bounded domains, under the assumption (3) on the exponent. He later extended the result to the case $\Omega = \mathbb{R}^d$ by supposing, in addition, that the exponent $p(\cdot)$ is constant outside some large fixed ball (see [8]). Diening's statement was independently improved by Nekvinda [19] and Cruz-Uribe, Fiorenza and Neugebauer [4] by obtaining the boundedness of $\mathcal{M}$ over general domains $\Omega$, for exponents not necessarily constant at infinity. In the former, some integral condition was imposed, while in the latter it was assumed that $p(\cdot)$ has the logarithmic decay at infinity (4).

For simplicity, the notation $\mathbb{P}_\varrho(\Omega)$ will stand for the class of all exponents $p(\cdot) \in \mathcal{P}(\Omega)$ for which the maximal operator $\mathcal{M}$ is bounded in $L^{p(\cdot)}(\Omega, \varrho)$, with $\varrho^{-1} \in L^{p'(\cdot)}_{\mathrm{loc}}(\Omega)$. We note that for weights of the form (7) the class $\mathbb{P}_\varrho(\Omega)$ is described by conditions (8) and (9).

*2.3. Approximate identities*

Let $\Omega = \mathbb{R}^d$. For dilations

$$\mathbb{K}_\varepsilon f(x) = \frac{1}{\varepsilon^d} \int_{\mathbb{R}^d} k\left(\frac{x-y}{\varepsilon}\right) f(y)\, dy, \quad \varepsilon > 0, \tag{10}$$

the following result was proved in [6].

**Proposition 4.** *Let $p \in \mathcal{P}_1(\mathbb{R}^d) \cap w_\infty$-Lip$(\mathbb{R}^d)$ and $k(x)$ be an integrable function on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} k(x)\, dx = 1$ and $\int_{\mathbb{R}^d} \sup_{|y|\geqslant|x|} |k(y)|\, dx < \infty$. Then*

(i) $\| \sup_{\varepsilon>0} \mathbb{K}_\varepsilon f \|_{L^{p(\cdot)}(\mathbb{R}^d)} \leqslant C \| f \|_{L^{p(\cdot)}(\mathbb{R}^d)}$,
(ii) $\mathbb{K}_\varepsilon f \to f$

*as $\varepsilon \to 0$ in $L^{p(\cdot)}(\mathbb{R}^d)$-norm and almost everywhere.*

It is also possible to have a similar result of Proposition 4 if we impose the boundedness of the maximal operator. This is well known for classical $L^p$-spaces (see e.g. [30, Chapter III]) and it was generalized to the non-weighted variable Lebesgue spaces in [7]. Recently the statement was extended to the weighted variable case in [22] as stated below.

**Proposition 5.** *Let $\varrho$ be a weight with $\varrho^{-1} \in L^{p'(\cdot)}_{\mathrm{loc}}(\mathbb{R}^d)$ and $p \in \mathcal{P}_1(\mathbb{R}^d)$. Let also $k$ be an integrable function on $\mathbb{R}^d$ such that $A := \int_{\mathbb{R}^d} \sup_{|y|\geqslant|x|} |k(y)|\, dx < \infty$. Then*

(i) $$\left| \sup_{\varepsilon>0} \mathbb{K}_\varepsilon f(x) \right| \leqslant A(\mathcal{M} f)(x) \quad \textit{for all } f \in L^{p(\cdot)}\big(\mathbb{R}^d, \varrho\big),$$

*so that*

(ii) $$\left\| \sup_{\varepsilon>0} \mathbb{K}_\varepsilon f \right\|_{L^{p(\cdot)}(\mathbb{R}^d, \varrho)} \leqslant C \| f \|_{L^{p(\cdot)}(\mathbb{R}^d, \varrho)} \quad \textit{when } p \in \mathbb{P}_\varrho\big(\mathbb{R}^d\big).$$

*If, in addition, $\int_{\mathbb{R}^d} k(y)\, dy = 1$, and $\varrho(x)$ satisfies condition* (5)*, then also*

(iii) $$\mathbb{K}_\varepsilon f \to f \quad \textit{as } \varepsilon \to 0$$

*in $L^{p(\cdot)}(\mathbb{R}^d, \varrho)$-norm and almost everywhere.*

**Remark 6.** Proposition 4 shows that the results given in Proposition 5, in the case $\varrho \equiv 1$, are valid even if $p(\cdot)$ reaches the value 1. Note that the condition $p^- > 1$ is implicit in the statements (ii), (iii) of Proposition 5, since the

boundedness of the maximal operator is required. The necessity of this lower bound condition was recently proved in [12] for general exponents, improving a former result from [4] where only lower semicontinuous exponents were considered.

### 2.4. Bessel potential operator

The Bessel potential operator $\mathcal{B}^\alpha$, of order $\alpha > 0$, is defined by

$$\mathcal{B}^\alpha \varphi(x) := \int_{\mathbb{R}^d} G_\alpha(x-y)\varphi(y)\,dy, \tag{11}$$

where $G_\alpha$ is the so-called *Bessel kernel* given through the Fourier transform

$$\widehat{G}_\alpha(x) = \left(1+|x|^2\right)^{-\frac{\alpha}{2}}, \tag{12}$$

and $\varphi$ is called the *density* of the Bessel potential.

It is known that

$$G_\alpha(x) = C(\alpha,d)\int_0^\infty e^{-\frac{\pi|x|^2}{t}-\frac{t}{4\pi}} t^{\frac{\alpha-d}{2}}\frac{dt}{t}, \quad x\in\mathbb{R}^d, \tag{13}$$

where $C(\alpha,d)>0$ is a certain constant (see, for example, [30, Section V.3.1]), so that $G_\alpha$ is a non-negative, radially decreasing function. Moreover, $G_\alpha$ is integrable with $\|G_\alpha\|_1 = \widehat{G}_\alpha(0) = 1$ and it can also be represented by means of the *McDonald function*:

$$G_\alpha(x) = C(\alpha,d)|x|^{\frac{\alpha-d}{2}} K_{\frac{d-\alpha}{2}}\left(|x|\right), \quad x\in\mathbb{R}^d.$$

From this we know that $G_\alpha$ behaves like the Riesz kernel at the origin (if $0<\alpha<d$) and it has an exponential decay at infinity (see, for example, [25, Lemma 27.1]). More precisely,

$$G_\alpha(x) \sim \begin{cases} c(\alpha,d)|x|^{\alpha-d} & \text{if } 0<\alpha<d,\\ c(d)\ln(\frac{1}{|x|}) & \text{if } \alpha = d,\\ c(\alpha,d) & \text{if } \alpha > d \end{cases} \tag{14}$$

as $|x|\to 0$, and

$$G_\alpha(x) \sim c(\alpha,d)|x|^{\frac{\alpha-d-1}{2}} e^{-|x|} \tag{15}$$

as $|x|\to\infty$.

## 3. The method of Approximative Inverse Operator (AIO)

The AIO method was first realized by Zavolzhenskii and Nogin (see [31], for instance) in the beginning of nineties. We refer also to the surveying papers [20,26] for details and to the monograph [28, Chapter 11], where several statements can be found in the classical $L^p$-spaces setting.

For completeness, let us briefly describe the idea behind this method. Given a convolution operator $A\varphi = a*\varphi$, its Fourier transform is given formally by

$$\mathcal{F}(A\varphi)(\xi) = \hat{a}(\xi)\cdot\hat{\varphi}(\xi). \tag{16}$$

Using (16) we find that, formally, our inverse operator is

$$A^{-1}\varphi = \mathcal{F}^{-1}\left(\frac{1}{\hat{a}(\xi)}\cdot\hat{\varphi}(\xi)\right). \tag{17}$$

Since the factor $\hat{a}(\xi)^{-1}$ increases at infinity, we introduce a "nice" factor $m_\varepsilon(\xi)$ depending on $\varepsilon>0$, so that $m_\varepsilon(\xi)\hat{a}(\xi)^{-1}$ vanishes at infinity (or other prescribed condition). Then (17) should be rewritten as

$$\begin{aligned}\big(A^{-1}\big)\varphi &= \lim_{\varepsilon\to 0}\big(A^{-1}\big)_\varepsilon\varphi \\ &= \lim_{\varepsilon\to 0}\mathcal{F}^{-1}\left(\frac{m_\varepsilon(\xi)}{\hat{a}(\xi)}\cdot\hat{\varphi}(\xi)\right).\end{aligned} \tag{18}$$

We only need to justify that (18) will really generate the inverse operator in the space under consideration.

In this paper, we are interested in approximating inverse operators to the Bessel operator (11), namely in the construction

$$\big(\mathcal{B}^\alpha\big)^{-1}\varphi = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^d} q_\alpha(y,\varepsilon)\varphi(x-y)\,dy. \tag{19}$$

Via Fourier transforms, we see that the kernels $q_\alpha(\cdot,\varepsilon)$ need to satisfy the relation

$$\lim_{\varepsilon\to 0}\widehat{q_\alpha}(\xi,\varepsilon)\cdot\widehat{G_\alpha}(\xi) = 1, \tag{20}$$

or

$$\widehat{q_\alpha}(\xi,\varepsilon) = \big(1+|\xi|^2\big)^{\frac{\alpha}{2}}\mathcal{K}_\varepsilon(\xi) \tag{21}$$

with $\mathcal{K}_\varepsilon(\xi)\to 1$ as $\varepsilon\to 0$. The auxiliary "nice" kernel $\mathcal{K}_\varepsilon(\xi)$ should compensate the increase of $(1+|\xi|^2)^{\alpha/2}$ at infinity. Additionally, we also require that $\mathcal{K}_\varepsilon(\xi) = \mathcal{K}(\varepsilon\xi)$, with $\mathcal{K}(0) = 1$.

## 4. Inversion of the Bessel potential operator

The next statement, on the continuity of the Bessel potential operator in $L^{p(\cdot)}(\mathbb{R}^d,\varrho)$, generalizes Theorem 5.1 in [3] to the weighted case.

**Theorem 7.** *If the exponent $p(\cdot)$ belongs to the class $\mathbb{P}_\varrho(\mathbb{R}^d)$, then the Bessel potential operator $\mathcal{B}^\alpha$ is bounded in $L^{p(\cdot)}(\mathbb{R}^d,\varrho)$.*

**Proof.** Since the kernel $G_\alpha$ is radially decreasing and integrable, then from Proposition 5(ii) there exists a constant $C>0$ such that

$$\big\|\mathcal{B}^\alpha\varphi\big\|_{p(\cdot),\varrho} = \|G_\alpha * \varphi\|_{p(\cdot),\varrho} \leqslant C\|\varphi\|_{p(\cdot),\varrho}$$

for all $\varphi\in L^{p(\cdot)}(\mathbb{R}^d,\varrho)$. □

Now we make use of the approximating inverse operators to get the inverse operator to $\mathcal{B}^\alpha$ as described above. As in [28], we shall consider two constructions corresponding to different auxiliary kernels $\mathcal{K}_\varepsilon$ from (21).

### 4.1. The choice $\mathcal{K}_\varepsilon(\xi) = e^{-\varepsilon|\xi|}$

Under this choice, we have

$$\widehat{q_\alpha}(\xi,\varepsilon) = \big(1+|\xi|^2\big)^{\frac{\alpha}{2}}e^{-\varepsilon|\xi|}$$

and by Bochner formula for the Fourier transform of radial functions (e.g. Lemma 2.1 in [28]), we obtain

$$q_\alpha(x,\varepsilon) = \frac{(2\pi)^{-d/2}}{|x|^{\frac{d}{2}-1}}\int_0^\infty t^{d/2}\big(1+t^2\big)^{\alpha/2}e^{-\varepsilon t}J_{\frac{d}{2}-1}\big(t|x|\big)\,dt, \tag{22}$$

where $J_\nu(z)$ is the *Bessel function of the first kind* (see, for example, [25]).

We already know that $q_\alpha(\cdot,\varepsilon)\in L^1(\mathbb{R}^d)$ for any $\varepsilon>0$ (see Lemma 1.22 in [28]). But we need to know more to use the continuity of the convolution with this kernel in the variable exponent setting. Having in mind Proposition 5, the next statement allow us to overcome this difficulty.

**Lemma 8.** *The kernel $q_\alpha(\cdot,\varepsilon)$ from* (22) *admits the bound*

$$\left|q_\alpha(x,\varepsilon)\right| \leqslant \frac{C(\alpha,d,\varepsilon)}{(1+|x|)^{d+\alpha}}, \quad x\in\mathbb{R}^d. \tag{23}$$

**Proof.** Taking into account the identity $\frac{d}{du}[u^\nu J_\nu(u)] = u^\nu J_{\nu-1}(u)$ and using integration by parts in (22), we arrive at the formula

$$q_\alpha(x,\varepsilon) = \frac{(2\pi)^{-d/2}}{|x|^{\frac{d}{2}-1}}\frac{(-1)^m}{|x|^m}\int_0^\infty g^{(m)}_{\alpha,\varepsilon}(t)t^{d/2}J_{\frac{d}{2}+m-1}\big(t|x|\big)\,dt, \quad x\neq 0 \tag{24}$$

(see (8.133) in [28]), where

$$g_{\alpha,\varepsilon}(t) = \big(1+t^2\big)^{\alpha/2}e^{-\varepsilon t}, \quad t>0,$$

and $m\geqslant m_0$ is an arbitrary integer (see (29) below). Obviously (24) holds if

$$g^{(k)}_{\alpha,\varepsilon}(t)t^{d/2}J_{\frac{d}{2}+k}\big(t|x|\big)\big|_0^\infty = 0, \quad k=0,1,\ldots,m-1. \tag{25}$$

To check this we first observe that (by Leibniz' formula)

$$g^{(k)}_{\alpha,\varepsilon}(t) = \sum_{j=0}^{k}\binom{k}{l}(-\varepsilon)^{k-j}e^{-\varepsilon t}\big(1+t^2\big)^{\alpha/2}\sum_{l=0}^{[j/2]}c_l(\alpha)\frac{t^{j-2l}}{(1+t^2)^{j-l}}$$

(some constants $c_l(\alpha)$ may vanish but not all simultaneously), which follows by straightforward calculations. Since $J_{\frac{d}{2}+k}(u)$ behaves like $u^{\frac{d}{2}+k}$ for small values of $u$ and like $u^{-1/2}$ for large ones, we obtain

$$g^{(k)}_{\alpha,\varepsilon}(t)t^{\frac{d}{2}}J_{\frac{d}{2}+k}\big(t|x|\big)\to 0, \quad k=0,1,\ldots,m-1,$$

as $t\to 0$ and as $t\to\infty$, which proves (25).

Let us show that $q_\alpha(\cdot,\varepsilon)\in L^\infty(B(0,1))$. Let $|x|<1$ (with $x\neq 0$). Having in mind the asymptotic behavior of $J_\nu(u)$, $\nu\geqslant 0$, at the origin we get

$$\begin{aligned}\int_0^1 \left|g^{(m)}_{\alpha,\varepsilon}(t)\right|t^{d/2}\left|J_{\frac{d}{2}+m-1}\big(t|x|\big)\right|dt &\leqslant C\int_0^1 t^{d/2}t^{\frac{d}{2}+m-1}|x|^{\frac{d}{2}+m-1}\,dt\\ &\leqslant C|x|^{\frac{d}{2}+m-1}\end{aligned} \tag{26}$$

assuming $m\geqslant 1$. As regards the integral at infinity, we split it into two integrals as follows:

$$\int_1^\infty \left|g^{(m)}_{\alpha,\varepsilon}(t)\right|t^{d/2}\left|J_{\frac{d}{2}+m-1}\big(t|x|\big)\right|dt = \int_1^{\frac{1}{|x|}}\cdots + \int_{\frac{1}{|x|}}^\infty\cdots.$$

Since $|J_\nu(u)|\leqslant c$, $\nu\geqslant 0$, then

$$\begin{aligned}\int_{\frac{1}{|x|}}^\infty \left|g^{(m)}_{\alpha,\varepsilon}(t)\right|t^{d/2}\left|J_{\frac{d}{2}+m-1}\big(t|x|\big)\right|dt &\leqslant C\int_{\frac{1}{|x|}}^\infty e^{-\varepsilon t}\big(1+t^2\big)^{\alpha/2}t^{d/2}\,dt\\ &\leqslant Ce^{-\frac{\varepsilon}{2|x|}}\int_{\frac{1}{|x|}}^\infty e^{-\frac{\varepsilon}{2}t}\big(1+t^2\big)^{\alpha/2}t^{d/2}\,dt\\ &\leqslant C(\alpha,\varepsilon,d)e^{-\frac{\varepsilon}{2|x|}}\\ &\leqslant C(\alpha,\varepsilon,d)|x|^{\frac{d}{2}+m-1}.\end{aligned} \tag{27}$$

On the other hand,

$$\int_1^{\frac{1}{|x|}} \left|g_{\alpha,\varepsilon}^{(m)}(t)\right| t^{d/2} \left|J_{\frac{d}{2}+m-1}\big(t|x|\big)\right| dt \leqslant C \int_1^{\frac{1}{|x|}} e^{-\varepsilon t}\big(1+t^2\big)^{\alpha/2} t^{d+m-1} |x|^{\frac{d}{2}+m-1}\, dt$$
$$\leqslant C(\alpha,\varepsilon,d)|x|^{\frac{d}{2}+m-1} \tag{28}$$

because

$$\int_1^{\infty} e^{-\varepsilon t}\big(1+t^2\big)^{\alpha/2+d+m-1}\, dt < \infty.$$

Gathering the estimates (26)–(28) in formula (24), we show that $|q_\alpha(x,\varepsilon)| \leqslant C(\alpha,\varepsilon,d)$, when $0 < |x| < 1$ (requiring $m \geqslant 1$).

Suppose now $|x| \geqslant 1$. Since $J_{\frac{d}{2}+m-1}$ is bounded (when $m \geqslant 1$), and since $|g_{\alpha,\varepsilon}^{(m)}(t)| \leqslant C$ if $t \leqslant 1$ and $|g_{\alpha,\varepsilon}^{(m)}(t)| \leqslant Ce^{-\varepsilon t}(1+t^2)^{\alpha/2}$ if $t > 1$, we get

$$\big|q_\alpha(x,\varepsilon)\big| \leqslant \frac{C}{|x|^{\frac{d}{2}+m-1}}\left(1+\int_1^{\infty} e^{-\varepsilon t}\big(1+t^2\big)^{\alpha/2+d/2}\, dt\right)$$
$$\leqslant \frac{C(\alpha,\varepsilon,d)}{|x|^{\frac{d}{2}+m-1}}$$

which allows to arrive at (23) if $m \geqslant \frac{d}{2}+1+\alpha$. Hence we may take

$$m_0 = \left[\frac{d}{2}+1+\alpha\right]+1 \tag{29}$$

in (24). □

Let $\mathcal{B}_\varepsilon^{-\alpha}$ be the convolution operators

$$\mathcal{B}_\varepsilon^{-\alpha} f(x) := \int_{\mathbb{R}^d} q_\alpha(y,\varepsilon) f(x-y)\, dy, \quad \varepsilon > 0\ (\alpha > 0). \tag{30}$$

**Proposition 9.** *The equality*

$$\mathcal{B}_\varepsilon^{-\alpha}\mathcal{B}^\alpha \varphi(x) = \int_{\mathbb{R}^d} P(y,\varepsilon)\varphi(x-y)\, dy, \tag{31}$$

*where*

$$P(y,\varepsilon) := \frac{c(d)\varepsilon}{(\varepsilon^2+|y|^2)^{\frac{d+1}{2}}}$$

*is the Poisson kernel, holds if one of the following conditions is satisfied*:

(i) $p \in \mathbb{P}_\varrho(\mathbb{R}^d)$ *and* $\varphi \in L^{p(\cdot)}(\mathbb{R}^d,\varrho)$;
(ii) $p \in \mathcal{P}_1(\mathbb{R}^d) \cap w_\infty$-$Lip(\mathbb{R}^d)$ *and* $\varphi \in L^{p(\cdot)}(\mathbb{R}^d)$.

**Proof.** We will only prove the first case, being the other proven in a similar way by using Proposition 4 below instead of Proposition 5.

Following [28], we first observe that identity (31) is valid for functions $\varphi \in \mathcal{S}(\mathbb{R}^d)$ (recall that $\widehat{P}(\xi,\varepsilon) = e^{-\varepsilon|\xi|}$, see [30]). Since $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^{p(\cdot)}(\mathbb{R}^d,\varrho)$ (see Theorem 1), we only need to prove the boundedness of all operators involved in (31).

The boundedness of $\mathcal{B}_\varepsilon^{-\alpha}$ in $L^{p(\cdot)}(\mathbb{R}^d, \varrho)$ follows from Proposition 5, since its kernel admits a radially decreasing and integrable dominant, which is clear from Lemma 8 (note that $\mathcal{B}^\alpha\varphi \in L^{p(\cdot)}(\mathbb{R}^d, \varrho)$ by Theorem 7). Similarly, the convolution operator of the right-hand side in (30) is also bounded in $L^{p(\cdot)}(\mathbb{R}^d, \varrho)$, since the Poisson kernel has the properties required in Proposition 5. In fact,

$$P(y, \varepsilon) = \varepsilon^{-d} P(y/\varepsilon, 1) \tag{32}$$

with

$$P(y, 1) \leqslant \frac{C}{(1 + |y|)^{d+1}}, \quad y \in \mathbb{R}^d. \qquad \square \tag{33}$$

**Remark 10.** Recently, it was proved in [10,11] that every convolution operator $Af = K * f$ is bounded in $L^{p(\cdot)}(\mathbb{R}^d)$ under the only assumptions $p \in \mathcal{P}_1(\mathbb{R}^d)$ and (4), if the kernel $K$ satisfies the estimate

$$\big|K(x)\big| \leqslant \frac{C}{(1 + |x|)^N}, \quad x \in \mathbb{R}^d,$$

with $N > d$. This is a case, for instance, of the Poisson kernel and the kernel $q_\alpha(\cdot, \varepsilon)$ above (see estimate (23)).

The following result provides the left-hand side inverse to the Bessel potential operator by using the approximative inverse operators (30).

**Theorem 11.** *Let $\alpha > 0$. The relation*

$$\lim_{\varepsilon \to 0} \mathcal{B}_\varepsilon^{-\alpha} \mathcal{B}^\alpha \varphi = \varphi$$

*holds, where the limit can be taken in the sense of norm or almost everywhere, if one of the conditions below is satisfied*:

(i) $p \in \mathbb{P}_\varrho(\mathbb{R}^d)$ *and* $\varphi \in L^{p(\cdot)}(\mathbb{R}^d, \varrho)$;
(ii) $p \in \mathcal{P}_1(\mathbb{R}^d) \cap w_\infty$-$Lip(\mathbb{R}^d)$ *and* $\varphi \in L^{p(\cdot)}(\mathbb{R}^d)$.

**Proof.** For the first case, the proof follows immediately from Proposition 5, taking into account relation (31) and the approximation identity properties of the Poisson kernel (see (32), (33)). The proof of the case (ii) is similar. $\square$

**Remark 12.** Theorem 11 allow us to recover the classical statement on the inversion of the Bessel potential operator in $L^p$ spaces with constant exponent (see, for example, [28, Theorem 11.25]), including the case $p = 1$, which follows from (ii).

*4.2. Approximative inverse under the choice $\mathcal{K}_\varepsilon(\xi) = (1 + \varepsilon^2|\xi|^2)^{-\alpha/2}$*

With this choice, our kernel $q_\alpha(\cdot, \varepsilon)$ is given by

$$\widehat{q_\alpha}(\xi, \varepsilon) = \left(\frac{1 + |\xi|^2}{1 + \varepsilon^2|\xi|^2}\right)^{\frac{\alpha}{2}}.$$

Since $\widehat{q_\alpha}(\xi, \varepsilon) \to \varepsilon^{-\alpha}$ as $|\xi| \to \infty$, the approximative inverse operators should include the identity operator. Hence, instead of (30), we have

$$\mathcal{B}_{\varepsilon,0}^{-\alpha} f(x) := \frac{1}{\varepsilon^\alpha} f(x) + \int_{\mathbb{R}^d} q_\alpha^0(y, \varepsilon) f(x - y)\, dy, \tag{34}$$

where

$$\widehat{q_\alpha^0}(\xi, \varepsilon) = \left(\frac{1 + |\xi|^2}{1 + \varepsilon^2|\xi|^2}\right)^{\frac{\alpha}{2}} - \frac{1}{\varepsilon^\alpha}.$$

By Lemma 11.26 in [28], the kernel $q_\alpha^0(\cdot,\varepsilon)$ is integrable and it can be written in terms of Bessel kernels as follows:

$$q_\alpha^0(x,\varepsilon) = \frac{1}{\varepsilon^{d+\alpha}} Q_{\alpha,\varepsilon}(x/\varepsilon), \tag{35}$$

where

$$Q_{\alpha,\varepsilon}(y) = \sum_{k=1}^{\infty} \binom{\alpha/2}{k} \left(\varepsilon^2 - 1\right)^k G_{2k}(y), \quad y \neq 0.$$

**Lemma 13.** *The kernel $q_\alpha^0(\cdot,\varepsilon)$ defined in* (35) *has an integrable and radially decreasing dominant, for every sufficiently small fixed value of $\varepsilon$.*

**Proof.** Indeed, such a dominant may have the form

$$M_{\alpha,\varepsilon}(x) := \frac{1}{\varepsilon^{d+\alpha}} \sum_{k=1}^{\infty} \left| \binom{\alpha/2}{k} \right| G_{2k}(x/\varepsilon)$$

for small values of $\varepsilon$, say $0 < \varepsilon < \sqrt{2}$, since $G_{2k}(x)$ is radial and decreasing by formula (13). Because $\int_{\mathbb{R}^d} G_{2k}(x)\,dx = 1$ for every $k$, then

$$\|M_{\alpha,\varepsilon}\|_1 = \frac{1}{\varepsilon^{\alpha}} \sum_{k=1}^{\infty} \left| \binom{\alpha/2}{k} \right| < \infty,$$

in view of the estimate $|\binom{\alpha/2}{k}| \leqslant \frac{C}{k^{1+\alpha/2}}$ as $k \to \infty$ (see [25, p. 14]). □

**Proposition 14.** *The equality*

$$\mathcal{B}_{\varepsilon,0}^{-\alpha} \mathcal{B}^{\alpha} \varphi(x) = \varepsilon^{-d} \int_{\mathbb{R}^d} G_\alpha(y/\varepsilon)\varphi(x-y)\,dy \tag{36}$$

*holds, if one of the following conditions is satisfied*:

(i) $p \in \mathbb{P}_\varrho(\mathbb{R}^d)$ *and* $\varphi \in L^{p(\cdot)}(\mathbb{R}^d, \varrho)$;
(ii) $p \in \mathcal{P}_1(\mathbb{R}^d) \cap w_\infty$-$Lip(\mathbb{R}^d)$ *and* $\varphi \in L^{p(\cdot)}(\mathbb{R}^d)$.

**Proof.** The proof follows from similar arguments used in Proposition 9. We only stress that the operator $\mathcal{B}_{\varepsilon,0}^{-\alpha}$ defined in (34) is bounded in $L^{p(\cdot)}(\mathbb{R}^d, \varrho)$, since the convolution kernel $q_\alpha^0(\cdot,\varepsilon)$ has a radially non-increasing and integrable dominant (see Lemma 13). □

From representation (36) and the well-known approximation identity property $\int_{\mathbb{R}^d} G_\alpha(x)\,dx = 1$ of the Bessel kernel, we obtain the following inversion statement.

**Theorem 15.** *Under the same assumptions of Proposition* 14*, we have*

$$\lim_{\varepsilon \to 0} \mathcal{B}_{\varepsilon,0}^{-\alpha} \mathcal{B}^{\alpha} \varphi = \varphi,$$

*where the limit can be taken in the sense of norm or almost everywhere.*

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## References


[1] E. Acerbi, G. Mingione, Regularity results for a class of functionals with non-standard growth, Arch. Ration. Mech. Anal. 156 (2001) 121–140.
[2] A. Almeida, Inversion of the Riesz potential operator on Lebesgue spaces with variable exponent, Fract. Calc. Appl. Anal. 6 (2003) 311–327.
[3] A. Almeida, S. Samko, Characterization of Riesz and Bessel potentials on variable Lebesgue spaces, J. Funct. Spaces Appl. 4 (2006) 113–144.
[4] D. Cruz-Uribe, A. Fiorenza, C.J. Neugebauer, The maximal function on variable $L^p$ spaces, Ann. Acad. Sci. Fenn. Math. 28 (2003) 223–238.
[5] D. Cruz-Uribe, A. Fiorenza, J.M. Martell, C. Pérez, The boundedness of classical operators on variable $L^p$ spaces, Ann. Acad. Sci. Fenn. Math. 31 (2006) 239–264.
[6] D. Cruz-Uribe, A. Fiorenza, Approximate identities in variable $L^p$ spaces, Math. Nachr. 280 (2007) 256–270.
[7] L. Diening, Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$, Math. Inequal. Appl. 7 (2004) 245–253.
[8] L. Diening, Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$, Math. Nachr. 268 (2004) 31–43.
[9] L. Diening, P. Hästö, A. Nekvinda, Open problems in variable exponent Lebesgue and Sobolev spaces, in: Function Spaces, Differential Operators and Nonlinear Analysis, Proc. Int. Conference, Milovy, Czech Rep., May 28–June 4, 2004, Math. Inst. Acad. Sci. Czech Rep., Prague, 2005, pp. 38–58.
[10] L. Diening, S. Samko, Hardy inequality in variable exponent Lebesgue spaces, preprint 01/2006, Math. Fak. Albert-Ludwigs-Universität, Freiburg, 2006.
[11] L. Diening, S. Samko, Hardy inequality in variable exponent Lebesgue spaces, Fract. Calc. Appl. Anal. 10 (2007) 1–18.
[12] L. Diening, P. Harjulehto, P. Hästö, Y. Mizuta, T. Shimomura, Maximal functions in variable exponent spaces: Limiting cases of the exponent, preprint, available at http://www.math.helsinki.fi/analysis/varsobgroup/, 2007.
[13] P. Gurka, P. Harjulehto, A. Nekvinda, Bessel potential spaces with variable exponent, Math. Inequal. Appl. 10 (2007) 661–676.
[14] M. Khabazi, Maximal operators in weighted $L^{p(x)}$ spaces, Proc. A. Razmadze Math. Inst. 135 (2004) 143–144.
[15] V. Kokilashvili, On a progress in the theory of integral operators in weighted Banach function spaces, in: Function Spaces, Differential Operators and Nonlinear Analysis, Proc. Int. Conference, Milovy, Czech Rep., May 28–June 4, 2004, Math. Inst. Acad. Sci. Czech Rep., Prague, 2005, pp. 152–175.
[16] V. Kokilashvili, S. Samko, Singular integrals in weighted Lebesgue spaces with variable exponent, Georgian Math. J. 10 (2003) 145–156.
[17] V. Kokilashvili, S. Samko, Maximal and fractional operators in weighted $L^{p(x)}$ spaces, Rev. Mat. Iberoamericana 20 (2004) 492–515.
[18] O. Kovăčík, J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czech. Math. J. 41 (116) (1991) 592–618.
[19] A. Nekvinda, Hardy–Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$, Math. Inequal. Appl. 7 (2004) 255–265.
[20] V.A. Nogin, S.G. Samko, Method of approximating inverse operators and its applications to inversion of potential type integral operators, Integral Transforms Spec. Funct. 8 (1999) 205–228.
[21] V.A. Nogin, S.G. Samko, Inversion and characterization of Riesz potentials in weighted spaces via approximative inverse operators, Proc. A. Razmadze Math. Inst. 129 (2002) 99–106.
[22] H. Rafeiro, S. Samko, Characterization of the range of one-dimensional fractional integration in the space with variable exponent, in: Proc. International Conference, WOAT 2006, Lisboa.
[23] H. Rafeiro, S. Samko, Approximative method for the inversion of the Riesz potential operator in $L^{p(\cdot)}$ spaces, manuscript, 2007.
[24] M. Růžička, Electrorheological Fluids: Modelling and Mathematical Theory, Lecture Notes in Math., vol. 1748, Springer-Verlag, Berlin, 2000.
[25] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon & Breach, New York, 1993; Russian edition: Fractional Integrals and Derivatives and Some of Their Applications, Nauka i Tekhnika, Minsk, 1987.
[26] S. Samko, Inversion theorems for potential-type integral transforms in $\mathbb{R}^n$ and on $S^{n-1}$, Integral Transforms Spec. Funct. 1 (1993) 145–163.
[27] S. Samko, Differentiation and integration of variable order and the spaces $L^{p(x)}$, in: Operator Theory and Complex and Hypercomplex Analysis, Proc. Int. Conference, Mexico City, December 12–17, 1994, in: Contemp. Math., vol. 212, 1998, pp. 203–219.
[28] S.G. Samko, Hypersingular Integrals and Their Applications, Taylor & Francis, London, 2002.
[29] S. Samko, On a progress in the theory of Lebesgue spaces with variable exponent: Maximal and singular operators, Integral Transforms Spec. Funct. 16 (2005) 461–482.
[30] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton, 1970.
[31] M.M. Zavolzhenskii, V.A. Nogin, Approximating approach to inversion of the generalized Riesz potentials, Dokl. Acad. Nauk 324 (1992) 738–741.