J. Math. Anal. Appl. 345 (2008) 964–974

Contents lists available at ScienceDirect

J. Math. Anal. Appl.

www.elsevier.com/locate/jmaa

On the functions with pseudoconvex sublevel sets and optimality conditions

Vsevolod I. Ivanov

Technical University of Varna, Department of Mathematics, 9010 Varna, Bulgaria

ARTICLE INFO

Article history:
Received 21 January 2008
Available online 10 May 2008
Submitted by A. Dontchev

Keywords:
Nonsmooth analysis
Nonsmooth optimization
Generalized convex functions
Optimality conditions
Lower Hadamard directional derivative

ABSTRACT

A new class of generalized convex functions, called the functions with pseudoconvex sublevel sets, is defined. They include quasiconvex ones. A complete characterization of these functions is derived. Further, it is shown that a continuous function admits pseudoconvex sublevel sets if and only if it is quasiconvex. Optimality conditions for a minimum of the nonsmooth nonlinear programming problem with inequality, equality and a set constraints are obtained in terms of the lower Hadamard directional derivative. In particular sufficient conditions for a strict global minimum are given where the functions have pseudoconvex sublevel sets.

1. Introduction

In this paper we define a new class of generalized convex functions called the functions with pseudoconvex sublevel set. This class includes quasiconvex functions and each continuous function, which admits pseudoconvex sublevel sets with respect to the pseudotangent cone, is quasiconvex. We obtain a complete characterization of the functions with pseudoconvex sublevel sets. Further, we derive necessary and sufficient optimality conditions of Fritz John type for isolated and local minimums of the nonsmooth problem with a set constraint, inequality constraints, and equality constraints in terms of the lower Hadamard directional derivative and the Bouligand tangent cone. In particular, we obtain sufficient conditions for a strict global minimum of a problem whose objective function and constraints admit pseudoconvex sublevel sets. It is a generalization of Theorem 3.1 in Bector, Chandra and Bector [3].

Optimality conditions play important role in optimization. Various sufficient conditions of Karush–Kuhn–Tucker type and Fritz John one were obtained for generalized convex functions which ensure that a given point is a global minimizer (see Arrow and Enthoven [2], Mangasarian [15, Theorem 10.1.2], Hanson [10, Theorem 2.1], Skarpness and Sposito [18], Bector, Chandra and Bector [3, Theorem 3.1], Giorgi [8, Theorem 2.2] and so on for the first-order case, Ginchev and Ivanov [7, Theorems 1, 3, 4] for the second-order one).

We cite the following papers where optimality conditions are given in terms of the lower Hadamard derivative or the respective subdifferential. Primal necessary conditions for efficiency of a multiobjective problem with inequality constraints, equality constraints and a set constraint were derived in Luu and Nguyen [14, Theorem 4.1]. Their dual conditions need Gâteaux differentiability. Necessary conditions for weak Pareto minimum of a vector problem with inequality constraints, equality constraints and a set constraint where the objective function is Hadamard differentiable with convex derivative, the inequality constraints are Dini differentiable with convex derivative, and the equality constraints are Dini differentiable with linear derivative were obtained in Jimenez and Novo [13, Theorems 4.1, 4.8]. Sufficient conditions for the problem with a set constraint can be found in Penot [17, Proposition 5.7(a)] and Ivanov [12, Theorem 7(a)]. We refer to Ioffe [11, Proposition 6]

E-mail address: vsevolodivanov@yahoo.com.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.05.010

where "fuzzy" Lagrange multiplier rule are given in terms of the lower Hadamard subdifferential under the name Dini subdifferential. Necessary optimality conditions were derived in Glover and Craven [9, Theorem 2.1] using the approximate subdifferential which is constructed with the help of the lower Hadamard derivative. Necessary and sufficient conditions for a local isolated minimum of the problem with a set constraint were obtained in Studniarski [19, Theorems 2.1, 2.2]. The reader can find sufficient conditions for a local isolated minimum of the problem with inequality constraints and a set constraint in Ward [20, Theorem 4.1]. Another Hadamard directional derivative is used in optimality conditions for set-valued optimization and vector one, which is constructed with the help of the limits of sets in the sense of Painlevé–Kuratowski. This derivative differs from the usual lower Hadamard derivative in the single-valued scalar case.

We introduce the basic definitions and notations.

Let S be a set in the Euclidean space $\mathbf{R}^n$. In the sequel we use the following notations: $:=$ for "for equal by definition," $\operatorname{cl} S$ for the closed hull of the set S, $\operatorname{conv} S$ for the convex hull of S, $\langle \mu, b\rangle := \sum_{i=1}^{p} \mu_i b_i$ for the scalar product in $\mathbf{R}^p$ between the vectors $\mu = (\mu_1, \ldots, \mu_p)$ and $b = (b_1, \ldots, b_p)$, B for the closed unit ball $\{x \in \mathbf{R}^n \mid \|x\| = 1\}$ in $\mathbf{R}^n$, $t \to +0$ for t approaches 0 with positive real values, K^* for the negative polar cone of the cone K defined by

$$K^* := \{v \in \mathbf{R}^n \mid \langle v, u\rangle \leqslant 0 \text{ for all } u \in K\}.$$

Denote by K^{**} the negative polar cone of the polar cone K^*. It is well known that the following relation holds for every nonempty cone K with a vertex at the origin 0: $K^{**} = \operatorname{cl}(\operatorname{co} K)$.

Besides the usual algebraic operations with infinities, we accept that

$$(\pm\infty) \cdot 0 = 0 \cdot (\pm\infty) = 0.$$

Let S be a convex set. A function $f : S \to \mathbf{R}$ is called quasiconvex on S if

$$x, \bar{x} \in S, \quad f(x) \leqslant f(\bar{x}) \quad \text{imply} \quad f\big(\bar{x} + t(x - \bar{x})\big) \leqslant f(\bar{x}), \quad \forall t \in [0, 1].$$

The following definition about quasiconvexity for differentiable functions is used in the sufficient conditions for a global minimum: Let S be open and f differentiable at $\bar{x} \in S$. Then f is said to be quasiconvex at $\bar{x}$ with respect to S if

$$x \in S, \quad f(x) \leqslant f(\bar{x}) \quad \text{imply} \quad \nabla f(\bar{x})(x - \bar{x}) \leqslant 0. \tag{1}$$

A differentiable function f is called quasiconvex on S if implication (1) holds for all $\bar{x} \in S$. If S is open and convex, and f differentiable on S, then both definitions are equivalent [15, Theorem 9.1.4].

A differentiable function f is called strictly pseudoconvex at $\bar{x} \in S$ on S if

$$x \in S, \quad f(x) \leqslant f(\bar{x}), \quad x \neq \bar{x} \quad \text{imply} \quad \nabla f(\bar{x})(x - \bar{x}) < 0.$$

Let $K(S, x)$ be a tangent cone to S at the point $x \in \operatorname{cl} S$. Recall that S is said to be pseudoconvex with respect to the cone $K(S, x)$ at the point x if $S \subset x + K(S, x)$.

We consider the following cones. The Bouligand tangent cone (or the contingent cone) of the set S at the point $x \in \operatorname{cl} S$ is defined as follows:

$$T(S, x) := \{u \in \mathbf{R}^n \mid \exists \{t_k\},\ t_k \to +0,\ \exists \{u_k\} \subset \mathbf{R}^n,\ u_k \to u \text{ such that } x + t_k u_k \in S \text{ for all positive integers } k\}.$$

The pseudotangent cone is the closed convex hull of the Bouligand tangent cone:

$$PT(S, x) := \operatorname{cl}\big(\operatorname{conv} T(S, x)\big).$$

It is obvious that each convex set S is pseudoconvex with respect to the Bouligand tangent cone and the pseudotangent one at each point $x \in S$.

Denote the sublevel set of the function f, defined on the set S, at $x \in S$ by

$$L(f; x; S) := \{y \in S \mid f(y) \leqslant f(x)\}.$$

Consider the normal cone of the sublevel set $L(f; x; S)$ of the function f at $x \in \operatorname{cl} S$:

$$N(x) := \{\xi \in \mathbf{R}^n \mid \langle \xi, y - x\rangle \leqslant 0 \text{ for all } y \in L(f; x; S)\}.$$

The lower Hadamard directional derivative of the function $f : \mathbf{R}^n \to \mathbf{R}$ at the point $x \in \mathbf{R}^n$ in direction $u \in \mathbf{R}^n$ is defined as follows:

$$f_H^{\downarrow}(x; u) := \liminf_{(t, u') \to (+0, u)} t^{-1}\big(f(x + tu') - f(x)\big).$$

Somewhere it is called the contingent derivative, or the lower Dini–Hadamard derivative, or the lower epiderivative. If there exists the limit

$$f'_H(x;u) := \lim_{(t,u')\to(+0,u)} t^{-1}\bigl(f(x+tu') - f(x)\bigr),$$

then the function is called Hadamard directionally differentiable and $f'_H(x;u)$ its Hadamard derivative at $x \in \mathbf{R}^n$ in direction $u \in \mathbf{R}^n$ (see, for instance, [5]).

The lower Hadamard conditional derivative (see Demyanov, Di Pillo and Facchinei [4]) of f with respect to the set S at the point $x \in S$ in direction $u \in \mathbf{R}^n$ is defined as follows:

$$f^{\downarrow}_H(x;u;S) := \liminf_{(t,u')\to(+0,u),\, x+tu'\in S} t^{-1}\bigl(f(x+tu') - f(x)\bigr).$$

By definition $f^{\downarrow}_H(x;u;S) = +\infty$ if $u \notin T(S,x)$.

Let $X \subset \mathbf{R}^n$ be a set and

$$f:\mathbf{R}^n \to \mathbf{R}, \qquad g_i:\mathbf{R}^n \to \mathbf{R}, \quad i = 1,2,\dots,m, \qquad h_j:\mathbf{R}^n \to \mathbf{R}, \quad j = 1,2,\dots,l,$$

given functions, defined on $\mathbf{R}^n$. Consider the nonlinear programming problem:

$$\text{Minimize } f(x) \text{ subject to } x \in S, \tag{P}$$

where $S = \{x \in X \mid g_i(x) \leqslant 0,\ i = 1,2,\dots,m,\ h_j(x) = 0,\ j = 1,2,\dots,l\}$ is the set of feasible points. If $\bar{x}$ is a feasible point of (P), we denote the set of active constraints by

$$I(\bar{x}) := \bigl\{i \in \{1,2,\dots,m\} \bigm| g_i(\bar{x}) = 0\bigr\}.$$

Suppose that $I(\bar{x}) = \{1,2,\dots,p\}$ where $p \leqslant m$.

Consider the vector functions $g = (g_1,\dots,g_m)$, $h = (h_1,\dots,h_l)$. We can rewrite the problem (P) in the following form:

$$\text{Minimize } f(x) \text{ s.t. } x \in X,\ g(x) \leqslant 0,\ h(x) = 0,$$

accepting that $g(x) \leqslant 0$ and $h(x) = 0$ means that the coordinates satisfy this inequality. If $\bar{x}$ is a point and u a direction we put

$$(g_{I(\bar{x})})^{\downarrow}_H(\bar{x};u) := \bigl((g_1)^{\downarrow}_H(\bar{x};u),\dots,(g_p)^{\downarrow}_H(\bar{x};u)\bigr),$$
$$h'_H(\bar{x};u) := \bigl((h_1)'_H(\bar{x};u),\dots,(h_l)'_H(\bar{x};u)\bigr),$$
$$S_{I(\bar{x})} := \bigl\{x \in X \bigm| g_i(x) \leqslant 0,\ i \in I(\bar{x}),\ h(x) = 0\bigr\},$$
$$C(\bar{x}) := \bigl\{u \in T(X,\bar{x}) \bigm| (g_{I(\bar{x})})^{\downarrow}_H(\bar{x};u) \leqslant 0,\ h'_H(\bar{x};u) = 0\bigr\}.$$

A feasible point $\bar{x} \in S$ is called an isolated local minimizer of (P) if there exist a positive real A and a neighborhood $N \ni \bar{x}$ such that $f(x) \geqslant f(\bar{x}) + A\|x - \bar{x}\|$ for all feasible points x with $x \in N$. If the inequality $f(x) \geqslant f(\bar{x})$ $(f(x) > f(\bar{x}))$ holds for all feasible $x \in N$, then $\bar{x}$ is called a (strict) local minimizer. In the case when the same inequality is fulfilled for all feasible points $x \in S$, then $\bar{x}$ is said to be a strict global minimizer.

The paper is organized as follows. In Section 2 we define the functions with pseudoconvex sublevel sets and some of their properties are derived. In Section 3 we obtain the optimality conditions with strict inequalities. In Section 4 we consider the optimality conditions with nonstrict inequalities.

2. Characterizations of the functions with pseudoconvex sublevel sets

For short, in this section we denote the sublevel set $L(f;x;S)$ by $L(x)$. We introduce the following notion.

Definition 1. We call the finite-valued real function f, defined on the set S, a function with pseudoconvex sublevel sets with respect to the tangent cone K if its sublevel sets are pseudoconvex with respect to K at every point $x \in S$, that is

$$L(x) \subset x + K\bigl(L(x),x\bigr) \quad \text{for all } x \in S. \tag{2}$$

Denote the set of functions, defined on S, which admit pseudoconvex sublevel sets with respect to the Bouligand tangent (pseudotangent) cone by $PCLS(T)$ (or $PCLS(PT)$ respectively).

It is well known that the normal cone of a convex set is the negative polar cone of the tangent cone. Therefore if f is quasiconvex, then $(T(L(x),x))^* = N(x)$ for all $x \in S$, because the sublevel sets of a quasiconvex function are convex. We extend this claim to the functions with pseudoconvex sublevel sets, and we show that this equality provides us with a complete characterization of the functions with pseudoconvex sublevel sets.

Theorem 1. *Let the function f be defined on the set S. Then $f \in PCLS(PT)$ if and only if*

$$\bigl(T\bigl(L(x),x\bigr)\bigr)^* = N(x) \quad \textit{for all } x \in S. \tag{3}$$

Proof. *Necessity*. Suppose that $f \in PCLS(PT)$. We prove that Eq. (3) holds. $L(x) \neq \emptyset$ because at least x belongs to this set.

If $x \in S$ is a strict global minimizer of f on S, then the claim is obvious, since the sets from both sides of (3) coincide with the whole space.

Let x be not a strict global minimizer. We prove that $(T(L(x), x))^* \subset N(x)$. We have $0 \in K^*$ for every cone K. Hence $(T(L(x), x))^* \neq \emptyset$. Suppose that $\xi \in (T(L(x), x))^*$ and $y \in L(x)$. By $f \in PCLS(PT)$ we have $y - x \in \mathrm{cl}(\mathrm{co}\, T(L(x), x))$. Since for every cone K it holds $K^* = (\mathrm{cl}(\mathrm{co}\, K))^*$, then $\langle \xi, y - x\rangle \leqslant 0$, i.e. $\xi \in N(x)$.

Suppose that $x \in S$ is not a strict global minimizer. We prove the inclusion $N(x) \subset (T(L(x), x))^*$. Assume that there exists $\xi \in N(x) \setminus (T(L(x), x))^*$. Hence, there is $u \in T(L(x), x)$ such that $\xi^T u > 0$. Since $u \in T(L(x), x)$ there exist sequences $\{t_k\}$, $t_k \to +0$, and $\{u_k\} \subset \mathbf{R}^n$, $u_k \to u$ such that $y_k = x + t_k u_k \in L(x)$. Thus $\xi^T(y_k - x) > 0$ for all sufficiently large k. On the other hand, by $\xi \in N(x)$, we have $\langle \xi, y_k - x\rangle \leqslant 0$ which is a contradiction.

Sufficiency. For the converse claim assume that $x \in S$ is arbitrary fixed and Eq. (3) is satisfied. We have

$$N(x) = \big(T\big(L(x), x\big)\big)^* \neq \emptyset.$$

Let $y \in L(x)$. Therefore $\langle \xi, y - x\rangle \leqslant 0$ for all $\xi \in N(x)$. Since $K^{**} = \mathrm{cl}(\mathrm{co}\, K)$ for every nonempty cone K and $0 \in T(L(x), x)$ for all $x \in S$, then

$$y - x \in \big(N(x)\big)^* = \big(T\big(L(x), x\big)\big)^{**} = \mathrm{cl}\big(\mathrm{co}\, T\big(L(x), x\big)\big) = PT\big(L(x), x\big).$$

Thus we obtain that $f \in PCLS(PT)$. $\square$

Each quasiconvex function is a function with pseudoconvex sublevel sets with respect to the pseudotangent or contingent cone since the sublevel sets of a quasiconvex function are convex. The converse does not hold as the following example shows.

Example 1. Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ defined by

$$f(x_1, x_2) = \begin{cases} x_1^2 + x_2^2 & \text{if } (x_1, x_2) \notin \{(\frac{1}{n}, 0) \mid n \text{ is positive integer}\}; \\ 1 & \text{if } (x_1, x_2) \in \{(\frac{1}{n}, 0) \mid n \text{ is positive integer}\}. \end{cases}$$

Here $S = \mathbf{R}^2$, $(x_1, x_2) = (0, 0)$ is a global minimizer, $f \in PCLS(T)$, $f \in PCLS(PT)$, but f is not quasiconvex on every convex neighborhood of the origin.

Theorem 2. *Let the function f be defined and continuous on the convex set S. Then $f \in PCLS(PT)$ if and only if f is quasiconvex on S.*

Proof. 1^0) We prove the sufficiency. Suppose that f is quasiconvex. The sublevel sets of f are convex. Each convex set is pseudoconvex with respect to the pseudotangent cone. Then every quasiconvex function is a function with pseudoconvex sublevel sets.

2^0) We prove the necessity. Let f be a function with pseudoconvex sublevel sets. Assume the contrary that f is not quasiconvex. Since the quasiconvex functions are the functions with convex sublevel sets, then there exist $x, y \in S$ with $f(y) \leqslant f(x)$ such that $[x, y] \not\subset L(x)$. Therefore there exists $z_0 \in (x, y]$ such that $z_0 \notin L(x)$. Owing to the continuity of f the set $L(x)$ is closed, and there exists $\delta > 0$ with $L(x) \cap (z_0 + \delta B) = \emptyset$. Denote $z_t = z_0 + t(x - z_0)$, $t \in [0, 1]$, and

$$s = \sup\big\{t \in (0, 1] \;\big|\; L(x) \cap (z_t + \delta B) = \emptyset\big\}.$$

It follows from $x \in L(x)$ and $L(x) \cap (z_0 + \delta B) = \emptyset$ that $0 < s < 1$. Denote $z = z_0 + s(x - z_0)$ and let p be arbitrary point from the set $L(x) \cap (z + \delta B)$. According to the continuity of f we have $f(p) = f(x)$. We prove that

$$\langle y - z, p - z\rangle \leqslant 0. \tag{4}$$

Assume the contrary. Consider the quadratic function $\varphi : \mathbf{R} \to \mathbf{R}$ defined as follows

$$\varphi(t) = \big\|z + t(y - z) - p\big\|^2.$$

Since the global minimum of φ is attained at the point

$$t = t_{\min} = \frac{\langle y - z, p - z\rangle}{\|y - z\|^2} > 0,$$

then φ is monotone decreasing when $t \in [0, t_{\min}]$ which contradicts the choice of z.

3^0) Since $\langle y - z, p - z\rangle = \langle y - p, p - z\rangle + \|p - z\|^2 > \langle y - p, p - z\rangle$, then it follows from (4) that

$$\langle y - p, p - z\rangle < 0. \tag{5}$$

4^0) We prove that $y \notin (PT(L(x), p) + p)$. Let u be an arbitrary point from $T(L(x), p)$. Therefore there exist a sequence $\{t_k\}$, $t_k \to +0$, and a sequence $\{u_k\} \subset \mathbf{R}^n$, $u_k \to u$ such that $p + t_k u_k \in L(x)$ for all positive integer k. Hence $\|z - p\|^2 \leqslant \|z - p - t_k u_k\|^2$. We obtain from here that $2\langle u_k, z - p\rangle \leqslant t_k \|u_k\|^2$. Taking the limits as $k \to +\infty$ we obtain that $\langle u, z - p\rangle \leqslant 0$. Therefore $\langle u, p - z\rangle \geqslant 0$ for all $u \in T(L(x), p)$. It follows from here that

$$\langle u, p - z\rangle \geqslant 0 \quad \text{for all } u \in \operatorname{cl}\bigl(\operatorname{conv} T\bigl(L(x), p\bigr)\bigr).$$

Taking into account inequality (5) we obtain that $y - p \notin PT(L(x), p)$.

5^0) On the other hand it follows from the pseudoconvexity of $L(x)$ and $f(x) = f(p)$ that $L(x) \subset (PT(L(x), p) + p)$. Therefore $y \in (PT(L(x), p) + p)$ which is a contradiction. □

Corollary 1. *Let the function f be defined and continuous on the convex set S. Then $f \in PCLS(T)$ if and only if f is quasiconvex on S.*

Conditions for a lower semicontinuous function to be quasiconvex are obtained in Diewert [6, Theorem 4], Yang and Liu [21, Theorem 2.1]. See also Mukherjee and Reddy [16, Theorem 3.2].

In the case when K is the Bouligand cone the functions with pseudoconvex sublevel sets can be defined in terms of the lower Hadamard conditional derivative.

Theorem 3. *Let K be the Bouligand tangent cone. Then condition* (2) *is equivalent to the following one*:

$$x \in S, \quad y \in L(x) \quad \textit{imply} \quad f_H^{\downarrow}\bigl(x; y - x; L(x)\bigr) \leqslant 0.$$

Proof. We prove that for all $x \in S$ the inclusion $y \in x + T(L(x), x)$ is equivalent to the inequality $f_H^{\downarrow}(x; y - x; L(x)) \leqslant 0$.

Let $y \in x + T(L(x), x)$. By the definition of the contingent cone there exist sequences $\{t_k\}$, $t_k > 0$, $t_k \to 0$, and $\{u_k\}$, $u_k \in \mathbf{R}^n$, $u_k \to (y - x)$ such that $x + t_k u_k \in L(x)$. We have

$$\begin{aligned} f_H^{\downarrow}\bigl(x; y - x; L(x)\bigr) &= \liminf_{(t,u') \to (+0, y-x),\, x+tu' \in L(x)} t^{-1}\bigl(f(x + tu') - f(x)\bigr) \\ &\leqslant \liminf_{k\to\infty} t_k^{-1}\bigl(f(x + t_k u_k) - f(x)\bigr) \leqslant 0. \end{aligned}$$

Let $f_H^{\downarrow}(x; y - x; L(x)) \leqslant 0$. The claim follows directly from the definition of the Hadamard conditional derivative because $f_H^{\downarrow}(x; y - x; L(x)) < +\infty$ and therefore $y - x \in T(L(x), x)$. □

It follows from Theorem 3 that Corollary 1 is a generalization of the following result due to Arrow and Enthoven [2].

Theorem 4. *Let S be an open convex set in the finite-dimensional space $\mathbf{R}^n$. Suppose that the differentiable function f is defined on S. Then f is quasiconvex on S if and only if the following implication is satisfied for all $x, y \in S$,*

$$x, y \in S, \quad f(y) \leqslant f(x) \quad \textit{imply} \quad \nabla f(x)(y - x) \leqslant 0.$$

3. Optimality conditions with strict inequalities

In this section we derive necessary and sufficient conditions for an isolated minimum and global one of the problem (P).

Lemma 1. *The following two conditions are equivalent*:

$$(a, b, c) \notin [-\infty, 0] \times [-\infty, 0]^p \times \{0\}^l \tag{6}$$

and

$$\begin{gathered} \exists(\lambda, \mu, \nu) \in [0, +\infty) \times [0, +\infty)^p \times (-\infty, +\infty)^l \setminus \bigl\{(0,0,0)\bigr\}: \\ \lambda a + \langle \mu, b\rangle + \langle \nu, c\rangle > 0, \qquad \lambda = 0 \quad \textit{if } a = -\infty, \quad \mu_i = 0 \quad \textit{if } b_i = -\infty. \end{gathered} \tag{7}$$

Proof. Let (6) holds. We prove condition (7). Assume the contrary that

$$\lambda a + \langle \mu, b\rangle + \langle \nu, c\rangle \leqslant 0$$

for all $(\lambda, \mu, \nu) \in [0, +\infty) \times [0, +\infty)^p \times (-\infty, +\infty)^l \setminus \{(0,0,0)\}$. Take $\mu = 0$ and $\nu = 0$. Then we have $\lambda a \leqslant 0$ for all $\lambda > 0$. Therefore $a \leqslant 0$. Using similar arguments we obtain that $b_i \leqslant 0$ for all $i = 1, 2, \ldots, p$. At last, we take $\lambda = 0$, $\mu = 0$ and $\nu_j = 0$, $j \neq s$. Therefore $\nu_s c_s \leqslant 0$ for all $\nu_s \neq 0$ which implies that $c_s = 0$. Applying these reasonings for $s = 1, 2, \ldots, l$ we get a contradiction with condition (6).

The claim is trivially satisfied if among a, b_i $(i = 1, 2, \dots, p)$, c_j $(j = 1, 2, \dots, l)$ there are infinities. For all a, b_i equal to $-\infty$ we put the respective multipliers equal to 0. Then we remove all these ones equal to $-\infty$ and apply the reasonings given above.

Let (7) be satisfied. We prove condition (6). Assume the contrary that

$$(a, b, c) \in [-\infty, 0] \times [-\infty, 0]^p \times \{0\}^l.$$

Therefore

$$\lambda a + \langle \mu, b\rangle + \langle \nu, c\rangle \leqslant 0 \quad \forall(\lambda, \mu, \nu) \in [0, +\infty) \times [0, +\infty)^p \times (-\infty, +\infty)^l$$

which contradicts (7). □

Lemma 2. *Let $\bar{x}$ be a feasible point of* (P) *and h be Hadamard differentiable at $\bar{x}$ in every direction. Then $T(S_{I(\bar{x})}, \bar{x}) \subset C(\bar{x})$.*

Proof. Let $u \in T(S_{I(\bar{x})}, \bar{x})$. By the definition of the tangent cone there are sequences $t_k \to +0$, $u_k \to u$ such that $\bar{x} + t_k u_k \in S_{I(\bar{x})}$. Therefore $\bar{x} + t_k u_k \in X$, $u \in T(X, \bar{x})$, $g_i(\bar{x} + t_k u_k) \leqslant 0 = g_i(\bar{x})$, $i \in I(\bar{x})$, and $h_j(\bar{x} + t_k u_k) = 0 = h_j(\bar{x})$, $j = 1, 2, \dots, l$. By the definition of the Hadamard derivative $u \in C(\bar{x})$. □

Notice that the inverse inclusion does not hold in general. In order to derive necessary conditions for optimality it is reasonable to introduce the following generalized Abadie constraint qualification $C(\bar{x}) = T(S_{I(\bar{x})}, \bar{x})$ which coincides with the Abadie constraint qualification [1] in the case when the problem is differentiable.

Theorem 5 *(Necessary condition for an isolated local minimum). Let $\bar{x}$ be an isolated local minimizer of* (P)*, the functions g_i, $i \notin I(\bar{x})$ be continuous at $\bar{x}$. Suppose that there exist the Hadamard directional derivatives $f'_H(\bar{x}; u)$ and $h'_H(\bar{x}; u)$ for all directions $u \in T(X, \bar{x})$ and the constraint qualification $C(\bar{x}) = T(S_{I(\bar{x})}, \bar{x})$ holds. Then the following conditions are satisfied for every $u \in T(X, \bar{x}) \setminus \{0\}$:*

$$\bigl(f'_H(\bar{x}; u), (g_{I(\bar{x})})^{\downarrow}_H(\bar{x}; u), h'_H(\bar{x}; u)\bigr) \notin [-\infty, 0] \times [-\infty, 0]^p \times \{0\}^l \tag{8}$$

and there exists $(\lambda, \mu, \nu) \in [0, +\infty) \times [0, +\infty)^p \times (-\infty, +\infty)^l \setminus \{(0, 0, 0)\}$ such that

$$\lambda f'_H(\bar{x}; u) + \bigl\langle \mu, (g_{I(\bar{x})})^{\downarrow}_H(\bar{x}; u)\bigr\rangle + \bigl\langle \nu, h'_H(\bar{x}; u)\bigr\rangle > 0 \tag{9}$$

where $I(\bar{x}) = \{1, 2, \dots, p\}$. If $f'_H(\bar{x}; u) = -\infty$, then inequality (9) *holds with $\lambda = 0$. If $(g_i)^{\downarrow}_H(\bar{x}; u) = -\infty$, $i \in I(\bar{x})$, then $\mu_i = 0$.*

Proof. Conditions (8) and (9) are equivalent according to Lemma 1. We prove (8). Assume the contrary that there exists $u \in T(X, \bar{x})$, $u \neq 0$ with

$$f'_H(\bar{x}; u) \leqslant 0, \qquad (g_i)^{\downarrow}_H(\bar{x}; u) \leqslant 0, \quad i \in I(\bar{x}) \quad \text{and} \quad h'_H(\bar{x}; u) = 0.$$

By the generalized Abadie constraint qualification $u \in T(S_{I(\bar{x})}, \bar{x})$. Therefore there are sequences $t_k \to +0$, $u_k \to u$ such that $\bar{x} + t_k u_k \in S_{I(\bar{x})}$. We have that

$$g_i(\bar{x} + t_k u_k) \leqslant 0 \quad \text{for all } i \notin I(\bar{x}),$$

and all sufficiently large integers k due to the assumed continuity. We obtain that the points $\bar{x} + t_k u_k$ are feasible for all sufficiently large k. Using that $\bar{x}$ is an isolated minimizer, there exist $A > 0$ with

$$f(\bar{x} + t_k u_k) \geqslant f(\bar{x}) + A t_k \|u_k\|.$$

It follows from here that $f'_H(\bar{x}; u) \geqslant A\|u\| > 0$ which contradicts our assumption. □

Remark 1. It is obvious that condition (9) is equivalent to the following one: for every $u \in T(X, \bar{x})$ there exists

$$(\lambda, \mu, \nu) \in [0, +\infty) \times [0, \infty)^m \times (-\infty, +\infty)^l \setminus \bigl\{(0, 0, 0)\bigr\}$$

such that

$$\lambda f'_H(\bar{x}; u) + \sum_{i=1}^{m} \mu_i (g_i)^{\downarrow}_H(\bar{x}; u) + \sum_{j=1}^{l} \nu_j (h_j)'_H(\bar{x}; u) > 0,$$
$$\mu_i g_i(\bar{x}) = 0, \quad i = 1, 2, \dots, m.$$

Condition (8) is known as primal one and condition (9) as dual.

The following example shows that the constraint qualification is essential for Theorem 5.

Example 2. Consider the problem (P) with an inequality constraint and a set constraint such that $X = [0, +\infty)$, $f, g : \mathbf{R} \to \mathbf{R}$ are given by $f(x) = -x$, $g(x) = x^2$. Let $\bar{x} = 0$. The point $\bar{x}$ is the only feasible one and it is an isolated local minimizer. We have $T(X, \bar{x}) = [0, +\infty)$, $f'_H(\bar{x}; u) \leqslant 0$ and $g^{\downarrow}_H(\bar{x}; u) = 0$ for all $u \in T(X, \bar{x})$, whence obviously condition (8) does not hold. The Abadie constraint qualification is not satisfied.

Theorem 6 *(Sufficient conditions for an isolated local minimum). Let $\bar{x}$ be a feasible point of* (P). *Suppose that the functions f, h_j, $j = 1, 2, \ldots, l$, are Hadamard differentiable at $\bar{x}$ in every direction $u \in T(X, \bar{x})$. Assume that for every $u \in T(X, \bar{x}) \setminus \{0\}$ condition* (8) *or condition* (9) *is satisfied. Then $\bar{x}$ is an isolated local minimizer.*

Proof. Since conditions (8) and (9) are equivalent by Lemma 1, it is enough to prove that condition (8) implies $\bar{x}$ is an isolated local minimizer.

Suppose that $\bar{x}$ is not an isolated minimizer. Therefore for every sequence $\varepsilon_k > 0$, $\varepsilon_k \to 0$ there exists a sequence $x_k \in X$ such that

$$\begin{aligned} &\|x_k - \bar{x}\| \leqslant \varepsilon_k, \qquad g_i(x_k) \leqslant 0, \quad i = 1, 2, \ldots, m, \\ &h_j(x_k) = 0, \quad j = 1, 2, \ldots, l, \qquad f(x_k) < f(\bar{x}) + \varepsilon_k \|x_k - \bar{x}\|. \end{aligned} \tag{10}$$

Denote $t_k = \|x_k - \bar{x}\|$. According to $t_k \leqslant \varepsilon_k$, $\varepsilon_k \to 0$ we have $t_k \to 0$. Denote $d_k = (x_k - \bar{x})/t_k$. Since $\|d_k\| = 1$ we can ensure that the sequence $\{d_k\}$ is convergent, that is $d_k \to d$. Using that $d \in T(X, \bar{x})$ we obtain from (10) that

$$\bigl(f'_H(\bar{x}; d), (g_{I(\bar{x})})^{\downarrow}_H(\bar{x}; d), h'_H(\bar{x}; d)\bigr) \in [-\infty, 0] \times [-\infty, 0]^p \times \{0\}^l$$

which contradicts (8). □

Denote

$$U(\bar{x}) := T\bigl(L(f; \bar{x}; X), \bar{x}\bigr) \cap \bigcap_{i \in I(\bar{x})} T\bigl(L(g_i; \bar{x}; X), \bar{x}\bigr) \cap \bigcap_{j=1}^{l} T\bigl(L(h_j; \bar{x}; X), \bar{x}\bigr) \cap \bigcap_{j=1}^{l} T\bigl(L(-h_j; \bar{x}; X), \bar{x}\bigr).$$

Theorem 7 *(Sufficient conditions for a strict global minimum). Let $\bar{x}$ be a feasible point of* (P). *Suppose that*

$$L(f; \bar{x}; X), \qquad L(g_i; \bar{x}; X),\ i \in I(\bar{x}), \qquad L(h_j; \bar{x}; X),\ L(-h_j; \bar{x}; X),\ j = 1, 2, \ldots, l,$$

are pseudoconvex, the functions f, h_j, $j = 1, 2, \ldots, l$, are Hadamard directionally differentiable at $\bar{x}$ in every direction $u \in U(\bar{x})$. Assume that for all $u \in U(\bar{x})$, $u \neq 0$ there exists

$$(\lambda, \mu, \nu) \in [0, +\infty) \times [0, \infty)^p \times (-\infty, +\infty)^l \setminus \bigl\{(0, 0, 0)\bigr\}$$

such that condition (8) *or condition* (9) *is satisfied. Then $\bar{x}$ is a strict global minimizer.*

Proof. According to Lemma 1 it is enough to prove that the statement $\bar{x}$ is a strict global minimizer follows from condition (8). Let condition (8) hold, but $\bar{x}$ is not a strict global minimizer. Therefore there exists $x \in S$, $x \neq \bar{x}$ such that $f(x) \leqslant f(\bar{x})$. It follows from the pseudoconvexity of $L(f; \bar{x}; X)$ that $x - \bar{x} \in T(L(f; \bar{x}; X), \bar{x})$. We conclude from here that there exist sequences $\{t_k\}$, $t_k \to +0$, and $\{u_k\}$, $u_k \to (x - \bar{x})$ with $f(\bar{x} + t_k u_k) \leqslant f(\bar{x})$. Hence

$$f'_H(\bar{x}; x - \bar{x}) = \lim_{k \to +\infty} \frac{f(\bar{x} + t_k u_k) - f(\bar{x})}{t_k} \leqslant 0. \tag{11}$$

Let $i \in I(\bar{x})$ be arbitrary. Since $g_i(x) \leqslant 0 = g_i(\bar{x})$ using similar arguments we obtain that $(g_i)^{\downarrow}_H(\bar{x}; x - \bar{x}) \leqslant 0$. Suppose that j is any integer from the set $\{1, 2, \ldots, l\}$. Using that $h_j(x) = h_j(\bar{x})$ we obtain from the pseudoconvexity of $L(h_j; \bar{x}; X)$ that $x - \bar{x} \in T(L(h_j; \bar{x}; X), \bar{x})$. Therefore $(h_j)'_H(\bar{x}; x - \bar{x}) \leqslant 0$. By the pseudoconvexity of $L(-h_j; \bar{x}; X)$ we have that $(h_j)'_H(\bar{x}; x - \bar{x}) \geqslant 0$ which is a contrary to condition (8). □

In the next theorem the Lagrange multipliers do not depend on the direction.

Theorem 8 *(Sufficient conditions for a strict global minimum). Let $\bar{x}$ be a feasible point of* (P). *Suppose that $L(f; \bar{x}; X)$, $L(g_i; \bar{x}; X)$, $i \in I(\bar{x})$, are pseudoconvex, the functions f, h_j, $j = 1, 2, \ldots, l$, are Hadamard directionally differentiable at $\bar{x}$ in every direction $u \in T(X, \bar{x})$. Assume that there exists $(\lambda, \mu, \nu) \in [0, +\infty) \times [0, \infty)^p \times (-\infty, +\infty)^l \setminus \{(0, 0, 0)\}$ such that condition* (9) *is verified for every $u \in T(X, \bar{x}) \setminus \{0\}$. Let the following sets be pseudoconvex for every $j \in \{1, 2, \ldots, l\}$: $L(h_j; \bar{x}; X)$ if $\nu_j > 0$, $L(-h_j; \bar{x}; X)$ if $\nu_j < 0$. Then $\bar{x}$ is a strict global minimizer of* (P).

Proof. The proof uses the arguments of Theorem 7. □

4. Optimality conditions with nonstrict inequalities

In this section we derive necessary conditions for a local minimum of the problem (P) and sufficient conditions for a global one of the same problem.

Lemma 3. *Let $a > -\infty$. Then the following two conditions are equivalent*:

$$(a, b, c) \notin (-\infty, 0) \times [-\infty, 0]^p \times \{0\}^l \tag{12}$$

and

$$\exists(\mu, \nu) \in [0, +\infty)^p \times (-\infty, +\infty)^l\colon \quad a + \langle \mu, b\rangle + \langle \nu, c\rangle \geqslant 0, \quad \mu_i = 0 \quad \text{if } b_i = -\infty. \tag{13}$$

Proof. Let (12) hold. We prove condition (13). Assume the contrary that

$$a + \langle \mu, b\rangle + \langle \nu, c\rangle < 0$$

for all $(\mu, \nu) \in [0, +\infty)^p \times (-\infty, +\infty)^l$. Take $\mu = 0$ and $\nu = 0$. Therefore $a < 0$. Take $\nu = 0$ and $\mu_i = 0$, $i \neq s$. Hence $a + \mu_s b_s < 0$ for all $\mu_s \geqslant 0$. Suppose that $b_s > 0$. If we take μ_s sufficiently large, then we obtain that $a + \mu_s b_s > 0$ which is a contradiction. Therefore $b_s \leqslant 0$ for all $s = 1, 2, \ldots, p$. At last, we take $\mu = 0$ and $\nu_j = 0$, $j \neq s$. Therefore $a + \nu_s c_s < 0$ for all $\nu_s \in \mathbf{R}$. If $c_s < 0$, then $a + \nu_s c_s > 0$ for all negative numbers ν_s with sufficiently large absolute value which is a contradiction. If $c_s > 0$, then $a + \nu_s c_s > 0$ for all sufficiently large positive numbers ν_s which is a contradiction again. Consequently $c_s = 0$ for all $s = 1, 2, \ldots, l$. Thus $(a, b, c) \in (-\infty, 0) \times [-\infty, 0]^p \times \{0\}^l$ which is a contrary to condition (12).

The claim is trivially satisfied if among a, b_i $(i = 1, 2, \ldots, p)$, c_j $(j = 1, 2, \ldots, l)$ there are infinities. For all b_i equal to $-\infty$ we put the respective multipliers μ_i equal to 0. Then we remove all these ones equal to $-\infty$ and apply the reasonings given above.

Let (13) hold. We prove condition (12). Assume the contrary that

$$(a, b, c) \in (-\infty, 0) \times [-\infty, 0]^p \times \{0\}^l.$$

Therefore

$$a + \langle \mu, b\rangle + \langle \nu, c\rangle < 0 \quad \text{for all } (\mu, \nu) \in [0, +\infty)^p \times (-\infty, +\infty)^l,$$

which contradicts (13). □

Theorem 9 *(Necessary conditions for a local minimum). Let $\bar{x}$ be a local minimizer of* (P)*. Suppose that the functions g_i, $i \notin I(\bar{x})$, are continuous at $\bar{x}$, there exist the Hadamard directional derivatives $f'_H(\bar{x}; u)$ and $h'_H(\bar{x}; u)$ for all directions $u \in T(X, \bar{x})$. Assume that the generalized Abadie constraint qualification $C(\bar{x}) = T(S_{I(\bar{x})}, \bar{x})$ holds. Then the following conditions are satisfied for every $u \in T(X, \bar{x})$*:

$$\bigl(f'_H(\bar{x}; u), (g_{I(\bar{x})})^{\downarrow}_H(\bar{x}; u), h'_H(\bar{x}; u)\bigr) \notin [-\infty, 0) \times [-\infty, 0]^p \times \{0\}^l \tag{14}$$

and there exists $(\lambda, \mu, \nu) \in [0, +\infty) \times [0, +\infty)^p \times (-\infty, +\infty)^l \setminus \{(0, 0, 0)\}$ such that

$$\lambda f'_H(\bar{x}; u) + \bigl\langle \mu, (g_{I(\bar{x})})^{\downarrow}_H(\bar{x}; u)\bigr\rangle + \bigl\langle \nu, h'_H(\bar{x}; u)\bigr\rangle \geqslant 0 \tag{15}$$

where $I(\bar{x}) = \{1, 2, \ldots, p\}$. If $f'_H(\bar{x}; u) > -\infty$, then inequality (15) *holds with $\lambda > 0$, if $f'_H(\bar{x}; u) = -\infty$, then $\lambda = 0$. If $(g_i)^{\downarrow}_H(\bar{x}; u) = -\infty$, $i \in I(\bar{x})$, then $\mu_i = 0$.*

Proof. 1^0) We prove (14). Assume the contrary that there exists $u \in T(X, \bar{x})$ with

$$f'_H(\bar{x}; u) < 0, \qquad (g_i)^{\downarrow}_H(\bar{x}; u) \leqslant 0, \quad i \in I(\bar{x}), \quad \text{and} \quad h'_H(\bar{x}; u) = 0.$$

By the generalized Abadie constraint qualification $u \in T(S_{I(\bar{x})}, \bar{x})$. Therefore there are sequences $t_k \to +0$, $u_k \to u$ such that $\bar{x} + t_k u_k \in S_{I(\bar{x})}$. We have that

$$g_i(\bar{x} + t_k u_k) \leqslant 0 \quad \text{for all } i \notin I(\bar{x}),$$

and all sufficiently large integers k by the assumed continuity. Hence the points $\bar{x} + t_k u_k$ are feasible for all sufficiently large k and $f(\bar{x} + t_k u_k) \geqslant f(\bar{x})$ since $\bar{x}$ is a local minimizer. Therefore $f'_H(\bar{x}; u) \geqslant 0$ which contradicts our assumption.

2^0) We prove condition (15). Assume that u is arbitrary direction from $T(X, \bar{x})$ and $f'_H(\bar{x}; u) > -\infty$. Then, by (14),

$$\bigl(f'_H(\bar{x}; u), (g_{I(\bar{x})})^{\downarrow}_H(\bar{x}; u), h'_H(\bar{x}; u)\bigr) \notin (-\infty, 0) \times [-\infty, 0]^p \times \{0\}^l.$$

Condition (15) holds with $\lambda = 1$ thanks to Lemma 3.

Consider the case when $f'_H(\bar{x}; u) = -\infty$. According to (14) we have that

$$\big((g_{I(\bar{x})})^{\downarrow}_H(\bar{x}; u), h'_H(\bar{x}; u)\big) \notin [-\infty, 0]^p \times \{0\}^l.$$

By Lemma 1 there exists $(\mu, \nu) \in [0, \infty)^p \times (-\infty, +\infty)^l \setminus \{(0, 0)\}$ such that

$$\big\langle \mu, (g_{I(\bar{x})})^{\downarrow}_H(\bar{x}; u)\big\rangle + \big\langle \nu, h'_H(\bar{x}; u)\big\rangle > 0.$$

Therefore condition (15) is fulfilled with $\lambda = 0$ and with a strict inequality. □

Corollary 2 *(Kuhn–Tucker necessary conditions). Let $\bar{x}$ be a local minimizer of* (P)*. Suppose that the functions g_i, $i \notin I(\bar{x})$ are continuous at $\bar{x}$, f is Lipschitz in a neighborhood of $\bar{x}$, there exist the Hadamard directional derivatives $f'_H(\bar{x}; u)$ and $h'_H(\bar{x}; u)$ for all directions $u \in T(X, \bar{x})$. Assume that the generalized Abadie constraint qualification $C(\bar{x}) = T(S_{I(\bar{x})}, \bar{x})$ holds. Then for every $u \in T(X, \bar{x})$ there exists $(\mu, \nu) \in [0, \infty)^p \times (-\infty, +\infty)^l$ such that*

$$f'_H(\bar{x}; u) + \big\langle \mu, (g_{I(\bar{x})})^{\downarrow}_H(\bar{x}; u)\big\rangle + \big\langle \nu, h'_H(\bar{x}; u)\big\rangle \geqslant 0.$$

Proof. This corollary follows from the proof of Theorem 9 using that the Hadamard directional derivative is finite in our case. □

Example 2 shows that the constraint qualification cannot be removed from Theorem 9 and Corollary 2. The point $\bar{x} = 0$ is a local minimizer, but $f'_H(\bar{x}; u) < 0$, $g^{\downarrow}_H(\bar{x}; u) = 0$ for all $u > 0$ and condition (14) does not hold.

It is well known that the constraint qualifications are essential in Fritz John optimality condition for nonsmooth programs (see [13, Example 4.4]). The following example shows that the generalized Abadie constraint qualification cannot be eliminated from Theorem 9.

Example 3. Consider the problem (P) with equality constraint only. The functions f, $h : \mathbf{R}^2 \to \mathbf{R}$ are given as follows: $f(x_1, x_2) = x_1^2 + x_2$ and

$$h(x_1, x_2) = \begin{cases} \sin x_1 & \text{if } x_1 \geqslant 0,\ x_2 = 0; \\ x_1 & \text{otherwise.} \end{cases}$$

Here $X = \mathbf{R}^2$. The point $\bar{x} = (0, 0)$ is a local minimizer. The functions f and h are differentiable at $\bar{x}$. However, the Fritz John conditions

$$\lambda \nabla f(\bar{x}) + \mu \nabla h(\bar{x}) = (0, 0) \quad \text{with } (\lambda, \mu) \neq (0, 0),\ \lambda \geqslant 0,$$

are not satisfied. It is immediately seen that the generalized Abadie constraint qualification does not hold.

In the following example f is not Hadamard differentiable and Theorem 9 is not satisfied.

Example 4. Consider the problem (P) with inequality constraint and a set constraint where $X = \{(0, 0)\}$. The functions f, $g : \mathbf{R}^2 \to \mathbf{R}$ are given as follows:

$$f(x_1, x_2) = g(x_1, x_2) = \begin{cases} -1 & \text{if } (x_1, x_2) \neq (0, 0); \\ 0 & \text{if } (x_1, x_2) = (0, 0). \end{cases}$$

The point $\bar{x} = (0, 0)$ is a local minimizer, because it is the only feasible point. The function f is not Hadamard differentiable and $f^{\downarrow}_H(\bar{x}; u) = g^{\downarrow}_H(\bar{x}; u) = -\infty$ for all $u \in \mathbf{R}^2$. All other hypotheses of Theorem 9 hold, but condition (14) is not satisfied. The only pair of Lagrange multipliers which fulfills condition (15) is $(\lambda, \mu) = (0, 0)$.

Remark 2. We can compare Theorem 9 with [13, Theorems 4.1, 4.8] and [14, Theorems 4.1, 5.2]. Dual conditions for the vector problem are obtained in [13, Theorem 4.1]. The objective function is Hadamard differentiable with a convex derivative, the inequality constraints are Dini differentiable with a convex derivative, the equality constraints are Dini differentiable with a linear derivative and some cones are convex or closed. The result is given in term of subdifferentials. The generalized Abadie constraint qualification is assumed. In [13, Theorem 4.8] the inequality constraints are Hadamard differentiable with a convex derivative, but there are no equality constraints in the problem. Only primal optimality condition for the vector problem is derived in [14, Theorem 4.1]. The objective function and the equality constraints are Hadamard differentiable and the result is given in terms of the Hadamard derivative and the lower Hadamard derivative. A dual condition of Kuhn–Tucker type is obtained in [14, Theorem 5.2]. All functions in this result are Gâteaux differentiable and some cones are supposed to be closed.

Definition 2. Let the function f be defined on the set X and $\bar{x} \in X$. We call the sublevel set $L(f; \bar{x}; X)$ strongly pseudoconvex if for all $x \in L(f; \bar{x}; X)$, $x \neq \bar{x}$ there are a number $\varepsilon > 0$ and sequences $t_k \to +0$, $u_k \to (x - \bar{x})$ such that

$$f(\bar{x} + t_k u_k) \leqslant f(\bar{x}) - \varepsilon t_k.$$

Theorem 10 *(Sufficient conditions for a strict global minimum). Let $\bar{x}$ be a feasible point of* (P)*, there exist the Hadamard derivatives $f'_H(\bar{x}; u)$, $h'_H(\bar{x}; u)$ for all $u \in T(X, \bar{x})$. Assume that there exists*

$$(\lambda, \mu, \nu) \in [0, +\infty) \times [0, +\infty)^p \times (-\infty, +\infty)^l \setminus \big\{(0, 0, 0)\big\}$$

such that the sets $L(f; \bar{x}; X)$, $L(g_i; \bar{x}; X)$, $i \in I(\bar{x})$, $L(h_j; \bar{x}; X)$ if $\nu_j > 0$, $L(-h_j; \bar{x}; X)$ if $\nu_j < 0$ are pseudoconvex, at least one of the sets $L(g_i; \bar{x}; X)$, $i \in I(\bar{x})$, where $\mu_i > 0$ or $L(f; \bar{x}; X)$ where $\lambda > 0$ is strongly pseudoconvex, condition (15) *is verified for every*

$$u \in T\big(L(f; \bar{x}; X), \bar{x}\big) \cap \bigcap_{i \in I(\bar{x})} T\big(L(g_i; \bar{x}; X), \bar{x}\big) \cap \bigcap_{j \in \{1,2,\ldots,l\},\, \nu_j > 0} T\big(L(h_j; \bar{x}; X), \bar{x}\big) \cap \bigcap_{j \in \{1,2,\ldots,l\},\, \nu_j < 0} T\big(L(-h_j; \bar{x}; X), \bar{x}\big).$$

Then $\bar{x}$ is a strict global minimizer of (P)*.*

Proof. Suppose that there exists $x \in S$, $x \neq \bar{x}$ such that $f(x) \leqslant f(\bar{x})$. It follows from the pseudoconvexity of $L(f; \bar{x}; X)$ that $x - \bar{x} \in T(L(f; \bar{x}; X), \bar{x})$. We conclude from here that there exist sequences $\{t_k\}$, $t_k \to +0$, and $\{u_k\}$, $u_k \to (x - \bar{x})$ with $f(\bar{x} + t_k u_k) \leqslant f(\bar{x})$. Hence inequality (11) holds. Let $i \in I(\bar{x})$ be arbitrary. Since $g_i(x) \leqslant 0 = g_i(\bar{x})$ using similar arguments we obtain that $x - \bar{x} \in T(L(g_i; \bar{x}; X), \bar{x})$ and

$$(g_i)^{\downarrow}_H(\bar{x}; x - \bar{x}) \leqslant 0. \tag{16}$$

Suppose that j is any integer from the set $\{1, 2, \ldots, l\}$ with $\nu_j > 0$. Using that $h_j(x) = h_j(\bar{x})$ we obtain from the pseudoconvexity of $L(h_j; \bar{x}; X)$ that $x - \bar{x} \in T(L(h_j; \bar{x}; X), \bar{x})$. Therefore

$$(h_j)'_H(\bar{x}; x - \bar{x}) \leqslant 0. \tag{17}$$

If $\nu_j < 0$ by the pseudoconvexity of $L(-h_j; \bar{x}; X)$ we have that $x - \bar{x} \in T(L(-h_j; \bar{x}; X), \bar{x})$ and

$$(h_j)'_H(\bar{x}; x - \bar{x}) \geqslant 0. \tag{18}$$

At least one of the inequalities (11), (16) is strict because of the strong pseudoconvexity. Inequalities (11), (16)–(18) contradict condition (15). □

Corollary 3. *Let $\bar{x}$ be a feasible point of* (P)*, X be open and the functions f, g, h be differentiable at $\bar{x}$. Suppose that there exist $(\lambda, \mu, \nu) \in [0, +\infty) \times [0, +\infty)^p \times (-\infty, +\infty)^l$ with*

$$\lambda \nabla f(\bar{x}) + \sum_{i \in I(\bar{x})} \mu_i \nabla g_i(\bar{x}) + \sum_{j=1}^{l} \nu_j \nabla h_j(\bar{x}) = 0.$$

Assume that, for $\bar{x}$:

(i) *f is quasiconvex;*
(ii) *for $i \in I(\bar{x})$, $i \neq s$, g_i is quasiconvex, g_s is strictly pseudoconvex with $\mu_s > 0$;*
(iii) *for all $j = 1, 2, \ldots, l$, h_j is quasiconvex if $\nu_j > 0$, h_j is quasiconcave if $\nu_j < 0$.*

Then $\bar{x}$ is a strict global solution of (P)*.*

Proof. The corollary is a consequence of Theorem 10 because the quasiconvexity (strict pseudoconvexity) at $\bar{x}$ of a differentiable function f defined on X follows from the (strong) pseudoconvexity of the sublevel set $L(f; \bar{x}; X)$. □

There are in literature several papers which concern with sufficient conditions for a global minimum (see [3,8,18] and references therein). Corollary 3 is a slight generalization of Theorem 3.1 in [3].

Theorem 11 *(Sufficient conditions for a strict global minimum). Let $\bar{x}$ be a feasible point of* (P)*. Suppose that $L(f; \bar{x}; X)$, $L(g_i; \bar{x}; X)$, $i \in I(\bar{x})$, $L(h_j; \bar{x}; X)$, $L(-h_j; \bar{x}; X)$, $j = 1, 2, \ldots, l$, are pseudoconvex, and $L(f; \bar{x}; X)$ is strongly pseudoconvex. Assume that the functions f, h_j, $j = 1, 2, \ldots, l$, are Hadamard directionally differentiable at $\bar{x}$ in every direction $u \in U(\bar{x})$ and for all $u \in U(\bar{x})$ condition* (14) *is satisfied. Then $\bar{x}$ is a strict global minimizer.*

Proof. The proof follows from the arguments of Theorem 10. □

The following example shows that the optimality conditions could be applied for solving problems whose objective function has pseudoconvex sublevel sets.

Example 5. Solve problem (P) with inequality constraint only, such that $X = \mathbf{R}^2$ and $f : \mathbf{R}^2 \to \mathbf{R}$, $g : \mathbf{R}^2 \to \mathbf{R}$ are given by

$$f(x_1, x_2) = \begin{cases} x_1^2 + x_2 & \text{if } (x_1, x_2) \notin N, \\ 12 + 4n^{-1} + n^{-3/2} & \text{if } (x_1, x_2) \in N, \end{cases}$$

where N is the set which consists of the points $(2 + \frac{1}{n}, 8)$, n arbitrary positive integer and

$$g(x_1, x_2) = -x_1 - x_2 + \sqrt{(x_1 - x_2)^2 + 64}.$$

The only point which satisfies the necessary conditions of Theorem 9 is $\bar{x} = (2, 8)$. Since f is a function with pseudoconvex sublevel sets, $L(f, \bar{x}, X)$ is strongly pseudoconvex because the function $f_1 = x_1^2 + x_2$ is strictly pseudoconvex, $L(g, \bar{x}, X)$ is convex and therefore pseudoconvex, then it follows from Theorem 10 that $\bar{x}$ is a strict global minimizer.

References

[1] J. Abadie, On the Kuhn–Tucker theorem, in: J. Abadie (Ed.), Nonlinear Programming, North-Holland, Amsterdam, 1967, pp. 19–36.
[2] K.J. Arrow, A.C. Enthoven, Quasiconvex programming, Econometrica 29 (1961) 779–800.
[3] C.R. Bector, S. Chandra, M.K. Bector, Sufficient optimality conditions and duality for a quasiconvex programming problem, J. Optim. Theory Appl. 59 (1988) 209–221.
[4] V.F. Demyanov, G. Di Pillo, F. Facchinei, Exact penalization via Dini and Hadamard conditional derivatives, Optim. Methods Softw. 9 (1998) 19–36.
[5] V.F. Demyanov, A.M. Rubinov, Constructive Nonsmooth Analysis, Peterlang, Frankfurt am Main, 1995.
[6] W.E. Diewert, Alternative characterizations of six kind of quasiconvexity in the nondifferentiable case with applications to nonsmooth programming, in: S. Schaible, W.T. Ziemba (Eds.), Generalized Concavity in Optimization and Economics, Academic Press, New York, 1981, pp. 51–95.
[7] I. Ginchev, V.I. Ivanov, Second-order optimality conditions for problems with C^1 data, J. Math. Anal. Appl. 340 (2008) 646–657.
[8] G. Giorgi, On sufficient optimality conditions for a quasiconvex programming problem, J. Optim. Theory Appl. 81 (1994) 401–405.
[9] B.M. Glover, B.D. Craven, A Fritz John optimality condition using the approximate subdifferential, J. Optim. Theory Appl. 82 (1994) 253–265.
[10] M.A. Hanson, On sufficiency of the Kuhn–Tucker conditions, J. Math. Anal. Appl. 80 (1981) 545–550.
[11] A.D. Ioffe, Calculus of Dini subdifferentials of functions and contingent coderivatives of set-valued maps, Nonlinear Anal. 8 (1984) 517–539.
[12] V.I. Ivanov, Second-order optimality conditions in nonsmooth optimization, C. R. Acad. Bulgare Sci. 60 (2007) 1053–1058.
[13] B. Jimenez, V. Novo, Alternative theorems and necessary optimality conditions for directionally differentiable multiobjective programs, J. Convex Anal. 9 (2002) 97–116.
[14] D.V. Luu, M.H. Nguyen, On alternative theorems and necessary conditions for efficiency, Preprint No. 19, Institute of Mathematics, Hanoi, Vietnam, 2005; available at http://mse.univ-paris1.fr/Publicat.htm.
[15] O.L. Mangasarian, Nonlinear Programming, Classics Appl. Math., vol. 10, SIAM, Philadelphia, 1994, repr. of the orig. 1969.
[16] R.N. Mukherjee, L.V. Reddy, Semicontinuity and quasiconvex function, J. Optim. Theory Appl. 94 (1997) 715–726.
[17] J.-P. Penot, Differentiability of relations and differential stability of perturbated optimization problems, SIAM J. Control Optim. 22 (1984) 529–551.
[18] B. Skarpness, V.A. Sposito, A modified Fritz John optimality criterion, J. Optim. Theory Appl. 31 (1980) 113–115.
[19] M. Studniarski, Necessary and sufficient conditions for isolated local minima of nonsmooth functions, SIAM J. Control Optim. 24 (1986) 1044–1049.
[20] D. Ward, Exact penalties and sufficient conditions for optimality in nonsmooth optimization, J. Optim. Theory Appl. 57 (1988) 485–499.
[21] X.M. Yang, S.Y. Liu, Three kinds of generalized convexity, J. Optim. Theory Appl. 86 (1995) 501–513.