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Actions of inverse semigroups arising from partial actions of groups

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ABSTRACT

In this work we present a definition of crossed product for actions of inverse semigroups on C^*-algebras, without resorting to covariant representations as done by Sieben in related work. We also show the existence of an isomorphism between the crossed product by a partial action of a group G and the crossed product by a related action of $S(G)$, an inverse semigroup associated to G introduced by the first named author.

1. Introduction

In his master thesis [5], N. Sieben introduced the notion of a crossed product by an action of an inverse semigroup on a C^*-algebra using covariant representations. We will present here another definition, inspired by the definition of the crossed product by a partial action. There is only one difference: we will need to take a quotient that isn't necessary in the case of partial actions of groups. And for this, we will show that, under certain conditions over the algebra, the algebraic crossed product is associative, and this part is very similar to what Dokuchaev and the first named author did in [1].

Associated to a partial action α of a group G and to the group itself, Sieben constructed a certain inverse semigroup S_G, and showed that the crossed product of α is isomorphic to the crossed product of a certain action of S_G. Here, we present an analogous result, but using an inverse semigroup that depends only of the group G. It is the semigroup $S(G)$ introduced by the first named author in [2].

Recall that a semigroup is a set S with an associative operation.

Definition 1.1. An inverse semigroup is a semigroup S such that, for any $r \in S$, exists a unique $r^* \in S$ such that $rr^*r = r$ and $r^*rr^* = r^*$. We call r^* the inverse of r.

Example 1.2. For any set X, the set $I(X)$ of partial bijections of X (that is, bijections between subsets of X), is an inverse semigroup. In fact, by the Wagner–Preston Theorem [4], any inverse semigroup is isomorphic to an inverse subsemigroup of $I(S)$.

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Definition 1.3. Let G be a group with unit e. We define $S(G)$ to be the universal semigroup generated by $\{[g]\colon g \in G\}$ under the following relations, for $g, h \in G$:

(i) $[g^{-1}][g][h] = [g^{-1}][gh]$,
(ii) $[g][h][h^{-1}] = [gh][h^{-1}]$,
(iii) $[g][e] = [g]$.

Note that $[e][g] = [gg^{-1}][g] = [g][g^{-1}][g] = [g][g^{-1}g] = [g][e] = [g]$, and then $S(G)$ is a semigroup with unit $[e]$.
The following is a universal property enjoyed by $S(G)$:

Proposition 1.4. *Let S be a semigroup and $f: G \to S$ be any function such that, for all $g, h \in G$:*

(i) $f(g^{-1})f(g)f(h) = f(g^{-1})f(gh)$,
(ii) $f(g)f(h)f(h^{-1}) = f(gh)f(h^{-1})$,
(iii) $f(g)f(e) = f(g)$.

Then there exists a unique homomorphism $\widehat{f}: S(G) \to S$ such that the diagram below commutes:

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "{[\cdot]}"'] & S \\ S(G) \arrow[ur, "\widehat{f}"'] & \end{tikzcd}$$

In what follows we will recall some facts about $S(G)$, referring the reader to [2] for more details. For $g \in G$, define $\varepsilon_g = [g][g^{-1}]$. Is not difficult to see that $\varepsilon_g = \varepsilon_g^2$ and $[h]\varepsilon_g = \varepsilon_{gh}[h]$, for $g, h \in G$. Another interesting propriety is that any element $s \in S(G)$ admits a decomposition

$$s = \varepsilon_{s_1} \ldots \varepsilon_{s_n}[g],$$

where $n \geqslant 0$ and $s_1, \ldots s_n$, $g \in G$. This decomposition is unique, except for the order of the s_j. We will call it the *standard form* of s.

We may construct an anti-automorphism $*$ on $S(G)$, such that $[g]^* := *([g]) = [g^{-1}]$ and, for any $s \in S(G)$, s^* is the inverse (in the sense of inverse semigroups) of s.

With this we may conclude that the inverse is unique, and so $S(G)$ is an inverse semigroup.

In the definition of crossed product by an action of an inverse semigroup that we will present, we will use the natural partial order of inverse semigroups, defined as follows: given elements s and t of an inverse semigroup S we say that

$$s \leqslant t \quad \Leftrightarrow \quad s = tf, \quad \text{for some idempotent } f \in S.$$

Example 1.5. Take an idempotent $l = \varepsilon_{l_1} \ldots \varepsilon_{l_n}[j] \in S(G)$, with $l_1, \ldots l_n$, $j \in G$. By the uniqueness of decomposition of $S(G)$, is not difficult to see that j must be the unit of the group. So $l = \varepsilon_{l_1} \ldots \varepsilon_{l_n}$. Now, for $r = \varepsilon_{r_1} \ldots \varepsilon_{r_n}[h]$ and $s = \varepsilon_{s_1} \ldots \varepsilon_{s_m}[g]$ in $S(G)$, if $s \leqslant r$ we have that $s = rf$, for some $f \in S(G)$ idempotent. Then $f = \varepsilon_{f_1} \ldots \varepsilon_{f_k}$ and we have that:

$$\varepsilon_{s_1} \ldots \varepsilon_{s_m}[g] = \varepsilon_{r_1} \ldots \varepsilon_{r_n}[h]\varepsilon_{f_1} \ldots \varepsilon_{f_k} = \varepsilon_{r_1} \ldots \varepsilon_{r_n}\varepsilon_{hf_1} \ldots \varepsilon_{hf_k}[h].$$

By the uniqueness of the decomposition in $S(G)$, it follows that $g = h$ and $\{s_1, \ldots, s_m\} = \{r_1, \ldots, r_n, hf_1, \ldots, hf_k\}$. So, the difference between s and r are some ε's.

2. Actions

Let G be a group.

Definition 2.1. A partial action α of G on an associative algebra A is a pair

$$\big(\{D_g\}_{g \in G}, \{\alpha_g\}_{g \in G}\big)$$

where, for each $g \in G$, D_g is a ideal of A and $\alpha_g: D_{g^{-1}} \to D_g$ is a isomorphism satisfying, for $g, h \in G$:

(i) $D_e = A$,
(ii) $\alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$,
(iii) $\alpha_g \circ \alpha_h(x) = \alpha_{gh}(x)$, $\forall x \in D_{h^{-1}} \cap D_{h^{-1}g^{-1}}$.

Definition 2.2. A partial action of G on a C^*-algebra A is a partial action such that, for all $g \in G$, D_g is a closed ideal of A and α_g is a $*$-isomorphism.

We call (A, G, α) a (C^*-)partial dynamical system if α is a partial action of the group G in the (C^*-)algebra A.

Definition 2.3. Let S be an inverse semigroup with unit e. We say that β is an action of S in the algebra A if for each $s \in S$ exists an ideal E_s of A and an isomorphism $\beta_s : E_{s^*} \to E_s$ such that, for all $r, s \in S$, $\beta_r \circ \beta_s = \beta_{rs}$ and $E_e = A$.

We note that the composition $\beta_r \circ \beta_s$ referred to above is only defined on the set of points x in the domain of β_s for which $\beta_s(x)$ lies in the domain of β_r. Is easy to verify that $(\beta_s)^{-1} = \beta_{s^*}$.

Proposition 2.4. *Let β be an action of the inverse semigroup S in the algebra A. Then, for $s, t \in S$, we have that $E_{st} \subseteq E_s$.*

Proof. By definition it follows that:

$$E_{st} = \mathrm{Dom}(\beta_{t^*s^*}) = \mathrm{Dom}(\beta_{t^*}\beta_{s^*}) = \beta_{s^*}^{-1}(E_t \cap E_{s^*}) = \beta_s(E_t \cap E_{s^*}) \subseteq E_s. \qquad \square$$

For the case of actions of $S(G)$ we have:

Proposition 2.5. *Let β be an action of $S(G)$ on the algebra A. Then, for $g, h \in G$, we have that $E_{[g][h]} = E_{[gh]} \cap E_{[g]}$.*

Proof. Using the last proposition, it follows that:

$$\begin{aligned} E_{[g][h]} &= E_{[g][g^{-1}][g][h]} = E_{[g][g^{-1}][gh]} = \beta_{[g]}(E_{[g^{-1}][gh]} \cap E_{[g^{-1}]}) \\ &= \beta_{[g]}\big(\beta_{[g^{-1}]}(E_{[gh]} \cap E_{[g]}) \cap E_{[g^{-1}]}\big) = \beta_{[g]}\big(\beta_{[g^{-1}]}(E_{[gh]} \cap E_{[g]})\big) = E_{[gh]} \cap E_{[g]}. \qquad \square \end{aligned}$$

Definition 2.6. An action of the inverse semigroup S on a C^*-algebra A is an action of S on the algebra A such that, for all $s \in S$, E_s is a closed ideal of A and β_s is a $*$-isomorphism.

In [2], the first named author shows that for any group G and any set X, there exists a bijection between the set of partial actions of G on X and the set of actions of $S(G)$ on X.

This result extends to the case of actions on an algebra or on a C^*-algebra as we shall now see. We begin with the following:

Proposition 2.7. *Let G be a group and let β be an action of $S(G)$ on the algebra A. Then $(\{E_{[g]}\}_{g\in G}, \{\beta_{[g]}\}_{g\in G})$ is a partial action of G on A.*

Proof. Item (i) and (iii) are obvious. Item (ii) follows from $\beta_{[g]} \circ \beta_{[h]} = \beta_{[g][h]}$ and Proposition 2.5. $\square$

The converse result follows by the universal propriety of $S(G)$.

Proposition 2.8. *Let $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial action of the group G on the algebra A. Then*

$$f : G \to I(A),$$
$$g \mapsto \alpha_g$$

satisfies (i)–(iii) *of Proposition* 1.4.

Proof. The result follows easily of items (ii) and (iii) of the definition of partial action. $\square$

With this, we have an unique homomorphism

$$\beta : S(G) \to I(A),$$
$$s \mapsto \beta_s$$

such that $\beta_{[g]} = \alpha_g$.

Now, for $s_1, \ldots, s_n, h \in G$, let $s = \varepsilon_{s_1} \ldots \varepsilon_{s_n}[h] = [h]\varepsilon_{h^{-1}s_1} \ldots \varepsilon_{h^{-1}s_n} \in S(G)$ be in the standard form. Note that

$$\beta_s = \beta_{[h]}\beta_{[h^{-1}s_1]}\beta_{[(h^{-1}s_1)^{-1}]} \ldots \beta_{[(h^{-1}s_n)^{-1}]} = \alpha_h\alpha_{h^{-1}s_1}\alpha_{(h^{-1}s_1)^{-1}} \ldots \alpha_{(h^{-1}s_n)^{-1}} = \alpha_h|_{D_{h^{-1}s_1}\cap\cdots\cap D_{h^{-1}s_n}}.$$

So we conclude that $E_{s^*} := \operatorname{dom}\beta_s = D_{h^{-1}} \cap D_{h^{-1}s_1} \cap \cdots \cap D_{h^{-1}s_n}$. Since $s^* = [h^{-1}]\varepsilon_{s_1}\ldots\varepsilon_{s_n}$, we have $E_s := \operatorname{dom}\beta_{s^*} = D_h \cap D_{s_1} \cap \cdots \cap D_{s_n}$.

Let us find the range of β_s:

$$\beta_s(E_{s^*}) = \alpha_h(D_{h^{-1}} \cap D_{h^{-1}s_1} \cap \cdots \cap D_{h^{-1}s_n}) \subseteq \alpha_h(D_{h^{-1}} \cap D_{h^{-1}s_i}) = D_h \cap D_{s_i}, \quad \forall i.$$

So $\beta_s(E_{s^*}) \subseteq D_h \cap D_{s_1} \cap \cdots \cap D_{s_n}$. If we change h by h^{-1} and s_i by $h^{-1}s_i$, we conclude that $\beta_s(E_{s^*}) \supseteq D_h \cap D_{s_1} \cap \cdots \cap D_{s_n}$, so

$$\beta_s(E_{s^*}) = \alpha_h(D_{h^{-1}} \cap D_{h^{-1}s_1} \cap \cdots \cap D_{h^{-1}s_n}) = D_h \cap D_{s_1} \cap \cdots \cap D_{s_n} = E_s.$$

So, for all $s \in S$, $\beta_s : E_{s^*} \to E_s$ is an isomorphism, E_s is an ideal and since $E_{[e]} = D_e = A$, we have that β is an action of $S(G)$ on A.

Observe that, for $r = [r_1][r_2]\ldots[r_n] \in S(G)$ (not necessarily in the standard form), $\beta_r = \alpha_{r_1}\alpha_{r_2}\ldots\alpha_{r_n}$, $E_{r^*} = D_{r_n^{-1}} \cap D_{r_n^{-1}r_{n-1}^{-1}} \cap \cdots \cap D_{r_n^{-1}r_{n-1}^{-1}\ldots r_1^{-1}}$ and $E_r = D_{r_1} \cap D_{r_1r_2} \cap \cdots \cap D_{r_1r_2\ldots r_n}$.

So, it is easy to see that the following theorem holds.

Theorem 2.9. *Let G be a group and A be an algebra. Then there is a bijection between the partial actions of G on A and the actions of $S(G)$ on A.*

Now, if we have an action β of $S(G)$ on the C^*-algebra A, Proposition 2.7 implies that $(\{E_{[g]}\}_{g\in G}, \{\beta_{[g]}\}_{g\in G})$ is a partial action of G on the algebra A. But $E_{[g]}$ is closed and $\beta_{[g]}$ preserves $*$. So $(\{E_{[g]}\}_{g\in G}, \{\beta_{[g]}\}_{g\in G})$ is a partial action of G on the C^*-algebra A.

Conversely, if $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is a partial action of G on the C^*-algebra A, we construct an action β of $S(G)$ on the algebra A. But intersection of closed ideals is a closed ideal, and α_g preserve $*$. So β is an action of $S(G)$ on the C^*-algebra A. So we may extend Theorem 2.9:

Theorem 2.10. *Let G be a group and A a C^*-algebra. There is a bijection between the partial actions of G on A and the actions of $S(G)$ on A.*

3. Algebraic crossed product

Consider a partial action α of the group G on the algebra A.

Definition 3.1. We define the algebraic crossed product of α as

$$A \rtimes_\alpha^a G = \left\{ \sum_{g\in G}^{\text{finite}} a_g\delta_g\colon\ a_g \in D_g \right\},$$

where δ_g are symbols, with addition defined in the obvious way and the product being the linear extension of

$$(a_g\delta_g)(a_h\delta_h) = \alpha_g\big(\alpha_{g^{-1}}(a_g)a_h\big)\delta_{gh}.$$

Notice that $\alpha_{g^{-1}}(a_g)a_h \in D_{g^{-1}} \cap D_h$, so $\alpha_g(\alpha_{g^{-1}}(a_g)a_h)\delta_{gh} \in D_g \cap D_{gh}$, so the multiplication is well defined.

For the case of actions of an inverse semigroups a similar construction may be performed, but it is just a first step in the definition of the algebraic crossed product.

Definition 3.2. Let β be an action of the inverse semigroup S on the algebra A. Define

$$L = \left\{ \sum_{s\in S}^{\text{finite}} a_s\delta_s\colon\ a_s \in E_s \right\},$$

where δ_s are symbols, with addition defined in the obvious way and the product being the linear extension of

$$(a_r\delta_r)(a_s\delta_s) = \beta_r\big(\beta_{r^{-1}}(a_r)a_s\big)\delta_{rs}.$$

In [1], Dokuchaev and the first named author found conditions under which $A \rtimes_\alpha G$ is associative, and they presented an example to show that associativity may fail (Proposition 3.6). Based on this example we may easily construct an action of $S(G)$ such that L is not associative. We therefore need to know under which conditions L is associative.

We begin by recalling the definition of the algebra of multipliers.

Definition 3.3. The algebra of multipliers of a K-algebra A is the set $M(A)$ of all ordered pairs (L, R), where L and R are linear operators on A such that, for $a, b \in A$:

(i) $L(ab) = L(a)b$,
(ii) $R(ab) = aR(b)$,
(iii) $R(a)b = aL(b)$.

For (L, R), $(L', R') \in M(A)$, $k \in K$, we define:

$$k(L,R) = (kL, kR),$$
$$(L,R) + (L',R') = (L + L', R + R'),$$
$$(L,R)(L',R') = (L \circ L', R' \circ R).$$

We say that L is a left multiplier and R is a right multiplier of A.

We will show that the operation of multiplication on L is associative when the ideals E_s associated with the action β are (L, R)-associative, that is, when $L \circ R' = R' \circ L$ for all $(L, R), (L', R') \in M(E_s)$ (for more details about the algebra of multipliers see [1]).

Theorem 3.4. *Let β be an action of the inverse semigroup S on the algebra A. If the ideals E_s are (L, R)-associative, then the operation of multiplication of the set L defined in Definition 3.2 is associative.*

Proof. Let $r, s, t \in S$ and $a_r \in E_r$, $a_s \in E_s$, $a_t \in E_t$.

We want to prove that

$$a_r\delta_r(a_s\delta_s a_t\delta_t) = (a_r\delta_r a_s\delta_s)a_t\delta_t,$$

that is:

$$\beta_r\big(\beta_{r^*}(a_r)\beta_s\big(\beta_{s^*}(a_s)a_t\big)\big)\delta_{rst} = \beta_{rs}\big(\beta_{s^*r^*}\big(\beta_r\big(\beta_{r^*}(a_r)a_s\big)\big)a_t\big)\delta_{rst}.$$

Analyzing the right side:

$$\begin{aligned}\beta_{rs}\big(\beta_{s^*r^*}\big(\beta_r\big(\beta_{r^*}(a_r)a_s\big)\big)a_t\big)\delta_{rst} &= \beta_{rs}\big(\beta_{s^*}\big(\beta_{r^*}\big(\beta_r\big(\beta_{r^*}(a_r)a_s\big)\big)\big)a_t\big)\delta_{rst}\\ &= \beta_{rs}\big(\beta_{s^*}\big(\beta_{r^*}(a_r)a_s\big)a_t\big)\delta_{rst}\\ &= \beta_r\big(\beta_s\big(\beta_{s^*}\big(\beta_{r^*}(a_r)a_s\big)a_t\big)\big)\delta_{rst}.\end{aligned}$$

So we need to prove that:

$$\beta_r\big(\beta_{r^*}(a_r)\beta_s\big(\beta_{s^*}(a_s)a_t\big)\big) = \beta_r\big(\beta_s\big(\beta_{s^*}\big(\beta_{r^*}(a_r)a_s\big)a_t\big)\big).$$

Applying β_{r^*} in both sides of equality by the left:

$$\beta_{r^*}\big(\beta_r\big(\beta_{r^*}(a_r)\beta_s\big(\beta_{s^*}(a_s)a_t\big)\big)\big) = \beta_{r^*}\big(\beta_r\big(\beta_s\big(\beta_{s^*}\big(\beta_{r^*}(a_r)a_s\big)a_t\big)\big)\big)$$

and it is equivalent to:

$$\beta_{r^*}(a_r)\beta_s\big(\beta_{s^*}(a_s)a_t\big) = \beta_s\big(\beta_{s^*}\big(\beta_{r^*}(a_r)a_s\big)a_t\big).$$

Because $\beta_{r^*}: E_r \to E_{r^*}$ is an isomorphism, the above condition is equivalent to:

$$a\beta_s\big(\beta_{s^*}(a_s)a_t\big) = \beta_s\big(\beta_{s^*}(aa_s)a_t\big), \quad \forall a \in E_{r^*},\ a_s \in E_s,\ a_t \in E_t.$$

Denoting $R_{a_t}: E_{s^*} \to E_{s^*}$ the right multiplier by a_t in E_{s^*} and $L_a: E_s \to E_s$ the left multiplier by a in E_s, the last equation is equivalent to:

$$L_a \circ \beta_s \circ R_{a_t} \circ \beta_{s^*}(a_s) = \beta_s \circ R_{a_t} \circ \beta_{s^*} \circ L_a(a_s), \quad \forall a \in E_{r^*},\ a_s \in E_s,\ a_t \in E_t.$$

Now, $\beta_s \circ R_{a_t} \circ \beta_{s^*}$ is a right multiplier of E_s, and because E_s is (L, R)-associative, the last equation holds.

So the multiplication of L is associative. $\square$

So, let us suppose that the ideals E_s related with the action β of S on A are (L, R)-associative.

Definition 3.5. Let β be an action of the inverse semigroup S on the algebra A. Consider $N = \langle a\delta_r - a\delta_t\colon\ a \in E_r, r \leqslant t\rangle$, that is, the ideal generated by $a\delta_r - a\delta_t$. We define the algebraic crossed product of β as

$$A \rtimes^a_\beta S = \frac{L}{N}.$$

Note that $r \leqslant t$ implies $r = ti$, for i idempotent. So, by Proposition 2.4, $E_r \subseteq E_t$.
We will denote the elements of $A \rtimes_\beta^a S$ by $\overline{a_s\delta_s}$, where $a_s\delta_s \in L$.

Lemma 3.6. *Let β be an action of S on A. For $r_1, \dots, r_n$, $g, h \in G$, we have*:

(1) $\overline{a\delta_{[g][h]}} = \overline{a\delta_{[gh]}}$, *for $a \in E_{[g][h]}$*,
(2) $\overline{a\delta_{\varepsilon_{r_1}\dots\varepsilon_{r_n}[g]}} = \overline{a\delta_{[g]}}$, *for $a \in E_{\varepsilon_{r_1}\dots\varepsilon_{r_n}[g]}$*.

Proof. (1): Well, $[g][h] = [g][h][h^{-1}][h] = [gh][h^{-1}][h]$. As $[h^{-1}][h]$ is idempotent, $[g][h] \leqslant [gh]$ and so $a\delta_{[g][h]} - a\delta_{[gh]} \in N$.
(2): Note that $\varepsilon_{r_1}\dots\varepsilon_{r_n}[g] = [g]\varepsilon_{g^{-1}r_1}\dots\varepsilon_{g^{-1}r_n}$ and the results follows because $\varepsilon_{g^{-1}r_1}\dots\varepsilon_{g^{-1}r_n}$ is idempotent. $\square$

So, we may now state the main result of this section:

Theorem 3.7. *Let α be a partial action of the group G on the algebra A. Consider the action β related with α by Theorem 2.9. Then $A \rtimes_\alpha^a G \cong A \rtimes_\beta^a S(G)$.*

Proof. Define

$$\varphi : A \rtimes_\alpha^a G \to A \rtimes_\beta^a S(G),$$
$$a\delta_g \mapsto \overline{a\delta_{[g]}}, \quad \text{linearly extended.}$$

Let us prove that φ is an isomorphism. It is well defined and using the lemma above we may prove that it is a homomorphism, because:

$$\varphi(a\delta_g)\varphi(b\delta_h) = (\overline{a\delta_{[g]}})(\overline{b\delta_{[h]}}) = \overline{\beta_{[g]}\big(\beta_{[g^{-1}]}(a)b\big)\delta_{[g][h]}} = \overline{\beta_{[g]}\big(\beta_{[g^{-1}]}(a)b\big)\delta_{[gh]}},$$
$$\varphi\big((a\delta_g)(b\delta_h)\big) = \varphi\big(\alpha_g\big(\alpha_{g^{-1}}(a)b\big)\delta_{gh}\big) = \overline{\alpha_g\big(\alpha_{g^{-1}}(a)b\big)\delta_{[gh]}} = \overline{\beta_{[g]}\big(\beta_{[g^{-1}]}(a)b\big)\delta_{[gh]}}.$$

To show that φ is bijective we will present an inverse for it. For $s_1, \dots s_n$, $g \in G$ consider $s = \varepsilon_{s_1}\dots\varepsilon_{s_n}[g] \in S(G)$ and the function γ such that $\gamma(s) = g$. Is very easy to show that γ is a homomorphism between the inverses semigroups $S(G)$ and G. Note that $\gamma(s^*) = g^{-1} = \gamma(s)^{-1}$. So we may define:

$$\psi : L \to A \rtimes_\alpha^a G,$$
$$a\delta_s \mapsto a\delta_{\gamma(s)}, \quad \text{linearly extended.}$$

Note that ψ is a homomorphism and using Example 1.5 we see that $\psi(N) = 0$.
So we may extend ψ to the homomorphism

$$\widetilde{\psi} : A \rtimes_\beta^a S(G) \to A \rtimes_\alpha^a G,$$
$$\overline{a\delta_s} \mapsto a\delta_{\gamma(s)}, \quad \text{linearly extended.}$$

As it is obvious that $\widetilde{\psi}$ and φ are inverses one each other, the theorem holds. $\square$

4. Crossed product

Let A be a C^*-algebra with unit and let α be a partial action of a group G on A. For $g \in G$ and $a_g \in D_g$, define in $A \rtimes_\alpha^a G$ the following operation *:

$$(a_g\delta_g)^* = \alpha_{g^{-1}}\big(a_g^*\big)\delta_{g^{-1}}, \quad \text{linearly extended:}$$
$$\left(\sum_{g\in G}^{\text{finite}} a_g\delta_g\right)^* = \sum_{g\in G}^{\text{finite}} (a_g\delta_g)^*.$$

It is easy to show that $A \rtimes_\alpha^a G$ with * is a $*$-algebra. Considering the following norm in $A \rtimes_\alpha^a G$, we have that it is a normed $*$-algebra:

$$\left\| \sum_{g\in G}^{\text{finite}} a_g\delta_g \right\|_1 = \sum_{g\in G}^{\text{finite}} \|a_g\|,$$

where the norm in the right side is the norm in A.

Given a Banach $*$-algebra B, its enveloping C^*-algebra is the completion of $B/\ker\rho_s$ with respect to $\rho_s(x) = \sup\{\|\rho(x)\|\colon \rho \text{ is a representation of } B\}$.

To define the crossed product by a partial action α of G on A, we want to take the enveloping C^*-algebra of $A \rtimes_\alpha^a G$. But this is not a Banach $*$-algebra. So we need to show that its representations are contractive. With this purpose, let π be a representation of $A \rtimes_\alpha^a G$. For $a_g \in D_g$:

$$\left\|\pi(a_g\delta_g)\right\|^2 = \left\|\pi(a_g\delta_g)^*\pi(a_g\delta_g)\right\| = \left\|\pi\left((a_g\delta_g)^*(a_g\delta_g)\right)\right\| = \left\|\pi\left(\alpha_{g^{-1}}\left(a_g^*a_g\right)\delta_e\right)\right\|.$$

Now, note that $\alpha_{g^{-1}}(a_g^*a_g)\delta_e \in A\delta_e$, and $A\delta_e$ is a C^*-algebra (isomorphic to A). Then:

$$\left\|\pi(a_g\delta_g)\right\|^2 = \left\|\pi\left(\alpha_{g^{-1}}\left(a_g^*a_g\right)\delta_e\right)\right\| \leqslant \left\|\alpha_{g^{-1}}\left(a_g^*a_g\right)\delta_e\right\|_1 = \left\|\alpha_{g^{-1}}\left(a_g^*a_g\right)\right\| = \left\|a_g^*a_g\right\| = \|a_g\|^2.$$

So, we have that:

$$\left\|\pi\left(\sum_{g\in G}^{\text{finite}} a_g\delta_g\right)\right\| \leqslant \sum_{g\in G}^{\text{finite}} \left\|\pi(a_g\delta_g)\right\| \leqslant \sum_{g\in G}^{\text{finite}} \|a_g\| = \left\|\sum_{g\in G}^{\text{finite}} a_g\delta_g\right\|_1.$$

So we may take the enveloping C^*-algebra of $A \rtimes_\alpha^a G$ to define the crossed product.

Definition 4.1. The crossed product algebra $A \rtimes_\alpha G$, is the enveloping C^*-algebra of the $*$-algebra $A \rtimes_\alpha^a G$.

Let us denote the elements of $A \rtimes_\alpha G$ as classes of the elements of $A \rtimes_\alpha^a G$, $\overline{a_g\delta_g}$.

To define the crossed product by an action β of an inverse semigroup S on A, we want to do the same, that is, take the enveloping C^*-algebra of $A \rtimes_\alpha^a S$.

So, for $r \in S$ and $a_r \in E_r$, define in $A \rtimes_\beta^a S$:

$$(\overline{a_r\delta_r})^* = \overline{\beta_{r^*}\left(a_r^*\right)\delta_{r^*}}, \quad \text{linearly extended.}$$

We may easily see that $A \rtimes_\beta^a S$ is a $*$-algebra. Also define a norm

$$\left\|\sum_{s\in S}^{\text{finite}} a_s\delta_s\right\|_2 = \sum_{s\in S}^{\text{finite}} \|a_s\|.$$

It is easily seen that $A \rtimes_\beta^a S$ is a normed $*$-algebra.

Proposition 4.2. *Every representation $\rho : A \rtimes_\beta^a S \to B(H)$ is contractive.*

Proof. A representation ρ of $A \rtimes_\beta^a S$ is one of L such that $\rho|_N \equiv 0$.

Let $s \in S$ and $a_s \in E_s$. As $s^*s \leqslant e$ (e the unit of S), it follows that

$$\begin{aligned}\left\|\rho(a_s\delta_s)\right\|^2 &= \left\|\rho(a_s\delta_s)^*\rho(a_s\delta_s)\right\| = \left\|\rho\left((a_s\delta_s)^*(a_s\delta_s)\right)\right\| = \left\|\rho\left(\beta_{s^*}\left(a_s^*a_s\right)\delta_{s^*s}\right)\right\| \\ &= \left\|\rho\left(\beta_{s^*}\left(a_s^*a_s\right)\delta_e\right)\right\| \leqslant \left\|a_s^*a_s\right\| = \|a_s\|^2.\end{aligned}$$

For any element $\sum_{s\in S}^{\text{finite}} a_s\delta_s \in L$:

$$\left\|\rho\left(\sum_{s\in S}^{\text{finite}} a_s\delta_s\right)\right\| \leqslant \sum_{s\in S}^{\text{finite}} \left\|\rho(a_s\delta_s)\right\| \leqslant \sum_{s\in S}^{\text{finite}} \|a_s\| = \left\|\sum_{s\in S}^{\text{finite}} a_s\delta_s\right\|_2.$$

So ρ is contractive. $\square$

Definition 4.3. The crossed product by the action β of the inverse semigroup S on the C^*-algebra A, denoted $A \rtimes_\beta S$, is the enveloping C^*-algebra of the $*$-algebra $A \rtimes_\beta^a S$.

Note that to construct $A \rtimes_\beta S$, we do two quotients. So, we will denote its elements like $\overline{\overline{\overline{a_r\delta_r}}}$, where $a_r\delta_r \in L$.

Also note that $A \rtimes_\beta S$ is just the completion of $(A \times_\beta^a S)/\ker\rho_s$, with respect to $\rho_s(x) = \sup\{\|\rho(x)\|\colon \rho \text{ representation of } A \times_\beta^a S\}$.

Let α be a partial action of the group G on the C^*-algebra A. Consider the action β of $S(G)$ on A related by the Theorem 2.10.

Theorem 4.4. *The C^*-algebras $A \rtimes_\alpha G$ and $A \rtimes_\beta S(G)$ are isomorphic.*

Proof. For $g \in G$ and $a_g \in D_g$, define

$$\begin{aligned} &\phi : A \rtimes_\alpha^a G \to A \rtimes_\beta S(G), \\ &\qquad a_g\delta_g \to \overline{\overline{a_g\delta_{[g]}}}, \quad \text{linearly extended.} \end{aligned}$$

Obviously ϕ is well defined and, using Lemma 3.6, we see that it is a homomorphism. Also it is easy to check that ϕ preserves $*$.

Then, by the universal property of $A \rtimes_\alpha G$, it follows that there exists a unique $*$-homomorphism $\varphi : A \rtimes_\alpha G \to A \rtimes_\beta S(G)$ such that the diagram below commutes, that is, $\varphi(\overline{a_g\delta_g}) = \overline{\overline{a_g\delta_{[g]}}}$.

$$\begin{tikzcd} A \rtimes_\alpha^a G \arrow[r, "f"] \arrow[d, "{[\cdot]}"'] & A \rtimes_\beta S(G) \\ A \rtimes_\alpha G \arrow[ur, "\widehat{f}"'] & \end{tikzcd}$$

Let

$$K = \left\{ \sum_{s \in S(G)}^{\text{finite}} a_s\delta_s \colon a_s \in E_s \right\}.$$

Using the homomorphism $\gamma : S(G) \to G$ that we define in Theorem 3.7, consider

$$\begin{aligned} &\omega : K \to A \rtimes_\alpha G, \\ &a_s\delta_s \mapsto \overline{a_s\delta_{\gamma(s)}}, \quad \text{linearly extended.} \end{aligned}$$

Now take M the ideal of K generated by $a\delta_r - a\delta_t$, where $a \in E_r$ and $r \leqslant t$. Is easy to see that ω is a homomorphism and, using Example 1.5, we see that $\omega(M) = 0$.

So we may define

$$\begin{aligned} &\widetilde{\phi} : A \rtimes_\beta^a S(G) \to A \rtimes_\alpha G, \\ &\qquad\qquad \overline{a_s\delta_s} \mapsto \overline{a_s\delta_{\gamma(s)}}, \quad \text{linearly extended.} \end{aligned}$$

By Lemma 3.6, $\widetilde{\phi}$ is a homomorphism easily checked to preserve $*$. Then, by the universal property of $A \rtimes_\beta S(G)$ there exists a unique $*$-homomorphism $\widetilde{\varphi} : A \rtimes_\beta S(G) \to A \rtimes_\alpha G$ such that the diagram below commutes,

$$\begin{tikzcd} A \rtimes_\beta^a S(G) \arrow[r, "f"] \arrow[d, "{[\cdot]}"'] & A \rtimes_\alpha G \\ A \rtimes_\beta S(G) \arrow[ur, "\widehat{f}"'] & \end{tikzcd}$$

that is, $\widetilde{\varphi}(\overline{\overline{a_s\delta_s}}) = \overline{a_s\delta_{\gamma(s)}}$.

Obviously $\varphi \circ \widetilde{\varphi} = Id_{A \rtimes_\beta S(G)}$ and $\widetilde{\varphi} \circ \varphi = Id_{A \rtimes_\alpha G}$, and the theorem is proved. $\square$

5. Covariant representations

The crossed product by an action of an inverse semigroup was introduced by Nándor Sieben in [5] in 1994. In this definition, he used covariant representations of an action. We will outline his definition and we will show that our definition is equivalent to his. See [5] for further details on Sieben's work.

Definition 5.1. Let β be an action of the inverse semigroup S on the C^*-algebra A. A covariant representation of β is a triple (π, ν, H) where $\pi : A \to B(H)$ is a representation of A on the Hilbert space H and $\nu : S \to PIso(H)$ is a map preserving products, such that, for $s \in S$:

(i) $\nu_s\pi(a)\nu_{s^*} = \pi(\beta_s(a))$ for all $a \in E_{s^*}$ (covariance condition),
(ii) ν_s has initial space $\overline{\operatorname{span}\{\pi(E_{s^*})H\}}$ and final space $\overline{\operatorname{span}\{\pi(E_s)H\}}$.

The set of covariant representations of (A, S, β) is denoted $\operatorname{CovRep}(A, S, \beta)$.

Let β be an action of the unital inverse semigroup S on the C^*-algebra A. Define

$$\widetilde{L} = \{x \in l^1(S, A)\colon\ x(s) \in E_s\},$$

with norm, scalar multiplication and addition inherited of $l^1(S, A)$.

For $x, y \in \widetilde{L}$, the product $x * y$ is defined by:

$$(x * y)(s) = \sum_{rt=s} \beta_r\big(\beta_{r^*}\big(x(r)\big)y(t)\big).$$

Also define x^* to be the element of $l^1(S, A)$ such that:

$$x^*(s) = \beta_s\big(x(s^*)^*\big).$$

The operations are well defined and $\widetilde{L}$ is a Banach $*$-algebra.

Definition 5.2. If $(\pi, \nu, H) \in \mathrm{CovRep}(A, S, \beta)$, define $\pi \times \nu : \widetilde{L} \to B(H)$ by

$$(\pi \times \nu)(x) = \sum_{s\in S} \pi\big(x(s)\big)\nu_s.$$

We have that $\pi \times \nu$ is a $*$-homomorphism.

Sieben defines the crossed product by an action of inverse semigroup as follows.

Definition 5.3. Let β be an action of the unital inverse semigroup S on the C^*-algebra A. Define a seminorm $\|.\|_c$ on $\widetilde{L}$ as

$$\|x\|_c = \sup\big\{\big\|(\pi \times \nu)(x)\big\|\colon\ (\pi, \nu, H) \in \mathrm{CovRep}(A, S, \beta)\big\}.$$

Consider $I = \{x \in \widetilde{L}\colon\ \|x\|_c = 0\}$. The crossed product of β is the C^*-algebra obtained by the completion of the quotient $\widetilde{L}/I$ with respect to $\|.\|_c$.

We want to prove that the above definition is equivalent to Definition 4.3. The first step is to show that in the above definition, we may take the L instead of $\widetilde{L}$.

Lemma 5.4. *Let $E \subseteq F$ be linear spaces and let $\|.\|_1$ and $\|.\|_2$ be two norms over F such that there exists $k > 0$ such that, for all $x \in F$, $\|x\|_2 \leqslant k\|x\|_1$. Suppose that E is dense in F with respect to $\|.\|_1$. Then the completions of E and F with respect to $\|.\|_2$ are isomorphic.*

Proof. For $i = 1, 2$ denote $\overline{E}^i$ the completion of E with respect to $\|.\|_i$, and the same for F. The function $T : (E, \|.\|_2) \to \overline{F}^2$, that includes an element of E in F and then put them in $\overline{F}^2$, is a linear isometry, and then is uniformly continuous. So we may extend it to the isometry $\widetilde{T} : \overline{E}^2 \to \overline{F}^2$. Is easy to see that $F \subseteq Ran(\widetilde{T})$.

Now, as $\widetilde{T}$ is an isometry and $\overline{E}^2$ is complete, $Ran(\widetilde{T})$ is complete with respect to $\|.\|_2$. Then, $Ran(\widetilde{T}) = \overline{F}^2$ and $\widetilde{T}$ is an isomorphism between $\overline{E}^2$ and $\overline{F}^2$. $\square$

So, consider $L \subseteq \widetilde{L}$. In $\widetilde{L}$ we have defined a norm

$$\|x\|_a = \sum_{s\in S} \big\|x(s)\big\|,$$

and a seminorm

$$\|x\|_c = \sup\big\{\big\|(\pi \times \nu)(x)\big\|\colon\ (\pi, \nu, H) \in \mathrm{CovRep}(A, S, \beta)\big\}.$$

Note that we may take the completion of $\widetilde{L}$ with respect to the seminorm $\|.\|_c$, and this is equal to the completion of $\widetilde{L}/(\ker\|.\|_c)$ with respect to the norm $\|.\|_c$.

Also note that, for $x \in \widetilde{L}$ and $(\pi, \nu, H) \in \mathrm{CovRep}(A, S, \beta)$:

$$\begin{aligned}\big\|(\pi \times \nu)(x)\big\| &= \Big\|\sum_{s\in S} \pi\big(x(s)\big)\nu_s\Big\| \leqslant \sum_{s\in S}\big\|\pi\big(x(s)\big)\nu_s\big\| \leqslant \sum_{s\in S}\big\|\pi\big(x(s)\big)\big\|\,\|\nu_s\| \\ &\leqslant \sum_{s\in S}\big\|\pi\big(x(s)\big)\big\| \leqslant \sum_{s\in S}\big\|x(s)\big\| = \|x\|_a,\end{aligned}$$

and then $\|x\|_c = \sup\|(\pi \times \nu)(x)\| \leqslant \|x\|_a$. As $\widetilde{L}$ is the completion of L with respect to $\|.\|_a$ it follows, by the previous lemma, that in Sieben's definition we may take L instead of $\widetilde{L}$.

Observe that by the Cohen–Hewitt Factorization Theorem [3, Theorem 32.22], it is easy to show that $\overline{\operatorname{span}\{\pi(I)H\}} = \pi(I)H$, for any closed ideal I of a C^*-algebra A.

So, if we show that

$$\|x\|_c = \rho_s(x),$$

for all $x \in L$ (remember that $\rho_s(x) = \sup\{\|\rho(x)\|\colon\ \rho \text{ representation of } A \times_\beta^a S\}$ is the seminorm used to define $A \times_\beta S$), we have proved that the two definitions of crossed product by an action of inverse semigroup are the same. To this, consider the next theorem.

Theorem 5.5. *Let ρ be a representation of L on the Hilbert space H. Then $\rho|_N \equiv 0 \Leftrightarrow \rho = \pi \times \nu$ for some $(\pi, \nu, H) \in$* $\operatorname{CovRep}(A, S, \beta)$.

Proof. ($\Leftarrow$) Let $\rho = \pi \times \nu$ and take $a\delta_r - a\delta_t$ a generator of N, that is, $r = ti$, for i idempotent. For $a \in E_r = E_{it} \subseteq E_i$ and $h \in H$:

$$\begin{aligned}\rho(a\delta_r - a\delta_t)(h) &= (\pi \times \nu)(a\delta_r - a\delta_t)(h) = \pi(a)\nu_r(h) - \pi(a)\nu_t(h)\\ &= \pi(a)\nu_{it}(h) - \pi(a)\nu_t(h) = \pi(a)\nu_i\nu_t(h) - \pi(a)\nu_t(h).\end{aligned}$$

Denote $K_i = \pi(E_i)H$. Then $H = K_i \oplus (K_i)^\perp$, and let us split the proof in cases:

$\underline{\nu_t(h) \in K_i}$: As i is idempotent, it follows that it is the identity in K_i. So $\nu_i(\nu_t(h)) = \nu_t(h)$ and then:

$$\pi(a)\nu_i\nu_t(h) - \pi(a)\nu_t(h) = \pi(a)\nu_t(h) - \pi(a)\nu_t(h) = 0.$$

$\underline{\nu_t(h) \in (K_i)^\perp}$: Note that $\pi(E_i) \equiv 0$ in $(K_i)^\perp$ and then:

$$\pi(a)\nu_i\nu_t(h) - \pi(a)\nu_t(h) = 0.$$

So $\rho(N) = 0$.

($\Rightarrow$) Suppose that $\rho|_N \equiv 0$. So, for $r \leqslant t$ and $a \in E_r$, $\rho(a\delta_r) = \rho(a\delta_t)$. Define

$$\begin{aligned}\pi : A &\to B(H), & \nu : S &\to B(H),\\ a &\mapsto \rho(a\delta_e), & s &\mapsto \lim_\lambda \rho(u_\lambda\delta_s), \quad \{u_\lambda\} \text{ approx. identity of } E_s,\end{aligned}$$

where $\lim_\lambda$ denotes the strong operator limit.

Let us prove that $(\pi, \nu, H) \in \operatorname{CovRep}(A, S, \beta)$. It is obvious that π is a representation.

To show that ν is well defined, let $s \in S$ and consider $\{u_\lambda\}$ an approximate identity for E_s. As $\beta_{s^*} : E_s \to E_{s^*}$ is an isomorphism, we know that $\{\beta_{s^*}(u_\lambda)\}$ is an approximate identity of E_{s^*}. As $H = \pi(E_{s^*})H \oplus (\pi(E_{s^*})H)^\perp$, we will split the proof. If $h \in \pi(E_{s^*})H$, then $h = \pi(a)k = \rho(a\delta_e)k$, for $a \in E_{s^*}, k \in H$. So:

$$\nu_s(h) = \lim_\lambda \rho(u_\lambda\delta_s)(h) = \lim_\lambda \rho(u_\lambda\delta_s)\rho(a\delta_e)(k) = \lim_\lambda \rho\big(\beta_s\big(\beta_{s^*}(u_\lambda)a\big)\delta_s\big)(k) = \rho\big(\beta_s(a)\delta_s\big)(k).$$

If $h \in (\pi(E_{s^*})H)^\perp$, we have that $\langle h, \rho(E_{s^*}\delta_e)H\rangle = \langle h, \pi(E_{s^*})H\rangle = 0$. So:

$$\big\langle \rho\big(\beta_{s^*}(\sqrt{u_\lambda})\delta_e\big)(h), H\big\rangle = \big\langle h, \rho\big(\beta_{s^*}(\sqrt{u_\lambda})\delta_e\big)H\big\rangle = 0,$$

that implies $\rho(\beta_{s^*}(\sqrt{u_\lambda})\delta_e)(h) = 0$. Then:

$$\begin{aligned}\lim_\lambda \rho(u_\lambda\delta_s)(h) &= \lim_\lambda \rho\big[\beta_s\big(\beta_{s^*}(\sqrt{u_\lambda})\beta_{s^*}(\sqrt{u_\lambda})\big)\delta_s\big](h) = \lim_\lambda \rho\big[(\sqrt{u_\lambda}\delta_s)\big(\beta_{s^*}(\sqrt{u_\lambda})\delta_e\big)\big](h)\\ &= \lim_\lambda \rho(\sqrt{u_\lambda}\delta_s)\rho\big(\beta_{s^*}(\sqrt{u_\lambda})\delta_e\big)(h) = 0.\end{aligned}$$

So ν_s is independent of the approximate identity taken. As ρ is contractive (Proposition 4.2):

$$\|\nu_s\| = \Big\|\lim_\lambda \rho(u_\lambda\delta_s)\Big\| = \lim_\lambda \big\|\rho(u_\lambda\delta_s)\big\| \leqslant \lim_\lambda \|u_\lambda\delta_s\| \leqslant \lim_\lambda \|u_\lambda\| \leqslant 1,$$

and then $\nu_s \in B(H)$. So it is well defined.

To show that ν_s is a partial isometry (with initial space $\pi(E_{s^*})H$ and final $\pi(E_s)H$), first let us show that $\nu_s^* = \nu_{s^*}$. Let $\{u_\lambda\}$ be an approximate identity of E_{s^*}. Then, for $k_1, k_2 \in H$:

$$\begin{aligned}\big\langle k_1, \nu_{s^*}(k_2)\big\rangle &= \big\langle k_1, \lim_\lambda \rho(u_\lambda\delta_{s^*})(k_2)\big\rangle = \lim_\lambda\big\langle k_1, \rho(u_\lambda\delta_{s^*})(k_2)\big\rangle = \lim_\lambda\big\langle \rho(u_\lambda\delta_{s^*})^*(k_1), k_2\big\rangle\\ &= \Big\langle \lim_\lambda \rho\big(\beta_s(u_\lambda)\delta_s\big)(k_1), k_2\Big\rangle = \big\langle \nu_s(k_1), k_2\big\rangle.\end{aligned}$$

So $\nu_s^* = \nu_{s^*}$.

Let us show that $\nu_s^*\nu_s$ is a projection over $\pi(E_{s^*})H$, because we already see that $\nu_s \equiv 0$ in $(\pi(E_{s^*})H)^{\perp}$. Then, let $h = \pi(a)k \in \pi(E_{s^*})H$ and $\{u_\gamma\}$ be an approximate identity of E_{s^*}:

$$\begin{aligned}\nu_s^*\nu_s(h) &= \nu_{s^*}\big(\rho(\beta_s(a)\delta_s)(k)\big) = \lim_\gamma \rho(u_\gamma\delta_{s^*})\rho\big(\beta_s(a)\delta_s\big)(k)\\ &= \lim_\gamma \rho\big(\beta_{s^*}\big(\beta_s(u_\gamma)\beta_s(a)\big)\delta_{s^*s}\big)(k) = \rho(a\delta_{s^*s})(k)\\ &= \rho(a\delta_e)(k) = h,\end{aligned}$$

because $\rho|_N = 0$. Then ν_s is a partial isometry with initial space $\pi(E_{s^*})H$. Doing the same to $\nu_s\nu_s^*$, we conclude that $\pi(E_s)H$ is the final space of ν_s.

Let us split the proof that ν is a homomorphism in two cases. Firstly take $h \in \pi(E_{t^*s^*})H$. Then $h = \rho(a\delta_e)(k)$, $a \in E_{t^*s^*}$ and $k \in H$. Let $\{u_\lambda\}$ be an approximate identity of E_s. Using the first part of the proof of that ν is well defined we have:

$$\begin{aligned}\nu_s\nu_t(h) &= \nu_s\nu_t\big(\rho(a\delta_e)(k)\big) = \nu_s\rho\big(\beta_t(a)\delta_t\big)(k) = \lim_\lambda \rho(u_\lambda\delta_s)\rho\big(\beta_t(a)\delta_t\big)(k)\\ &= \lim_\lambda \rho\big(\beta_s\big(\beta_{s^*}(u_\lambda)\beta_t(a)\big)\delta_{st}\big)(k) = \lim_\lambda \rho\big(u_\lambda\beta_s\big(\beta_t(a)\big)\delta_{st}\big)(k)\\ &= \rho\big(\beta_{st}(a)\delta_{st}\big)(k) = \nu_{st}(h).\end{aligned}$$

Now let $h \in (\pi(E_{t^*s^*})H)^{\perp}$. We have that $\nu_{st}(h) = 0$. Let us show that $\nu_s\nu_t(h) = 0$. Take $\{u_\lambda\}$ an approximate identity of E_s and $\{u_\gamma\}$ of E_t. Well, $\beta_s(\beta_{s^*}(u_\lambda)u_\gamma) \in \beta_s(E_{s^*} \cap E_t) = E_{st}$ and by the Cohen–Hewitt Factorization Theorem, $\beta_s(\beta_{s^*}(u_\lambda)u_\gamma) = xy, x, y \in E_{st}$. By hypothesis:

$$\big\langle \rho\big(\beta_{t^*s^*}(y)\delta_e\big)(h), H\big\rangle = \big\langle h, \rho\big(\beta_{t^*s^*}(y)\delta_e\big)H\big\rangle = \big\langle h, \pi\big(\beta_{t^*s^*}(y)\big)H\big\rangle = 0,$$

that implies $\rho(\beta_{t^*s^*}(y)\delta_e)(h) = 0$. As

$$\rho\big(\beta_s\big(\beta_{s^*}(u_\lambda)u_\gamma\big)\delta_{st}\big)(h) = \rho(xy\delta_{st})(h) = \rho(x\delta_{st})\rho\big(\beta_{t^*s^*}(y)\delta_e\big)(h) = 0,$$

taking $\{u_\lambda\} \subset E_s$ and $\{u_\omega\} \subset E_{s^*}$ their approximate identities, it follows that:

$$\nu_s\nu_t(h) = \lim_\lambda \rho(u_\lambda\delta_s)\lim_\omega \rho(u_\omega\delta_t)(h) = \lim_{\lambda,\omega} \rho\big(\beta_s\big(\beta_{s^*}(u_\lambda)u_\omega\big)\delta_{st}\big)(h) = 0.$$

So $\nu_{st} = \nu_s\nu_t$ and ν is a homomorphism.

Finally, we need to prove the covariance condition, that is, that $\nu_s\pi(a)\nu_{s^*} = \pi(\beta_s(a))$. Let $a \in E_{s^*}$ and $\{u_\lambda\}, \{u_\gamma\}$ be approximate identities of E_s. Then:

$$\begin{aligned}\nu_s\pi(a)\nu_{s^*} &= \lim_\lambda \rho(u_\lambda\delta_s)\rho(a\delta_e)\lim_\gamma \rho\big(\beta_{s^*}(u_\gamma)\delta_{s^*}\big) = \lim_{\lambda,\gamma} \rho(u_\lambda\delta_s)\rho\big(a\beta_{s^*}(u_\gamma)\delta_{s^*}\big)\\ &= \lim_{\lambda,\gamma} \rho\big(\beta_s\big(\beta_{s^*}(u_\lambda)a\beta_{s^*}(u_\gamma)\big)\delta_{ss^*}\big) = \lim_{\lambda,\gamma} \rho\big(u_\lambda\beta_s(a)u_\gamma\delta_{ss^*}\big)\\ &= \rho\big(\beta_s(a)\delta_{ss^*}\big) = \rho\big(\beta_s(a)\delta_e\big) = \pi\big(\beta_s(a)\big),\end{aligned}$$

because $\rho|_N \equiv 0$. □

By the previous theorem we have, for $x \in L$:

$$\begin{aligned}\|x\|_c &= \sup\big\{\big\|(\pi \times \nu)(x)\big\|:\ (\pi, \nu, H) \in \mathrm{CovRep}(A, S, \beta)\big\}\\ &= \sup\big\{\big\|\rho(x)\big\|:\ \rho \text{ representation of } L \text{ equals zero in } N\big\}\\ &= \sup\big\{\big\|\rho(x)\big\|:\ \rho \text{ representation of } A \times_\beta^a S\big\} = \rho_s(x).\end{aligned}$$

So we conclude that:

Theorem 5.6. *The definition of crossed product by an action of inverse semigroup in a C^*-algebra that we present is equivalent to that introduced by Sieben in* [5].

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