J. Math. Anal. Appl. 365 (2010) 120–134

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

L^p estimates for Feynman–Kac propagators with time-dependent reference measures ☆

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ARTICLE INFO

Article history:
Received 27 May 2009
Available online 13 October 2009
Submitted by M. Ledoux

Keywords:
Time-inhomogeneous Markov processes
Feynman–Kac formula
Dirichlet forms
Poincaré inequalities
Logarithmic Sobolev inequalities
Markov semigroups
L^p estimates
Markov Chain Monte Carlo
Sequential Monte Carlo
Importance sampling

ABSTRACT

We introduce a class of time-inhomogeneous transition operators of Feynman–Kac type that can be considered as a generalization of symmetric Markov semigroups to the case of a time-dependent reference measure. Applying weighted Poincaré and logarithmic Sobolev inequalities, we derive $L^p \to L^p$ and $L^p \to L^q$ estimates for the transition operators. Since the operators are not Markovian, the estimates depend crucially on the value of p. Our studies are motivated by applications to sequential Markov Chain Monte Carlo methods.

1. Introduction

The purpose of this work is to derive $L^p \to L^p$ and $L^p \to L^q$ bounds for a class of non-Markovian time inhomogeneous transition operators $q_{s,t}$. These Feynman–Kac type transition operators play a rôle in the analysis of sequential MCMC methods, see [5,11]. In the time-homogeneous case, the class of operators considered here is precisely that of transition functions of symmetric Markov processes.

In general, let

$$\mu_t(x) = \frac{1}{Z_t} \exp\bigl(-\mathcal{H}_t(x)\bigr)\mu_0(x), \quad t \geqslant 0,$$

denote a family of mutually absolutely continuous probability measures on a finite set S. Here Z_t is a normalization constant, and $(t,x) \mapsto \mathcal{H}_t(x)$ is a given function on $[0,\infty) \times S$ that is continuously differentiable in the first variable. For instance, if $\mathcal{H}_t(x) = t\mathcal{H}(x)$ for some function $\mathcal{H} : S \to \mathbb{R}$, then $(\mu_t)_{t \geqslant 0}$ is the exponential family corresponding to $\mathcal{H}$ and μ_0. We assume that S is finite to keep the presentation as simple and non-technical as possible, although most results of this paper extend to continuous state spaces under standard regularity assumptions.

☆ We thank the referee for careful reading and very valuable suggestions on the first version of this paper. The work for this paper was carried out while the second-named author was visiting the Department of Statistics of Purdue University supported by a MOIF fellowship. Both authors were partially supported by the Sonderforschungsbereich 611, Bonn.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.10.019

Note that if $\mathcal{H}_t \equiv 0$ for all $t \geqslant 0$, then $\mu_t = \mu_0$ for all $t \geqslant 0$. In this case, the measures are invariant for a Markov transition semigroup $(p_t)_{t\geqslant 0}$, i.e.

$$p^*_{t-s}\mu_s = \mu_t, \quad \forall t \geqslant s \geqslant 0,$$

if, for example, the generator satisfies a detailed balance condition w.r.t. μ_0. Here p^*_{t-s} stands for the adjoint of the matrix p_{t-s}, i.e.

$$\big(p^*_{t-s}\mu_s\big)(y) := (\mu_s p_{t-s})(y) = \sum_{x\in S} \mu_s(x) p_{t-s}(x, y).$$

It is well known that in this time-homogeneous case, L^p and $L^p \to L^q$ bounds for the transition operators p_t follow from Poincaré inequalities (i.e. spectral gap estimates) and logarithmic Sobolev inequalities w.r.t. the measure μ_0, respectively. We refer to [18] and references therein for more background and results on corresponding bounds for time-homogeneous Markov chains (see also [1–3,8,10,16]). Such bounds are exploited in the mathematical analysis of Markov Chain Monte Carlo (MCMC) methods for approximating expectation values w.r.t. the measure μ_0, see e.g. [7,9,17] and references therein, as well as the above references.

We now introduce the class of non-Markovian, time-inhomogeneous transition operators for which we will prove corresponding L^p and $L^p \to L^q$ bounds. Let $\mathcal{L}_t$, $t \geqslant 0$, be generators (Q-matrices) of Markov processes on S satisfying the detailed balance conditions

$$\mu_t(x)\mathcal{L}_t(x, y) = \mu_t(y)\mathcal{L}_t(y, x), \quad \forall t \geqslant 0,\ x, y \in S. \tag{1}$$

In particular, $\mathcal{L}^*_t \mu_t = 0$, i.e.,

$$\int \mathcal{L}_t f \, d\mu_t = \sum_{x\in S} (\mathcal{L}_t f)(x)\mu_t(x) = 0 \quad \text{for all } f : S \to \mathbb{R} \text{ and } t \geqslant 0, \tag{2}$$

where

$$(\mathcal{L}_t f)(x) := \sum_{y\in S} \mathcal{L}_t(x, y) f(y).$$

We assume that $\mathcal{L}_t(x, y)$ depends continuously on t, and we fix a continuous positive function $t \mapsto \lambda_t$. For $0 \leqslant s \leqslant t < \infty$, let $q_{s,t}(x, y)$, $x, y \in S$, denote the solutions of the backward equations

$$-\frac{\partial}{\partial s} q_{s,t}(x, y) = \lambda_s(\mathcal{L}_s q_{s,t})(x, y) - H_s(x) q_{s,t}(x, y), \quad s \in [0, t],$$

with terminal condition $q_{t,t}(x, y) = \delta_{x,y}$, where

$$H_t(x) := -\frac{\partial}{\partial t} \log \mu_t(x) = \frac{\partial}{\partial t}\mathcal{H}_t - \int \frac{\partial}{\partial t}\mathcal{H}_t \, d\mu_t$$

denotes the negative logarithmic time derivative of the measures μ_t. Since the state space is finite, the solutions are unique. For $f : S \to \mathbb{R}$, $q_{s,t} f$ satisfies the backward equation

$$-\frac{\partial}{\partial s} q_{s,t} f = \lambda_s \mathcal{L}_s q_{s,t} f - H_s q_{s,t} f, \quad s \in [0, t], \tag{3}$$

with terminal condition $q_{t,t} f = f$.

As a consequence of the detailed balance condition (1) and the backward equation (3), it is not difficult to verify that the invariance property

$$q^*_{s,t}\mu_s = \mu_t \tag{4}$$

holds for all $t \geqslant s \geqslant 0$, see Proposition 2 below.

Moreover, it can be shown that $q_{s,t} f$ is also the unique solution of the corresponding forward equation

$$\frac{\partial}{\partial t} q_{s,t} f = q_{s,t}(\lambda_t \mathcal{L}_t f - H_t f), \quad t \in [s, \infty), \tag{5}$$

with initial condition $q_{s,s} f = f$. As a consequence, a probabilistic representation of $q_{s,t}$ is given by the Feynman–Kac formula

$$(q_{s,t} f)(x) = \mathbb{E}_{s,x}\big[e^{-\int_s^t H_r(X_r)\,dr} f(X_t)\big] \quad \text{for all } x \in S, \tag{6}$$

where $(X_t)_{t\geqslant s}$ is a time-inhomogeneous Markov process w.r.t. $\mathbb{P}_{s,x}$ with generators $\lambda_t \mathcal{L}_t$ and initial condition $X_s = x$ $\mathbb{P}_{s,x}$-a.s., see e.g. [12,14]. Let

$$(p_{s,t}f)(x) = \mathbb{E}_{s,x}\big[f(X_t)\big] \tag{7}$$

denote the transition operators of this process.

In Theorems 6, 8 and 11, and Corollary 10 below, we derive L^p and $L^p \to L^q$ bounds for the non-Markovian operators $q_{s,t}$. This is partially similar to the case of time-homogeneous Markov semigroups, but some important differences occur. In particular, since the operators $q_{s,t}$ in general are not contractions on L^∞, the resulting L^p bounds depend crucially on the value of p.

Remark 1. (i) The non-Markovian transition operators $q_{s,t}$ arise naturally in the analysis of sequential Markov Chain Monte Carlo methods. For a detailed description of sequential MCMC methods and related stochastic processes we refer to [5,11].

(ii) Evolution operators such as $q_{s,t}$, also in continuous time and space, have been investigated intensively (see e.g. the monograph [14] and references therein). However, they are usually considered in L^p spaces with respect to a *fixed* reference measure. In the applications to sequential MCMC methods we are interested in, the time-varying measures μ_t are given a priori, and the analysis on the corresponding L^p spaces is crucial. Moreover, a setup with time-varying reference measure is more natural in many respects. In particular it provides a generalization of the L^p theory of symmetric Markov semigroups.

(iii) There are several generalizations of symmetric time-homogeneous Markov semigroups to the case of time-dependent reference measures. One possibility is to consider time-inhomogeneous Markov semigroups with infinitesimal generators satisfying (1). However, these semigroups usually do not satisfy the invariance property (4). Alternatively, there exist time-inhomogeneous Markov processes with transition semigroup satisfying (4). However, the corresponding generators depend on H_t in a non-local way, since increasing the mass at one point and decreasing the mass at another point requires an additional drift of the process between the points. This illustrated in the example below. The third possibility, that we consider here, is to replace the Markov semigroup by a Feynman–Kac semigroup as defined above. Although these semigroups do not correspond to a classical Markov process, they can be approximated by a stochastic approach combining Markov Chain Monte Carlo and importance sampling concepts, cf. [5,11].

Example. To illustrate our setup and, in particular, the last remark, we consider a simple situation where the weights of the underlying measure μ_t vary only at two points: suppose that $S = \{0, 1, 2, \dots, n\}$ for some $n \in \mathbb{N}$, μ_0 is the uniform distribution on S, and

$$\mu_t(i) = \mu_0(i) = \frac{1}{n+1}$$

for all $t \geqslant 0$ and $i = 1, 2, \dots, n-1$. Hence only the weights $\mu_t(0)$ and $\mu_t(n)$ are not constant in t and $\frac{d}{dt}\mu_t(n) = -\frac{d}{dt}\mu_t(0)$, i.e. mass is transferred from 0 to n and viceversa. We describe three types of transition operators satisfying the invariance property (4) in this situation.

(i) Suppose that (in contrast to our setup above) $q_{s,t}$ are the transition functions of an ordinary time-inhomogeneous Markov process on S with generators $\mathcal{L}_t$ satisfying $\mathcal{L}_t(x,y) = 0$ whenever $|x-y| > 1$, i.e. the process only jumps to neighbor sites. An elementary computation based on the forward equation shows that in this case the invariance property (4) holds for all $0 \leqslant s \leqslant t$ if and only if

$$\mathcal{L}_t(y-1, y) - \mathcal{L}_t(y, y-1) = (n+1)\frac{d}{dt}\mu_t(0)$$

for all $y \in \{1, 2, \dots, n\}$ and $t \geqslant 0$. Hence to compensate for the change of measure at two points, a global drift growing linearly with the distance of the two points is required. This is inconvenient for the numerical applications we are interested in, see [11].

(ii) A second possibility (consistent with our setup) would be to choose

$$q_{s,t}(x,y) = \frac{\mu_t(x)}{\mu_s(x)}\delta_{x,y}.$$

This corresponds to the case $\lambda_t = 0$ for all $t \geqslant 0$ in the framework introduced above, i.e. the underlying Markov process does not move at all. In this case L^p bounds for the operators $q_{s,t}$ depend on the L^∞ norm of the relative density $\mu_t(x)/\mu_s(x)$, which is also inconvenient for the applications we are interested in.

(iii) A third possibility (again consistent with our setup above) is to choose for $\mathcal{L}_t$ the generator of a Random Walk Metropolis Chain with respect to the measure μ_t, i.e.

$$\mathcal{L}_t(x,y) = \begin{cases} \frac{1}{2}\min\big(\frac{\mu_t(x)}{\mu_s(x)}, 1\big), & \text{if } |x-y| = 1, \\ 0, & \text{if } |x-y| > 1, \end{cases} \tag{8}$$

and to define $q_{s,t}$ by (3). Our first main result, Theorem 6 below, shows that in this case for $p \geqslant 2$, the L^p bound

$$\|q_{s,t} f\|_{L^p(\mu_s)} \leqslant 2^{1/4} \|f\|_{L^p(\mu_t)}$$

holds for all $0 \leqslant s \leqslant t$ and $f : S \to \mathbb{R}$ provided λ_t is large enough.

The remaining content of the paper is organized as follows: in Section 2 we collect some properties of the propagators $q_{s,t}$ that are frequently used in subsequent sections. Sections 3 and 4 deal with L^p bounds under the assumption that global Poincaré inequalities hold. In Section 5 we apply the results to derive L^p estimates on a subset that is invariant w.r.t. the underlying dynamics from Poincaré inequalities on this subset. Finally, in Section 6 we prove an $L^p \to L^q$ estimate assuming that a (time-dependent) logarithmic Sobolev inequality is satisfied.

2. Preliminaries and notation

We shall denote throughout the paper the expectation value of a function $f : S \to \mathbb{R}$ with respect to a measure ν on S by

$$\langle f, \nu \rangle := \int f \, d\nu = \sum_{x \in S} f(x) \nu(x).$$

The positive and negative part of a function f are defined, respectively, by

$$f^+ = \max(f, 0), \qquad f^- = \max(-f, 0),$$

so that $f = f^+ - f^-$.

For $t \geqslant 0$ and $f, g : S \to \mathbb{R}$, the Dirichlet form $\mathcal{E}_t(f, g)$ corresponding to the self-adjoint operator $\mathcal{L}_t$ on $L^2(S, \mu_t)$ is given by

$$\mathcal{E}_t(f, g) = -\int f \mathcal{L}_t g \, d\mu_t = \frac{1}{2} \sum_{x, y \in S} \big(f(y) - f(x)\big)\big(g(y) - g(x)\big) \mathcal{L}_t(x, y) \mu_t(x). \tag{9}$$

We shall also use the shorthand notation $\mathcal{E}_t(f) = \mathcal{E}_t(f, f)$.

Note that by the definition of H_t one immediately has

$$\mu_t(x) = \exp\left(-\int_0^t H_s(x) \, ds\right) \mu_0(x), \tag{10}$$

and

$$\langle H_t, \mu_t \rangle = 0, \quad \forall t \geqslant 0. \tag{11}$$

In fact, since $\mu_t(S) = 1$ for all $t \geqslant 0$, one has

$$\langle H_t, \mu_t \rangle = -\sum_{x \in S} \mu_t(x) \frac{\partial}{\partial t} \log \mu_t(x) = -\frac{\partial}{\partial t} \sum_{x \in S} \mu_t(x) = 0.$$

As a consequence of (11),

$$H_t = \frac{\partial}{\partial t} \mathcal{H}_t - \left\langle \frac{\partial}{\partial t} \mathcal{H}_t, \mu_t \right\rangle.$$

In the following proposition we collect some properties of the operators $q_{s,t}$ which will be used throughout the paper.

Proposition 2. *For all* $0 \leqslant s \leqslant t \leqslant u$,

(i) $q_{s,t} q_{t,u} = q_{s,u}$ (*Chapman–Kolmogorov equation*).
(ii) *If* $f \geqslant 0$, *then* $q_{s,t} f \geqslant 0$ (*positivity preserving property*).
(iii) *If* $f \geqslant 0$, *then* $q_{s,t} f \leqslant \exp(\int_s^t \max_{x \in S} H_r^-(x) \, dr) p_{s,t} f$ (*pointwise estimate*).
(iv) $q_{s,t}^* \mu_s = \mu_t$ (*invariance*).
(v) $\|q_{s,t} f\|_{L^1(\mu_s)} \leqslant \|f\|_{L^1(\mu_t)}$ (L^1 *bound*).
(vi) $\|q_{s,t} f\|_{L^p(\mu_s)} \leqslant \exp(\frac{p-1}{p} \int_s^t \max_{x \in S} H_r^-(x) \, dr) \|f\|_{L^p(\mu_t)}$ (*rough* L^p *bound*).

Proof. (i) is a consequence of the Markov property of the process $(X_t, \mathbb{P}_{s,x})$, and (ii), (iii) are immediate by the Feynman–Kac representation (6). Similarly, (iv) is an elementary consequence of (10), (3), and (2), which imply

$$\frac{\partial}{\partial s}\langle q_{s,t} f, \mu_s\rangle = -\langle H_s q_{s,t} f, \mu_s\rangle - \lambda_s\langle \mathcal{L}_s q_{s,t} f, \mu_s\rangle + \langle H_s q_{s,t} f, \mu_s\rangle = 0,$$

and hence $\langle q_{s,t} f, \mu_s\rangle = \langle q_{t,t} f, \mu_t\rangle = \langle f, \mu_t\rangle$ for all $f : S \to \mathbb{R}$ and $s \in [0,t]$.

In order to prove (v), take $f \geqslant 0$. Then, by (ii) and (iv),

$$\|q_{s,t} f\|_{L^1(\mu_s)} = \langle q_{s,t} f, \mu_s\rangle = \langle f, \mu_s q_{s,t}\rangle = \langle f, \mu_t\rangle = \|f\|_{L^1(\mu_t)}.$$

The general case follows by the decomposition $f = f^+ - f^-$ with $f^+, f^- \geqslant 0$, and using the linearity of $q_{s,t}$.

Finally, (vi) is a consequence of (iii) when $p = \infty$, and of (v) when $p = 1$. The assertion for general $p \in [1,\infty]$ then follows by the Riesz–Thorin interpolation theorem, see e.g. [4, §1.1.5]. □

3. Global L^p estimates

Setting

$$\mathcal{K}_t := \left\{ f : S \to \mathbb{R}\colon \int f\, d\mu_t = 0,\ f \not\equiv 0 \right\},$$

let

$$C_t := \sup_{f \in \mathcal{K}_t} \frac{1}{\mathcal{E}_t(f)} \int f^2\, d\mu_t$$

denote the (possibly infinite) inverse spectral gap of $\mathcal{L}_t$, and define

$$A_t := \sup_{f \in \mathcal{K}_t} \frac{1}{\mathcal{E}_t(f)} \int (-H_t) f^2\, d\mu_t, \qquad B_t := \sup_{f \in \mathcal{K}_t} \frac{1}{\mathcal{E}_t(f)} \left| \int_S H_t f\, d\mu_t \right|^2.$$

Thus C_t, A_t and B_t are the optimal constants in the global Poincaré inequalities

$$\mathrm{Var}_{\mu_t}(f) \leqslant C_t \cdot \mathcal{E}_t(f), \quad \forall f : S \to \mathbb{R}, \tag{12}$$

$$-\int H_t \left(f - \int f\, d\mu_t \right)^2 d\mu_t \leqslant A_t \cdot \mathcal{E}_t(f), \quad \forall f : S \to \mathbb{R}, \tag{13}$$

$$\left| \int H_t f\, d\mu_t \right|^2 \leqslant B_t \cdot \mathcal{E}_t(f), \quad \forall f : S \to \mathbb{R}, \tag{14}$$

where Var_{μ_t} denotes the variance w.r.t. μ_t.

Our aim in this section is to bound the $L^p \to L^p$ norms of the operators $q_{s,t}$ in terms of the constants A_t, B_t and C_t.

Remark 3. (i) There exist efficient techniques to obtain upper bounds for C_t, for example the method of canonical paths, comparison methods (see e.g. [18]), as well as decomposition methods (see e.g. [15]). Variants of these techniques can be applied to estimate A_t and B_t as well.

(ii) Clearly, one has

$$A_t \leqslant C_t \cdot \max_{x \in S} H_t^-(x), \tag{15}$$

$$B_t \leqslant C_t \cdot \mathrm{Var}_{\mu_t}(H_t), \tag{16}$$

so an upper bound on C_t yields upper bounds on A_t and B_t.

Example (continued). In the situation of example (iii) above, suppose that

$$\left|H_t(x)\right| = \frac{|\frac{d}{dt}\mu_t(x)|}{\mu_t(x)} \leqslant 1$$

for all $x \in S$. Then one can prove the upper bounds

$$A_t \leqslant 4(n+1), \qquad B_t \leqslant 8(n+1)$$

for all $t \geqslant 0$ (see Appendix A). On the other hand, in this case the inverse spectral gap C_t is of order n^2.

Let us start with a basic estimate.

Lemma 4. *For all $s \geqslant 0$ and $f : S \to \mathbb{R}$, one has*

$$-\int H_s f^2 \, d\mu_s \leqslant A_s \mathcal{E}_s(f) + 2B_s^{1/2} \big|\langle f, \mu_s\rangle\big| \mathcal{E}_s(f)^{1/2}. \tag{17}$$

Proof. Set $\bar{f}_s = f - \langle f, \mu_s\rangle$. Then, observing that $\langle H_s, \mu_s\rangle = 0$ and $\mathcal{E}_s(\bar{f}_s) = \mathcal{E}_s(f)$, (13) and (14) imply

$$\begin{aligned} -\int H_s f^2 \, d\mu_s &= -\int H_s\big(\bar{f}_s^2 + \langle f, \mu_s\rangle^2 + 2\bar{f}_s \langle f, \mu_s\rangle\big) \, d\mu_s \\ &\leqslant A_s \mathcal{E}_s(f) + 2\big|\langle f, \mu_s\rangle\big| B_s^{1/2} \mathcal{E}_s(f)^{1/2}, \end{aligned}$$

which proves the claim. □

In the following proposition we establish an integral inequality for the $L^p(\mu_s)$ norm of $q_{s,t} f$.

Proposition 5. *Let $p \geqslant 2$ and assume that*

$$\lambda_s > pA_s/4, \quad \forall s \in [0, t].$$

Then for all $s \in [0, t]$ and for all $f : S \to \mathbb{R}$,

$$\big\langle |q_{s,t} f|^p, \mu_s\big\rangle \leqslant \big\langle |f|^p, \mu_t\big\rangle + p(p-1) \int_s^t \frac{B_r}{4\lambda_r - pA_r} \big\langle \big(q_{r,t}|f|\big)^{p/2}, \mu_r\big\rangle^2 \, dr. \tag{18}$$

Proof. Recalling (10), the backward equation (3) allows us to write, for $f : S \to \mathbb{R}_+$ and $r \in [0, t]$,

$$\begin{aligned} -\frac{\partial}{\partial r} \int (q_{r,t} f)^p \, d\mu_r &= p \int (q_{r,t} f)^{p-1} (\lambda_r \mathcal{L}_r q_{r,t} f - H_r q_{r,t} f) \, d\mu_r + \int H_r (q_{r,t} f)^p \, d\mu_r \\ &= -p\lambda_r \mathcal{E}_r\big(q_{r,t} f, (q_{r,t} f)^{p-1}\big) - (p-1) \int H_r (q_{r,t} f)^p \, d\mu_r, \end{aligned}$$

where we have used the definition of the Dirichlet form $\mathcal{E}_r$ in the second step. Applying the inequality

$$\mathcal{E}_r\big(\phi, \phi^{p-1}\big) \geqslant \frac{4(p-1)}{p^2} \mathcal{E}_r\big(\phi^{p/2}\big), \quad \forall \phi : S \to \mathbb{R}^+ \tag{19}$$

(see e.g. [6, p. 242]), we obtain

$$-\frac{\partial}{\partial r} \int (q_{r,t} f)^p \, d\mu_r \leqslant -\frac{4(p-1)}{p} \lambda_r \mathcal{E}_r\big((q_{r,t} f)^{p/2}, (q_{r,t} f)^{p/2}\big) - (p-1) \int H_r (q_{r,t} f)^p \, d\mu_r.$$

Estimate (17) combined with the previous inequality yields

$$-\frac{\partial}{\partial r} \int (q_{r,t} f)^p \, d\mu_r \leqslant -\frac{(p-1)}{p} (4\lambda_r - pA_r) \mathcal{E}_r\big((q_{r,t} f)^{p/2}\big) + 2(p-1) B_r^{1/2} \big|\big\langle (q_{r,t} f)^{p/2}, \mu_r\big\rangle\big| \mathcal{E}_r\big((q_{r,t} f)^{p/2}\big)^{1/2},$$

hence also, using the elementary inequality $-ax + 2bx^{1/2} \leqslant b^2/a$, where $a, b, x \geqslant 0$,

$$-\frac{\partial}{\partial r} \int (q_{r,t} f)^p \, d\mu_r \leqslant \frac{p(p-1)B_r}{4\lambda_r - pA_r} \big\langle (q_{r,t} f)^{p/2}, \mu_r\big\rangle^2.$$

Integrating this inequality from s to t with respect to r we get, recalling that $q_{t,t} f = f$,

$$\big\langle (q_{s,t} f)^p, \mu_s\big\rangle \leqslant \big\langle f^p, \mu_t\big\rangle + p(p-1) \int_s^t \frac{B_r}{4\lambda_r - pA_r} \big\langle (q_{r,t} f)^{p/2}, \mu_r\big\rangle^2 \, dr.$$

Since $|q_{s,t} f| \leqslant q_{s,t}|f|$, the claim is obtained applying the above inequality to the positive function $|f|$. □

We can now prove our first main result.

Theorem 6. *Let $t \geqslant 0$ and $p \geqslant 2$. Assume that*

$$\lambda_s \geqslant \frac{p}{4} A_s + \frac{p(p+3)}{4} t B_s \tag{20}$$

for all $s \in [0,t]$. Then we have, for all $s \in [0,t]$ and $f : S \to \mathbb{R}$,

(i) $\|q_{s,t} f\|_{L^p(\mu_s)} \leqslant 2^{1/4} \|f\|_{L^p(\mu_t)}$;
(ii) $\|q_{s,t} f\|_{L^p(\mu_s)} \leqslant \|f\|_{L^p(\mu_t)} + 2^{1/4} \|f\|_{L^{p/2}(\mu_t)}$.

Proof. By Proposition 2(v) we have that (i) always holds for $p = 1$. Let us now prove that, for $p \geqslant 2$, (i) holds provided (20) is satisfied and (i) holds with p replaced by $p/2$. In fact, in this case we have

$$\big\langle (q_{r,t}|f|)^{p/2}, \mu_r \big\rangle \leqslant 2^{p/8} \big\langle |f|^{p/4}, \mu_t \big\rangle^2 \leqslant 2^{p/8} \big\langle |f|^{p/2}, \mu_t \big\rangle, \quad \forall r \in [0,t].$$

Then (18) implies

$$\big\langle |q_{s,t} f|^p, \mu_s \big\rangle \leqslant \big\langle |f|^p, \mu_t \big\rangle + 2^{p/4} p(p-1) \int_s^t \frac{B_r}{4\lambda_r - pA_r} \big\langle |f|^{p/2}, \mu_t \big\rangle^2 \, dr.$$

A simple calculation shows that, by (20),

$$\int_s^t \frac{pB_r}{4\lambda_r - pA_r} \, dr \leqslant \frac{1}{p+3}. \tag{21}$$

Therefore, by the elementary inequality

$$2^{-p/4} + \frac{p-1}{p+3} \leqslant 1, \quad \forall p \geqslant 2,$$

we obtain

$$\begin{aligned} 2^{-p/4} \big\langle |q_{s,t} f|^p, \mu_s \big\rangle &\leqslant 2^{-p/4} \big\langle |f|^p, \mu_t \big\rangle + \frac{p-1}{p+3} \big\langle |f|^{p/2}, \mu_t \big\rangle^2 \\ &\leqslant 2^{-p/4} \big\langle |f|^p, \mu_t \big\rangle + \frac{p-1}{p+3} \big\langle |f|^p, \mu_t \big\rangle \\ &\leqslant \big\langle |f|^p, \mu_t \big\rangle, \end{aligned}$$

thus proving our claim.

If (20) is satisfied for $p = 2$, then (i) holds with $p = 1$, hence with $p = 2$. The Riesz–Thorin interpolation theorem then allows to conclude that (i) holds for all $p \in [1,2]$.

If (20) is satisfied for some $p \geqslant 2$, then it also holds for all smaller values of p, including $p = 2$. Choosing $n \in \mathbb{N}$ such that $\tilde{p} := p2^{-n} \in [1,2]$, we conclude that (i) holds for $\tilde{p}$, hence by induction it also holds for $p = \tilde{p}2^n$. The proof of (i) is thus complete.

Let us now prove (ii): by (18), (21) and (i) we have, noting that $(p-1)/(p+3) \leqslant 1$ for all $p \geqslant 2$,

$$\begin{aligned} \big\langle |q_{s,t} f|^p, \mu_s \big\rangle &\leqslant \big\langle |f|^p, \mu_t \big\rangle + p(p-1) \int_s^t \frac{B_r}{4\lambda_r - pA_r} \, dr \sup_{r \in [s,t]} \big\langle (q_{r,t}|f|)^{p/2}, \mu_r \big\rangle^2 \\ &\leqslant \big\langle |f|^p, \mu_t \big\rangle + \frac{p-1}{p+3} \sup_{r \in [s,t]} \big\langle (q_{r,t}|f|)^{p/2}, \mu_r \big\rangle^2 \\ &\leqslant \big\langle |f|^p, \mu_t \big\rangle + 2^{p/4} \big\langle |f|^{p/2}, \mu_t \big\rangle^2, \end{aligned}$$

which implies (ii), in view of the elementary inequality $(a+b)^{1/p} \leqslant a^{1/p} + b^{1/p}$, with $a, b \geqslant 0$. $\square$

Example (continued). In the situation of example (iii) above, provided the assumption on H_t made above is satisfied, condition (20) is fulfilled if

$$\lambda_s \geqslant \big(1 + 2(p+3)t\big) p(n+1).$$

4. Improved L^p bounds for functions with μ_t mean zero

It is well known (see e.g. [18]) that for a time-homogeneous reversible Markov semigroup p_t with stationary distribution μ, one has

$$\|p_t f\|_{L^2(\mu)} \leqslant e^{-t/C}\|f\|_{L^2(\mu)}$$

for all $f : S \to \mathbb{R}$ with $\langle f, \mu\rangle = 0$, where the exponential decay rate C is the inverse spectral gap. Similar bounds hold for other L^p norms with $p \in [2, \infty)$. The purpose of this section is to prove related bounds for $q_{s,t} f$ if $\langle f, \mu_t\rangle = 0$.

Let us start with a proposition, which can be seen as an analogue of Proposition 5, asserting that an exponential decay of order γ of $\langle |q_{s,t} f|^{p/2}, \mu_s\rangle$ implies exponential decay of the same order for $\langle |q_{s,t} f|^p, \mu_s\rangle$.

Proposition 7. *Let $p \geqslant 2$ and $\gamma \geqslant 0$, and assume that*

$$\lambda_s \geqslant \frac{p}{4} A_s + \kappa \frac{p(p-1)}{4} B_s + \frac{\gamma}{2} C_s, \quad \forall s \in [0, t] \tag{22}$$

for some $\kappa > 0$. Then we have, for all $s \in [0, t]$ and $f : S \to \mathbb{R}$,

$$\left\langle |q_{s,t} f|^2, \mu_s\right\rangle \leqslant e^{-\gamma(t-s)}\left\langle |f|^2, \mu_t\right\rangle + \left(1 + \frac{1}{\kappa\gamma}\right)\left(1 - e^{-\gamma(t-s)}\right)\langle f, \mu_t\rangle^2 \tag{23}$$

and

$$\left\langle |q_{s,t} f|^p, \mu_s\right\rangle \leqslant e^{-\gamma(t-s)}\left(\left\langle |f|^p, \mu_t\right\rangle + \left(\frac{1}{\kappa} + \gamma\right)\int_s^t e^{\gamma(t-r)}\left\langle |q_{r,t} f|^{p/2}, \mu_r\right\rangle^2 dr\right). \tag{24}$$

Proof. By (3), (10), and the definition of $\mathcal{E}_t$, we obtain, similarly as above,

$$-\frac{\partial}{\partial s} e^{\gamma(t-s)} \int |q_{s,t} f|^2 \, d\mu_s = -2e^{\gamma(t-s)}\lambda_s \mathcal{E}_s(q_{s,t} f) - e^{\gamma(t-s)} \int H_s \cdot (q_{s,t} f)^2 \, d\mu_s + \gamma e^{\gamma(t-s)} \int (q_{s,t} f)^2 \, d\mu_s. \tag{25}$$

By the Poincaré inequality (12), we have

$$\int (q_{s,t} f)^2 \, d\mu_s \leqslant C_s \cdot \mathcal{E}_s(q_{s,t} f) + \langle q_{s,t} f, \mu_s\rangle^2.$$

Moreover, by (17),

$$-\int H_s (q_{s,t} f)^2 \, d\mu_s \leqslant A_s \mathcal{E}_s(q_{s,t} f) + 2B_s^{1/2}\left|\langle q_{s,t} f, \mu_s\rangle\right| \cdot \mathcal{E}_s(q_{s,t} f)^{1/2},$$

hence, by (22),

$$\begin{aligned}
-\frac{\partial}{\partial s} e^{\gamma(t-s)} \int |q_{s,t} f|^2 \, d\mu_s &\leqslant -e^{\gamma(t-s)}(2\lambda_s - \gamma C_s - A_s)\mathcal{E}_s(q_{s,t} f) + 2e^{\gamma(t-s)} B_s^{1/2}\left|\langle q_{s,t} f, \mu_s\rangle\right|\mathcal{E}_s(q_{s,t} f)^{1/2} \\
&\quad + \gamma e^{\gamma(t-s)}\langle q_{s,t} f, \mu_s\rangle^2 \\
&\leqslant \frac{B_s}{2\lambda_s - \gamma C_s - A_s} e^{\gamma(t-s)}\langle q_{s,t} f, \mu_s\rangle^2 + \gamma e^{\gamma(t-s)}\langle q_{s,t} f, \mu_s\rangle^2 \\
&\leqslant \left(\frac{1}{\kappa} + \gamma\right) e^{\gamma(t-s)}\langle f, \mu_t\rangle^2.
\end{aligned}$$

Here we have used that $\langle q_{s,t} f, \mu_s\rangle = \langle f, \mu_t\rangle$ as in Proposition 2. We obtain (23) integrating the previous inequality with respect to s.

Let us now prove (24): appealing again to (3) and (10), we obtain, in analogy to the derivation of (25),

$$\begin{aligned}
-\frac{\partial}{\partial s} e^{\gamma(t-s)} \int |q_{s,t} f|^p \, d\mu_s &= -pe^{\gamma(t-s)}\lambda_s \mathcal{E}_s\left(|q_{s,t} f|^{p-1}\operatorname{sgn}(q_{s,t} f), q_{s,t} f\right) - (p-1)e^{\gamma(t-s)} \int H_s |q_{s,t} f|^p \, d\mu_s \\
&\quad + \gamma e^{\gamma(t-s)} \int |q_{s,t} f|^p \, d\mu_s.
\end{aligned}$$

Since for all $\phi : S \to \mathbb{R}$ and $x, y \in S$,

$$\left(\phi(x) - \phi(y)\right)\left(\left|\phi(x)\right|^{p-1}\operatorname{sgn}\phi(x) - \left|\phi(y)\right|^{p-1}\operatorname{sgn}\phi(y)\right) \geqslant \left(\left|\phi(x)\right| - \left|\phi(y)\right|\right)\left(\left|\phi(x)\right|^{p-1} - \left|\phi(y)\right|^{p-1}\right),$$

taking into account that the off-diagonal terms of $\mathcal{L}_s(x, y)$ are nonnegative, we obtain by (9) that

$$\mathcal{E}_s\big(|q_{s,t}f|^{p-1}\operatorname{sgn}(q_{s,t}f), q_{s,t}f\big) \geqslant \mathcal{E}_s\big(|q_{s,t}f|^{p-1}, |q_{s,t}f|\big),$$

hence also, thanks to (19),

$$\mathcal{E}_s\big(|q_{s,t}f|^{p-1}\operatorname{sgn}(q_{s,t}f), q_{s,t}f\big) \geqslant \frac{4(p-1)}{p^2}\mathcal{E}_s\big(|q_{s,t}f|^{p/2}\big).$$

Proceeding now as in the proof of Proposition 5 we get

$$\begin{aligned}
-\frac{\partial}{\partial s}e^{\gamma(t-s)}\int |q_{s,t}f|^p\,d\mu_s &\leqslant -e^{\gamma(t-s)}\left(\frac{4(p-1)}{p}\lambda_s - \gamma C_s - (p-1)A_s\right)\mathcal{E}_s\big(|q_{s,t}f|^{p/2}\big) \\
&\quad + 2(p-1)e^{\gamma(t-s)}B_s^{1/2}\big\langle|q_{s,t}f|^{p/2}, \mu_s\big\rangle\mathcal{E}_s\big(|q_{s,t}f|^{p/2}\big)^{1/2} + \gamma e^{\gamma(t-s)}\big\langle|q_{s,t}f|^{p/2}, \mu_s\big\rangle^2 \\
&\leqslant \frac{(p-1)^2B_s}{4\lambda_s(p-1)/p - \gamma C_s - (p-1)A_s}e^{\gamma(t-s)}\big\langle|q_{s,t}f|^{p/2}, \mu_s\big\rangle^2 + \gamma e^{\gamma(t-s)}\big\langle|q_{s,t}f|^{p/2}, \mu_s\big\rangle^2 \\
&\leqslant \left(\frac{1}{\kappa} + \gamma\right)e^{\gamma(t-s)}\big\langle|q_{s,t}f|^{p/2}, \mu_s\big\rangle^2.
\end{aligned}$$

The last estimate holds by (22), since $2(p-1)/p \geqslant 1$. We obtain (24) integrating the previous inequality with respect to s. $\square$

As a consequence we obtain the following result.

Theorem 8. *Let $t, \alpha, \beta \geqslant 0$. Then for all $f : S \to \mathbb{R}$ such that $\langle f, \mu_t\rangle = 0$, we have*:

(i) *If $\lambda_s \geqslant \frac{1}{2}A_s + \alpha C_s$ for all $s \in [0, t]$, then*

$$\|q_{s,t}f\|_{L^2(\mu_s)} \leqslant e^{-\alpha(t-s)}\|f\|_{L^2(\mu_t)}.$$

(ii) *If $p = 2^n$ for some $n \in \mathbb{N}$ and*

$$\lambda_s \geqslant \frac{p}{4}A_s + \beta\frac{p-1}{4}B_s + \alpha\frac{p}{2}C_s, \quad \forall s \in [0, t],$$

then

$$\|q_{s,t}f\|_{L^p(\mu_s)} \leqslant e^{-\alpha(t-s)}\sqrt{2 + (\alpha\beta)^{-1}}\|f\|_{L^p(\mu_t)}.$$

Proof. Since $\langle q_{s,t}f, \mu_s\rangle = \langle f, \mu_t\rangle = 0$, assertion (i) follows from (23) in the limit $\kappa \downarrow 0$.

(ii) We shall prove by induction on n that if

$$\lambda_s \geqslant \frac{p}{4}A_s + \kappa\frac{p(p-1)}{4}B_s + \frac{\gamma}{2}C_s, \quad \forall s \in [0, t]$$

for some $\kappa, \gamma \geqslant 0$, then

$$\|q_{s,t}f\|_{L^p(\mu_s)} \leqslant e^{-\gamma(t-s)/p}\left(2 + \frac{1}{\kappa\gamma}\right)^{1/2-1/p}\|f\|_{L^p(\mu_t)}. \tag{26}$$

This implies (ii) by choosing $\gamma = \alpha p$ and $\kappa = \beta/p$. For $n = 1$, i.e. $p = 2$, (26) holds by (i). Now suppose (26) holds for $n - 1$. Then for $r \in [0, t]$,

$$\big\langle|q_{r,t}f|^{p/2}, \mu_r\big\rangle \leqslant e^{-\gamma(t-r)}\left(2 + \frac{1}{\kappa\gamma}\right)^{p/4-1}\big\langle|f|^{p/2}, \mu_t\big\rangle,$$

hence (24) yields

$$\begin{aligned}\langle |q_{s,t}f|^p,\mu_s\rangle &\leqslant e^{-\gamma(t-s)}\left(\langle |f|^p,\mu_t\rangle+\left(\kappa^{-1}+\gamma\right)\int_s^t e^{\gamma(t-r)}\langle |q_{r,t}f|^{p/2},\mu_r\rangle^2\,dr\right)\\ &\leqslant e^{-\gamma(t-s)}\langle |f|^p,\mu_t\rangle\left(1+\frac{\kappa^{-1}+\gamma}{\gamma}\left(1-e^{-\gamma(t-s)}\right)\left(2+\kappa^{-1}\gamma^{-1}\right)^{p/2-2}\right)\\ &\leqslant e^{-\gamma(t-s)}\left(2+(\kappa\gamma)^{-1}\right)^{p/2-1}\langle |f|^p,\mu_t\rangle,\end{aligned}$$

and thus (26). □

Remark 9. For general $p \geqslant 2$, exponential decay of the $L^p(\mu_s)$ norm of $q_{s,t}f$ with $\langle f,\mu_t\rangle = 0$ follows from the above result by the Riesz–Thorin interpolation theorem, in analogy to situations already encountered before.

Example (continued). In the situation of example (iii) above, provided the assumption on H_t made above is satisfied, one has to choose λ_s of order n^2 in order to guarantee exponential decay of $\|q_{s,t}f\|_{L^p(\mu_s)}$.

5. L^p estimates on invariant subsets

The aim of this section is to show that one can still obtain L^p estimates for the transitions operators $q_{s,t}$ on a subset $\tilde{S} \subseteq S$ that is invariant w.r.t. the underlying Markovian dynamics, i.e.

$$\mathcal{L}_t(x,y) = 0, \quad \forall (x,y) \in \tilde{S} \times \tilde{S}^c. \tag{27}$$

Instead of Poincaré inequalities on S, we then only have to assume corresponding inequalities on the subset $\tilde{S}$. The results stated below are then a consequence of the global bounds derive above, and they are relevant for the applications studied in [11].

Let us define, for $t \geqslant 0$, the conditional measure

$$\tilde{\mu}_t(x) = \mu_t(x|\tilde{S}) := \frac{\mu_t(\{x\}\cap\tilde{S})}{\mu(\tilde{S})},$$

and set $\tilde{H}_t := H_t - \langle H_t, \tilde{\mu}_t\rangle$. Note that $\langle \tilde{H}_t, \tilde{\mu}_t\rangle = 0$, and

$$\tilde{\mu}_t \propto \mu_t \propto \exp\left(-\int_0^t H_s\,ds\right)\mu_0 \propto \exp\left(-\int_0^t \tilde{H}_s\,ds\right)\mu_0 \quad \text{on } \tilde{S},$$

where "∝" means that the functions agree up to a multiplicative constant. Thus the conditional measure $\tilde{\mu}_t$ can be represented in the same way as μ_t with H_t replaced by $\tilde{H}_t$. Assumption (27) implies that $\mathcal{L}_t$ also satisfies the detailed balance condition with respect to $\tilde{\mu}_t$:

$$\tilde{\mu}_t(x)\mathcal{L}_t(x,y) = \tilde{\mu}_t(y)\mathcal{L}_t(y,x), \quad \forall t \geqslant 0,\ x, y \in S. \tag{28}$$

Let

$$\tilde{\mathcal{E}}_t(f) = -\int f\mathcal{L}_t f\,d\tilde{\mu}_t = \frac{1}{2}\sum_{x,y\in S}\left(f(y)-f(x)\right)^2\tilde{\mu}_t\mathcal{L}_t(x,y)$$

denote the corresponding Dirichlet form on $L^2(S,\tilde{\mu}_t)$. Note that, by (27), only the summands for $x,\ y \in \tilde{S}$ contribute to the sum.

As a consequence of Theorem 6(i) and Theorem 8 we obtain:

Corollary 10. *Assume that* (27) *holds and that $\mathcal{L}_t$ satisfies the inequalities*

$$\begin{aligned}&\mathrm{Var}_{\tilde{\mu}_t}(f) \leqslant \tilde{C}_t\tilde{\mathcal{E}}_t(f),\\ &-\int \tilde{H}_t\left(f-\langle f,\tilde{\mu}_t\rangle\right)^2 d\tilde{\mu}_t \leqslant \tilde{A}_t\tilde{\mathcal{E}}_t(f),\\ &\left|\int \tilde{H}_t f\,d\tilde{\mu}_t\right|^2 \leqslant \tilde{B}_t\tilde{\mathcal{E}}_t(f)\end{aligned}$$

for all $f: S \to \mathbb{R}$. Then the following assertions hold true for all $t, \alpha, \beta \geqslant 0$:

(i) *Let* $p \geqslant 2$. *If*

$$\lambda_s \geqslant \frac{p}{4}\tilde{A}_s + \frac{p(p+3)}{4}t\tilde{B}_s$$

for all $s \in [0,t]$, *then*

$$\|q_{s,t}f\|_{L^p(\tilde{\mu}_s)} \leqslant 2^{1/4}\frac{\mu_t(\tilde{S})}{\mu_s(\tilde{S})}\|f\|_{L^p(\tilde{\mu}_t)}$$

for all $f : S \to \mathbb{R}$ *and* $s \in [0,t]$.

(ii) *If*

$$\lambda_s \geqslant \frac{1}{2}\tilde{A}_s + \alpha\tilde{C}_s$$

for all $s \in [0,t]$, *then*

$$\|q_{s,t}f\|_{L^2(\tilde{\mu}_s)} \leqslant e^{-\alpha(t-s)}\frac{\mu_t(\tilde{S})}{\mu_s(\tilde{S})}\|f\|_{L^2(\tilde{\mu}_t)}$$

for all $f : S \to \mathbb{R}$ *with* $\langle f, \tilde{\mu}_t\rangle = 0$.

(iii) *If* $p = 2^n$ *and*

$$\lambda_s \geqslant \frac{p}{4}\tilde{A}_s + \beta\frac{p-1}{4}\tilde{B}_s + \alpha\frac{p}{2}\tilde{C}_s$$

for all $s \in [0,t]$, *then*

$$\|q_{s,t}f\|_{L^p(\tilde{\mu}_s)} \leqslant e^{-\alpha(t-s)}(2+1/\alpha\beta)^{1/2}\frac{\mu_t(\tilde{S})}{\mu_s(\tilde{S})}\|f\|_{L^p(\tilde{\mu}_t)}$$

for all $f : S \to \mathbb{R}$ *with* $\langle f, \tilde{\mu}_t\rangle = 0$.

Proof. Let $\tilde{q}_{s,t}$ denote the transition operators defined via the backward equation (3) with $\tilde{H}_t$ replacing H_t. By the detailed balance condition (28) and the assumptions, the operators $\tilde{q}_{s,t}$ satisfy the L^p bounds from the previous sections with $\tilde{H}_t$ replacing H_t, under the conditions on λ_s, $\tilde{A}_s$, $\tilde{B}_s$ and $\tilde{C}_s$ stated above. Now note that by a simple calculation based on (10),

$$H_t(x) = \tilde{H}_t(x) + \langle H_t, \tilde{\mu}_t\rangle = \tilde{H}_t(x) + h_t(\tilde{S}), \quad \forall x \in \tilde{S},$$

where

$$h_t(\tilde{S}) = -\frac{d}{dt}\log\mu_t(\tilde{S}).$$

Hence for any function $f : S \to \mathbb{R}$ and for all $x \in \tilde{S}$, we have

$$q_{s,t}f(x) = e^{-\int_s^t h_r(\tilde{S})\,dr}\tilde{q}_{s,t}f(x) = \frac{\mu_t(\tilde{S})}{\mu_s(\tilde{S})}\tilde{q}_{s,t}f(x).$$

The assertions now follow applying Theorem 6(i) and Theorem 8 to $\tilde{q}_{s,t}f$. □

In particular, it is worth pointing out that sufficiently strong mixing properties on the component can make up for an increase of the weight of the component as long as one is only looking for bounds for $q_{s,t}f$ on functions f such that $\langle f, \tilde{\mu}_t\rangle = 0$.

6. Logarithmic Sobolev inequalities and $L^p \to L^q$ estimates

We finally obtain an $L^p \to L^q$ estimate for $q_{s,t}$ from logarithmic Sobolev inequalities for the Dirichlet forms $\mathcal{E}_t$, by an adaptation of the classical argument that a log Sobolev inequality implies hypercontractivity (see e.g. [13] or [6, §6.1.14]). This generalizes well-known results for time-homogeneous Markov chains, for which we refer to e.g. [8,18], to the time-inhomogeneous setting.

Theorem 11. *Suppose that each of the measures μ_t, $t \geqslant 0$, satisfies a logarithmic Sobolev inequality with constant $C_t^{LS} > 0$, i.e.*

$$\int f^2 \log\left(\frac{f}{\|f\|_{L^2(\mu_t)}}\right)^2 d\mu_t \leqslant C_t^{LS} \cdot \mathcal{E}_t(f) \tag{29}$$

for all $t \geqslant 0$ and $f : S \to \mathbb{R}$. Then, for $1 < p \leqslant q < \infty$, one has

$$\|q_{s,t} f\|_{L^q(\mu_s)} \leqslant \exp\left(\int_s^t \max H_r^- \, dr\right) \|f\|_{L^p(\mu_t)}$$

for all $f : S \to \mathbb{R}$ and $0 \leqslant s \leqslant t$ such that

$$\int_s^t \frac{\lambda_r}{C_r^{LS}} \, dr \geqslant \frac{1}{4} \log \frac{q-1}{p-1}.$$

Proof. Let us set

$$\bar{q}_{s,t} f(x) := e^{-\int_s^t \max_{x \in S} H_r^-(x)\, dr} q_{s,t} f(x) \quad 0 \leqslant s \leqslant t,$$

which satisfies the backward equation

$$-\frac{\partial}{\partial s} \bar{q}_{s,t} f = \lambda_s \mathcal{L}_s \bar{q}_{s,t} f - \left(H_s + \max H_s^-\right) \bar{q}_{s,t} f.$$

Let $p : [0,t] \to]1, +\infty[$ be a continuously differentiable function. By computations similar to those carried out in the proof of Proposition 5, we obtain, noting that $H_s + \max H_s^- \geqslant 0$,

$$\begin{aligned} -p_s \|\bar{q}_{s,t} f\|_{L^{p_s}(\mu_s)}^{p_s - 1} \frac{\partial}{\partial s} \|\bar{q}_{s,t} f\|_{L^{p_s}(\mu_s)} &= -\frac{\partial}{\partial s} \int (\bar{q}_{s,t} f)^{p_s} \, d\mu_s \\ &= -p_s \lambda_s \cdot \mathcal{E}_s\left((\bar{q}_{s,t} f)^{p_s - 1}, \bar{q}_{s,t} f\right) - (p_s - 1) \int \left(H_s + \max H_s^-\right) (\bar{q}_{s,t} f)^{p_s} \, d\mu_s \\ &\quad - p_s' \int (\bar{q}_{s,t} f)^{p_s} \log(\bar{q}_{s,t} f) \, d\mu_s \leqslant -4 \frac{p_s - 1}{p_s} \lambda_s \mathcal{E}_s\left((\bar{q}_{s,t} f)^{p_s/2}\right) \\ &\quad - \frac{p_s'}{p_s} \int (\bar{q}_{s,t} f)^{p_s} \log(\bar{q}_{s,t} f)^{p_s} \, d\mu_s \end{aligned}$$

for all $f : S \to \mathbb{R}_+$, where $p_s' := dp_s/ds$. Choosing

$$p_s = 1 + (p-1) \exp\left(4 \int_s^t \lambda_r / C_r^{LS} \, dr\right),$$

we have $p_s' = -4(p_s - 1)\lambda_s / C_s^{LS}$, hence the log Sobolev inequality (29) implies

$$-\frac{\partial}{\partial s} \|\bar{q}_{s,t} f\|_{L^{p_s}(\mu_s)} \leqslant 0$$

for all $s \in]0, t[$. Therefore we can conclude

$$\|q_{s,t} f\|_{L^{p_s}(\mu_s)} = e^{\int_s^t \max H_r^- \, dr} \|\bar{q}_{s,t} f\|_{L^{p_s}(\mu_s)} \leqslant e^{\int_s^t \max H_r^- \, dr} \|f\|_{L^p(\mu_t)}$$

for all $s \in [0, t]$. □

Appendix A

In this Appendix A we prove bounds for the constants A_t, B_t, C_t in the situation of example (iii), for a fixed $t \geqslant 0$. Let us briefly recall the setup: we have $S = \{0, 1, \ldots, n\}$, μ_0 is the uniform distribution on S, $\mu_t(i) = \mu_0(i)$ for all $1 \leqslant 1 \leqslant n-1$, and $\mathcal{L}_t$ is defined by (8). Denoting the derivative of μ_t with respect to time by μ_t', we have $\mu_t'(i) = 0$ for all $1 \leqslant 1 \leqslant n-1$ and $\mu_t'(n) = -\mu_t'(0)$. We are going to assume, without loss of generality, that $\mu_t'(0) \geqslant 0$. Then we have

$$-H_t(i) = \frac{\mu_t'(i)}{\mu_t(i)} = \begin{cases} 0, & 1 \leqslant i \leqslant n-1, \\ \frac{\mu_t'(0)}{\mu_t(0)} \geqslant 0, & i = 0, \\ \frac{\mu_t'(n)}{\mu_t(n)} \leqslant 0, & i = n. \end{cases} \tag{30}$$

In this situation we can prove the following estimates for $n \in \mathbb{N}$.

Lemma 12. *Let A_t, B_t and C_t be the constants defined in* (12)–(14)*. Then one has*

$$\begin{aligned} &A_t \leqslant -4H_t(0)(n+1), \\ &B_t \leqslant 4\big(H_t(0)^2 + H_t(n)^2\big)(n+1), \\ &\frac{(n-4)^4}{48(n+1)^2} \leqslant C_t \leqslant n \max\left(\frac{n+1}{2}, 2\right), \quad \forall n \geqslant 4. \end{aligned}$$

Proof. To derive the upper bound for A_t, we observe that by (9) and (1) we have

$$\mathcal{E}_t(f) = \sum_{i=0}^{n-1} \big(f(i+1) - f(i)\big)^2 a_t(i) \tag{31}$$

for all $f : S \to \mathbb{R}$ and $t \geqslant 0$, where

$$a_t(i) = \mu_t(i)\mathcal{L}_t(i, i+1) = \frac{1}{2}\min\big(\mu_t(i), \mu_t(i+1)\big),$$

and, by (30),

$$-\int H_t\big(f - \langle f, \mu_t\rangle\big)^2 \, d\mu_t \leqslant -H_t(0)\big(f(0) - \langle f, \mu_t\rangle\big)^2 \mu_t(0). \tag{32}$$

Moreover, by (31), we have

$$\begin{aligned} \big(f(0) - \langle f, \mu_t\rangle\big)^2 &= \left(\sum_{k=0}^{n} \big(f(k) - f(0)\big)\mu_t(k)\right)^2 \\ &= \left(\sum_{i=0}^{n-1} \big(f(i+1) - f(i)\big) \sum_{k=i+1}^{n} \mu_t(k)\right)^2 \\ &\leqslant \mathcal{E}_t(f) \sum_{i=0}^{n-1} \frac{1}{a_t(i)} \left(\sum_{k=i+1}^{n} \mu_t(k)\right)^2. \end{aligned} \tag{33}$$

Noting that

$$a_t(i) = \begin{cases} \frac{1}{2}\min(\frac{1}{n+1}, \mu_t(0)), & i = 0, \\ \frac{1}{2(n+1)}, & i = 1, \dots, n-2, \\ \frac{1}{2}\min(\frac{1}{n+1}, \mu_t(n)), & i = n-1, \end{cases} \tag{34}$$

(32) and (33) imply

$$\begin{aligned} A_t &\leqslant -H_t(0) \sum_{i=0}^{n-1} \frac{1}{a_t(i)} \left(\sum_{k=i+1}^{n} \mu_t(k)\right)^2 \mu_t(0) \\ &\leqslant -2H_t(0)\mu_t(0)\big(\max\big(n+1, \mu_t(0)^{-1}\big) + (n-2)(n+1) + \max\big(n+1, \mu_t(n)^{-1}\big)\mu_t(n)\big) \\ &\leqslant -2H_t(0)\big(n(n+1)\mu_t(0) + 2\big) \leqslant -4H_t(0)(n+1). \end{aligned}$$

The upper bound for B_t can be obtained in a similar way: since $\langle H_t, \mu_t\rangle = 0$, we have

$$\begin{aligned}
\left|\int H_t f\,d\mu_t\right| &= \left|\int H_t\big(f-\langle f,\mu_t\rangle\big)\,d\mu_t\right| \\
&\leqslant \left|\int H_t^2\big(f-\langle f,\mu_t\rangle\big)^2\,d\mu_t\right| \\
&= H_t(0)^2\big(f(0)-\langle f,\mu_t\rangle\big)^2\mu_t(0)+H_t(n)^2\big(f(n)-\langle f,\mu_t\rangle\big)^2\mu_t(0) \\
&\leqslant 4\big(H_t(0)^2+H_t(n)^2\big)(n+1)
\end{aligned}$$

by an analogous computation as above.

To prove the upper bound for C_t note that, for $f : S \to \mathbb{R}$ and $0 \leqslant k \leqslant \ell \leqslant n$, we have

$$\big(f(\ell)-f(k)\big)^2 = \left(\sum_{i=k}^{\ell-1}\big(f(i+1)-f(i)\big)\right)^2 \leqslant (\ell-k)\sum_{i=k}^{\ell-1}\big(f(i+1)-f(i)\big)^2.$$

Hence, for $t \geqslant 0$,

$$\begin{aligned}
\mathrm{Var}_{\mu_t}(f) &= \frac{1}{2}\sum_{k,\ell=0}^{n}\big(f(\ell)-f(k)\big)^2\mu_t(k)\mu_t(\ell) \\
&= \sum_{k<\ell}\big(f(\ell)-f(k)\big)^2\mu_t(k)\mu_t(\ell) \\
&\leqslant \sum_{i=0}^{n-1}\big(f(i+1)-f(i)\big)^2\sum_{k=0}^{i}\sum_{\ell=i+1}^{n}(\ell-k)\mu_t(k)\mu_t(\ell) \\
&\leqslant n\sum_{i=0}^{n-1}\big(f(i+1)-f(i)\big)^2\mu_t\big(\{0,1,\dots,i\}\big)\mu_t\big(\{i+1,i+2,\dots,n\}\big) \\
&\leqslant n\max\big((n+1)/2,2\big)\mathcal{E}_t(f).
\end{aligned}$$

The last estimate holds by (31), (34), and because

$$\mu_t\big(\{0,1,\dots,i\}\big)\mu_t\big(\{i+1,i+2,\dots,n\}\big) \leqslant \frac{1}{4}, \quad \forall 0 \leqslant i \leqslant n.$$

We have thus proved that $C_t \leqslant n\max((n+1)/2,2)$.

Conversely, choosing $f(i) = i$ for $1 \leqslant i \leqslant n-1$, $f(0) = 1$, and $f(n) = n-1$, we have

$$\mathcal{E}_t(f) = \sum_{i=1}^{n-1} a_t(i) = \frac{n-1}{2(n+1)} \leqslant \frac{1}{2}$$

by (31) and (34), and

$$\begin{aligned}
\mathrm{Var}_{\mu_t}(f) &\geqslant \sum_{k=1}^{n-1}\sum_{\ell=k+1}^{n-1}(\ell-k)^2\mu_t(k)\mu_t(\ell) \\
&\geqslant \frac{1}{8(n+1)^2}\sum_{k=1}^{n-2}\sum_{m=1}^{n}m^2 \geqslant \frac{(n-4)^4}{96(n+1)^2},
\end{aligned}$$

which proves the lower bound for C_t. □

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