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Some types of solutions and generalized binary Darboux transformation for the mKP equation with self-consistent sources

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ABSTRACT

In this paper, an mKP equation with self-consistent sources (mKPESCSs) is structured in the framework of the constrained mKP equation. Based on the conjugate Lax pairs, we construct the generalized binary Darboux transformation and the N-times repeated Darboux transformation with arbitrary functions at time t for the mKPESCSs which offers a non-auto-Bäcklund transformation between two mKPESCSs with different degrees of sources. With the help of these transformations, some new solutions for the mKPESCSs such as soliton solutions, rational solutions, breather type solutions and exponential solutions are found by taking the special initial solution for auxiliary linear problems and the special functions of t-time.

1. Introduction

Since being introduced by Darboux in 1882 as a transformation which generates a new Sturm–Liouville differential equation from an old one giving a relation between two solutions [1], the Darboux transformation (DT) has been widely used. DT method based on Lax pairs has been proved to be one of the most fruitful algorithmic procedures to get explicit solutions of nonlinear evolution equations. The key for constructing Darboux transformation is to expose a kind of covariant properties that the corresponding spectral problems possess. In 1990, Matveev and Salle [2] first investigated the DT in integral form and presented binary Darboux transformation (BDT). Nimmo [3–6] has carried out a lot of excellent work about BDT: in Ref. [4], the general construction of BDT for KP hierarchy preserving certain properties of the operator, such as self-adjointness, is given; the BDT of two-dimensional Zakharov–Shabati/AKNS spectral problem [5] is obtained by composing the elementary transformation, for one solution matrix, with its inverse for another solution matrix.

Recently, more and more physicists and mathematicians are interested in studying soliton equations with self-consistent. They constitute an important class of integrable equations and serve as important models fields of physics, such as hydrodynamics, solid state physics, plasma physics, etc. [7–15]. In the past few years, in the $(1+1)$-dimensional case, many SESCSs have been constructed, various approaches for solving such system, for example, $\bar{\partial}$-method [14,15], the inverse scattering method [16,17], Darboux transformations (DT) [18–20], bilinear Bäcklund transformations and Hirota bilinear method [21], hodograph transformations [22], have been used. New approach also appears in systematic construction of SESCSs. Generalized binary Darboux transformations with arbitrary functions in time t for some $(1+1)$-dimensional SESCSs, which offer a non-auto-Bäcklund transformation between two SESCSs with different degrees of sources, have been constructed and can be used to obtain the N-soliton solution. But in the $(2+1)$-dimensional case, fewer results for the SESCSs have been

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obtained. The KP equation with self-consistent sources (KPESCS) arose in some physical models describing the interaction of long and short waves and the soliton solution of it was first found by Mel'nikov [23,24]. Zeng [18] and Hu [25] has carried out a lot of excellent work. Due to the important role played by the soliton equations with self-consistent sources (SESCSs) in many fields of physics, we have presented a method to find the explicit time part of the Lax representation for mKPSESCSs and to construct generalized binary Darboux transformations with arbitrary functions at time t for mKPSESCSs which, in contrast with the Darboux transformation for soliton equations [2,26], offer a non-auto-Bäcklund transformation between two SESCSs with different degrees of sources and can be used to obtain N-soliton, rational, breather type and exponential solutions [27,28].

The paper will be organized as follows. We recall some facts about the Darboux transformation for the mKP equation in the next section. In Section 3, through the pseudo-differential operator (PDO) formalism we introduces an mKPESCS and the conjugate Lax pairs of the mKP hierarchy with self-consistent sources briefly. Using the conjugate Lax pairs, we can construct the generalized Darboux transformations with arbitrary functions in time t for mKPESCS. In Section 4, by the nature of Wronskian, the N-times repeated generalized binary Darboux transformation for the mKPESCS will be constructed. In Section 5, with the generalized Darboux transformations and the N-times repeated generalized binary Darboux transformation, some new solutions such as soliton, rational, breather type and exponential solutions and some other solutions can be obtained.

2. The Darboux transformation for mKP equation

Based on the framework of the Sato theory, we first give a simple description of the mKP hierarchy [29–31]. Let us consider the following pseudo-differential operator (PDO)

$$L=\partial+H_0+H_1\partial^{-1}+H_2\partial^{-2}+H_3\partial^{-3}+\cdots, \tag{2.1}$$

where ∂ denotes $\frac{\partial}{\partial_x}$ and H_i $(i=0,1,2,\ldots)$ are functions of infinitely many variables $t=(t_1,t_2,\ldots)$ with $t_1=x$. Denote $B_m=(L_m)_+$ for $\forall m\in N$, where $B_m=(L_m)_+$ denote the differential of L_m whose order is more than 1. When $H_0=0$, L is reduced to the operator for the mKP hierarchy. Then the mKP hierarchy is written in the Lax representation

$$L_{t_m}=[B_m,L],\quad m=1,2,\ldots, \tag{2.2}$$

or the equivalent form

$$L^n_{t_m}=\left[B_m,L^n\right],\quad m,n=1,2,\ldots. \tag{2.3}$$

The mKP hierarchy (2.2) can also be written in the zero-curvature form

$$(B_n)_{t_k}-(B_k)_{t_n}+[B_n,B_k]=0,\quad n,k=2,3,\ldots. \tag{2.4}$$

Eq. (2.4) has a pair of conjugate Lax pairs as follows

$$\Phi^-_{t_k}=B_k\Phi^-, \tag{2.5a}$$

$$\Phi^-_{t_n}=B_n\Phi^-, \tag{2.5b}$$

and

$$\Phi^+_{t_k}=-(\bar{B}_k)\Phi^+, \tag{2.6a}$$

$$\Phi^+_{t_n}=-(\bar{B}_n)\Phi^+, \tag{2.6b}$$

where $\bar{B}_k=(\partial B_k\partial^{-1})^*$, $k\geqslant 2$. When $n=2$, $m=3$ we get the mKP equation as follows

$$4H_{t_3}-H_{xxx}+6H^2H_x-3\left(\partial^{-1}H_{t_2t_2}\right)-6H_x\left(\partial^{-1}H_{t_2}\right)=0. \tag{2.7}$$

Under the following transformation:

$$u=-H,\qquad t=-t_3,\qquad y=-\alpha t_2, \tag{2.8}$$

we get the simplest and the most important equation in the hierarchy (2.4), the mKP equation

$$4u_t+u_{xxx}-6u^2u_x+6\alpha u_x\left(\partial^{-1}u_y\right)+3\alpha^2\left(\partial^{-1}u_{yy}\right)=0, \tag{2.9}$$

which is called the mKPI equation when $\alpha=i$ and mKPII equation when $\alpha=1$. From (2.5) and (2.6), we will get the conjugate Lax pairs of (2.9), respectively, as follows

$$\alpha\Phi^-_y=\Phi^-_{xx}+2u\Phi^-_x, \tag{2.10a}$$

$$\Phi^-_t=A^-(u)\Phi^-,\quad A^-(u)=-\partial_{xxx}-3u\partial_{xx}-\frac{3}{2}u_x\partial-\frac{3}{2}u^2\partial-\frac{3}{2}\partial^{-1}u_y\partial, \tag{2.10b}$$

and

$$\alpha\Phi_y^+ = -\Phi_{xx}^+ + 2u\Phi_x^+, \tag{2.11a}$$

$$\Phi_t^+ = A^+(u)\Phi^+, \quad A^+(u) = -\partial_{xxx} + 3u\partial_{xx} + \frac{3}{2}u_x\partial - \frac{3}{2}u^2\partial - \frac{3}{2}\partial^{-1}u_y\partial. \tag{2.11b}$$

From (2.10) and (2.11), we can construct three types of Darboux transformations for the mKP equation (2.9).

2.1. The forward Darboux transformation for the mKP equation

If u is a solution of mKP equation (2.9) and $\Phi_1^- = \Phi_1^-(x, y, t)$ is a solution of (2.10). The forward Darboux transformation for the lax pairs (2.10) is defined by [32]

$$u[-1] = u + \partial \ln\left(\frac{\Phi_{1,x}^-}{\Phi_1^-}\right), \tag{2.12a}$$

$$\Phi^-[-1] = \Phi^- - \frac{\Phi_1^-}{\Phi_{1,x}^-}\Phi_x^-, \tag{2.12b}$$

where $u[-1]$ is a new solution of the mKP equation (2.9). Substituting (2.12) into (2.10b), we have

$$\begin{aligned} A^-\big(u[-1]\big)\Phi^-[-1] &= \left(\Phi^- - \frac{\Phi_1^-}{\Phi_{1,x}^-}\Phi_x^-\right)_t \\ &= A^-(u)\Phi^- - \frac{(A^-(u)\Phi_1^-)\Phi_x^- + \Phi_1^-(A^-(u)\Phi^-)_x}{\Phi_{1,x}^-} + \frac{\Phi_1^-(A^-(u)\Phi_1^-)_x\Phi_x^-}{(\Phi_{1,x}^-)^2}. \end{aligned} \tag{2.13}$$

2.2. The backward Darboux transformation for the mKP equation

If u is a solution of mKP equation (2.9) and $\Phi_2^+ = \Phi_2^+(x, y, t)$ is a solution of (2.11). The backward Darboux transformation for the lax pairs (2.10) is defined by

$$u[+1] = u + \partial \ln\left(\frac{\Phi_{2,x}^+}{\Phi_2^+}\right), \tag{2.14a}$$

$$\Phi^-[+1] = \frac{C_1 + \int \Phi^-\Phi_{2,x}^+\,dx}{\Phi_2^+}, \tag{2.14b}$$

where C_1 is an arbitrary constant. We point out that throughout the paper, the integral operation $\int P_1P_2\,dx$ (such as $\int \Phi^-\Phi_{2,x}^+\,dx$) here means $\int_{-\infty}^x P_1P_2\,dx$ and contains no arbitrary function of y and t, only numerical constant if we impose some suitable boundary condition on the integrand functions P_1 and P_2 at $x = -\infty$ or $x = \infty$. For arbitrariness of the constants in the Darboux transformations such as C_1 here, in our computation later, the integral constants are taken to be zero. Substituting (2.14) into (2.10b), we get the following equality:

$$\begin{aligned} A^-\big(u[+1]\big)\Phi^-[+1] &= \left(\frac{C_1 + \int \Phi^-\Phi_{2,x}^+\,dx}{\Phi_2^+}\right)_t \\ &= \frac{\int[(A^-(u)\Phi^-)\Phi_{2,x}^+ + \Phi^-(A^+(u)\Phi_2^+)_x]\,dx - A^+(u)(C_1 + \int \Phi^-\Phi_{2,x}^+\,dx)}{\Phi_2^+}. \end{aligned} \tag{2.15}$$

2.3. The binary Darboux transformation for the mKP equation

Applied the forward Darboux transformation and the backward Darboux transformation to the system (2.10), we can get the binary DT as follows

$$u[+1,-1] = u[+1] + \partial \ln\left(\frac{\Phi_1^-[+1]_x}{\Phi_1^-[+1]}\right) = u + \partial \ln\left(\frac{\Phi_1^-\Phi_{2,x}^+}{C_2 + \int \Phi_1^-\Phi_{2,x}^+\,dx}\right), \tag{2.16a}$$

$$\Phi^-[+1,-1] = \Phi^-[+1] - \frac{\Phi_1^-[+1]}{\Phi_1^-[+1]_x}\Phi^-[+1]_x = \Phi^- - \frac{\Phi^-\Phi_1^-\Phi_2^+ - \Phi_1^-(C_1 + \int \Phi^-\Phi_{2,x}^+\,dx)}{\Phi_1^-\Phi_2^+ - (C_2 + \int \Phi_1^-\Phi_{2,x}^+\,dx)}, \tag{2.16b}$$

where $\Phi_1^-[+1] = \frac{C_2+\int \Phi_1^- \Phi_2^+ dx}{\Phi_{2,x}^+}$, C_i $(i = 1, 2)$ are arbitrary constants. Similarly, we can get the forward DT, backward DT and binary DT for system (2.11). Specific formula is as follows:

The forward Darboux transformation for the system (2.11),

$$u[+1] = u + \partial \ln\left(\frac{\Phi_{2,x}^+}{\Phi_2^+}\right), \tag{2.17a}$$

$$\Phi^+[+1] = \Phi^+ - \frac{\Phi_2^+}{\Phi_{2,x}^+}\Phi_x^+, \tag{2.17b}$$

where $u[+1]$ is a new solution of the mKP equation (2.9).

The backward Darboux transformation for the system (2.11),

$$u[-1] = u + \partial \ln\left(\frac{\Phi_{1,x}^-}{\Phi_1^-}\right), \tag{2.18a}$$

$$\Phi^+[-1] = \frac{C_3 + \int \Phi^+ \Phi_{1,x}^- dx}{\Phi_1^-}. \tag{2.18b}$$

The binary Darboux transformation for the system (2.11),

$$u[-1,+1] = u[-1] + \partial \ln\left(\frac{\Phi_2^+[-1]_x}{\Phi_2^+[-1]}\right) = u + \partial \ln\left(\frac{\Phi_2^+ \Phi_{1,x}^-}{C_3 + \int \Phi_2^+ \Phi_{1,x}^- dx}\right), \tag{2.19a}$$

$$\Phi^+[-1,+1] = \Phi^+[-1] - \frac{\Phi_2^+[-1]}{\Phi_2^+[-1]_x}\Phi^+[-1]_x = \Phi^+ - \frac{\Phi^+\Phi_1^-\Phi_2^+ - \Phi_2^+(C_3 + \int \Phi^+\Phi_{1,x}^- dx)}{\Phi_1^-\Phi_2^+ - (C_4 + \int \Phi_2^+ \Phi_{1,x}^- dx)}, \tag{2.19b}$$

where $\Phi_2^+[-1] = \frac{C_4+\int \Phi_2^+ \Phi_1^- dx}{\Phi_{1,x}^-}$, C_i $(i = 1, 2, 3, 4)$ are arbitrary constants.

3. The binary Darboux transformation for the mKP equation with self-consistent sources

Based on the framework of the references [31–36], we will get the mKP equation with self-consistent sources and the corresponding lax pairs. For the pseudo-differential operator L given by (2.1)

$$L = \partial + H_0 + H_1\partial^{-1} + H_2\partial^{-2} + H_3\partial^{-3} + \cdots,$$

and we consider a constraint as follows

$$L^n = B_n + \sum_{i=1}^{m} q_i\partial^{-1} r_i\partial \quad \text{or} \quad \left(L^n\right)_- = \sum_{i=1}^{m} q_i\partial^{-1} r_i\partial, \tag{3.1}$$

where $B_n = (L_n)_+$, $q_{i,t_k} = B_k q_i$, $r_{i,t_k} = -\bar{B}_k q_i$ $(i = 1, \dots, m,\ k \geqslant 2)$ and $(L^n)_-$ denote the residual parts of L^n. The n-constrained mKP hierarchy is defined as follows:

$$\left(L^n\right)_{t_k} = \left[\left(L^k\right)_+, L^n\right] = \left[B_k, L^n\right], \tag{3.2a}$$

$$q_{t_k} = B_k q, \tag{3.2b}$$

$$r_{t_k} = -\bar{B}_k r. \tag{3.2c}$$

Adding the term $(B_k)_{t_n}$ to the right-hand side of (3.2a), we can define the mKP hierarchy with self-consistent sources as follows:

$$(B_k)_{t_n} - \left(L^n\right)_{t_k} + \left[B_k, L^n\right] = 0, \tag{3.3a}$$

$$q_{i,t_k} = B_k q_i, \tag{3.3b}$$

$$r_{i,t_k} = -\bar{B}_k r_i, \quad i = 1, 2, \dots, m. \tag{3.3c}$$

So if the variable t_n is viewed as the evolution variable, the n-constrained mKP hierarchy may be regarded as the stationary hierarchy of the mKP hierarchy with self-consistent sources. Under conditions (3.1), (3.3b) and (3.3c), we naturally get the conjugate Lax pairs of (3.3a) as follows

$$\Phi^-_{t_k} = B_k \Phi^-, \tag{3.4a}$$

$$\Phi^-_{t_n} = L^n \Phi^- = B_n \Phi^- + \sum_{i=1}^{m} q_i \int r_i \Phi^-_x \, dx, \tag{3.4b}$$

and

$$\Phi^+_{t_k} = -\bar{B}_k \Phi^+, \tag{3.5a}$$

$$\Phi^+_{t_n} = -\big(L^n\big)^* \Phi^+ = -\bar{B}_n \Phi^+ - \left[\partial\left(\sum_{i=1}^{m} q_i \int r_i \partial \, dx\right)\partial^{-1}\right]^* \Phi^+ = -\bar{B}_n \Phi^+ - \sum_{i=1}^{m} r_i \int q_i \Phi^+_x \, dx. \tag{3.5b}$$

When $k = 2$, $n = 3$, under transformation (2.8) and setting

$$\psi_i = q_i, \qquad \phi_i = -r_i,$$

we will get the mKPESCS and its conjugate Lax pairs respectively from (3.3)–(3.5). The mKPESCS is

$$4u_t + u_{xxx} - 6u^2 u_x + 6\alpha u_x\big(\partial^{-1} u_y\big) + 3\alpha^2\big(\partial^{-1} u_{yy}\big) - \sum_{i=1}^{m} (\psi_i \phi_i)_x = 0, \tag{3.6a}$$

$$\alpha \psi_{i,y} = \psi_{i,xx} + 2u \psi_{i,x}, \tag{3.6b}$$

$$\alpha \phi_{i,y} = -\phi_{i,xx} + 2u \phi_{i,x}, \quad i = 1, 2, \ldots, m, \tag{3.6c}$$

which is called the mKPIESCS when $\alpha = i$ and mKPIIESCS when $\alpha = 1$. Under conditions the system of (3.4), (3.5) and (3.6b), (3.6c) the Lax pairs for (3.6a) are

$$\alpha \Phi^-_y = \Phi^-_{xx} + 2u \Phi^-_x, \tag{3.7a}$$

$$\Phi^-_t = A^-(u)\Phi^- + F^-_m(\psi, \phi)\Phi^-, \tag{3.7b}$$

where $F^-_m(\psi, \phi)\Phi^- = -\sum_{i=1}^{m} \psi_i \int \phi_i \Phi^-_x \, dx$. The conjugate Lax pairs for (3.6a) are

$$\alpha \Phi^+_y = -\Phi^+_{xx} + 2u \Phi^+_x, \tag{3.8a}$$

$$\Phi^+_t = A^+(u)\Phi^+ + F^+_m(\psi, \phi)\Phi^+, \tag{3.8b}$$

where $F^+_m(\psi, \phi)\Phi^+ = \sum_{i=1}^{m} \phi_i \int \psi_i \Phi^+_x \, dx$. With the system of (3.6b) and (3.6c), we can construct the forward DT, backward DT and binary DT for the mKPESCS.

Theorem 3.1. *If u, $\psi_1, \ldots, \psi_m$, $\phi_1, \ldots, \phi_m$ be the solution of the mKPESCS* (3.6) *and Φ^-_1 satisfies* (3.7), *then the forward Darboux transformation for the lax pairs* (3.7) *can be defined by*

$$u[-1] = u + \partial \ln\left(\frac{\Phi^-_{1,x}}{\Phi^-_1}\right), \tag{3.9a}$$

$$\Phi^-[-1] = \Phi^- - \frac{\Phi^-_1}{\Phi^-_{1,x}} \Phi^-_x, \tag{3.9b}$$

$$\phi_i[-1] = \phi_i - \frac{\int \Phi^-_1 \phi_{i,x} \, dx}{\Phi^-_1}, \tag{3.9c}$$

$$\psi_i[-1] = \psi_i - \frac{\Phi^-_1}{\Phi^-_{1,x}} \psi_{i,x}, \tag{3.9d}$$

namely, $u[-1]$, $\Phi^-[-1]$, $\phi_i[-1]$, $\psi_i[-1]$, $i = 1, 2, \ldots, m$, satisfy (3.6) *and* (3.7).

Proof. From above, it is seen that $u[-1]$, $\Phi^-[-1]$, $\psi_i[-1]$, $\phi_i[-1]$ $(i = 1, 2, \ldots, m)$ satisfy (3.6b), (3.6c) and (3.7a). So we only need to prove (3.7b), which is the following equality

$$\begin{aligned}
\Phi^-[-1]_t &= \left(\Phi^- - \frac{\Phi^-_1}{\Phi^-_{1,x}} \Phi^-_x\right)_t = A^-(u)\Phi^- - \frac{(A^-(u)\Phi^-_1)\Phi^-_x + \Phi^-_1 (A^-(u)\Phi^-)_x}{\Phi^-_{1,x}} + \frac{\Phi^-_1 (A^-(u)\Phi^-_1)_x \Phi^-_x}{(\Phi^-_{1,x})^2} \\
&\quad + F^-_m(\psi, \phi)\Phi^- - \frac{(F^-_m(\psi, \phi)\Phi^-_1)\Phi^-_x}{\Phi^-_{1,x}} - \frac{\Phi^-_1 (F^-_m(\psi, \phi)\Phi^-)_x}{\Phi^-_{1,x}} + \frac{\Phi^-_1 (F^-_m(\psi, \phi)\Phi^-_1)_x \Phi^-_x}{(\Phi^-_{1,x})^2} \\
&= A^-\big(u[-1]\big)\Phi^-[-1] + F^-_m\big(\psi[-1], \phi[-1]\big)\Phi^-[-1].
\end{aligned} \tag{3.10}$$

It is easy to verify that (2.13)

$$A^-\big(u[-1]\big)\Phi^-[-1]=A^-(u)\Phi^- - \frac{(A^-(u)\Phi_1^-)\Phi_x^- + \Phi_1^-(A^-(u)\Phi^-)_x}{\Phi_{1,x}^-}+\frac{\Phi_1^-(A^-(u)\Phi_1^-)_x\Phi_x^-}{(\Phi_{1,x}^-)^2},$$

still holds now. So we only need to prove the following identity:

$$F_m^-\big(\psi[-1],\phi[-1]\big)\Phi^-[-1]=F_m^-(\psi,\phi)\Phi^- - \frac{(F_m^-(\psi,\phi)\Phi_1^-)\Phi_x^-}{\Phi_{1,x}^-}-\frac{\Phi_1^-(F_m^-(\psi,\phi)\Phi^-)_x}{\Phi_{1,x}^-} + \frac{\Phi_1^-(F_m^-(\psi,\phi)\Phi_1^-)_x\Phi_x^-}{(\Phi_{1,x}^-)^2}. \tag{3.11}$$

Substituting the expression of $F_m^-(\psi,\phi)\Phi^- = -\sum_{i=1}^m \psi_i\int\phi_i\Phi_x^-\,dx$ into (3.11), we can easily verify that the identity (3.11) is hold. This completes the proof. □

Theorem 3.2. *If u, $\psi_1,\dots,\psi_m$, $\phi_1,\dots,\phi_m$ be the solution of the mKPESCS* (3.6) *and Φ_2^+ satisfies* (3.8), *then the backward Darboux transformation for the lax pairs* (3.8) *can be defined by*

$$u[+1]=u+\partial\ln\left(\frac{\Phi_{2,x}^+}{\Phi_2^+}\right), \tag{3.12a}$$

$$\Phi^-[+1]=\frac{C_4+\int\Phi^-\Phi_{2,x}^+\,dx}{\Phi_2^+}, \tag{3.12b}$$

$$\phi_i[+1]=\phi_i-\frac{\Phi_2^+}{\Phi_{2,x}^+}\phi_{i,x}, \tag{3.12c}$$

$$\psi_i[+1]=\psi_i-\frac{\int\Phi_2^+\psi_{i,x}\,dx}{\Phi_2^+}, \tag{3.12d}$$

where C_4 is an arbitrary constant. This theorem can be expressed as $u[+1]$, $\Phi^-[+1]$, $\phi_i[+1]$, $\psi_i[+1]$ $(i=1,\dots,m)$ satisfy (3.6) *and* (3.7).

Proof. The same as above, it is seen that $u[+1]$, $\Phi^-[+1]$, $\phi_i[+1]$, $\psi_i[+1]$ $(i=1,\dots,m)$ satisfy (3.6b), (3.6c) and (3.7a). So we only need to prove (3.7b), which is the following equality

$$\begin{aligned}\Phi^-[+1]_t &= \left(\frac{C_4+\int\Phi^-\Phi_{2,x}^+\,dx}{\Phi_2^+}\right)_t = \frac{\int[(A^-(u)\Phi^-)\Phi_{2,x}^+ + \Phi^-(A^+(u)\Phi_2^+)_x]\,dx - A^+(u)(C_1+\int\Phi^-\Phi_{2,x}^+\,dx)}{\Phi_2^+}\\ &\quad + \frac{\int[(F_m^-(\psi,\phi)\Phi^-)\Phi_{2,x}^+ + \Phi^-(F_m^+(\psi,\phi)\Phi_2^+)_x]\,dx}{\Phi_2^+} - \frac{F_m^+(\psi,\phi)\Phi_2^+}{(\Phi_2^+)^2}\left(C_1+\int\Phi^-\Phi_{2,x}^+\,dx\right)\\ &= A^-\big(u[+1]\big)\Phi^-[+1]+F_m^-\big(\psi[+1],\phi[+1]\big)\Phi^-[+1].\end{aligned} \tag{3.13}$$

Similarly, it is easy to verify that (2.15)

$$A^-\big(u[+1]\big)\Phi^-[+1]=\frac{\int[(A^-(u)\Phi^-)\Phi_{2,x}^+ + \Phi^-(A^+(u)\Phi_2^+)_x]\,dx - A^+(u)(C_1+\int\Phi^-\Phi_{2,x}^+\,dx)}{\Phi_2^+},$$

still holds now. So we only need to prove the following identity:

$$F_m^-\big(\psi[+1],\phi[+1]\big)\Phi^-[+1]=\frac{\int[(F_m^-(\psi,\phi)\Phi^-)\Phi_{2,x}^+ + \Phi^-(F_m^+(\psi,\phi)\Phi_2^+)_x]\,dx}{\Phi_2^+} - \frac{F_m^+(\psi,\phi)\Phi_2^+}{(\Phi_2^+)^2}\left(C_1+\int\Phi^-\Phi_{2,x}^+\,dx\right). \tag{3.14}$$

Substituting the expression of $F_m^-(\psi,\phi)\Phi^- = -\sum_{i=1}^m\psi_i\int\phi_i\Phi_x^-\,dx$ and $F_m^+(\psi,\phi)\Phi^+=\sum_{i=1}^m\phi_i\int\psi_i\Phi_x^+\,dx$ into (3.14), we can easily verify that the identity (3.14) is hold. This completes the proof. □

Applied the forward Darboux transformation (3.9) (Theorem 3.1) and the backward Darboux transformation (3.12) (Theorem 3.2) to the system (3.7), we can get the binary Darboux transformation for the system (3.7).

Theorem 3.3. *If $u, \psi_1, \ldots, \psi_m, \phi_1, \ldots, \phi_m$ be the solution of the mKPESCS* (3.6) *and let Φ_1^- and Φ_2^+ satisfy the system* (3.7) *and* (3.8) *respectively, then the binary Darboux transformation for the system* (3.7) *by choosing $C_4 = 0$, $\Phi_1^-[+1] = \frac{C+\int \Phi_1^- \Phi_{2,x}^+ dx}{\Phi_2^+}$ (C is a constant) as follows*

$$u[+1,-1] = u + \partial \ln\left(\frac{\Phi_1^- \Phi_{2,x}^+}{C + \int \Phi_1^- \Phi_{2,x}^+ dx}\right), \tag{3.15a}$$

$$\Phi^-[+1,-1] = \Phi^- - \frac{\Phi_1^- \int \Phi_2^+ \Phi_x^- dx}{\int \Phi_{1,x}^- \Phi_2^+ dx - C}, \tag{3.15b}$$

$$\phi_i[+1,-1] = \phi_i - \frac{\Phi_2^+ \int \Phi_1^- \phi_{i,x} dx}{C + \int \Phi_1^- \Phi_{2,x}^+ dx}, \tag{3.15c}$$

$$\psi_i[+1,-1] = \psi_i - \frac{\Phi_1^- \int \Phi_2^+ \psi_{i,x} dx}{\int \Phi_{1,x}^- \Phi_2^+ dx - C}. \tag{3.15d}$$

Proof. With the forward Darboux transformation (3.9) and the backward Darboux transformation (3.12), we can get the following equations:

$$u[+1,-1] = u[+1] + \partial \ln\left(\frac{\Phi_1^-[+1]_x}{\Phi_1^-[+1]}\right) = u + \partial \ln\left(\frac{\Phi_1^- \Phi_{2,x}^+}{C + \int \Phi_1^- \Phi_{2,x}^+ dx}\right),$$

$$\Phi^-[+1,-1] = \Phi^-[+1] - \frac{\Phi_1^-[+1]}{\Phi_1^-[+1]_x}\Phi^-[+1]_x = \Phi^- - \frac{\Phi_1^- \int \Phi_2^+ \Phi_x^- dx}{\int \Phi_{1,x}^- \Phi_2^+ dx - C},$$

$$\phi_i[+1,-1] = \phi_i[+1] - \frac{\int \Phi_1^-[+1]\phi_i[+1]_x dx}{\Phi_1^-[+1]} = \phi_i - \frac{\Phi_2^+ \int \Phi_1^- \phi_{i,x} dx}{C + \int \Phi_1^- \Phi_{2,x}^+ dx},$$

$$\psi_i[+1,-1] = \psi_i[+1] - \frac{\Phi_1^-[+1]}{\Phi_1^-[+1]_x}\psi_i[+1]_x = \psi_i - \frac{\Phi_1^- \int \Phi_2^+ \psi_{i,x} dx}{\int \Phi_{1,x}^- \Phi_2^+ dx - C},$$

where $C_4 = 0$, $\Phi_1^-[+1] = \frac{C+\int \Phi_1^- \Phi_{2,x}^+ dx}{\Phi_2^+}$ (C is a constant). This completes the proof. □

If C is replaced by $C(t)$, an arbitrary function in time t in (3.15), then (3.6b), (3.6c) and (3.7a) are also covariant w.r.t. (3.15). In the following, substituting (3.15) into (3.7b), we can obtain the equality

$$\Phi^-[+1,-1]_t = A^-\big(u[+1,-1]\big)\Phi^-[+1,-1] + F_m^-\big(\psi[+,-1], \phi[+1,-1]\big)\Phi^-[+1,-1]. \tag{3.16}$$

With $F_m^-(\psi[+1,-1], \phi[+1,-1])\Phi^-[+1,-1] = -\sum_{i=1}^m \psi_i[+1,-1]\int \phi_i[+1,-1]\Phi^-[+1,-1]_x dx$ and (3.15), we have the left-hand side of (3.16)

$$\Phi^-[+1,-1]_t = \left(\Phi^- - \frac{\Phi_1^- \int \Phi_2^+ \Phi_x^- dx}{\int \Phi_{1,x}^- \Phi_2^+ dx - C}\right)_t = \Phi_t^- - \left(\frac{\Phi_1^- \int \Phi_2^+ \Phi_x^- dx}{\int \Phi_{1,x}^- \Phi_2^+ dx - C}\right)_t. \tag{3.17}$$

By a tedious computation, the right-hand side of (3.16) does not contain derivatives of t. From above, it is obvious that (3.7b) is not covariant w.r.t. (3.15) any longer when we replace C in (3.15) by $C(t)$. In fact, we have the following theorem.

Theorem 3.4. *If $u, \psi_1, \ldots, \psi_m, \phi_1, \ldots, \phi_m$ be the solution of the mKPESCS* (3.6) *and let Φ_1^- and Φ_2^+ satisfy the system* (3.7) *and* (3.8) *respectively, then the binary Darboux transformation for the system* (3.7) *by choosing $C_4 = 0$, $\Phi_1^-[+1] = \frac{C(t)+\int \Phi_1^- \Phi_{2,x}^+ dx}{\Phi_2^+}$ ($C(t)$ is an arbitrary function in t) defined by*:

$$u[+1,-1] = u + \partial \ln\left(\frac{\Phi_1^- \Phi_{2,x}^+}{C(t) + \int \Phi_1^- \Phi_{2,x}^+ dx}\right), \tag{3.18a}$$

$$\Phi^-[+1,-1] = \Phi^- - \frac{\Phi_1^- \int \Phi_2^+ \Phi_x^- dx}{\int \Phi_{1,x}^- \Phi_2^+ dx - C(t)}, \tag{3.18b}$$

$$\phi_i[+1,-1] = \phi_i - \frac{\Phi_2^+ \int \Phi_1^- \phi_{i,x} dx}{C(t) + \int \Phi_1^- \Phi_{2,x}^+ dx}, \tag{3.18c}$$

$$\psi_i[+1,-1] = \psi_i - \frac{\Phi_1^- \int \Phi_2^+ \psi_{i,x}\,dx}{\int \Phi_{1,x}^- \Phi_2^+\,dx - C(t)}, \tag{3.18d}$$

and

$$\phi_{m+1}[+1,-1] = \frac{\sqrt{\dot C(t)}\Phi_2^+}{C(t) + \int \Phi_1^- \Phi_{2,x}^+\,dx}, \qquad \psi_{m+1}[+1,-1] = -\frac{\sqrt{\dot C(t)}\Phi_1^-}{\int \Phi_{1,x}^- \Phi_2^+\,dx - C(t)}, \tag{3.18e}$$

transforms (3.6b), (3.6c) *and* (3.7), *respectively, into*

$$\alpha\psi_i[+1,-1]_y = \psi_i[+1,-1]_{xx} + 2u[+1,-1]\psi_i[+1,-1]_x, \tag{3.19a}$$

$$\alpha\phi_i[+1,-1]_y = -\phi_i[+1,-1]_{xx} + 2u[+1,-1]\phi_i[+1,-1]_x, \quad i = 1,2,\dots,m, \tag{3.19b}$$

$$\alpha\Phi^-[+1,-1]_y = \Phi^-[+1,-1]_{xx} + 2u[+1,-1]\Phi^-[+1,-1]_x, \tag{3.19c}$$

$$\Phi^-[+1,-1]_t = A^-\big(u[+1,-1]\big)\Phi^-[+1,-1] + F_m^-\big(\psi[+1,-1],\phi[+1,-1]\big)\Phi^-[+1,-1]. \tag{3.19d}$$

So $u[+1,-1]$, $\psi_i[+1,-1]$, $\phi_i[+1,-1]$ $(i = 1,\dots,m+1)$ *is a new solution of the mKPESCS* (3.6) *with degree* $m+1$.

Proof. Eqs. (3.19a)–(3.19c) hold obviously. We only need to prove (3.19d). Substituting (3.18b) into the left-hand side of (3.19d) and using the result of the previous theorem, we have

$$\begin{aligned}
\Phi^-[+1,-1]_t &= \left(\Phi^- - \frac{\Phi_1^- \int \Phi_2^+ \Phi_x^-\,dx}{\int \Phi_{1,x}^- \Phi_2^+\,dx - C(t)}\right)_t = \Phi_t^- - \left(\frac{\Phi_1^- \int \Phi_2^+ \Phi_x^-\,dx}{\int \Phi_{1,x}^- \Phi_2^+\,dx - C(t)}\right)_t \\
&= A^-\big(u[+1,-1]\big)\Phi^-[+1,-1] + F_m^-\big(\psi[+1,-1],\phi[+1,-1]\big)\Phi^-[+1,-1] \\
&\quad - \frac{\dot C(t)\Phi_1^- \int \Phi_2^+ \Phi_x^-\,dx}{(\int \Phi_{1,x}^- \Phi_2^+\,dx - C(t))^2}.
\end{aligned} \tag{3.20}$$

So we only need to prove

$$-\psi_{m+1}[+1,-1]\int \phi_{m+1}[+1,-1]\Phi^-[+1,-1]_x\,dx = -\frac{\dot C(t)\Phi_1^- \int \Phi_2^+ \Phi_x^-\,dx}{(\int \Phi_{1,x}^- \Phi_2^+\,dx - C(t))^2},$$

i.e., to prove

$$\int \frac{\sqrt{\dot C(t)}\Phi_2^+}{C(t) + \int \Phi_1^- \Phi_{2,x}^+\,dx}\left(\Phi^- - \frac{\Phi_1^- \int \Phi_2^+ \Phi_x^-\,dx}{\int \Phi_{1,x}^- \Phi_2^+\,dx - C(t)}\right)_x dx = -\frac{\int \Phi_2^+ \Phi_x^-\,dx}{\int \Phi_{1,x}^- \Phi_2^+\,dx - C(t)}, \tag{3.21}$$

the left-hand side of (3.21)

$$\begin{aligned}
&= \int \frac{\sqrt{\dot C(t)}\Phi_2^+}{C(t) + \int \Phi_1^- \Phi_{2,x}^+\,dx}\left[\Phi_x^- - \frac{\Phi_{1,x}^- \int \Phi_2^+ \Phi_x^-\,dx + \Phi_1^- \Phi_2^+ \Phi_x^-}{\int \Phi_{1,x}^- \Phi_2^+\,dx - C(t)} + \frac{\Phi_1^- \Phi_2^+ \Phi_{1,x}^- \int \Phi_2^+ \Phi_x^-\,dx}{(\int \Phi_{1,x}^- \Phi_2^+\,dx - C(t))^2}\right]dx \\
&= \int \left[-\frac{\Phi_2^+ \Phi_x^-}{\int \Phi_{1,x}^- \Phi_2^+\,dx - C(t)} + \frac{\Phi_2^+ \Phi_{1,x}^- \int \Phi_2^+ \Phi_x^-\,dx}{(\int \Phi_{1,x}^- \Phi_2^+\,dx - C(t))^2}\right]dx = -\frac{\int \Phi_2^+ \Phi_x^-\,dx}{\int \Phi_{1,x}^- \Phi_2^+\,dx - C(t)},
\end{aligned} \tag{3.22}$$

= the right-hand side of (3.21). This completes the proof. □

Remark. For system (3.8), the DTs described in this section can also be constructed, we omit the results here. If $C(t)$ is not a constant, the DT (3.18) provides a non-auto-Bäcklund transformation between two mKPESCSs (3.6) with degrees m and $m+1$, respectively.

4. The *N*-times repeated generalized binary Darboux transformation for the mKPESCS

In this section, we will used Theorem 3.4 to construct the N-times repeated generalized binary Darboux transformation for the mKPESCS. By the nature of Wronskian, we have the following lemma:

Lemma 4.1. *If* $\Phi^-, \Phi_1^-, \dots, \Phi_N^-$ *are solutions of* (3.7) *and* $\Phi^+, \Phi_{N+1}^+, \dots, \Phi_{N+N}^+$ *are solutions of* (3.8). *For notational simplicity, we denote* $\Phi_k^-[i] = \Phi_k^-[+i,-i]$, $\Phi_{N+k}^+[i] = \Phi_{N+k}^+[+i,-i]$, $k = 1,\dots,N$, $i = 0,1,\dots,N$, $m = 2,\dots,N$. *Then we have*

$$W_1\big(\Phi_m^-[m-1],\ldots,\Phi_{m+k}^-[m-1];\Phi_{N+m}^+[m-1],\ldots,\Phi_{N+m+k}^+[m-1];C_m(t),\ldots,C_{m+k}(t)\big)$$
$$=\frac{W_1(\Phi_{m-1}^-[m-2],\ldots,\Phi_{m+k}^-[m-2];\Phi_{N+m-1}^+[m-2],\ldots,\Phi_{N+m+k}^+[m-2];C_{m-1}(t),\ldots,C_{m+k}(t))}{C_{m-1}(t)+\int\Phi_{m-1}^-[m-2]\Phi_{N+m-1}^+[m-2]_x\,dx}, \tag{4.1a}$$

$$W_2\big(\Phi_m^-[m-1],\ldots,\Phi_{m+k}^-[m-1];\Phi_{N+m}^+[m-1],\ldots,\Phi_{N+m+k}^+[m-1];C_m(t),\ldots,C_{m+k}(t)\big)$$
$$=\frac{W_2(\Phi_{m-1}^-[m-2],\ldots,\Phi_{m+k}^-[m-2];\Phi_{N+m-1}^+[m-2],\ldots,\Phi_{N+m+k}^+[m-2];C_{m-1}(t),\ldots,C_{m+k}(t))}{\int\Phi_{m-1}^-[m-2]_x\Phi_{N+m-1}^+[m-2]\,dx-C_{m-1}(t)}, \tag{4.1b}$$

$$W_3\big(\Phi_m^-[m-1],\ldots,\Phi_{m+k}^-[m-1];\Phi_{N+m}^+[m-1],\ldots,\Phi_{N+m+k-1}^+[m-1],\Phi^+[m-2];C_m(t),\ldots,C_{m+k-1}(t)\big)$$
$$=\frac{W_3(\Phi_{m-1}^-[m-2],\ldots,\Phi_{m+k}^-[m-2];\Phi_{N+m-1}^+[m-2],\ldots,\Phi_{N+m+k-1}^+[m-2],\Phi^+[m-2];C_{m-1}(t),\ldots,C_{m+k-1}(t))}{C_{m-1}(t)+\int\Phi_{m-1}^-[m-2]\Phi_{N+m-1}^+[m-2]_x\,dx}, \tag{4.1c}$$

$$W_4\big(\Phi_m^-[m-1],\ldots,\Phi_{m+k-1}^-[m-1],\Phi^-[m-2];\Phi_{N+m}^+[m-1],\ldots,\Phi_{N+m+k}^+[m-1];C_m(t),\ldots,C_{m+k-1}(t)\big)$$
$$=\frac{W_4(\Phi_{m-1}^-[m-2],\ldots,\Phi_{m+k-1}^-[m-2],\Phi^-[m-2];\Phi_{N+m-1}^+[m-2],\ldots,\Phi_{N+m+k}^+[m-2];C_{m-1}(t),\ldots,C_{m+k-1}(t))}{\int\Phi_{m-1}^-[m-2]_x\Phi_{N+m-1}^+[m-2]\,dx-C_{m-1}(t)}, \tag{4.1d}$$

where

$$W_1\big(\Phi_m^-[m-1],\ldots,\Phi_{m+k}^-[m-1];\Phi_{N+m}^+[m-1],\ldots,\Phi_{N+m+k}^+[m-1];C_m(t),\ldots,C_{m+k}(t)\big)$$
$$=\det\left(\delta_{i,j}C_{m-1+i}(t)+\int\Phi_{m-1+i}^-[m-1]\Phi_{N+m-1+j}^+[m-1]_x\,dx\right),\quad i,j=1,\ldots,k+1, \tag{4.2a}$$

$$W_2\big(\Phi_m^-[m-1],\ldots,\Phi_{m+k}^-[m-1];\Phi_{N+m}^+[m-1],\ldots,\Phi_{N+m+k}^+[m-1];C_m(t),\ldots,C_{m+k}(t)\big)$$
$$=\det\left(-\delta_{i,j}C_{m-1+i}(t)+\int\Phi_{m-1+i}^-[m-1]_x\Phi_{N+m-1+j}^+[m-1]\,dx\right),\quad i,j=1,\ldots,k+1, \tag{4.2b}$$

$$W_3\big(\Phi_m^-[m-1],\ldots,\Phi_{m+k}^-[m-1];\Phi_{N+m}^+[m-1],\ldots,\Phi_{N+m+k-1}^+[m-1],\Phi^+[m-2];C_m(t),\ldots,C_{m+k-1}(t)\big)$$
$$=\det\left(\delta_{i,j}C_{m-1+i}(t)+\int\Phi_{m-1+i}^-[m-1]\Phi_{N+m-1+j}^+[m-1]_x\,dx\right)$$

and

$$\delta_{i,j}C_{m-1+i}(t)+\int\Phi_{m-1+i}^-\Phi_{N+m-1+j,x}^+\,dx=\Phi_{N+m-1+j}^+,\quad i=1,\ldots,k,\ j=1,1,\ldots,k+1, \tag{4.2c}$$

$$W_4\big(\Phi_m^-[m-1],\ldots,\Phi_{m+k-1}^-[m-1],\Phi^-[m-2];\Phi_{N+m}^+[m-1],\ldots,\Phi_{N+m+k}^+[m-1];C_m(t),\ldots,C_{m+k-1}(t)\big)$$
$$=\det\left(-\delta_{i,j}C_{m-1+i}(t)+\int\Phi_{m-1+j}^-[m-1]_x\Phi_{N+m-1+i}^+[m-1]\,dx\right)$$

and

$$-\delta_{i,j}C_{m-1+i}(t)+\int\Phi_{m-1+j,x}^-\Phi_{N+m-1+i}^+\,dx=\Phi_{m-1+j}^-,\quad i=1,\ldots,k,\ j=1,1,\ldots,k+1. \tag{4.2d}$$

Proof. With Theorem 3.4, we have

$$\Phi_{m-1+i}^-[m-1]=\Phi_{m-1+i}^-[m-2]-\frac{\Phi_{m-1}^-[m-2]\int\Phi_{N+m-1}^+[m-2]\Phi_{m-1+i}^-[m-2]_x\,dx}{\int\Phi_{m-1}^-[m-2]_x\Phi_{N+m-1}^+[m-2]\,dx-C_{m-1}(t)}, \tag{4.3a}$$

$$\Phi_{N+m-1+j}^+[m-1]=\Phi_{N+m-1+j}^+[m-2]-\frac{\Phi_{N+m-1}^+[m-2]\int\Phi_{m-1}^-[m-2]\Phi_{N+m-1+j}^+[m-2]_x\,dx}{\int\Phi_{m-1}^-[m-2]\Phi_{N+m-1}^+[m-2]_x\,dx-C_{m-1}(t)}, \tag{4.3b}$$

where $i,j=1,\ldots,k+1$. According to (4.2a) and (4.3), we have

$$\delta_{i,j}C_{m-1+i}(t)+\int\Phi_{m-1+i}^-[m-1]\Phi_{N+m-1+j}^+[m-1]_x\,dx$$
$$=\delta_{i,j}C_{m-1+i}(t)+\int\Phi_{m-1+i}^-[m-2]\Phi_{N+m-1+j}^+[m-2]_x\,dx$$

$$
\begin{aligned}
&= \delta_{i,j} C_{m-1+i}(t) + \int \Phi^-_{m-1+i}[m-1]\Phi^+_{N+m-1+j}[m-1]_x\,dx \\
&\quad - \frac{\int \Phi^-_{m-1+i}[m-2]_x \Phi^+_{N+m-1}[m-2]\,dx \int \Phi^-_{m-1}[m-2]\Phi^+_{N+m-1+j}[m-2]_x\,dx}{\int \Phi^-_{m-1}[m-2]\Phi^+_{N+m-1}[m-2]_x\,dx - C_{m-1}(t)} \\
&= \delta_{i,j} C_{m-1+i}(t) + a_{i,j} - d a_{0,j} b_{i,0},
\end{aligned}
\tag{4.4}
$$

where $a_{i,j} = \int \Phi^-_{m-1+i}[m-1]\Phi^+_{N+m-1+j}[m-1]_x\,dx$, $b_{i,0} = \int \Phi^-_{m-1+i}[m-2]_x \Phi^+_{N+m-1}[m-2]\,dx$,

$$
d = \frac{1}{\int \Phi^-_{m-1}[m-2]\Phi^+_{N+m-1}[m-2]_x\,dx - C_{m-1}(t)}.
$$

With the system of (4.2) and (4.4), we can obtain

$$
W_1\big(\Phi^-_m[m-1], \ldots, \Phi^-_{m+k}[m-1]; \Phi^+_{N+m}[m-1], \ldots, \Phi^+_{N+m+k}[m-1]; C_m(t), \ldots, C_{m+k}(t)\big)
$$

$$
= \begin{vmatrix}
C_m(t) + a_{1,1} - d a_{0,1} b_{1,0} & a_{1,2} - d a_{0,2} b_{1,0} & \cdots & a_{1,k+1} - d a_{0,k+1} b_{1,0} \\
a_{2,1} - d a_{0,1} b_{2,0} & C_{m+1}(t) + a_{2,2} - d a_{0,2} b_{2,0} & \cdots & a_{2,k+1} - d a_{0,k+1} b_{2,0} \\
\vdots & \vdots & \vdots & \vdots \\
a_{k+1,1} - d a_{0,1} b_{k+1,0} & a_{k+1,2} - d a_{0,2} b_{k+1,0} & \cdots & C_{m+k}(t) + a_{k+1,k+1} - d a_{0,k+1} b_{k+1,0}
\end{vmatrix}
$$

$$
\begin{aligned}
&= d \begin{vmatrix}
C_{m-1}(t) + a_{0,0} & a_{0,1} & a_{0,2} & \cdots & a_{0,k+1} \\
b_{1,0} & C_m(t) + a_{1,1} & a_{1,2} & \cdots & a_{1,k+1} \\
b_{2,0} & a_{2,1} & C_{m+1}(t) + a_{2,2} & \cdots & a_{2,k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
b_{k+1,0} & a_{k+1,1} & a_{k+1,2} & \cdots & C_{m+k}(t) + a_{k+1,k+1}
\end{vmatrix} \\
&= \frac{W_1(\Phi^-_{m-1}[m-2], \ldots, \Phi^-_{m+k}[m-2]; \Phi^+_{N+m-1}[m-2], \ldots, \Phi^+_{N+m+k}[m-2]; C_{m-1}(t), \ldots, C_{m+k}(t))}{C_{m-1}(t) + \int \Phi^-_{m-1}[m-2]\Phi^+_{N+m-1}[m-2]_x\,dx}.
\end{aligned}
$$

As can be seen from the above, formula (4.2a) is proved. Similarly, formulae (4.2b), (4.2c) and (4.3d) can be proved. This completes the proof. □

With Theorem 3.4 and Lemma 4.1, we can construct the N-times repeated generalized binary Darboux transformation for the mKPESCS.

Theorem 4.2. *If $u, \psi_1, \ldots, \psi_m, \phi_1, \ldots, \phi_m$ be the solution of the mKPESCS* (3.6)*, let $\Phi^-_1, \ldots, \Phi^-_N$ and $\Phi^+_{N+1}, \ldots, \Phi^+_{N+N}$ satisfy the system* (3.7) *and* (3.8) *respectively, $C_1(t), \ldots, C_N(t)$ are N arbitrary functions in t. Then the N-times repeated generalized binary Darboux transformation for the system* (3.7) *is given by*

$$
u[+N, -N] = u + \partial \ln\left(\frac{W_1(\Phi^-_1, \ldots, \Phi^-_N; \Phi^+_{N+1}, \ldots, \Phi^+_{N+N}; C_1(t), \ldots, C_N(t))_x}{W_1(\Phi^-_1, \ldots, \Phi^-_N; \Phi^+_{N+1}, \ldots, \Phi^+_{N+N}; C_1(t), \ldots, C_N(t))}\right), \tag{4.5a}
$$

$$
\Phi^-[+N, -N] = \frac{W_4(\Phi^-_1, \ldots, \Phi^-_N, \Phi^-; \Phi^+_{N+1}, \ldots, \Phi^+_{N+N}; C_1(t), \ldots, C_N(t))}{W_2(\Phi^-_1, \ldots, \Phi^-_N; \Phi^+_{N+1}, \ldots, \Phi^+_{N+N}; C_1(t), \ldots, C_N(t))}, \tag{4.5b}
$$

$$
\phi_i[+N, -N] = \frac{W_3(\Phi^-_1, \ldots, \Phi^-_N; \Phi^+_{N+1}, \ldots, \Phi^+_{N+N}, \phi_i; C_1(t), \ldots, C_N(t))}{W_1(\Phi^-_1, \ldots, \Phi^-_N; \Phi^+_{N+1}, \ldots, \Phi^+_{N+N}; C_1(t), \ldots, C_N(t))}, \tag{4.5c}
$$

$$
\psi_i[+N, -N] = \frac{W_4(\Phi^-_1, \ldots, \Phi^-_N, \psi_i; \Phi^+_{N+1}, \ldots, \Phi^+_{N+N}; C_1(t), \ldots, C_N(t))}{W_2(\Phi^-_1, \ldots, \Phi^-_N; \Phi^+_{N+1}, \ldots, \Phi^+_{N+N}; C_1(t), \ldots, C_N(t))}, \tag{4.5d}
$$

$$
\phi_{m+j}[+N, -N] \tag{4.5e}
$$

$$
= \sqrt{\dot{C}(j,t)}\,\frac{W_3(\Phi^-_1, \ldots, \Phi^-_{j-1}, \Phi^-_{j+1}, \ldots, \Phi^-_N; \Phi^+_{N+1}, \ldots, \Phi^+_{N+j-1}, \Phi^+_{N+j+1}, \ldots, \Phi^+_{N+N}, \Phi^+_{N+j}; C_1(t), \ldots, C_{j-1}(t), C_{j+1}(t), \ldots, C_N(t))}{W_1(\Phi^-_1, \ldots, \Phi^-_N; \Phi^+_{N+1}, \ldots, \Phi^+_{N+N}; C_1(t), \ldots, C_N(t))}, \tag{4.5f}
$$

$$\psi_{m+j}[+N,-N] = -\sqrt{\dot{C}(j,t)}\frac{W_4(\Phi_1^-,\dots,\Phi_{j-1}^-,\Phi_{j+1}^-,\dots,\Phi_N^-,\Phi_j^-;\Phi_{N+1}^+,\dots,\Phi_{N+j-1}^+,\Phi_{N+j+1}^+,\dots,\Phi_{N+N}^+;C_1(t),\dots,C_{j-1}(t),C_{j+1}(t),\dots,C_N(t))}{W_2(\Phi_1^-,\dots,\Phi_N^-;\Phi_{N+1}^+,\dots,\Phi_{N+N}^+;C_1(t),\dots,C_N(t))},$$
$$i=1,2,\dots,m,\quad j=1,2,\dots,N \tag{4.5g}$$

transforms (3.6b), (3.6c) *and* (3.7), *respectively, into*

$$\alpha\psi_i[+N,-N]_y = \psi_i[+N,-N]_{xx} + 2u[+N,-N]\psi_i[+N,-N]_x, \tag{4.6a}$$

$$\alpha\phi_i[+N,-N]_y = -\phi_i[+N,-N]_{xx} + 2u[+N,-N]\phi_i[+N,-N]_x, \quad i=1,2,\dots,m, \tag{4.6b}$$

$$\alpha\Phi^-[+N,-N]_y = \Phi^-[+N,-N]_{xx} + 2u[+N,-N]\Phi^-[+N,-N]_x, \tag{4.6c}$$

$$\Phi^-[+N,-N]_t = A^-\big(u[+N,-N]\big)\Phi^-[+N,-N] + F_m^-\big(\psi[+N,-N],\phi[+N,-N]\big)\Phi^-[+N,-N]. \tag{4.6d}$$

So $u[+N,-N]$, $\psi_i[+N,-N]$, $\phi_i[+N,-N]$ $(i=1,\dots,N+m)$ *is a new solution of the mKPESCS* (3.6) *with degree* $N+m$.

Proof. By the system of (3.18) and (4.1), we can obtain

$$\begin{aligned}
u[N] &= u[N-1] + \partial\ln\left(\frac{\Phi_N^-[N-1]\Phi_{N+N}^+[N-1]_x}{C_N(t)+\int\Phi_N^-[N-1]\Phi_{N+N}^+[N-1]_x\,dx}\right)\\
&= u[N-1] + \partial\ln\left(\frac{W_1(\Phi_N^-[N-1];\Phi_{N+N}^+[N-1];C_N(t))_x}{W_1(\Phi_N^-[N-1];\Phi_{N+N}^+[N-1];C_N(t))}\right)\\
&= u[N-2] + \partial\ln\left(\frac{W_1(\Phi_{N-1}^-[N-2],\Phi_N^-[N-2];\Phi_{N+N-1}^+[N-2],\Phi_{N+N}^+[N-2];C_{N-1}(t),C_N(t))_x}{W_1(\Phi_{N-1}^-[N-2],\Phi_N^-[N-2]);\Phi_{N+N-1}^+[N-2],\Phi_{N+N}^+[N-2];C_{N-1}(t),C_N(t)}\right)\\
&= \dots = u + \partial\ln\left(\frac{W_1(\Phi_1^-,\dots,\Phi_N^-;\Phi_{N+1}^+,\dots,\Phi_{N+N}^+;C_1(t),\dots,C_N(t))_x}{W_1(\Phi_1^-,\dots,\Phi_N^-;\Phi_{N+1}^+,\dots,\Phi_{N+N}^+;C_1(t),\dots,C_N(t))}\right).
\end{aligned} \tag{4.7}$$

So formula (4.5a) is proved.

$$\begin{aligned}
\Phi^-[N] &= \Phi^-[N-1] - \frac{\Phi_N^-[N-1]\int\Phi_{N+N}^+[N-1]\Phi^-[N-1]_x\,dx}{\int\Phi_N^-[N-1]_x\Phi_{N+N}^+[N-1]\,dx - C_N(t)}\\
&= \frac{W_4(\Phi_N^-[N-1],\Phi^-[N-1];\Phi_{N+N}^+[N-1];C_N(t))}{W_2(\Phi_N^-[N-1];\Phi_{N+N}^+[N-1];C_N(t))}\\
&= \frac{W_4(\Phi_{N-2}^-[N-2],\Phi_N^-[N-2],\Phi^-[N-2];\Phi_{N+N-1}^+[N-2],\Phi_{N+N}^+[N-2];C_{N-1}(t),C_N(t))}{W_2(\Phi_{N-1}^-[N-2],\Phi_N^-[N-2];\Phi_{N+N-1}^+[N-2],\Phi_{N+N}^+[N-2];C_{N-1}(t),C_{N-1}(t))}\\
&= \dots = \frac{W_4(\Phi_1^-,\dots,\Phi_N^-,\Phi^-;\Phi_{N+1}^+,\dots,\Phi_{N+N}^+;C_1(t),\dots,C_N(t))}{W_2(\Phi_1^-,\dots,\Phi_N^-[m-1];\Phi_{N+1}^+,\dots,\Phi_{N+N}^+;C_1(t),\dots,C_N(t))}.
\end{aligned} \tag{4.8}$$

So formula (4.5b) is proved. Similarly, we can prove (4.5c) and (4.5d) hold.

With the (3.18b) and (3.18e), we have

$$\psi_{m+j}[j] = -\frac{\sqrt{\dot{C}_j(t)}\Phi_j^-[j-1]}{\int\Phi_j^-[j-1]_x\Phi_{N+j}^+[j-1]\,dx - C_j(t)},$$

$$\Phi_j^-[j] = \Phi_j^-[j-1] - \frac{\Phi_j^-[j-1]\int\Phi_{N+j}^+[j-1]\Phi_j^-[j-1]_x\,dx}{\int\Phi_j^-[j-1]_x\Phi_{N+j}^+[j-1]\,dx - C_j(t)} = -\frac{C_j(t)\Phi_j^-[j-1]}{\int\Phi_j^-[j-1]_x\Phi_{N+j}^+[j-1]\,dx - C_j(t)},$$

then

$$\psi_{m+j}[j] = \frac{\sqrt{\dot{C}_j(t)}}{C_j(t)}\Phi_j^-[j].$$

So, we have

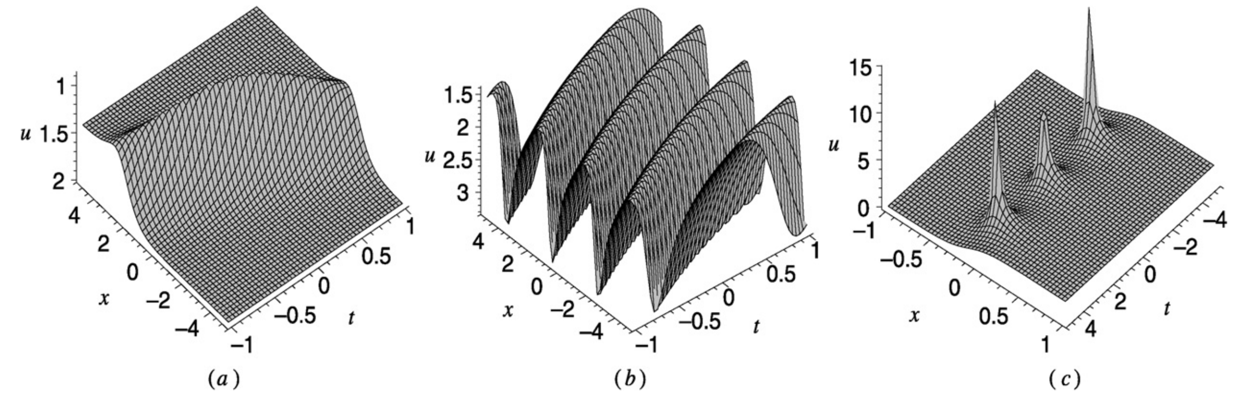

Fig. 1. (a), (b) and (c) showing the module of $u[+1,-1]$ at $y=0$ are the soliton solutions of Eqs. (5.1) with $\phi_1[+1,-1]=\frac{\sqrt{t}e^{(1+i)x-2y+\frac{t^2}{2i}+2(1+i)t}}{(1+i)e^{2x-4y+2(1+i)t}+2ie^{\frac{t^2}{i}}}$ and $\psi_1[+1,-1]=\frac{2\sqrt{t}e^{(1-i)x-2y+\frac{t^2}{2i}+2(1-i)t}}{2ie^{\frac{t^2}{i}}-(1-i)e^{2x-4y+4t}}$, $\phi_1[+1,-1]=\frac{\sqrt{t}e^{(i-1)x+2y+\frac{t^2}{2i}+6t}}{(i-1)e^{2ix+4y+2(3+i)t}-2e^{\frac{t^2}{i}}}$ and $\psi_1[+1,-1]=-\frac{2i\sqrt{t}e^{(1+i)x+2y+\frac{t^2}{2i}+2it}}{2e^{\frac{t^2}{i}}+(1+i)e^{2ix+4y+2(3+i)t}}$, $\phi_1[+1,-1]=\frac{\sqrt{t}e^{2ix-4iy+\frac{t^2}{2i}-(7-i)t}}{2ie^{2ix-4iy-8t}-2e^{\frac{t^2}{i}}}$ and $\psi_1[+1,-1]=-i\sqrt{t}e^{-(1+i)t}$, respectively.

$$\begin{aligned}
\psi_{m+j}[N] &= \frac{W_4(\Phi_N^-[N-1],\psi_{m+j}[N-1];\Phi_{N+N}^+[N-1];C_N(t))}{W_2(\Phi_N^-[N-1];\Phi_{N+N}^+[N-1];C_N(t))} \\
&= \frac{W_4(\Phi_{j+1}^-[j],\dots,\Phi_N^-[j],\psi_{m+j}[j];\Phi_{N+j+1}^+[j],\dots,\Phi_{N+N}^+[j];C_{j+1}(t),\dots,C_N(t))}{W_2(\Phi_{j+1}^-[j],\dots,\Phi_N^-[j];\Phi_{N+j+1}^+[j],\dots,\Phi_{N+N}^+[j];C_{j+1}(t),\dots,C_N(t))} \\
&= \frac{\sqrt{\dot{C}_j(t)}}{C_j(t)}\frac{W_4(\Phi_1^-,\dots,\Phi_N^-,\Phi_j^-;\Phi_{N+1}^-,\dots,\Phi_{N+N}^-;C_1(t),\dots,C_N(t))}{W_2(\Phi_1^-,\dots,\Phi_N^-;\Phi_{N+1}^-,\dots,\Phi_{N+N}^-;C_1(t),\dots,C_N(t))} \\
&= -\sqrt{\dot{C}_j(t)}\frac{W_4(\Phi_1^-,\dots,\Phi_{j-1}^-,\Phi_{j+1}^-,\dots,\Phi_N^-,\Phi_j^-;\Phi_{N+1}^+,\dots,\Phi_{N+j-1}^+,\Phi_{N+j+1}^+,\dots,\Phi_{N+N}^+;C_1(t),\dots,C_{j-1}(t),C_{j+1}(t),\dots,C_N(t))}{W_2(\Phi_1^-,\dots,\Phi_N^-;\Phi_{N+1}^+,\dots,\Phi_{N+N}^+;C_1(t),\dots,C_N(t))}.
\end{aligned} \tag{4.9}$$

So formula (4.5f) is proved. Similarly, we can prove (4.5e). This completes the proof. □

Remark. If $C_j(t)$ $(j=1,\dots,N)$ are constants, we can obtain the N-times repeated Darboux transformation for (3.14). If $C_j(t)$ $(j=1,\dots,N)$ are not constants, the DT (4.5) provides a non-auto-Bäcklund transformation between two mKPESCSs (3.6) with degrees m and $N+m$, respectively.

5. Some examples of solutions for the mKPESCS

5.1. Soliton solution

Example 1 *(Single-soliton solution for mKPIESCS).* If we set $\alpha=i$ in system of (3.6), we get the mKPIESCS

$$4u_t+u_{xxx}-6u^2u_x+6iu_x\left(\partial^{-1}u_y\right)-3\left(\partial^{-1}u_{yy}\right)+4\sum_{i=1}^m(\psi_i\phi_i)_x=0, \tag{5.1a}$$

$$i\psi_{i,y}=\psi_{i,xx}+2u\psi_{i,x}, \tag{5.1b}$$

$$i\phi_{i,y}=-\phi_{i,xx}+2u\phi_{i,x},\quad i=1,2,\dots,m. \tag{5.1c}$$

We take $u=0$ as the initial solution for the mKPIESCS with $m=0$ and let

$$\Phi_1^-=e^{-ikx+ik^2y-k^3t},\qquad \Phi_2^+=e^{i\bar{k}x-i\bar{k}^2y+\bar{k}^3t},\qquad C(t)=ie^{2f(t)}, \tag{5.2}$$

where $k=a+ib$, $a,b\in\mathbb{R}$, $b\neq 0$, $f(t)$ is an arbitrary function in t. Then

$$\Phi_1^-=e^{\alpha+\beta},\qquad \Phi_2^+=e^{\alpha-\beta},$$

where

$$\alpha=bx-2aby-a\left(a^2-3b^2\right)t,\qquad \beta=-iax+i\left(a^2-b^2\right)y-ib\left(3a^2-b\right)t.$$

With the system of (3.18), we can get the 1-soliton solution of single-soliton the mKPIESCS (5.1) with $m=1$ (see Fig. 1).

$$u[+1,-1]=\partial\ln\left(\frac{\Phi_1^-\Phi_{2,x}^+}{C(t)+\int\Phi_1^-\Phi_{2,x}^+\,dx}\right)=\frac{be^{2\alpha}+4ib^2e^{2f(t)}-iae^{2\alpha}}{2ibe^{2f(t)}+e^{2\alpha}},\tag{5.3a}$$

$$\phi_1[+1,-1]=\frac{\sqrt{\dot C(t)}\Phi_2^+}{C(t)+\int\Phi_1^-\Phi_{2,x}^+\,dx}=\frac{\sqrt{2i\dot f(t)}e^{f(t)+\alpha-\beta}}{(b+ia)e^{2\alpha}+2ibe^{2f(t)}},\tag{5.3b}$$

$$\psi_1[+1,-1]=-\frac{\sqrt{\dot C(t)}\Phi_1^-}{\int\Phi_{1,x}^-\Phi_2^+\,dx-C(t)}=\frac{2b\sqrt{2i\dot f(t)}e^{f(t)+\alpha+\beta}}{2ibe^{2f(t)}-(b-ia)e^{2\alpha}}.\tag{5.3c}$$

From graph of the module $u[+1,-1]$ at $y=0$ is plotted in Fig. 1, we find that different sources $\{\phi_1[+1,-1],\psi_1[+1,-1]\}$ have different soliton wave solutions which mainly affect the shape of the soliton wave solutions. From Eq. (5.3a), we can find that the solution only have some singularities when $2ibe^{2(f(t)-\alpha)}=-1$. Otherwise, the solution $u[+1,-1]$ is always free of singularities.

More generally, if we take

$$\Phi_j^-=e^{-ik_jx+ik_j^2y-k_j^3t},\qquad \Phi_{N+j}^+=e^{i\bar k_jx-i\bar k_j^2y+\bar k_j^3t},\qquad C_j(t)=ie^{2f_j(t)},\quad j=1,2,\ldots,N,\tag{5.4}$$

where $k_j=a_j+ib_j$, $\alpha_j,\beta_j\in\mathbb{C}$, $k_i\neq\bar k_j$, $\forall i,j$, $\beta_j\neq 0$, $f_j(t)$ $(j=1,\ldots,N)$ is an arbitrary function in t. So, with (4.5a), (4.5e) and (4.5f), we can obtain the n-soliton solution for the mKPIESCS with $m=N$.

Example 2 *(Two-soliton solution for the mKPIIESCS).* If we set $\alpha=1$ in system of (3.6), we get the mKPIIESCS

$$4u_t+u_{xxx}-6u^2u_x+6u_x\big(\partial^{-1}u_y\big)+3\big(\partial^{-1}u_{yy}\big)+4\sum_{i=1}^m(\psi_i\phi_i)_x=0,\tag{5.5a}$$

$$\psi_{i,y}=\psi_{i,xx}+2u\psi_{i,x},\tag{5.5b}$$

$$\phi_{i,y}=-\phi_{i,xx}+2u\phi_{i,x},\quad i=1,2,\ldots,m.\tag{5.5c}$$

We take $u=0$ as the initial solution for the mKPIESCS with $m=0$ and let

$$\begin{aligned}&\Phi_i^-=e^{\alpha_i},\qquad \Phi_{2+i}^+=e^{\beta_i},\qquad C_1(t)=\frac{l_1e^{2f(t)}}{k_1+l_1},\qquad C_2(t)\equiv\frac{l_2C}{k_2+l_2},\\&\alpha_i=k_ix+k_i^2y-k_i^3t,\qquad \beta_i=l_ix+l_i^2y-l_i^3t,\quad i=1,2;\end{aligned}\tag{5.6}$$

where k_i, $l_i\in\mathbb{C}$, $f(t)$ is an arbitrary function in t and C is an arbitrary constant. From (4.5a), (4.5e) and (4.5f), we get the 2-soliton solution of the mKPIIESCS (5.5) with $m=1$ (see Fig. 2) as follows:

$$\begin{aligned}u[+2,-2]&=\partial\ln\left(\frac{\begin{vmatrix}C_1(t)+\int\Phi_1^-\Phi_{3,x}^+\,dx & \int\Phi_1^-\Phi_{4,x}^+\,dx\\ \int\Phi_2^-\Phi_{3,x}^+\,dx & C_2(t)+\int\Phi_2^-\Phi_{4,x}^+\,dx\end{vmatrix}_x}{\begin{vmatrix}C_1(t)+\int\Phi_1^-\Phi_{3,x}^+\,dx & \int\Phi_1^-\Phi_{4,x}^+\,dx\\ \int\Phi_2^-\Phi_{3,x}^+\,dx & C_2(t)+\int\Phi_2^-\Phi_{4,x}^+\,dx\end{vmatrix}}\right)\\
&=\partial\ln\left(\frac{\begin{vmatrix}\frac{l_1e^{2f(t)}}{k_1+l_1}+\frac{l_1e^{\alpha_1+\beta_1}}{k_1+l_1} & \frac{l_2e^{\alpha_1+\beta_2}}{k_1+l_2}\\ \frac{l_1e^{\alpha_2+\beta_1}}{k_2+l_1} & \frac{l_2C}{k_2+l_2}+\frac{l_2e^{\alpha_2+\beta_2}}{k_2+l_2}\end{vmatrix}_x}{\begin{vmatrix}\frac{l_1e^{2f(t)}}{k_1+l_1}+\frac{l_1e^{\alpha_1+\beta_1}}{k_1+l_1} & \frac{l_2e^{\alpha_1+\beta_2}}{k_1+l_2}\\ \frac{l_1e^{\alpha_2+\beta_1}}{k_2+l_1} & \frac{l_2C}{k_2+l_2}+\frac{l_2e^{\alpha_2+\beta_2}}{k_2+l_2}\end{vmatrix}}\right)\\
&=\partial\ln\left(\frac{(k_1+l_1)e^{\alpha_1+\beta_1-2f(t)}+(k_2+l_2)e^{\alpha_2+\beta_2-C_0}+\frac{(l_1-l_2)(k_1-k_2)(k_1+l_1+k_2+l_2)}{(k_1+l_2)(k_2+l_1)}e^{\alpha_1+\beta_1+\alpha_2+\beta_2-2f(t)-C_0}}{1+e^{\alpha_1+\beta_1-2f(t)}+e^{\alpha_2+\beta_2-C_0}+\frac{(l_1-l_2)(k_1-k_2)}{(k_1+l_2)(k_2+l_1)}e^{\alpha_1+\beta_1+\alpha_2+\beta_2-2f(t)-C_0}}\right),\end{aligned}\tag{5.7a}$$

$$\begin{aligned}\phi_1[+2,-2]&=\frac{\sqrt{\dot C_1(t)}W_4(\Phi_2^-;\Phi_4^+,\Phi_3^+;C_2(t))}{W_1(\Phi_1^-,\Phi_2^-;\Phi_3^+,\Phi_4^+;C_1(t),C_2(t))}\\
&=\frac{\sqrt{\frac{2l_1\dot f(t)}{k_1+l_1}}e^{2f(t)}\big(1-\frac{k_1e^{\alpha_1+\beta_1-2f(t)}}{l_1}-\frac{k_2e^{\alpha_2+\beta_2-2f(t)-C_0}}{l_2}+\frac{k_0}{k_1+l_2}e^{\alpha_1+\beta_1+\alpha_2+\beta_2-2f(t)-C_0}\big)}{\big(1+e^{\alpha_1+\beta_1-2f(t)}+e^{\alpha_2+\beta_2-C_0}+\frac{(l_1-l_2)(k_1-k_2)}{(k_1+l_2)(k_2+l_1)}e^{\alpha_1+\beta_1+\alpha_2+\beta_2-2f(t)-C_0}\big)},\end{aligned}\tag{5.7b}$$

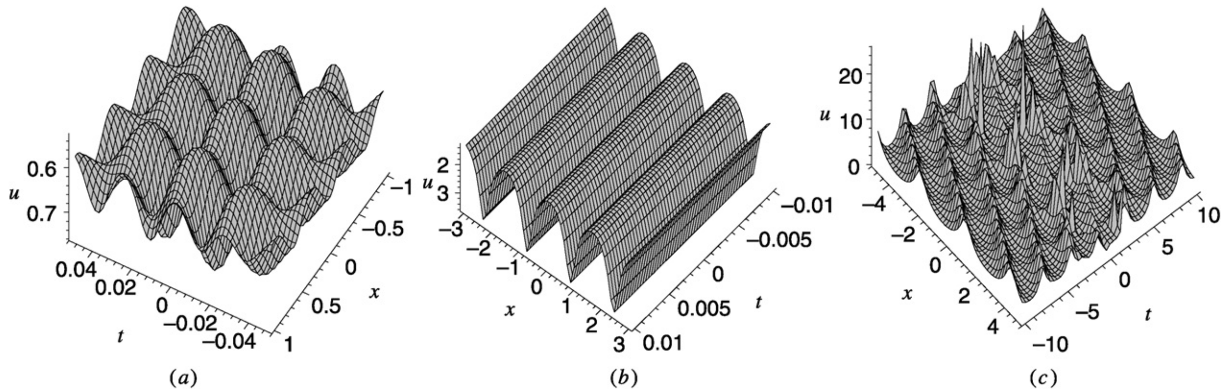

Fig. 2. (a), (b) and (c) showing the module of $u[+2,-2]$ at $y=0$ are the soliton solutions of Eqs. (5.5) with $\phi_2[+2,-2]=-\frac{\sqrt{-6t}e^{-6t^2}(-1+e^{\xi_1^{(1)}}+e^{\xi_1^{(2)}}-32ie^{\xi_1^{(3)}})}{1+e^{\xi_1^{(1)}}+e^{\xi_1^2}+16e^{\xi_1^3}}$ and $\psi_2[+2,-2]=\frac{12\sqrt{it}e^{-3t^2}(-1-30e^{\xi_1^{(2)}})}{1-e^{\xi_1^{(1)}}+e^{\xi_1^{(2)}}+64e^{e^{\xi_1^{(3)}}}}$, $\phi_2[+2,-2]=\frac{-3\sqrt{2t}e^{2t^2}(-3+3e^{\xi_2^{(1)}}+3e^{\xi_2^{(2)}}-ie^{\xi_2^{(3)}})}{9+9e^{\xi_2^{(1)}}+9e^{\xi_2^{(2)}}+e^{\xi_2^{(3)}}}$ and $\psi_2[+2,-2]=4\frac{\sqrt{it}e^{t^2}(1-4e^{\xi_2^{(2)}})}{-1+e^{\xi_2^{(1)}}-e^{\xi_2^{(2)}}-e^{e^{\xi_2^{(3)}}}}$, $\phi_2[+2,-2]=-\frac{\sqrt{2}\sqrt{t}e^{2t^2}(-1+e^{\xi_2^{(1)}}+e^{\xi_3^{(1)}}+9ie^{\xi_3^{(2)}})}{1+e^{\xi_2^{(1)}}+e^{\xi_3^{(1)}}+9e^{\xi_3^{(2)}}}$ and $\psi_2[+2,-2]=\frac{4\sqrt{it}e^{t^2}(-1-4e^{\xi_3^{(1)}})}{1-e^{\xi_2^{(1)}}+e^{\xi_3^{(1)}}-9i^2e^{e^{\xi_3^{(2)}}}}$ where $\xi_1^{(1)}=-6ix-18y-54it+6t^2$, $\xi_1^{(2)}=10ix-50y+250it+6t^2$, $\xi_1^{(3)}=4ix-68y+196it+6t^2$, $\xi_2^{(1)}=2ix-2y+2it-2t^2$, $\xi_2^{(2)}=4ix-8y+16it-2t^2$, $\xi_2^{(3)}=6ix-10y+18it-2t^2$, respectively.

$$\psi_1[+2,-2]=-\frac{\sqrt{\dot{C}_1(t)}W_3(\Phi_2^-,\Phi_1^-;\Phi_4^+;C_2(t))}{W_2(\Phi_1^-,\Phi_2^-;\Phi_3^+,\Phi_4^+;C_1(t),C_2(t))}$$
$$=-\frac{\sqrt{2l_1\dot{f}(t)}e^{f(t)}(k_2+l_2)(1+\frac{l_2(k_1+l_1)}{l_1(k_2+l_2)}e^{\alpha_2+\beta_2-2f(t)})}{l_2C(1-\frac{k_1e^{\alpha_1+\beta_1-2f(t)}}{l_1}-\frac{k_2e^{\alpha_2+\beta_2-2f(t)-C_0}}{l_2}+\frac{k_0}{k_1+l_2}e^{\alpha_1+\beta_1+\alpha_2+\beta_2-2f(t)-C_0})},\tag{5.7c}$$

$\phi_2[+2,-2]=\psi_2[+2,-2]=0$, where $C_0=\ln C$, $k_0=\frac{k_1^2k_2^2+k_1k_2^2l_2+k_1^2k_2l_1-k_1^2k_2l_2-k_1k_2^2l_1-k_1^2l_2^2}{l_1l_2}$.

From graph of the module $u[+2,-2]$ at $y=0$ is plotted in Fig. 2, we find that different sources $\{\phi_2[+2,-2],\psi_2[+2,-2]\}$ have different soliton wave solutions which mainly affect the shape of the soliton solutions. From Eq. (5.7a), we can find that the solution only have some singularities when $e^{\alpha_1+\beta_1-2f(t)}+e^{\alpha_2+\beta_2-C_0}+\frac{(l_1-l_2)(k_1-k_2)}{(k_1+l_2)(k_2+l_1)}e^{\alpha_1+\beta_1+\alpha_2+\beta_2-2f(t)-C_0}=-1$. Otherwise, the solution $u[+2,-2]$ is always free of singularities.

In a similar way, for $\forall m$, $N\in\mathbb{N}$, $N>m$, when $C_j(t)$, $j=1,\dots,m$, are taken to be arbitrary functions in t and $C_j(t)$, $j=m+1,\dots,N$, are taken to be numerical constants, we can get the N-soliton solution of (5.5) with degree m.

5.2. Rational solution

Example 3 *(Lump solution for the mKPIESCS).* We take $u=0$, $\phi_1=ae^{ikx+ik^2y}$, $\psi_1=be^{-ikx-ik^2y}$ as the initial solution of (5.1) with $N=1$ and let

$$\Phi_1^-=\left(x-2ly+3l^2t-\frac{abkti}{(k+l)^2}\right)e^{-ilx+il^2y-il^3t-\frac{ablt}{k+l}},\qquad \Phi_2^+=e^{-ilx-il^2y-il^3t+\frac{ablt}{l-k}},\tag{5.8}$$

where $l\neq\pm k$, $a,b,k,l\in\mathbb{R}$ and $C(t)=0$; then by DT (3.18), we get the 1-lump solution for the mKPIESCS (5.1) with $m=1$ as follows

$$u[+1,-1]=\partial\ln\left(\frac{\Phi_1^-\Phi_{2,x}^+}{C(t)+\int\Phi_1^-\Phi_{2,x}^+dx}\right)=1-il-l^2-\frac{il}{x-2ly+3l^2t-\frac{i}{l}-\frac{abkti}{(k+l)^2}},\tag{5.9a}$$

$$\phi_1[+1,-1]=ae^{ikx+ik^2y}-\left(1-\frac{k-2l}{(k-l)(1-2iyl^2+3il^3t-ilx+\frac{abktl}{(k+l)^2})}\right)e^{ikx+ik^2y-ilx-il^2y-il^3t+\frac{ablt}{l-k}},\tag{5.9b}$$

$$\psi_1[+1,-1]=be^{-ikx-ik^2y}+\frac{4ibkl(x-2ly+3l^2t-\frac{abkti}{(k+l)^2}e^{-ilx+il^2y-il^3t-\frac{ablt}{k+l}})}{(k+l)(1-4il^2y+6il^3t+\frac{2abktl}{(k+l)^2})}.\tag{5.9c}$$

From Eq. (5.9a), we can find that the solution only have some singularities when $x-2ly+3l^2t-\frac{i}{l}-\frac{abkti}{(k+l)^2}=0$. Otherwise, the solution $u[+2,-2]$ is always free of singularities.

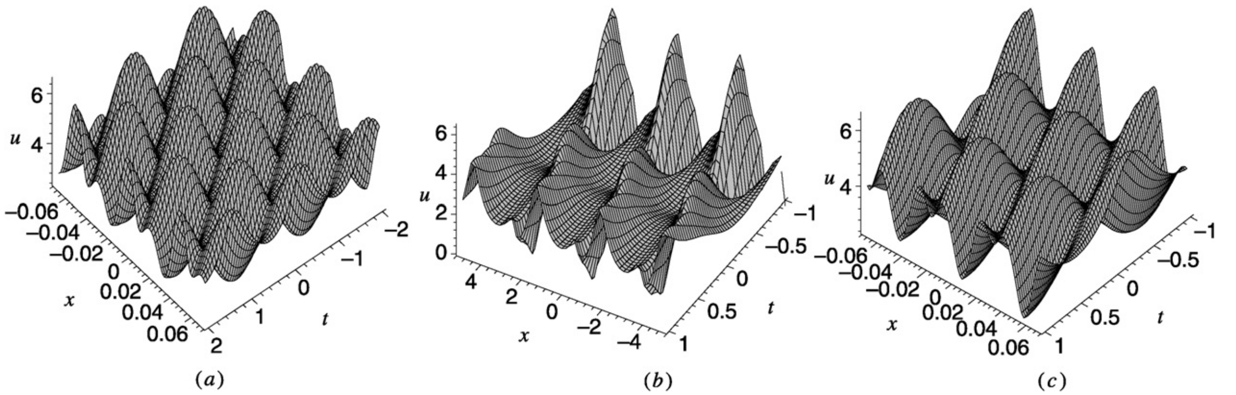

Fig. 3. (a), (b) and (c) showing the module of $u[+2,-2]$ at $y=0$ are the solutions of breather type of Eqs. (5.1) with $\phi_2[+2,-2]=\frac{4e^{-ix+y+it}}{4t+e^{-4ix+2y+28it}}$ and $\psi_2[+2,-2]=\frac{4e^{-3i(x+iy-9t)}}{4t-3e^{-4ix+2y+28it}}$, $\phi_2[+2,-2]=\frac{2e^{-3ix-3y+27it}}{2t-3e^{2ix-8iy-98it}}$ and $\psi_2[+2,-2]=\frac{2e^{5ix-5y-125it}}{2t-5e^{2ix-8iy-98it}}$, $\phi_2[+2,-2]=\frac{4e^{5ix+5y-125it}}{4t+5e^{4ix+6y-124it}}$ and $\psi_2[+2,-2]=\frac{4e^{-ix+y+it}}{4t+e^{4ix+6y-124it}}$, respectively.

More generally, if we take

$$\Phi_j^- = \left(x - 2l_j y + 3l_j^2 t - \frac{abk_j ti}{(k_j+l_j)^2}\right) e^{-il_j x + il_j^2 y - il_j^3 t - \frac{abl_j t}{k_j+l_j}}, \qquad \Phi_2^+ = e^{-il_j x - il_j^2 y - il_j^3 t + \frac{abl_j t}{l_j - k_j}}, \tag{5.10}$$

where $l_j \neq \pm k_j$, $a, b, k_j, l_j \in \mathbb{R}$ and $C_j(t)=0$, $j=1,\dots,N$. Then we will give the multi-lump solution for the mKPIESCS with $N=1$. Using the same method, we can get a similar Lump solution for the mKPIIESCS.

5.3. Solutions of breather type

Example 4 *(Solutions of breather type for the mKPIESCS).* We take $u=0$ as the initial solution for the mKPIESCS (5.1) with $N=0$. If we take

$$\Phi_i^- = e^{-i\xi_i x + i\xi_i^2 y - i\xi_i^3 t}, \qquad \Phi_{2+i}^+ = e^{i\eta_i x - i\eta_i^2 y - i\eta_i^3 t}, \qquad C_i(t) = ie^{2f_i(t)}, \quad i=1,2; \tag{5.11}$$

where $(\xi_1,\xi_2)=(k_1,l_1)$, $(\eta_1,\eta_2)=(\bar{l}_1,\bar{k}_1)$, $k_1,l_1\in\mathbb{C}$, $Im(k_1)\neq 0$, $Im(l_1)\neq 0$. (Here we take $k_1=-ai$, $l_1=-bi$, $f_1(t)=f_2(t)t$.) We will get the solutions of breather type for the mKPIESCS by (4.5a), (4.5e) and (4.5f) as follows:

$$u[+2,-2] = -\frac{(2ta+2tb+e^{-bx-biy+b^3t-ax+aiy-a^3t}a+e^{-ax-aiy+a^3t-bx-biy-b^3t}b)t(a+b)^2}{(ta+tb+e^{-bx-biy+b^3t-ax+aiy-a^3t}a)(ta+tb+e^{-ax-aiy+a^3t-bx-biy-b^3t}b)}, \tag{5.12a}$$

$$\phi_1[+2,-2] = \frac{e^{-b(x+iy+b^2t)}(a+b)}{ta+tb+\mathrm{e}^{-ax-aiy+a^3t-bx-biy-b^3t}b}, \tag{5.12b}$$

$$\psi_1[+2,-2] = -\frac{e^{a(-x-iy+a^2t)}(a+b)}{e^{-ax-aiy+a^3t-bx-biy-b^3t}a-ta-tb}, \tag{5.12c}$$

$$\phi_2[+2,-2] = \frac{e^{-a(x-iy+a^2t)}(a+b)}{ta+tb+e^{-bx-biy-b^3t-ax+aiy-a^3t}a}, \tag{5.12d}$$

$$\psi_2[+2,-2] = -\frac{e^{-b(x+iy+b^2t)}(a+b)}{e^{-bx-biy-b^3t-ax+aiy-a^3t}b-ta-tb}. \tag{5.12e}$$

From graph of the module $u[+2,-2]$ at $y=0$ is plotted in Fig. 3, we find that different sources $\{\phi_2[+2,-2],\psi_2[+2,-2]\}$ have different the breather types of solutions which mainly affect the shape of the breather solutions. From Eq. (5.12a), we can find that the solution only have some singularities when $ae^{-(a+b)x-i(a+b)y-(a^3-b^3)t}=-(a+b)t$ and

$$be^{-(a+b)x-i(a+b)y+(a^3-b^3)t}=-(a+b)t.$$

Otherwise, the solution $u[+2,-2]$ is always free of singularities.

More generally, if we take

$$\Phi_i^- = e^{-i\xi_i x + i\xi_i^2 y - i\xi_i^3 t}, \qquad \Phi_{2+i}^+ = e^{i\eta_i x - i\eta_i^2 y - i\eta_i^3 t}, \qquad C_i(t) = ie^{2f_i(t)}, \quad i=1,\dots,2N; \tag{5.13}$$

where $(\xi_1,\dots,\xi_{2N})=(k_1,\dots,k_N;l_1,\dots,l_N)$, $(\eta_1,\dots,\eta_{2N})=(\bar{l}_1,\dots,\bar{l}_N;\bar{k}_1,\dots,\bar{k}_N)$, $k_i,l_i\in\mathbb{C}$, $Im(k_i)\neq 0$, $Im(l_i)\neq 0$. Similarly, we can get the solution of breather type for the mKPIIESCS.

5.4. Mixture of the exponential solutions

Example 5 *(Mixture of the exponential for the mKPIIESCS).* We take $u = 0$ as the initial solution for the mKPIIESCS (5.5) with $N = 0$. If we take

$$\Phi_1^- = \left(x + 2py - 3p^2t\right)e^{\xi}, \qquad \Phi_2^- = e^{\xi}, \qquad \Phi_3^+ = \left(x - 2qy - 3q^2t\right)e^{\eta},$$
$$\Phi_4^+ = e^{\eta}, \qquad C_1(t) = \frac{e^{2f(t)}}{p+q}, \qquad C_2(t) \equiv \frac{C}{p+q}, \tag{5.14}$$

where $\xi = px + p^2y - p^3t$, $\eta = qx - q^2y - q^3t$ $(p, q \in \mathbb{C})$, $f(t)$ is an arbitrary function in t and C is a constant, we will obtain mixture of the exponential solution for the KPIIESCS with degree $m = 1$ as follows

$$u[+2,-2] = \partial \ln\left(\frac{x + 2py - 3p^2t}{1 + (p+q)(x + 2py - 3p^2t) + \frac{qx+2pqy-3p^2qt+e^{2f(x)-\xi-\eta}}{p+q}}\right), \tag{5.15a}$$

$$\phi_1[+2,-2] = \frac{\sqrt{\frac{2\dot f(t)}{p+q}}(p+q)^3B_0e^{f(t)}}{(p+q)^2e^{2f(t)-\eta} + [4q - (qC_0+1)(p+q) + A_0(p+q)^2 + qA_0B_0(p+q)^2]}, \tag{5.15b}$$

$$\psi_1[+2,-2] = \frac{\sqrt{\frac{2\dot f(t)}{p+q}}A_0(p+q)^2e^{f(t)-\eta}}{1 - pC_0 + (p+q)(1 + pA_0B_0 - B_0 + e^{2f(t)-\xi-\eta})}, \tag{5.15c}$$

where $A_0 = x + 2py - 3p^2t$, $B_0 = x - 2qy - 3q^2t$, $C_0 = 4x - 2qy - 3q^2t = 2ky - 3k^2t$. From Eq. (5.15a), we can find that the solution only have some singularities when $x + 2py - 3p^2t = 0$ and $1 + (p+q)(x + 2py - 3p^2t) + \frac{qx+2pqy-3p^2qt+e^{2f(x)-\xi-\eta}}{p+q} = 0$. Otherwise, the solution $u[+2,-2]$ is always free of singularities.

Similarly, we can get the solution of mixture of the exponential for the mKPIESCS.

6. Conclusion

With the constrained flows of mKP hierarchy, constructed mKP hierarchy with self-consistent sources and their Lax representation. Based on the conjugate Lax pairs, we construct the generalized binary Darboux transformation and the N-times repeated Darboux transformation with arbitrary functions at time t for the mKPESCS which offers a non-auto-Bäcklund transformation between two mKPESCSs with different degrees of sources. With the help of these transformations, some new solutions for the mKPESCSs such as soliton solutions, rational solutions, breather type solutions and exponential solutions are found by taking the special initial solution for auxiliary linear problems and the special functions of t-time. And we also have some description of these solutions. It is convinced that this approach for constructing systems with self-consistent sources and generalized Darboux transformation technique are available for other nonlinear evolution equations in mathematical physics.

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