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A sum operator equation and applications to nonlinear elastic beam equations and Lane–Emden–Fowler equations

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ABSTRACT

This paper is concerned with an operator equation $Ax + Bx + Cx = x$ on ordered Banach spaces, where A is an increasing α-concave operator, B is an increasing sub-homogeneous operator and C is a homogeneous operator. The existence and uniqueness of its positive solutions is obtained by using the properties of cones and a fixed point theorem for increasing general β-concave operators. As applications, we utilize the fixed point theorems obtained in this paper to study the existence and uniqueness of positive solutions for two classes nonlinear problems which include fourth-order two-point boundary value problems for elastic beam equations and elliptic value problems for Lane–Emden–Fowler equations.
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1. Introduction and preliminaries

Because of the crucial role played by nonlinear equations in the applied science as well as in mathematics, nonlinear functional analysis has been an active area of research, and nonlinear operators which arise in the connection with nonlinear differential and integral equations have been extensively studied over the past several decades (see, for instance, [2,3,5,23, 35–37,40]). The existence and uniqueness of positive solutions to nonlinear operator equations is very important in theory and applications. Many authors have studied this problem, for a small sample of such work, we refer the reader to works [4,6,9,21,22,27,30,38,42]. This paper considers the existence and uniqueness of positive solutions to the following operator equation on ordered Banach spaces

$$Ax + Bx + Cx = x, \tag{1.1}$$

where A is an increasing α-concave operator, B is an increasing sub-homogeneous operator and C is a homogeneous operator. To our knowledge, the fixed point results on the operator equation (1.1) with α-concave, sub-homogeneous and homogeneous operators are still very few. So it is worthwhile to investigate the operator equation (1.1). In this paper, using the properties of cones and a fixed point theorem for increasing general β-concave operators, we obtain some existence and uniqueness results of positive solutions for the operator equation (1.1). To demonstrate the applicability of our abstract results, we give, in the last two sections of the paper, some simple applications to nonlinear elastic beam equations and Lane–Emden–Fowler equations.

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In the following two subsections, we state some definitions, notations and known results. For convenience of readers, we suggest that one refer to [4,12,19,39] for details.

1.1. Some basic definitions and notations

Suppose that E is a real Banach space which is partially ordered by a cone $P \subset E$, i.e., $x \leqslant y$ if and only if $y - x \in P$. If $x \leqslant y$ and $x \neq y$, then we denote $x < y$ or $y > x$. By θ we denote the zero element of E. Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies (i) $x \in P$, $\lambda \geqslant 0 \Rightarrow \lambda x \in P$; (ii) $x \in P$, $-x \in P \Rightarrow x = \theta$.

Putting $\mathring{P} = \{x \in P \mid x \text{ is an interior point of } P\}$, a cone P is said to be solid if $\mathring{P}$ is non-empty. Moreover, P is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E$, $\theta \leqslant x \leqslant y$ implies $\|x\| \leqslant N\|y\|$; in this case N is called the normality constant of P. If $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E \mid x_1 \leqslant x \leqslant x_2\}$ is called the order interval between x_1 and x_2. We say that an operator $A : E \to E$ is increasing (decreasing) if $x \leqslant y$ implies $Ax \leqslant Ay$ ($Ax \geqslant Ay$).

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leqslant y \leqslant \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (i.e., $h \geqslant \theta$ and $h \neq \theta$), we denote by P_h the set $P_h = \{x \in E \mid x \sim h\}$. It is easy to see that $P_h \subset P$.

1.2. Some definitions of operators and properties

Definition 1.2.1. Let $D = P$ or $D = \mathring{P}$ and α be a real number with $0 \leqslant \alpha < 1$. An operator $A : P \to P$ is said to be α-concave if it satisfies

$$A(tx) \geqslant t^{\alpha} Ax, \quad \forall t \in (0, 1),\ x \in D. \tag{1.2.1}$$

Notice that the definition of an α-concave operator mentioned above is different from that in [19], because we need not require the cone to be solid in general.

Definition 1.2.2. An operator $A : E \to E$ is said to be homogeneous if it satisfies

$$A(\lambda x) = \lambda Ax, \quad \forall \lambda > 0,\ x \in E.$$

An operator $A : P \to P$ is said to be sub-homogeneous if it satisfies

$$A(tx) \geqslant tAx, \quad \forall t \in (0, 1),\ x \in P. \tag{1.2.2}$$

Theorem 1.2.3. *(See [39].) Assume that cone P is normal and operator T satisfies*:

(D_1) *$T : P_h \to P_h$ is increasing in P_h*;
(D_2) *for any $x \in P_h$ and $t \in (0, 1)$, there exists $\beta(t) \in (0, 1)$ such that $T(tx) \geqslant t^{\beta(t)} Tx$*;
(D_3) *for every $x_0 \in P$, there is a constant $l \geqslant 0$ such that $x_0 \in [\theta, lh]$.*

Then operator equation $x = Tx + x_0$ has a unique solution in P_h.

Remark 1.2.1. An operator T is said to be **general β-concave** if T satisfies (D_2).

Lemma 1.2.4. *(See [19].) Let P be a normal cone in E. The following conclusion holds*:

for every sequence $x_n, y_n, z_n \in E$ satisfying $x_n \leqslant z_n \leqslant y_n$ ($n = 1, 2, \ldots$) and $\|x_n - x\| \to 0, \|y_n - x\| \to 0$ ($n \to \infty$), then $\|z_n - x\| \to 0$ ($n \to \infty$).

2. Fixed point theorems

In this section we consider the existence and uniqueness of positive solutions for the operator equation (1.1). We always assume that E is a real Banach space with a partial order introduced by a **normal** cone P of E. Take $h \in E$, $h > \theta$, P_h is given as in Section 1.1.

Theorem 2.1. *Let $A : P \to P$ be an increasing α-concave operator, $B : P \to P$ be an increasing sub-homogeneous operator and $C : P \to P$ be an increasing homogeneous operator. Assume that*

(i) *there is $h > \theta$ such that $Ah \in P_h$, $Bh \in P_h$ and $Ch \in P_h$*;
(ii) *there exist two constants $\delta_1, \delta_2 > 0$ such that $Ax \geqslant \delta_1 Bx + \delta_2 Cx$, $\forall x \in P$.*

Then the operator equation (1.1) has a unique solution x^* in P_h. Moreover, constructing successively the sequence $y_n = Ay_{n-1} + By_{n-1} + Cy_{n-1}$, $n = 1, 2, \ldots$ for any initial value $y_0 \in P_h$, we have $y_n \to x^*$ as $n \to \infty$.

Proof. Since $Ah, Bh, Ch \in P_h$, there exist constants $\lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2 > 0$ such that

$$\lambda_1 h \leqslant Ah \leqslant \lambda_2 h, \qquad \mu_1 h \leqslant Bh \leqslant \mu_2 h, \qquad \nu_1 h \leqslant Ch \leqslant \nu_2 h.$$

Then we have

$$(\lambda_1 + \mu_1 + \nu_1)h \leqslant Ah + Bh + Ch \leqslant (\lambda_2 + \mu_2 + \nu_2)h.$$

So $Ah + Bh + Ch \in P_h$. Define an operator $T = A + B + C$ by $Tx = Ax + Bx + Cx$. Then $T : P \to P$ and $Th \in P_h$. Next we show that $T : P_h \to P_h$. From (1.2.1) and (1.2.2), we have

$$A\left(\frac{1}{t}x\right) \leqslant \frac{1}{t^\alpha}Ax \quad \text{and} \quad B\left(\frac{1}{t}x\right) \leqslant \frac{1}{t}Bx \quad \text{for } t \in (0,1),\ x \in P.$$

For any $x \in P_h$, we can choose a sufficiently small number $t_0 \in (0,1)$ such that

$$t_0 h \leqslant x \leqslant \frac{1}{t_0}h. \tag{2.1}$$

Note that $T : P \to P$ is increasing and from (2.1), we have

$$\begin{aligned} Tx &= Ax + Bx + Cx \leqslant A\left(\frac{1}{t_0}h\right) + B\left(\frac{1}{t_0}h\right) + C\left(\frac{1}{t_0}h\right) \\ &\leqslant \frac{1}{{t_0}^\alpha}Ah + \frac{1}{t_0}Bh + \frac{1}{t_0}Ch \leqslant \left(\frac{\lambda_2}{t_0^\alpha} + \frac{\mu_2}{t_0} + \frac{\nu_2}{t_0}\right)h, \\ Tx &= Ax + Bx + Cx \geqslant A(t_0 h) + B(t_0 h) + C(t_0 h) \\ &\geqslant t_0^\alpha Ah + t_0 Bh + t_0 Ch \geqslant \left(t_0^\alpha \lambda_1 + t_0\mu_1 + t_0\nu_1\right)h. \end{aligned}$$

Evidently, $\frac{\lambda_2}{t_0^\alpha} + \frac{\mu_2}{t_0} + \frac{\nu_2}{t_0}$, $t_0^\alpha\lambda_1 + t_0\mu_1 + t_0\nu_1 > 0$. Thus $Tx \in P_h$; that is, $T : P_h \to P_h$. Moreover, $A : P_h \to P_h$, $B : P_h \to P_h$ and $C : P_h \to P_h$. In the following, we show that there exists $\beta_0(t) \in (\alpha, 1)$ with respect to t such that

$$T(tx) \geqslant t^{\beta_0(t)}Tx, \quad \forall t \in (0,1),\ x \in P_h. \tag{2.2}$$

Let $\delta_0 = \min\{\delta_1, \delta_2\}$. Then from (ii), $Ax \geqslant \delta_0(Bx + Cx)$, $x \in P$.

Consider the following function:

$$f(t) = \frac{t^\beta - t}{t^\alpha - t^\beta}, \quad \forall t \in (0,1), \text{ where } \beta \in (\alpha, 1).$$

It is easy to prove that f is increasing in $(0,1)$ and

$$\lim_{t\to 0^+} f(t) = 0, \qquad \lim_{t\to 1^-} f(t) = \frac{1-\beta}{\beta-\alpha}.$$

Further, fixing $t \in (0,1)$, we have

$$\lim_{\beta\to 1^-} f(t) = \lim_{\beta\to 1^-} \frac{t^\beta - t}{t^\alpha - t^\beta} = 0.$$

So there exists $\beta_0(t) \in (\alpha, 1)$ with respect to t such that

$$\frac{t^{\beta_0(t)} - t}{t^\alpha - t^{\beta_0(t)}} \leqslant \delta_0, \quad t \in (0,1).$$

Hence we have

$$Ax \geqslant \delta_0(Bx + Cx) \geqslant \frac{t^{\beta_0(t)} - t}{t^\alpha - t^{\beta_0(t)}}(Bx + Cx), \quad \forall t \in (0,1),\ x \in P_h.$$

Then we obtain

$$t^\alpha Ax + tBx + tCx \geqslant t^{\beta_0(t)}(Ax + Bx + Cx), \quad \forall t \in (0,1),\ x \in P_h.$$

Consequently, for any $t \in (0,1)$ and $x \in P_h$,

$$\begin{aligned} T(tx) &= A(tx) + B(tx) + C(tx) \geqslant t^\alpha Ax + tBx + tCx \\ &\geqslant t^{\beta_0(t)}(Ax + Bx + Cx) = t^{\beta_0(t)}Tx. \end{aligned}$$

Let $x_0 = \theta$. An application of Theorem 1.2.3 implies that the equation $Tx = x$ has a unique solution x^* in P_h. That is, the operator equation (1.1) has a unique solution x^* in P_h.

Now we construct successively the sequence $y_n = Ay_{n-1} + By_{n-1} + Cy_{n-1}$, $n = 1, 2\ldots$ for any initial point $y_0 \in P_h$. Since $y_0 \in P_h$ and $Ty_0 \in P_h$, we can choose a sufficiently small number $t_0 \in (0, 1)$ such that

$$t_0 y_0 \leqslant Ty_0 \leqslant \frac{1}{t_0} y_0. \tag{2.3}$$

Note that $0 < \beta_0(t_0) < 1$, we can also take a positive integer k such that

$$k > \frac{1}{1 - \beta_0(t_0)}. \tag{2.4}$$

Put $u_0 = t_0^k y_0$, $v_0 = \frac{1}{t_0^k} y_0$. Evidently, $u_0, v_0 \in P_h$ and $u_0 \leqslant y_0 \leqslant v_0$.

By the monotonicity of T, we have $Tu_0 \leqslant Tv_0$. Further, combining condition (2.2) with (2.3), we have

$$\begin{aligned} Tu_0 &= T\left(t_0^k y_0\right) = T\left(t_0 \cdot t_0^{k-1} y_0\right) \geqslant t_0^{\beta_0(t_0)} T\left(t_0^{k-1} y_0\right) \\ &= t_0^{\beta_0(t_0)} T\left(t_0 \cdot t_0^{k-2} y_0\right) \geqslant t_0^{\beta_0(t_0)} \cdot t_0^{\beta_0(t_0)} T\left(t_0^{k-2} y_0\right) \geqslant \cdots \\ &\geqslant \left(t_0^{\beta_0(t_0)}\right)^k T y_0 \geqslant \left(t_0^{\beta_0(t_0)}\right)^k t_0 y_0 = t_0^{k\beta_0(t_0)+1} y_0. \end{aligned}$$

By (2.4), one obtains $k - k\beta_0(t_0) > 1$, that is, $k\beta_0(t_0) + 1 < k$. Thus

$$t_0^{k\beta_0(t_0)+1} > t_0^k. \tag{2.5}$$

Therefore,

$$Tu_0 \geqslant t_0^{k\beta_0(t_0)+1} y_0 > t_0^k y_0 = u_0.$$

By (2.2), we get

$$T\left(\frac{1}{t}x\right) \leqslant \frac{1}{t^{\beta_0(t)}} Tx, \quad \forall t \in (0, 1),\ x \in P_h.$$

Thus we have

$$\begin{aligned} Tv_0 &= T\left(\frac{1}{t_0^k} y_0\right) = T\left(\frac{1}{t_0} \cdot \frac{1}{t_0^{k-1}} y_0\right) \\ &\leqslant \frac{1}{t_0^{\beta_0(t_0)}} T\left(\frac{1}{t_0^{k-1}} y_0\right) = \frac{1}{t_0^{\beta_0(t_0)}} T\left(\frac{1}{t_0} \cdot \frac{1}{t_0^{k-2}} y_0\right) \\ &\leqslant \frac{1}{t_0^{\beta_0(t_0)}} \cdot \frac{1}{t_0^{\beta_0(t_0)}} T\left(\frac{1}{t_0^{k-2}} y_0\right) \leqslant \cdots \\ &\leqslant \frac{1}{\left(t_0^{\beta_0(t_0)}\right)^k} T y_0 \leqslant \frac{1}{t_0^{k\beta_0(t_0)+1}} y_0. \end{aligned}$$

An application of (2.5) implies

$$Tv_0 \leqslant \frac{1}{t_0^{k\beta_0(t_0)+1}} y_0 \leqslant \frac{1}{t_0^k} y_0 = v_0.$$

Thus we have $u_0 \leqslant Tu_0 \leqslant Tv_0 \leqslant v_0$. Let $u_n = Tu_{n-1}$, $v_n = Tv_{n-1}$, $n = 1, 2, \ldots$, then $u_n \leqslant y_n \leqslant v_n$. Using the same proof of Theorem 1.3 in [39], we can easily obtain that there exists $y^* \in P_h$ such that

$$Ty^* = y^*, \qquad \lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = y^*.$$

Thus from Lemma 1.2.4, $y_n \to y^*$ $(n \to \infty)$. By the uniqueness of fixed points of the sum operator T in P_h, we get $x^* = y^*$. □

From the proof of Theorem 2.1, we can easily prove the following conclusion.

Theorem 2.2. *Let $A : P \to P$ be an increasing α-concave operator and $B : P \to P$ be an increasing sub-homogeneous operator. Assume that*

(i) *there is $h > \theta$ such that $Ah \in P_h$ and $Bh \in P_h$;*
(ii) *there exists a constant $\delta_0 > 0$ such that $Ax \geqslant \delta_0 Bx$, $\forall x \in P$.*

Then operator equation $Ax + Bx = x$ has a unique solution x^* in P_h. Moreover, constructing successively the sequence $y_n = Ay_{n-1} + By_{n-1}$, $n = 1, 2, \ldots$ for any initial value $y_0 \in P_h$, we have $y_n \to x^*$ as $n \to \infty$.

Remark 2.1. (1) In Theorem 2.1, if the operator C satisfies that $C : P \to P$ is linear, then C is an increasing sub-homogeneous operator. (2) Generally speaking, a sub-homogeneous operator does not guarantee a fixed point. In [31], Nussbaum discussed the operator which is the sum of a sub-homogeneous operator and an α-concave operator, and proved the convergence property under the assumption of the existence of a normalized eigenvector. When E is a finite dimensional space, he proved the existence of the unique normalized eigenvector. Our aim here is to discuss the conditions which guarantee the existence and uniqueness of a fixed point for the sum of an α-concave operator, a sub-homogeneous operator and a homogeneous operator in a general ordered Banach space. (3) Under the conditions of Theorem 2.1, from the proof, we can prove the existence of upper and lower solutions for the sum operator $A + B + C$. Take $h \in \mathring{P}$, then $P_h = \mathring{P}$. If we suppose that operator $A : P_h \to P_h$ or $A : \mathring{P} \to \mathring{P}$ with P as a solid cone, then $Ah \in P_h$ is automatically satisfied.

This shows the following corollaries.

Corollary 2.1. *Let $h > \theta$ and $A : P_h \to P_h$ be an increasing α-concave operator, $B : P_h \to P_h$ be an increasing sub-homogeneous operator and $C : P_h \to P_h$ be an increasing homogeneous operator. Assume that there exist two constants $\delta_1, \delta_2 > 0$ such that $Ax \geqslant \delta_1 Bx + \delta_2 Cx$, $\forall x \in P_h$. Then the operator equation (1.1) has a unique solution x^* in P_h. Moreover, constructing successively the sequence $y_n = Ay_{n-1} + By_{n-1} + Cy_{n-1}$, $n = 1, 2, \ldots$ for any initial value $y_0 \in P_h$, we have $y_n \to x^*$ as $n \to \infty$.*

Corollary 2.2. *Let P be a solid cone and $A : \mathring{P} \to \mathring{P}$ be an increasing α-concave operator, $B : \mathring{P} \to \mathring{P}$ be an increasing sub-homogeneous operator and $C : \mathring{P} \to \mathring{P}$ be an increasing homogeneous operator. Assume that there exist two constants $\delta_1, \delta_2 > 0$ such that $Ax \geqslant \delta_1 Bx + \delta_2 Cx$, $\forall x \in \mathring{P}$. Then the operator equation (1.1) has a unique solution x^* in $\mathring{P}$. Moreover, constructing successively the sequence $y_n = Ay_{n-1} + By_{n-1} + Cy_{n-1}$, $n = 1, 2, \ldots$ for any initial value $y_0 \in \mathring{P}$, we have $y_n \to x^*$ as $n \to \infty$.*

Corollary 2.3. *Let $A : P \to P$ be an increasing α-concave operator, $B : P \to P$ be an increasing sub-homogeneous operator and $C : P \to P$ be an increasing homogeneous operator. Assume that*

(i) *there is $h > \theta$ such that $Ah \in P_h$, $Bh \in P_h$ and $Ch \in P_h$;*
(ii) *there exist two constants $\delta_1, \delta_2 > 0$ such that $Ax \geqslant \delta_1 Bx + \delta_2 Cx$, $\forall x \in P$.*

Then the operator equation $Ax + Bx + Cx = \lambda x$ has a unique solution x^*_λ in P_h for any given $\lambda > 0$. Moreover, constructing successively the sequence $y_n = \frac{1}{\lambda}(Ay_{n-1} + By_{n-1} + Cy_{n-1})$, $n = 1, 2, \ldots$ for any initial value $y_0 \in P_h$, we have $y_n \to x^*_\lambda$ as $n \to \infty$.

Corollary 2.4. *Let $h > \theta$ and $A : P_h \to P_h$ be an increasing α-concave operator and $B : P_h \to P_h$ be an increasing sub-homogeneous operator. Assume that there exists a constant $\delta_0 > 0$ such that $Ax \geqslant \delta_0 Bx$, $\forall x \in P_h$. Then operator equation $Ax + Bx = x$ has a unique solution x^* in P_h. Moreover, constructing successively the sequence $y_n = Ay_{n-1} + By_{n-1}$, $n = 1, 2, \ldots$ for any initial value $y_0 \in P_h$, we have $y_n \to x^*$ as $n \to \infty$.*

Corollary 2.5. *Let P be a solid cone and $A : \mathring{P} \to \mathring{P}$ be an increasing α-concave operator and $B : \mathring{P} \to \mathring{P}$ be an increasing sub-homogeneous operator. Assume that there exists a constant $\delta_0 > 0$ such that $Ax \geqslant \delta_0 Bx$, $\forall x \in \mathring{P}$. Then operator equation $Ax + Bx = x$ has a unique solution x^* in $\mathring{P}$. Moreover, constructing successively the sequence $y_n = Ay_{n-1} + By_{n-1}$, $n = 1, 2, \ldots$ for any initial value $y_0 \in \mathring{P}$, we have $y_n \to x^*$ as $n \to \infty$.*

Corollary 2.6. *Let $A : P \to P$ be an increasing α-concave operator and $B : P \to P$ be an increasing sub-homogeneous operator. Assume that*

(i) *there is $h > \theta$ such that $Ah \in P_h$ and $Bh \in P_h$;*
(ii) *there exists a constant $\delta_0 > 0$ such that $Ax \geqslant \delta_0 Bx$, $\forall x \in P$.*

Then operator equation $Ax + Bx = \lambda x$ has a unique solution x^*_λ in P_h for any given $\lambda > 0$. Moreover, constructing successively the sequence $y_n = \frac{1}{\lambda}(Ay_{n-1} + By_{n-1})$, $n = 1, 2, \ldots$ for any initial value $y_0 \in P_h$, we have $y_n \to x^*_\lambda$ as $n \to \infty$.

3. Applications to nonlinear elastic beam equations

We now illustrate how the results of Section 2 can be used in the study of certain boundary value problems. We study the existence and uniqueness of positive solutions for the following fourth-order two-point boundary value problem for elastic beam equations

$$\begin{cases} u''''(t) = f\big(t, u(t)\big), \quad 0 \leqslant t \leqslant 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = g\big(u(1)\big), \end{cases} \tag{3.1}$$

where $f \in C([0,1] \times \mathbf{R})$ and $g \in C(\mathbf{R})$ are real functions. The problem (3.1) models an elastic beam of length 1 subject to a nonlinear foundation given by the function f. The first boundary condition $u(0) = u'(0) = 0$ means that the left end of the beam is fixed. The boundary condition $u''(1) = 0$, $u'''(1) = g(u(1))$ means that the right end of the beam is attached to a bearing device, given by the function g. Fourth-order two-point boundary value problems are useful for mechanics of materials because the problems usually characterize the deformations of an elastic beam. The existence and multiplicity of positive solutions for the elastic beam equations have been studied extensively, see for example [3,7,18,28,29,35,37,40] and references therein. But in the existing literature, there are few papers concerned with the uniqueness of positive solutions. In this section, we apply Theorem 2.2 to study the fourth-order two-point boundary value problem for elastic beam equations (3.1).

In what follows, for the sake of convenience, set $E = C[0,1]$, the Banach space of continuous functions on $[0,1]$ with the norm $\|y\| = \max\{|y(t)|\colon\ t \in [0,1]\}$. $P = \{y \in C[0,1] \mid y(t) \geqslant 0,\ t \in [0,1]\}$.

$$G(t,s) = \frac{1}{6} \begin{cases} s^2(3t - s), & 0 \leqslant s \leqslant t \leqslant 1, \\ t^2(3s - t), & 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{3.2}$$

It is clear that P is a normal cone of which the normality constant is 1, P_h is given as in Section 1.1. Moreover, it is easy to prove that $G(t,s) \geqslant 0$ and

$$\frac{1}{3}t^2 s^2 \leqslant G(t,s) \leqslant \frac{1}{2}t^2 s \quad \text{for } (t,s) \in [0,1] \times [0,1]. \tag{3.3}$$

Theorem 3.1. *Assume that*

(H_1) $f(t,u) : [0,1] \times [0,+\infty) \to [0,+\infty)$ *is continuous and* $g : [0,+\infty) \to (-\infty, 0]$ *is continuous, decreasing*;
(H_2) $f(t,u)$ *is increasing in* $u \in [0,+\infty)$ *for fixed* $t \in [0,1]$ *and there exists a constant* $\alpha \in (0,1)$ *such that*

$$f(t,\lambda u) \geqslant \lambda^{\alpha} f(t,u), \quad \forall t \in [0,1],\ \lambda \in (0,1),\ u \in [0,+\infty);$$

(H_3) *for any* $\lambda \in (0,1)$ *and* $u \in [0,+\infty)$, $g(\lambda u) \leqslant \lambda g(u)$;
(H_4) $g(1) < 0$ *and there exists a constant* $\zeta > 0$ *such that*

$$-g(u) \leqslant \zeta \leqslant \frac{2}{3}\int_0^1 s^2 f(s,0)\,ds, \quad u \geqslant 0.$$

Then the problem (3.1) *has a unique positive solution* u^* *in* P_h, *where* $h(t) = t^2$, $t \in [0,1]$. *Moreover, for any* $u_0 \in P_h$, *constructing successively the sequence*

$$u_{n+1}(t) = \int_0^1 G(t,s) f\big(s, u_n(s)\big)\,ds - g\big(u_n(1)\big)\left(\frac{1}{2}t^2 - \frac{1}{6}t^3\right), \quad n = 0, 1, 2, \ldots,$$

we have $\|u_n - u^*\| \to 0$ *as* $n \to \infty$.

Proof. To begin with, it is easy to see that the problem (3.1) has an integral formulation given by

$$u(t) = \int_0^1 G(t,s) f\big(s, u(s)\big)\,ds - g\big(u(1)\big)\left(\frac{1}{2}t^2 - \frac{1}{6}t^3\right),$$

where $G(t,s)$ is given as in (3.2).

Define two operators $A : P \to E$ and $B : P \to E$ by

$$(Au)(t) = \int_0^1 G(t,s) f\big(s, u(s)\big)\, ds, \qquad (Bu)(t) = -g\big(u(1)\big)\left(\frac{1}{2}t^2 - \frac{1}{6}t^3\right).$$

It is easy to prove that u is the solution of the problem (3.1) if and only if $u = Au + Bu$. From (H_1), we know that $A : P \to P$ and $B : P \to P$. Further, it follows from (H_1), (H_2) that the two operators A and B are increasing in P. For any $\lambda \in (0,1)$ and $u \in P$, by (H_2) we obtain

$$\begin{aligned} A(\lambda u)(t) &= \int_0^1 G(t,s) f\big(s, \lambda u(s)\big)\, ds \\ &\geqslant \lambda^{\alpha} \int_0^1 G(t,s) f\big(s, u(s)\big)\, ds = \lambda^{\alpha} Au(t). \end{aligned}$$

That is, $A(\lambda u) \geqslant \lambda^{\alpha} Au$ for $\lambda \in (0,1)$, $u \in P$. So the operator A is α-concave. Also, for any $\lambda \in (0,1)$ and $u \in P$, from (H_3) we know that

$$B(\lambda u)(t) = -g\big(\lambda u(1)\big)\left(\frac{1}{2}t^2 - \frac{1}{6}t^3\right) \geqslant -\lambda g\big(u(1)\big)\left(\frac{1}{2}t^2 - \frac{1}{6}t^3\right) = \lambda Bu(t),$$

that is, $B(\lambda u) \geqslant \lambda Bu$ for $\lambda \in (0,1)$, $u \in P$. So the operator B is sub-homogeneous. Next we show that $Ah \in P_h$ and $Bh \in P_h$. In fact, from (3.3)

$$\begin{aligned} Ah(t) &= \int_0^1 G(t,s) f\big(s, h(s)\big)\, ds = \int_0^1 G(t,s) f\big(s, s^2\big)\, ds \\ &\leqslant \int_0^1 \frac{1}{2}t^2 s f(s,1)\, ds = t^2 \cdot \frac{1}{2}\int_0^1 s f(s,1)\, ds, \end{aligned}$$

and

$$Ah(t) \geqslant \int_0^1 \frac{1}{3}t^2 s^2 f(s,0)\, ds = t^2 \cdot \frac{1}{3}\int_0^1 s^2 f(s,0)\, ds.$$

From (H_4), we have

$$\int_0^1 s f(s,1)\, ds \geqslant \int_0^1 s^2 f(s,0)\, ds > 0,$$

and thus $Ah \in P_h$. Since $g(1) < 0$ and

$$\begin{aligned} Bh(t) &= -g\big(h(1)\big)\left(\frac{1}{2}t^2 - \frac{1}{6}t^3\right) \leqslant -g(1) \cdot \frac{1}{2}t^2, \\ Bh(t) &= -g\big(h(1)\big)\left(\frac{1}{2}t^2 - \frac{1}{6}t^3\right) \geqslant -g(1) \cdot \frac{1}{3}t^2, \end{aligned}$$

we have $Bh \in P_h$. Hence the condition (i) of Theorem 2.2 is satisfied. In the following we show the condition (ii) of Theorem 2.2 is also satisfied. For $u \in P$, from (3.3) and (H_4),

$$\begin{aligned} (Au)(t) &= \int_0^1 G(t,s) f\big(s, u(s)\big)\, ds \geqslant \frac{1}{3}t^2 \int_0^1 s^2 f(s,0)\, ds \\ &\geqslant \frac{1}{2}t^2 \zeta \geqslant -g\big(u(1)\big)\left(\frac{1}{2}t^2 - \frac{1}{6}t^3\right) = (Bu)(t). \end{aligned}$$

Take $\delta_0 \in (0,1]$. Then we get $Au \geqslant \delta_0 Bu$, $u \in P$. So the conclusion of Theorem 3.1 follows from Theorem 2.2. □

Remark 3.1. (1) There exist many functions which satisfy (H_1)–(H_4). For example, (i) let $f(t,u) = u^\alpha + \varphi(t)$ and

$$g(u) = \begin{cases} -u - a, & 0 \leqslant u \leqslant b, \\ -a - b, & u \geqslant b, \end{cases}$$

where $\alpha \in (0,1)$, $a, b > 0$ and $\varphi : [0,1] \to [0,+\infty)$ is continuous, and satisfies

$$\int_0^1 s^2 \varphi(s)\, ds \geqslant \frac{3}{2}(a+b).$$

Here we take $\zeta = a + b$, then

$$-g(u) \leqslant \zeta \leqslant \frac{2}{3}\int_0^1 s^2 \varphi(s)\, ds = \frac{2}{3}\int_0^1 s^2 f(s,0)\, ds, \quad u \geqslant 0.$$

Evidently, functions $f(t,u) = u^{\frac{1}{2}} + 4t + 18$, $(t,u) \in [0,1] \times [0,+\infty)$ and

$$g(u) = \begin{cases} -u - 1, & 0 \leqslant u \leqslant 3, \\ -4, & u \geqslant 3, \end{cases}$$

satisfy the above conditions with $\alpha = \frac{1}{2}$, $a = 1$, $b = 3$ and $\varphi(t) = 4t + 18$; (ii) let $f(t,u) = u^{\frac{1}{4}} + 36$, $u \in [0,+\infty)$ and

$$g(u) = \begin{cases} -u, & 0 \leqslant u \leqslant 1, \\ -u^{\frac{1}{3}}, & 1 \leqslant u \leqslant 512, \\ -8, & u \geqslant 512. \end{cases}$$

Here we take $\alpha = \frac{1}{3}$, $\zeta = 8$, then (H_1)–(H_4) are satisfied.

(2) We can give a simple example to illustrate Theorem 3.1. Let $f(t,u) \equiv 1$, $g(u) \equiv -\frac{1}{9}$. Then the problem (3.1) has a unique solution

$$u(t) = t^2\left(\frac{1}{24}t^2 - \frac{5}{27}t + \frac{11}{36}\right), \quad t \in [0,1].$$

We can obtain that

$$\frac{35}{216}t^2 \leqslant u(t) \leqslant \frac{11}{36}t^2, \quad t \in [0,1].$$

So the unique solution u is a positive solution and satisfies $u \in P_h = P_{t^2}$.

4. Applications to nonlinear Lane–Emden–Fowler equations

Let Ω be a bounded domain with smooth boundary in $\mathbf{R}^N$ $(N \geqslant 1)$. Consider the following singular Dirichlet problem for the Lane–Emden–Fowler equation:

$$\begin{cases} -\Delta u = f(x,u) + g(x,u), & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4.1}$$

where the nonlinear terms $f(x,u)$, $g(x,u)$ are allowed to be singular on $\partial\Omega$.

The problem (4.1) arises in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, as well as in the theory of heat conduction in electrically materials (see [11,14,15,17]). The theory of singular elliptic boundary value problems for partial differential equations has become an important area of investigation in the past three decades, see [10,11,13–17,20,24,26,32,33,41] and references therein. By means of sub-supersolutions and various techniques related to the maximum principle for elliptic equations, some existence and nonexistence results, a unique positive solution are established. However, to our knowledge, the results on the existence and uniqueness of positive solutions for the general singular elliptic equation are still very few. The purpose here is to establish the existence and uniqueness of positive solutions to the singular Dirichlet problem for the Lane–Emden–Fowler equation (4.1). Different from the above works mentioned, we will use Corollary 2.4 to show the existence and uniqueness of positive solutions for the problem (4.1).

Throughout this section, denote by $W^{k,l}(\Omega)$ the Sobolev space (see [1]), where $l > 1$ and k is a nonnegative integer. And denote by φ_1 and λ_1 the first eigenfunction and the first eigenvalue of the following eigenvalue problem $-\Delta\varphi = \lambda\varphi$ in Ω,

and $\varphi|_{\partial\Omega} = 0$. For convenience, we assume that $\varphi_1(x) \geqslant 0$ in $\bar{\Omega}$. Moreover, it is well known that (see for instance [34]) there exist two positive constants C_2, C_3 such that the first eigenvalue function satisfies

$$0 < C_2 \leqslant \varphi_1(x)\big[d(x)\big]^{-1} \leqslant C_3, \quad x \in \Omega, \tag{4.2}$$

where $d(x) = dist(x, \partial\Omega)$.

Lemma 4.1. *(See [8].) Let Ω be a bounded domain with smooth boundary in $\mathbf{R}^N$ $(N \geqslant 1)$. If the operator $-\Delta + k$ is coercive (that is, $k > -\lambda_1$) and $u \in L^1_{loc}(\Omega)$ satisfies*

$$\begin{cases} -\Delta u + ku \geqslant 0, & x \in \Omega, \\ u(x) \geqslant 0, & x \in \bar{\Omega}, \end{cases}$$

then either $u(x) \equiv 0$ or $u(x) \geqslant C_0 d(x)$, $x \in \Omega$, where $d(x) = dist(x, \partial\Omega)$ and C_0 is a positive constant depending only upon N, Ω and k.

The above result is originally due to G. Stampacchia (see [8]), which plays an important role in the proof of the following main result.

Theorem 4.1. *Assume that*

- (S_1) *$f(x,u)$, $g(x,u)$ are nonnegative on $\Omega \times (0,+\infty)$ and there exists a constant $\delta > 0$ such that $f(x,u) \geqslant \delta g(x,u)$ for $\Omega \times (0,+\infty)$;*
- (S_2) *$f(x,u)$, $g(x,u)$ are Hölder continuous in the variable x with the Hölder exponent $\gamma \in (0,1)$ for each $u \in (0,+\infty)$ and are continuous in the variable u for each $x \in \Omega$;*
- (S_3) *$f(x,u)$, $g(x,u)$ are increasing in u for each x and, for any $t \in (0,1)$, there exists a constant $\alpha \in (0,1)$ such that*

$$f(x,tu) \geqslant t^\alpha f(x,u), \qquad g(x,tu) \geqslant t g(x,u) \quad \text{for any } u \in (0,+\infty); \tag{4.3}$$

- (S_4) *$f(x,u)$, $g(x,u)$ satisfy the conditions of integrability, i.e.,*

$$\int_\Omega f\big(x,\varphi_1(x)\big)^l dx < +\infty, \qquad \int_\Omega g\big(x,\varphi_1(x)\big)^l dx < +\infty \quad \text{for some } l > N. \tag{4.4}$$

Then the problem (4.1) *has a unique positive solution $u^* \in C^{1,\beta}(\bar{\Omega})$ with $\beta = 1 - \frac{N}{l}$.*

Proof. For the sake of convenience, set $E = C(\bar{\Omega})$, the Banach space of continuous functions on $\bar{\Omega}$ with the norm $\|u\| = \max\{|u(x)|\colon\ x \in \bar{\Omega}\}$. Set $P = \{u \in C(\bar{\Omega}) \mid u(x) \geqslant 0,\ x \in \bar{\Omega}\}$, the standard cone. It is clear that P is a normal cone in E and the normality constant is 1, P_{φ_1} is given as in Section 1.1. We divide the proof into several steps.

Step 1. We consider the following linear elliptic boundary value problem

$$\begin{cases} -\Delta w = f(x,u), & x \in \Omega, \\ w(x) > 0, & x \in \Omega, \\ w(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4.5}$$

where $u \in P_{\varphi_1}$. Since $u \in P_{\varphi_1}$, we can choose a sufficiently small number $r_0 \in (0,1)$ such that

$$r_0\varphi_1(x) \leqslant u(x) \leqslant \frac{1}{r_0}\varphi_1(x), \quad x \in \bar{\Omega}. \tag{4.6}$$

Then from (S_3),

$$f\big(x,u(x)\big) \geqslant f\big(x, r_0\varphi_1(x)\big) \geqslant {r_0}^\alpha f\big(x,\varphi_1(x)\big), \quad x \in \Omega, \tag{4.7}$$

$$f\big(x,u(x)\big) \leqslant f\left(x, \frac{1}{r_0}\varphi_1(x)\right) \leqslant \frac{1}{{r_0}^\alpha} f\big(x,\varphi_1(x)\big), \quad x \in \Omega. \tag{4.8}$$

Thus we get by applying the integrability condition (4.4) that

$$\int_\Omega \big[f\big(x,u(x)\big)\big]^l dx < +\infty,$$

namely, $f(x,u) \in L^l(\Omega)$. By the classical theory of linear elliptic equations (see [25]), the problem (4.5) admits a unique strong solution $w_u \in W^{2,l}(\Omega) \cap W_0^{1,l}(\Omega)$. Recall that $l > N$. Using the Sobolev imbedding theory, $w_u \in C^{1,\beta}(\bar{\Omega})$ with $\beta = 1 - \frac{N}{l}$. Now we define an operator $A : P_{\varphi_1} \to E$ by

$$Au(x) = w_u(x), \quad u \in P_{\varphi_1},$$

where w_u is the unique strong solution of (4.5) for $u \in P_{\varphi_1}$. Next we prove that $A : P_{\varphi_1} \to P_{\varphi_1}$. Suppose that ϕ is the solution of (4.5) with $u = \varphi_1$, then $A\varphi_1 = \phi \in C^{1,\beta}(\bar{\Omega})$. From (4.7), (4.8) and applying the comparison principle, we conclude that

$$r_0{}^{\alpha}\phi(x) \leqslant w_u(x) \leqslant \frac{1}{r_0{}^{\alpha}}\phi(x), \quad x \in \bar{\Omega}. \tag{4.9}$$

We now show a useful conclusion: if $w \in W^{2,l}(\Omega)$ and $w(x) = 0$ for $x \in \partial\Omega$, then there exists a constant $M_1 > 0$ such that

$$\big|w(x)\big| \leqslant M_1\varphi_1(x), \quad x \in \Omega, \tag{4.10}$$

where M_1 depends only upon N and Ω. In fact, by the Sobolev imbedding theory, $w \in C^{1,\beta}(\bar{\Omega})$. Clearly, w is Lipschitz continuous in $\bar{\Omega}$, and hence

$$\big|w(x) - w(y)\big| \leqslant C_1|x - y| \quad \text{for any } y \in \partial\Omega.$$

Since $w(y) = 0$ for $y \in \partial\Omega$, we deduce that

$$\big|w(x)\big| \leqslant C_1 \inf_{y\in\partial\Omega} |x - y| = C_1\, dist(x, \partial\Omega) = C_1 d(x).$$

Further, from (4.2) we obtain

$$\big|w(x)\big| \leqslant C_1 d(x) \leqslant C_1 \cdot \frac{1}{C_2}\varphi_1(x) = \frac{C_1}{C_2}\varphi_1(x) \equiv M_1\varphi_1(x), \quad x \in \bar{\Omega}.$$

Recall that the Lipschitz constant C_1, the constants C_2 and C_3 in (4.2) depend only upon N and Ω. Therefore, (4.10) holds. Then from (4.10), there exists a constant $C_\phi > 0$ such that

$$\phi(x) \leqslant C_\phi\varphi_1(x), \quad x \in \bar{\Omega}. \tag{4.11}$$

Note that $f(x, \varphi_1(x)) \geqslant 0$. By the maximal principle, $\phi(x) \geqslant 0$. Since $\phi(x) > 0$ for $x \in \Omega$, an application of Lemma 4.1 implies that

$$\phi(x) \geqslant C_0 d(x), \quad x \in \bar{\Omega}. \tag{4.12}$$

Combining (4.2) and (4.12), there exists a constant $c_\phi > 0$ such that

$$\phi(x) \geqslant c_\phi\varphi_1(x), \quad x \in \bar{\Omega}. \tag{4.13}$$

It follows from (4.9), (4.11) and (4.13) that

$$r_0{}^{\alpha}c_\phi\varphi_1(x) \leqslant w_u(x) \leqslant \frac{1}{r_0{}^{\alpha}}C_\phi\varphi_1(x), \quad x \in \bar{\Omega}.$$

Therefore, $Au = w_u \in P_{\varphi_1}$ for any $u \in P_{\varphi_1}$. So we obtain that $A : P_{\varphi_1} \to P_{\varphi_1}$ is well defined. In the following we prove that $A : P_{\varphi_1} \to P_{\varphi_1}$ is increasing. Suppose that $u_1, u_2 \in P_{\varphi_1}$ with $u_1(x) \geqslant u_2(x)$ for any $x \in \Omega$, then we have

$$\begin{cases} -\Delta(Au_1 - Au_2) = f(x,u_1) - f(x,u_2), & x \in \Omega, \\ Au_1(x) - Au_2(x) = 0, & x \in \partial\Omega. \end{cases}$$

In view of $f(x,u_1) \geqslant f(x,u_2)$ and applying the comparison principle, we have $Au_1 \geqslant Au_2$.

Next we prove that $A(tu) \geqslant t^{\alpha}Au$ for any $u \in P_{\varphi_1}$ and $t \in (0,1)$. For any $u \in P_{\varphi_1}$ and $t \in (0,1)$, we have

$$\begin{cases} -\Delta A(tu) = f(x,tu), & x \in \Omega, \\ A(tu)(x) = 0, & x \in \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta t^{\alpha}Au = t^{\alpha}f(x,u), & x \in \Omega, \\ t^{\alpha}Au(x) = 0, & x \in \partial\Omega. \end{cases}$$

From (4.3) we also get $f(x,tu(x)) - t^\alpha f(x,u(x)) \geqslant 0$ for any $x \in \bar{\Omega}$. Therefore,

$$\begin{cases} -\Delta\big(A(tu) - t^\alpha Au\big) \geqslant 0, & x \in \Omega, \\ A(tu)(x) - t^\alpha Au(x) = 0, & x \in \partial\Omega. \end{cases}$$

Using the comparison principle again, we can obtain $A(tu) \geqslant t^\alpha Au$ immediately. So we have $A(tu) \geqslant t^\alpha Au$ for $t \in (0,1)$, $u \in P_{\varphi_1}$.

Step 2. We consider the following linear elliptic boundary value problem

$$\begin{cases} -\Delta w = g(x,u), & x \in \Omega, \\ w(x) > 0, & x \in \Omega, \\ w(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4.14}$$

where $u \in P_{\varphi_1}$. From (4.6) and (S_3),

$$g\big(x,u(x)\big) \geqslant g\big(x, r_0\varphi_1(x)\big) \geqslant r_0 g\big(x,\varphi_1(x)\big), \quad x \in \Omega,$$
$$g\big(x,u(x)\big) \leqslant g\left(x, \frac{1}{r_0}\varphi_1(x)\right) \leqslant \frac{1}{r_0} g\big(x,\varphi_1(x)\big), \quad x \in \Omega.$$

Thus we get by applying the integrability condition (4.4) that

$$\int_\Omega \big[g\big(x,u(x)\big)\big]^l dx < +\infty,$$

that is, $g(x,u) \in L^l(\Omega)$. By the classical theory of linear elliptic equations, the problem (4.14) also admits a unique strong solution $w_u \in W^{2,l}(\Omega) \cap W_0^{1,l}(\Omega)$. Recall that $l > N$. Using the Sobolev imbedding theory, $w_u \in C^{1,\beta}(\bar{\Omega})$ with $\beta = 1 - \frac{N}{l}$. We define an operator $B : P_{\varphi_1} \to E$ by

$$Bu(x) = w_u(x), \quad u \in P_{\varphi_1},$$

where w_u is the unique strong solution of (4.14) for $u \in P_{\varphi_1}$. Suppose that ϕ is the solution of (4.14) with $u = \varphi_1$, then $B\varphi_1 = \phi \in C^{1,\beta}(\bar{\Omega})$. By applying the comparison principle, we conclude that

$$r_0\phi(x) \leqslant w_u(x) \leqslant \frac{1}{r_0}\phi(x), \quad x \in \bar{\Omega}.$$

Similar to Step 1, we can prove that there exist constants $C_\phi, c_\phi > 0$ such that

$$c_\phi\varphi_1(x) \leqslant \phi(x) \leqslant C_\phi\varphi_1(x), \quad x \in \bar{\Omega}.$$

So we have

$$r_0 c_\phi\varphi_1(x) \leqslant w_u(x) \leqslant \frac{1}{r_0} C_\phi\varphi_1(x), \quad x \in \bar{\Omega}.$$

Therefore, $Bu = w_u \in P_{\varphi_1}$ for any $u \in P_{\varphi_1}$. So we obtain that $B : P_{\varphi_1} \to P_{\varphi_1}$ is well defined. Using the comparison principle again, we can easily obtain that $B : P_{\varphi_1} \to P_{\varphi_1}$ is increasing and $B(tu) \geqslant tBu$ for any $u \in P_{\varphi_1}$ and $t \in (0,1)$.

Step 3. In the following we show that $Au \geqslant \delta Bu$ for any $u \in P_{\varphi_1}$. Note that

$$\begin{cases} -\Delta Au = f(x,u), & x \in \Omega, \\ Au(x) = 0, & x \in \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta\delta Bu = \delta g(x,u), & x \in \Omega, \\ \delta Bu(x) = 0, & x \in \partial\Omega. \end{cases}$$

From (S_1),

$$\begin{cases} -\Delta(Au - \delta Bu) \geqslant 0, & x \in \Omega, \\ Au(x) - \delta Bu(x) = 0, & x \in \partial\Omega. \end{cases}$$

Using the comparison principle again, we can obtain $Au \geqslant \delta Bu$ for any $u \in P_{\varphi_1}$.

Finally, using Corollary 2.4, operator equation $Au + Bu = u$ has a unique fixed point u^* in P_{φ_1}, i.e., $Au^* + Bu^* = u^*$. This implies that the problem (4.1) admits a unique solution $u^* \in P_{\varphi_1}$. By the theory of the linear elliptic equation, the problem

$$\begin{cases} -\Delta w = f\big(x, u^*\big) + g\big(x, u^*\big), & x \in \Omega, \\ w(x) > 0, & x \in \Omega, \\ w(x) = 0, & x \in \partial\Omega, \end{cases}$$

admits a unique solution $\bar{u}^* \in W^{2,l}(\Omega) \cap W_0^{1,l}(\Omega)$, and hence $\bar{u}^* \in C^{1,\beta}(\Omega)$. Recalling the uniqueness of the solution of (4.1), one can see easily that $\bar{u}^* = u^*$. Thus the problem (4.1) has a unique classical solution $u^* \in C^{1,\beta}(\bar{\Omega})$. □

Remark 4.1. Some examples of the functions which satisfy the conditions (S_1)–(S_4) are:

(1) $f(x,u) = a(x)u^\alpha$, $g(x,u) = b(x)u^\alpha$, where $\alpha \in (0,1)$, $a(x), b(x) \geqslant 0$ and $a(x) \geqslant \delta b(x)$ for $x \in \bar{\Omega}$; $a(x), b(x) \in C^\gamma(\bar{\Omega})$ with $\gamma \in (0,1)$, $\delta > 0$.

(2) $f(x,u) = c(x)u^{\frac{1}{2}}$, $g(x,u) = c(x)g_1(u)$, where

$$g_1(u) = \begin{cases} u, & 0 \leqslant u \leqslant 1, \\ u^{\frac{1}{3}}, & u \geqslant 1, \end{cases}$$

and c is a Hölder continuous function in Ω such that $c_1 d(x)^q \leqslant c(x) \leqslant c_2 d(x)^q$ in Ω, here $c_1, c_2 > 0$, q is a real number and $d(x) = dist(x, \partial\Omega)$. Moreover, $N(\frac{1}{2} + q) > -1$.

(3) $f(x,u)$ is as in (2) and $g(x,u) = c(x)g_2(u)$, where c is as in (2) and

$$g_2(u) = \begin{cases} u, & 0 \leqslant u \leqslant 1, \\ 1, & u \geqslant 1. \end{cases}$$

Remark 4.2. In [15,16], Ghergu and Rădulescu studied the existence or the nonexistence of solutions for the following two elliptic problems with parameters

$$\begin{cases} -\Delta u + K(x)g(u) = \lambda f(x,u) + \mu g(x,u), & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4.15}$$

and

$$\begin{cases} -\Delta u = \lambda f(u) + a(x)g(u), & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4.16}$$

where λ, μ are positive real numbers. Under reasonable conditions, we can establish the existence and uniqueness of positive solutions for (4.15) and (4.16) by using Theorem 4.1. Moreover, the method used here is new to the literature and so is the existence–uniqueness result to the singular Dirichlet problem for the Lane–Emden–Fowler equation. This is also the main motivation for the study of (4.1) in the present work.

References

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[2] R.P. Agarwal, D. O'Regan, A multiplicity result for second order impulsive differential equations via the Leggett Williams fixed point theorem, Appl. Math. Comput. 161 (2005) 433–439.
[3] E. Alves, T.F. Ma, M.L. Pelicer, Monotone positive solutions for a fourth order equation with nonlinear boundary conditions, Nonlinear Anal. 71 (2009) 3834–3841.
[4] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. 18 (4) (1976) 620–709.
[5] R.I. Avery, J. Henderson, Three symmetric positive solutions for a second order boundary value problem, Appl. Math. Lett. 13 (3) (2000) 1–7.
[6] R.I. Avery, A.C. Peterson, Three positive fixed points of nonlinear operators on ordered Banach spaces, Comput. Math. Appl. 42 (2001) 313–322.
[7] Z. Bai, The upper and lower solution method for some fourth-order boundary value problems, Nonlinear Anal. 67 (2007) 1704–1709.
[8] H. Brezis, L. Nirenberg, Minima locaux relatifs a C^1 et H^1, C. R. Acad. Sci. Paris 317 (1993) 465–472.
[9] Y.Z. Chen, Stability of positive fixed points of nonlinear operators, Positivity 6 (2002) 47–57.
[10] M.M. Coclite, G. Palmieri, On a singular nonlinear Dirichlet problem, Comm. Partial Differential Equations 14 (1989) 1315–1327.
[11] M.G. Crandall, P.H. Rabinowitz, L. Tartar, On a Dirichlet problem with a singular nonlinearity, Comm. Partial Differential Equations 2 (2) (1977) 193–222.
[12] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, Berlin, 1985.
[13] L. Dupaigne, M. Ghergu, V. Rădulescu, Lane–Emden–Fowler equations with convection and singular potential, J. Math. Pures Appl. 87 (2007) 563–581.
[14] W. Fulks, J.S. Maybee, A singular nonlinear elliptic equation, Osaka J. Math. 12 (1960) 1–19.
[15] M. Ghergu, V. Rădulescu, Sublinear singular elliptic problems with two parameters, J. Differential Equations 195 (2003) 520–536.
[16] M. Ghergu, V. Rădulescu, Ground state solutions for the singular Lane–Emden–Fowler equation with sublinear convection term, J. Math. Anal. Appl. 333 (2007) 265–273.
[17] M. Ghergu, V. Rădulescu, Singular Elliptic Problems. Bifurcation and Asymptotic Analysis, Oxford Lecture Ser. Math. Appl., vol. 37, Oxford University Press, 2008.
[18] J.R. Graef, B. Yang, Positive solutions of a nonlinear fourth order boundary value problem, Comm. Appl. Nonlinear Anal. 14 (1) (2007) 61–73.
[19] D.J. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press Inc., Boston, New York, 1988.
[20] J. Hernández, F. Mancebo, J.M. Vega, Positive solutions for singular nonlinear elliptic equations, Proc. Roy. Soc. Edinburgh Sect. A 137 (2007) 41–62.

[21] D. Ilic, V. Rakocevic, Common fixed points for maps on cone metric space, J. Math. Anal. Appl. 341 (2008) 876–882.
[22] J. Jachymski, The contraction principle for mappings on a metric space with a graph, Proc. Amer. Math. Soc. 136 (2008) 1359–1373.
[23] I.Y. Karaca, Nonlinear triple-point problems with change of sign, Comput. Math. Appl. 55 (14) (2008) 691–703.
[24] A. Kristăly, V. Radulescu, Cs. Varga, Variational Principles in Mathematical Physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems, Encyclopedia Math. Appl., vol. 136, Cambridge University Press, Cambridge, 2010.
[25] O.A. Ladyzhenskaya, N.N. Ural'ceva, Linear and Quasilinear Elliptic Equations, Academic Press, New York, 1968 (English transl.).
[26] A.C. Lazer, P.J. Mckenna, On a singular nonlinear elliptic boundary value problem, Proc. Amer. Math. Soc. 111 (1991) 721–730.
[27] K. Li, J. Liang, T.J. Xiao, A fixed point theorem for convex and decreasing operators, Nonlinear Anal. 63 (2005) e209–e216.
[28] Y. Li, Two-parameter nonresonance condition for the existence of fourth-order boundary value problems, J. Math. Anal. Appl. 308 (2005) 121–128.
[29] B. Liu, Positive solutions of fourth-order two-point boundary value problems, Appl. Math. Comput. 148 (2004) 407–420.
[30] J.J. Nieto, R.L. Pouso, R. Rodriguez-Lopez, Fixed point theorems in ordered abstract spaces, Proc. Amer. Math. Soc. 135 (2007) 2505–2517.
[31] R.D. Nussbaum, Iterated nonlinear maps and Hilbert's projective metric, Mem. Amer. Math. Soc. 75 (391) (1988).
[32] J. Shi, M. Yao, On a singular nonlinear semilinear elliptic problem, Proc. Roy. Soc. Edinburgh Sect. A 128 (1998) 1389–1401.
[33] C.A. Stuart, Existence and approximation of solutions of nonlinear elliptic equations, Math. Z. 147 (1976) 53–63.
[34] M. Wiegner, A degenerate diffusion equation with a nonlinear source term, Nonlinear Anal. 28 (12) (1997) 1977–1995.
[35] B. Yang, Positive solutions for the beam equation under certain boundary value problems, Electron. J. Differential Equations 2005 (78) (2005) 1–8.
[36] C. Yang, C.B. Zhai, J.R. Yan, Positive solutions of three-point boundary value problem for second order differential equations with an advanced argument, Nonlinear Anal. 65 (2006) 2013–2023.
[37] Q. Yao, Positive solutions for eigenvalue problems of fourth-order elastic beam equations, Appl. Math. Lett. 17 (2004) 237–243.
[38] C.B. Zhai, C.M. Guo, On α-convex operators, J. Math. Anal. Appl. 316 (2006) 556–565.
[39] C.B. Zhai, C. Yang, C.M. Guo, Positive solutions of operator equation on ordered Banach spaces and applications, Comput. Math. Appl. 56 (2008) 3150–3156.
[40] X.P. Zhang, Existence and iteration of monotone positive solutions for an elastic beam with a corner, Nonlinear Anal. Real World Appl. 10 (2009) 2097–2103.
[41] Z.J. Zhang, The asymptotic behaviour of the unique solution for the singular Lane–Emden–Fowler equation, J. Math. Anal. Appl. 312 (2005) 33–43.
[42] Z. Zhao, Multiple fixed points of a sum operator and applications, J. Math. Anal. Appl. 360 (2009) 1–6.