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Proofs of two conjectures on the Landau constants

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ABSTRACT

In this paper we derive a family of asymptotic expansions for the nth Landau constant G_n. The proof is based on a new general integral representation of the constants in terms of a Hypergeometric Function. As a consequence we prove two conjectures related to earlier asymptotic developments of the G_n's.

1. Introduction and previous results

Let $T_n(f)$ be the polynomial operator, that associates to each function f, holomorphic on the open unit disk, the partial sum of order n of its Taylor series $f(z) = \sum_{k \geqslant 0} a_k z^k$. In 1913, Landau [3] showed that if $|f(z)| < 1$ for $|z| < 1$, then

$$\left| \sum_{k=0}^{n} a_k \right| \leqslant G_n$$

and $\|T_n\| = G_n$, where

$$G_n = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 + \cdots + \left(\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}\right)^2$$

is the nth Landau constant [7,9].

The investigation of the asymptotic behavior of G_n was begun by Landau, who proved that

$$G_n \sim \frac{1}{\pi} \log n,$$

then Watson [9] established the following asymptotic formula

$$G_n = \frac{1}{\pi} \log(n+1) + \frac{1}{\pi}(\gamma + 4 \log 2) + o(1).$$

Here $\gamma = 0.577215\ldots$ is the Euler–Mascheroni constant. Recently, Zhao [10] gave a much better approximation as follows

$$G_n = \frac{1}{\pi} \log(n+1) + \frac{1}{\pi}(\gamma + 4 \log 2) - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2} + \mathcal{O}\left(\frac{1}{(n+1)^3}\right).$$

More recently, Mortici [4] extended this asymptotic formula to

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$$G_n = \frac{1}{\pi}\log(n+1) + \frac{1}{\pi}(\gamma + 4\log 2) - \frac{1}{4\pi(n+1)} + \frac{5}{192\pi(n+1)^2} + \frac{3}{128\pi(n+1)^3} - \frac{341}{122880\pi(n+1)^4} + \mathcal{O}\left(\frac{1}{(n+1)^5}\right).$$

Finally, A. Nemes and the author (hereinafter called "they") established a complete asymptotic expansion for G_n in terms of $1/(n+1)$ [5]. First, they showed that

$$G_n \sim \frac{1}{\pi}\log\left(n+\frac{1}{2}\right) + \frac{1}{\pi}(\gamma + 4\log 2) - \sum_{k\geqslant 1}\frac{\alpha_k}{(n+\frac{1}{2})^k} \tag{1.1}$$

as $n \to +\infty$, where the α_k's are certain computable constants. From this expansion it followed that the Landau constants G_n have the following asymptotic expansion

$$G_n \sim \frac{1}{\pi}\log(n+1) + \frac{1}{\pi}(\gamma + 4\log 2) - \sum_{k\geqslant 1}\frac{\beta_k}{(n+1)^k} \tag{1.2}$$

as $n \to +\infty$, where β_k's, again, are some computable constants. The computation of the first few coefficients α_k and β_k led them to the conjecture that $\beta_k = (-1)^k\alpha_k$ for every $k \geqslant 1$. In this paper we show that such an identity indeed holds by proving the following more general result:

Theorem 1.1. *Let $0 < h < 3/2$. The Landau constants G_n have the following asymptotic expansions*

$$G_n \sim \frac{1}{\pi}\log(n+h) + \frac{1}{\pi}(\gamma + 4\log 2) - \sum_{k\geqslant 1}\frac{g_k(h)}{(n+h)^k} \tag{1.3}$$

as $n \to +\infty$, where the coefficients $g_k(h)$ are certain computable constants that satisfy $g_k(h) = (-1)^k g_k(3/2-h)$ for every $k \geqslant 1$.

By the uniqueness theorem on asymptotic series $\alpha_k = g_k(1/2)$, $\beta_k = g_k(1)$, therefore we have indeed proved the above conjecture. For the sake of completeness, we obtain an explicit expression for the coefficients $g_k(h)$ in Appendix A.

In the same paper they derived the first few terms of an asymptotic series for G_n in terms of $1/(n+3/4)$,

$$G_n \sim \frac{1}{\pi}\log\left(n+\frac{3}{4}\right) + \frac{1}{\pi}(\gamma + 4\log 2) + \frac{11}{192\pi(n+\frac{3}{4})^2} - \frac{1541}{122880\pi(n+\frac{3}{4})^4} + \frac{63433}{8257536\pi(n+\frac{3}{4})^6} - \frac{9199901}{1006632960\pi(n+\frac{3}{4})^8} + \frac{317959723}{17716740096\pi(n+\frac{3}{4})^{10}} - \cdots.$$

This suggested that there might exist an alternating even type asymptotic series for the G_n's in terms of $1/(n+3/4)$. We show that such an asymptotic expansion really exists by proving the following theorem:

Theorem 1.2. *The Landau constants G_n have the following asymptotic expansion*

$$G_n \sim \frac{1}{\pi}\log\left(n+\frac{3}{4}\right) + \frac{1}{\pi}(\gamma + 4\log 2) - \sum_{k\geqslant 1}\frac{(-1)^k\gamma_k}{\pi(n+\frac{3}{4})^{2k}}$$

as $n \to +\infty$, where the coefficients γ_k are positive rational numbers.

Note that we must have $\gamma_k = (-1)^k\pi g_{2k}(3/4)$.

The proofs are based on a new general integral representation of the G_n constants.

2. A general integral representation

For later use, we need the following lemma:

Lemma 2.1. *Fix $0 < h < 3/2$ and let*

$$F(t) = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;t\right) = \sum_{k\geqslant 0}\binom{2k}{k}^2\frac{1}{2^{4k}}t^k, \quad |t| < 1, \tag{2.1}$$

where ${}_2F_1(a,b;c;t)$ is the Hypergeometric Function and t is complex. The function

$$\Gamma_h(x) = \begin{cases} 1 - \frac{xe^{(h-1/2)x}}{e^x-1} F(1-e^{-x}), & x > -\log 2 \text{ and } x \neq 0, \\ 0, & x = 0 \end{cases}$$

is continuous on $(-\log 2, +\infty)$ *and can be extended into a complex analytic function near the origin.*

Proof. The function $1 - e^{-x}$ maps $(-\log 2, +\infty)$ onto $(-1, 1)$. Since the function F is analytic on $(-1, 1)$, we conclude that $F(1 - e^{-x})$ is a well-defined continuous function on $(-\log 2, +\infty)$. The function

$$\begin{cases} \frac{x}{e^x-1}, & x \neq 0, \\ 1, & x = 0 \end{cases} \tag{2.2}$$

is continuous for every real x, hence Γ_h is continuous on $(-\log 2, +\infty)$. Now we regard x as a complex variable. It is clear that $F(1 - e^{-x})$ is a well-defined complex analytic function near the origin. The function (2.2) is complex analytic near $x = 0$ since it has the power series

$$\sum_{k \geqslant 0} \frac{B_k}{k!} x^k$$

converges for $|x| < 2\pi$. Here B_k is the kth Bernoulli Number. Hence, the Γ_h can be extended into a complex analytic function near the origin. □

We are in position to state the main result of this section.

Theorem 2.1. *Fix* $0 < h < 3/2$. *The nth Landau constant* G_n *can be expressed in the form*

$$G_n = \frac{1}{\pi} \log(n+h) + \frac{1}{\pi}(\gamma + 4\log 2) + \frac{1}{\pi} \int_0^{+\infty} e^{-(n+h)x} \frac{\Gamma_h(x)}{x}\, dx. \tag{2.3}$$

It is interesting to note that the function F defined by (2.1) appears in the ordinary generating function of the Landau constants since

$$\frac{F(x)}{1-x} = \sum_{n \geqslant 0} G_n x^n.$$

We shall use the following representation of the ψ Digamma Function (for details about this function see, e.g., [6]):

Lemma 2.2. *Suppose that* $t > 0$ *and* $a > 0$, *a being fixed. Then*

$$\psi(t+a) = \log t + \int_0^{+\infty} \left(\frac{1}{x} - \frac{e^{(1-a)x}}{e^x - 1} \right) e^{-tx}\, dx.$$

Proof. Our starting point is the representation

$$\psi(s) = \int_0^{+\infty} \left(\frac{e^{-x}}{x} - \frac{e^{(1-s)x}}{e^x - 1} \right) dx,$$

valid for $s > 0$. Suppose that $t > 0$ and $a > 0$, a being fixed. The substitution $s = t + a$ yields

$$\psi(t+a) = \int_0^{+\infty} \left(\frac{e^{-x}}{x} - \frac{e^{(1-(t+a))x}}{e^x - 1} \right) dx.$$

The formula

$$\log t = \int_0^{+\infty} \frac{e^{-x} - e^{-tx}}{x}\, dx,$$

and a straightforward calculation gives

$$\begin{aligned}\psi(t+a) &= \int_0^{+\infty}\left(\frac{e^{-x}}{x}-\frac{e^{-tx}}{x}-\frac{e^{(1-(t+a))x}}{e^x-1}+\frac{e^{-tx}}{x}\right)dx\\ &= \int_0^{+\infty}\frac{e^{-x}-e^{-tx}}{x}\,dx+\int_0^{+\infty}\left(\frac{1}{x}-\frac{e^{(1-a)x}}{e^x-1}\right)e^{-tx}\,dx\\ &= \log t+\int_0^{+\infty}\left(\frac{1}{x}-\frac{e^{(1-a)x}}{e^x-1}\right)e^{-tx}\,dx. \qquad \square\end{aligned}$$

Proof of Theorem 2.1. Cvijović and Klinowski [2] showed that

$$G_n=\frac{1}{\pi}\psi\left(n+\frac{3}{2}\right)+\frac{1}{\pi}(\gamma+4\log 2)-\sum_{k\geqslant 1}\binom{2k}{k}^2\frac{1}{2^{4k}\pi}\frac{(k-1)!}{(n+\frac{3}{2})((n+\frac{3}{2})+1)\cdots((n+\frac{3}{2})+k-1)}. \tag{2.4}$$

By the Beta integral we have

$$\frac{(k-1)!}{(n+\frac{3}{2})((n+\frac{3}{2})+1)\cdots((n+\frac{3}{2})+k-1)}=\int_0^{+\infty}e^{-(n+3/2)x}\left(1-e^{-x}\right)^{k-1}dx=\int_0^{+\infty}\frac{e^{-(n+1/2)x}}{e^x-1}\left(1-e^{-x}\right)^k dx.$$

Hence

$$\begin{aligned}\sum_{k\geqslant 1}\binom{2k}{k}^2\frac{1}{2^{4k}\pi}\frac{(k-1)!}{(n+\frac{3}{2})((n+\frac{3}{2})+1)\cdots((n+\frac{3}{2})+k-1)} &= \frac{1}{\pi}\sum_{k\geqslant 1}\binom{2k}{k}^2\frac{1}{2^{4k}}\int_0^{+\infty}\frac{e^{-(n+1/2)x}}{e^x-1}\left(1-e^{-x}\right)^k dx\\ &= \frac{1}{\pi}\int_0^{+\infty}\frac{e^{-(n+1/2)x}}{e^x-1}\sum_{k\geqslant 1}\binom{2k}{k}^2\frac{1}{2^{4k}}\left(1-e^{-x}\right)^k dx\\ &= \frac{1}{\pi}\int_0^{+\infty}e^{-(n+1/2)x}\frac{F(1-e^{-x})-1}{e^x-1}\,dx.\end{aligned}$$

Plugging this into (2.4) yields

$$G_n=\frac{1}{\pi}\psi\left(n+\frac{3}{2}\right)+\frac{1}{\pi}(\gamma+4\log 2)-\frac{1}{\pi}\int_0^{+\infty}e^{-(n+1/2)x}\frac{F(1-e^{-x})-1}{e^x-1}\,dx. \tag{2.5}$$

Using the lemma with $t=n+h$ and $a=3/2-h$ $(0<h<3/2)$ we get

$$\frac{1}{\pi}\psi\left(n+\frac{3}{2}\right)=\frac{1}{\pi}\log(n+h)+\frac{1}{\pi}\int_0^{+\infty}\left(\frac{1}{x}-\frac{e^{(h-1/2)x}}{e^x-1}\right)e^{-(n+h)x}\,dx.$$

Plugging this into (2.5) and manipulating the terms under the integral we deduce (2.3). $\square$

3. The proof of Theorem 1.1

Since $\Gamma_h(0)=0$, from Lemma 2.1 we conclude that a power series

$$\Gamma_h(x)/x=\sum_{k\geqslant 0}\tilde{g}_k(h)x^k \tag{3.1}$$

holds as $x\to 0+$. From the asymptotic formula (see formula (2.8) in [8])

$${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-t\right)=-\frac{1}{\pi}\log t+\mathcal{O}(1), \quad t\to 0+,$$

we have

$$F\left(1-e^{-x}\right)=\frac{x}{\pi}+\mathcal{O}(1)$$

as $x \to +\infty$. Hence, $\Gamma_h(x)/x = o(e^x)$ as $x \to +\infty$. These imply that the integral in (2.3) satisfies the conditions of Watson's Lemma (see, e.g., [6]), and we obtain the asymptotic expansion

$$G_n \sim \frac{1}{\pi}\log(n+h) + \frac{1}{\pi}(\gamma + 4\log 2) + \frac{1}{\pi}\sum_{k\geqslant 1}\frac{(k-1)!\tilde{g}_{k-1}(h)}{(n+h)^k} \tag{3.2}$$

as $n \to +\infty$. If we set

$$g_k(h) = -\frac{(k-1)!\tilde{g}_{k-1}(h)}{\pi}, \tag{3.3}$$

we get the form as it appears in Theorem 1.1. In order to prove the relation $g_k(h) = (-1)^k g_k(3/2-h)$, it is enough to show that $\tilde{g}_{k-1}(h) = (-1)^k \tilde{g}_{k-1}(3/2-h)$. This is equivalent to the assertion $\Gamma_h(x) = \Gamma_{3/2-h}(-x)$ near $x = 0$. From the well-known relation (see, e.g., [1, p. 560])

$${}_2F_1(a,b;c;t) = (1-t)^{-a}{}_2F_1\left(a, c-b; c; \frac{t}{t-1}\right), \quad -1 < t < \frac{1}{2},$$

it follows that

$$F\left(1-e^{-x}\right) = e^{x/2}F\left(1-e^x\right) \tag{3.4}$$

near $x = 0$. Hence, as $x \to 0$

$$\begin{aligned}\Gamma_h(x) &= 1 - \frac{xe^{(h-1/2)x}}{e^x-1}F\left(1-e^{-x}\right) = 1 - \frac{-xe^{(h-1/2)x}}{1-e^x}e^{x/2}F\left(1-e^x\right) \\ &= 1 - \frac{-xe^{(h-1)x}}{e^{-x}-1}F\left(1-e^x\right) = 1 - \frac{-xe^{((3/2-h)-1/2)(-x)}}{e^{-x}-1}F\left(1-e^x\right) = \Gamma_{3/2-h}(-x).\end{aligned}$$

4. The proof of Theorem 1.2

By Theorem 1.1 it is obvious that in the case $h = 3/4$ the asymptotic expansion (1.3) is even-type, since $g_k(3/4) = (-1)^k g_k(3/4)$. Using the notations of the previous section, the coefficients γ_k of Theorem 1.2 can be written as $\gamma_k = (-1)^k \pi g_{2k}(3/4) = (-1)^{k-1}(2k-1)!\tilde{g}_{2k-1}(3/4)$. It remains to show that the γ_k's are positive rational numbers. This is equivalent to the rationality of the $\tilde{g}_{2k-1}(3/4)$'s and the assertion $(-1)^{k-1}\tilde{g}_{2k-1}(3/4) > 0$ for $k \geqslant 1$. Hereinafter t is complex and x is real from a small neighborhood of 0. Now we use the fact that $\Gamma_{3/4}$ is complex analytic around the origin. In the imaginary axis, near $x = 0$,

$$\Gamma_{3/4}(ix) = \sum_{k\geqslant 1}(-1)^k \tilde{g}_{2k-1}(3/4)x^{2k}.$$

From the transformation formula (see, e.g., [1, p. 560])

$${}_2F_1(a,b;2b;t) = (1-t)^{-\frac{a}{2}}{}_2F_1\left(a, 2b-a; b+\frac{1}{2}; -\frac{(1-\sqrt{1-t})^2}{4\sqrt{1-t}}\right),$$

it follows that

$$F\left(1-e^{-x}\right) = e^{x/4}F\left(-\frac{(1-e^{-x/2})^2}{4e^{-x/2}}\right) = e^{x/4}F\left(-\sinh^2\left(\frac{x}{4}\right)\right).$$

This gives

$$-\Gamma_{3/4}(ix) = \frac{ixe^{ix/4}}{e^{ix}-1}F\left(1-e^{-ix}\right) - 1 = \frac{ixe^{ix/4}}{e^{ix}-1}e^{ix/4}F\left(-\sinh^2\left(\frac{ix}{4}\right)\right) - 1 = \frac{\frac{x}{2}}{\sin(\frac{x}{2})}F\left(\sin^2\left(\frac{x}{4}\right)\right) - 1.$$

Now, we use the relation [1, p. 560]

$${}_1F_2(a,b;2b;t) = \left(\frac{1+\sqrt{1-t}}{2}\right)^{-2a}{}_2F_1\left(a, a-b+\frac{1}{2}; b+\frac{1}{2}; \left(\frac{1-\sqrt{1-t}}{1+\sqrt{1-t}}\right)^2\right)$$

to obtain

$$F\left(\sin^2\left(\frac{x}{4}\right)\right) = \left(\frac{1+\sqrt{1-\sin^2(\frac{x}{4})}}{2}\right)^{-1}F\left(\left(\frac{1-\sqrt{1-\sin^2(\frac{x}{4})}}{1+\sqrt{1-\sin^2(\frac{x}{4})}}\right)^2\right) = \frac{1}{\cos^2(\frac{x}{8})}F\left(\tan^4\left(\frac{x}{8}\right)\right).$$

This gives

$$-\Gamma_{3/4}(ix) = \frac{\frac{x}{2}}{\sin(\frac{x}{2})}\frac{1}{\cos^2(\frac{x}{8})}F\left(\tan^4\left(\frac{x}{8}\right)\right) - 1,$$

and therefore

$$\frac{\frac{x}{2}}{\sin(\frac{x}{2})}\frac{1}{\cos^2(\frac{x}{8})}F\left(\tan^4\left(\frac{x}{8}\right)\right) - 1 = \sum_{k\geqslant 1}(-1)^{k-1}\tilde{g}_{2k-1}(3/4)x^{2k}. \tag{4.1}$$

The series

$$\frac{\frac{x}{2}}{\sin(\frac{x}{2})} = \sum_{k\geqslant 0}\frac{(-1)^{k-1}(2^{2k-1}-1)B_{2k}}{2^{2k-1}(2k)!}x^{2k},$$

$$\frac{1}{\cos^2(\frac{x}{8})} = \sum_{k\geqslant 0}\frac{(-1)^{k}(2^{2k+2}-1)B_{2k+2}}{2^{4k-2}(2k+2)(2k)!}x^{2k},$$

$$\tan\left(\frac{x}{8}\right) = \sum_{k\geqslant 1}\frac{(-1)^{k-1}(2^{2k}-1)B_{2k}}{2^{4k-3}(2k)!}x^{2k-1},$$

$$F\left(x^4\right) = \sum_{k\geqslant 0}\binom{2k}{k}^2\frac{1}{2^{4k}}x^{4k}$$

all have positive rational coefficients and thus the series in (4.1) too.

Appendix A

In view of expressions (3.1) and (3.3), we have the exponential generating function

$$-\frac{1}{\pi}\Gamma_h(x) = \sum_{k\geqslant 1}kg_k(h)\frac{x^k}{k!}. \tag{A.1}$$

We shall use the exponential generating functions of the $S(k,m)$ Stirling Numbers of the Second Kind and the $B_k(t)$ Bernoulli Polynomials:

$$\left(e^x-1\right)^m = m!\sum_{k\geqslant m}S(k,m)\frac{x^k}{k!}, \qquad \frac{xe^{tx}}{e^x-1} = \sum_{k\geqslant 0}B_k(t)\frac{x^k}{k!}.$$

A simple series transformation gives

$$\begin{aligned}F\left(1-e^{-x}\right) &= \sum_{m\geqslant 0}\binom{2m}{m}^2\frac{1}{2^{4m}}\left(1-e^{-x}\right)^m \\ &= \sum_{m\geqslant 0}\binom{2m}{m}^2\frac{(-1)^m}{2^{4m}}m!\sum_{k\geqslant m}(-1)^kS(k,m)\frac{x^k}{k!} \\ &= \sum_{k\geqslant 0}\left(\sum_{m=0}^{k}(-1)^{k+m}\binom{2m}{m}^2\frac{m!S(k,m)}{2^{4m}}\right)\frac{x^k}{k!},\end{aligned}$$

which yields

$$\begin{aligned}-\frac{1}{\pi}\Gamma_h(x) &= \frac{1}{\pi}\frac{xe^{(h-1/2)x}}{e^x-1}F\left(1-e^{-x}\right) - \frac{1}{\pi} \\ &= \frac{1}{\pi}\sum_{k\geqslant 0}B_k\left(h-\frac{1}{2}\right)\frac{x^k}{k!}\times\sum_{k\geqslant 0}\left(\sum_{m=0}^{k}(-1)^{k+m}\binom{2m}{m}^2\frac{m!S(k,m)}{2^{4m}}\right)\frac{x^k}{k!} - \frac{1}{\pi} \\ &= \sum_{k\geqslant 1}\left(\frac{1}{\pi}\sum_{j=0}^{k}\left(\binom{k}{j}B_{k-j}\left(h-\frac{1}{2}\right)\sum_{m=0}^{j}(-1)^{j+m}\binom{2m}{m}^2\frac{m!S(j,m)}{2^{4m}}\right)\right)\frac{x^k}{k!}.\end{aligned}$$

Comparing this with (A.1) we obtain the relatively simple representation

$$g_k(h) = \frac{1}{\pi k}\sum_{j=0}^{k}\left(\binom{k}{j}B_{k-j}\left(h-\frac{1}{2}\right)\sum_{m=0}^{j}(-1)^{j+m}\binom{2m}{m}^2\frac{m!S(j,m)}{2^{4m}}\right).$$

The first few are given by

$$g_1(h) = \frac{4h-3}{4\pi},$$
$$g_2(h) = \frac{96h^2-144h+43}{192\pi},$$
$$g_3(h) = \frac{128h^3-288h^2+172h-21}{384\pi},$$
$$g_4(h) = \frac{30720h^4-92160h^3+82560h^2-20160h-619}{122880\pi},$$
$$g_5(h) = \frac{24576h^5-92160h^4+110080h^3-40320h^2-2476h+1425}{122880\pi}.$$

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