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Lie isomorphisms of nest algebras on Banach spaces ☆

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ABSTRACT

We show that a Lie isomorphism ψ between nest algebras on Banach spaces can be decomposed as $\psi = \phi + \tau$, where ϕ is an (algebraic) isomorphism or a negative of an (algebraic) anti-isomorphism, and τ is a linear map with image in the center vanishing on each commutator.

1. Introduction

A nest algebra is an operator algebra whose invariant subspace lattice is a nest. Since Ringrose introduced them in 1965, nest algebras have been widely studied, see [3,11]. The most extensive results known are obtained in the Hilbert space case. It is natural and interesting to extend these results to the Banach space case. The purpose of this paper is to extend to the Banach space setting the result about Lie isomorphisms between nest algebras on Hilbert spaces by Marcoux and Sourour [12]. The techniques here are quite different from those in Hilbert space case. Motivation for this is the recent trend to study the well-known and important results to spaces lacking inner products and projections (see for example [4,5,11,16,17]).

Let $\mathcal{A}$ be an associative algebra. Then $\mathcal{A}$ becomes a Lie algebra under the *Lie product* $[A, B] = AB - BA$. A *Lie homomorphism* ϕ of $\mathcal{A}$ into another associative algebra is a linear map which preserves the Lie product, that is, $\phi([A, B]) = [\phi(A), \phi(B)]$ for all $A, B \in \mathcal{A}$. As usual, a bijective Lie homomorphism is called a *Lie isomorphism*. The study of Lie isomorphisms of associative algebras and operator algebras, primarily focusing upon their relations to associative (anti-) isomorphisms, has a long history. See [1,2,6,7,10,12–14] and the references therein.

The main result in this paper is as follows.

Theorem 1.1. *Let X and Y be Banach spaces. Let $\mathcal{N}$ and $\mathcal{M}$ be nests on X and Y, respectively. Suppose that ψ is a Lie isomorphism from the nest algebra $\mathrm{Alg}\,\mathcal{N}$ onto the nest algebra $\mathrm{Alg}\,\mathcal{M}$. Then one of the following holds.*

(1) *There exist an invertible operator T in $B(Y, X)$ and a linear functional τ on $\mathrm{Alg}\,\mathcal{N}$ vanishing on each commutator such that*

$$\psi(A) = T^{-1}AT + \tau(A)I$$

for all $A \in \mathrm{Alg}\,\mathcal{N}$.

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(2) *There exist an invertible operator T in $B(Y, X^*)$ and a linear functional τ on* $\mathrm{Alg}\mathcal{N}$ *vanishing on each commutator such that*

$$\psi(A) = -T^{-1}A^*T + \tau(A)I$$

for all $A \in \mathrm{Alg}\mathcal{N}$.

For the proof of our results, in Section 3 we describe the structure of commutative Lie ideals. This characterization makes it possible to identify the behavior of the Lie isomorphism on some special sets of operators. In fact, we note that the introduction of techniques of commutative Lie ideals in the treatment of Lie isomorphisms is perhaps the most interesting novelty in this paper.

2. Preliminaries

Throughout, all algebras and vector spaces will be over $\mathbb{F}$, where $\mathbb{F}$ is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Given a Banach space X with the topological dual X^*, by $B(X)$ we mean the algebra of all bounded linear operators on X. The terms *operator* on X and *subspace* of X will mean bounded linear map of X into itself and norm closed linear manifold of X, respectively. For $A \in B(X)$, denote by A^* the adjoint of A. For any non-empty subset $L \subseteq X$, $L^\perp$ denotes its annihilator, that is, $L^\perp = \{f \in X^*\colon\ f(x) = 0 \text{ for all } x \in L\}$. For $x \in X$ and $f \in X^*$, the rank-one operator $x \otimes f$ is defined by $(x \otimes f)z = f(z)x$.

A *nest* $\mathcal{N}$ on a Banach space X is a totally ordered family of subspaces of X which contains (0) and X, and is complete in the sense that it is closed under the formation of arbitrary closed linear spans (denoted by $\bigvee$) and intersections (denoted by $\bigwedge$). For $E \in \mathcal{N}$, we define

$$E_- = \bigvee\{F \in \mathcal{N}\colon\ F < E\}$$

and

$$E_+ = \bigwedge\{F \in \mathcal{N}\colon\ F > N\}.$$

It is not difficult to verify that $X = \bigvee\{E \in \mathcal{N}\colon\ E_- < X\} = \bigvee\{E_+\colon\ E \in \mathcal{N},\ \ E < X\}$ and $(0) = \bigwedge\{E \in \mathcal{N}\colon\ E_+ > (0)\} = \bigwedge\{E_-\colon\ E \in \mathcal{N},\ \ E > (0)\}$. The *nest algebra* $\mathrm{Alg}\mathcal{N}$ associated to the nest $\mathcal{N}$ is the set of operators on X leaving every subspace in $\mathcal{N}$ invariant, that is,

$$\mathrm{Alg}\mathcal{N} = \bigl\{A \in B(X)\colon\ Ax \in E \text{ for every } x \in E \text{ and for every } E \in \mathcal{N}\bigr\}.$$

Lemma 2.1. *(See [8,16].) Let $\mathcal{N}$ be a nest. Then the rank-one operator $x \otimes f$ belongs to* $\mathrm{Alg}\mathcal{N}$ *if and only if there exists a subspace E in $\mathcal{N}$ such that $x \in E$ and $f \in E_-^\perp$. Here $E_-^\perp$ means $(E_-)^\perp$.*

From the lemma, one can see that nest algebras are rich in rank-one operators. Using this property, there is a characterization of idempotents. By $\mathrm{idem}(\mathcal{N})$ we denote the set of all idempotent operators in the nest algebra $\mathrm{Alg}\mathcal{N}$.

Lemma 2.2. *(See [12, Lemma 3.1].) Let $\mathcal{N}$ be a nest and A be in* $\mathrm{Alg}\mathcal{N}$.

(1) $A \in \mathbb{F}I + \mathrm{idem}(\mathcal{N})$ *if and only if* $[A,[A,[A,T]]] = [A,T]$ *for all* $T \in \mathrm{Alg}\mathcal{N}$.
(2) *A is the sum of a scalar operator and an idempotent operator whose range belongs to $\mathcal{N}$ if and only if* $[A,[A,T]] = [A,T]$ *for all* $T \in \mathrm{Alg}\mathcal{N}$.

We remark that this result was proved in [12] for the Hilbert space case but the proof is valid for the Banach space case.

Lemma 2.3. *(See [9].) Let $\mathcal{N}$ be a nest and A be in* $\mathrm{Alg}\mathcal{N}$.

(1) *If $AT = TA$ for all $T \in \mathrm{Alg}\mathcal{N}$, then $A = \lambda I$ for some $\lambda \in \mathbb{F}$.*
(2) *If $ATA = 0$ for all $T \in \mathrm{Alg}\mathcal{N}$, then there is an E in $\mathcal{N}$ such that $AE = 0$ and $A^*E^\perp = 0$.*

We close this section with a proposition about rank-preserving linear maps, whose proof can be found in many papers, for example, [15]. Let E and F be subspaces of X and X^* respectively. By $E \otimes F$ we denote the set $\{x \otimes f\colon\ x \in E,\ \ f \in F\}$.

Proposition 2.4. *Let X_i be an infinite-dimensional Banach space, $i = 1, 2$. Let E_i and F_i be linear subspaces of X_i and X_i^* respectively. Let $\mathcal{A}_i$ be a unital subalgebra of $B(X_i)$ containing $E_i \otimes F_i$. Suppose that ϕ is a linear bijective map from $\mathcal{A}_1$ onto $\mathcal{A}_2$ which satisfies $\phi(\mathbb{F}I) = \mathbb{F}I$ and carries $\mathbb{F}I + E_1 \otimes F_1$ onto $\mathbb{F}I + E_2 \otimes F_2$. Then one of the following holds.*

(1) *There are a bilinear map* $\gamma : E_1 \times F_1 \to \mathbb{F}$, *linear bijective maps* $C : E_1 \to E_2$ *and* $D : F_1 \to F_2$ *such that* $\phi(x \otimes f) = \gamma(x, f)I + Cx \otimes Df$ *for all* $x \in E_1$ *and* $f \in F_1$.
(2) *There are a bilinear map* $\gamma : E_1 \times F_1 \to \mathbb{F}$, *linear bijective maps* $C : E_1 \to F_2$ *and* $D : F_1 \to E_2$ *such that* $\phi(x \otimes f) = \gamma(x, f)I + Df \otimes Cx$ *for all* $x \in E_1$ *and* $f \in F_1$.

3. Commutative Lie ideals

In this section, we shall describe commutative Lie ideals in a nest algebra. Throughout this section, $\mathcal{N}$ is a nest in a Banach space X and $\mathcal{N}^0 = \mathcal{N} \setminus \{(0), X\} \neq \emptyset$.

We begin with an example.

Example 3.1. Let E be in $\mathcal{N}$. Define

$$\mathfrak{I}(\mathcal{N}, E) = \{A \in \operatorname{Alg}\mathcal{N}\colon\ AE = 0 \text{ and } A^*E^\perp = 0\}.$$

In other words, $\mathfrak{I}(\mathcal{N}, E)$ is the set of all operators in $\operatorname{Alg}\mathcal{N}$ whose ranges are contained in E and whose kernels contain E. It is not difficult to verify that $\mathfrak{I}(\mathcal{N}, E)$ is an ideal. Moreover, the product of any pair of operators in $\mathfrak{I}(\mathcal{N}, E)$ is zero. So $\mathfrak{I}(\mathcal{N}, E)$ and $\mathbb{F}I + \mathfrak{I}(\mathcal{N}, E)$ are commutative Lie ideals.

We shall show that $\mathbb{F}I + \mathfrak{I}(\mathcal{N}, E)$ is a maximal commutative Lie ideal and any maximal commutative Lie ideals arise in this way. To do this, we need a deeper understanding of $\mathfrak{I}(\mathcal{N}, E)$.

Remark 3.2. The following properties can be straightforward verified. Let E and F be in $\mathcal{N}^0$.

(1) The typical operators in $\mathfrak{I}(\mathcal{N}, E)$ are rank-one operators $x \otimes f$ with $x \in E$ and $f \in E^\perp$.
(2) If $A\mathfrak{I}(\mathcal{N}, E)B = 0$ with $A, B \in B(X)$, then either $AE = 0$ or $B^*E^\perp = 0$.
(3) $\mathfrak{I}(\mathcal{N}, E) = \mathfrak{I}(\mathcal{N}, F)$ if and only if $E = F$.
(4) $\mathfrak{I}(\mathcal{N}, E)\mathfrak{I}(\mathcal{N}, F) = 0$ if and only if $F \leqslant E$.

Lemma 3.3. *Let* A *be in* $\operatorname{Alg}\mathcal{N}$. *Then* $A \in \mathbb{F}I + \mathfrak{I}(\mathcal{N}, E)$ *for some* $E \in \mathcal{N}$ *if and only if*

$$[A, [A, T]] = 0 \quad \text{for each } T \in \operatorname{Alg}\mathcal{N}. \tag{3.1}$$

Proof. The necessity is obvious. To verify the sufficiency, we first expand Eq. (3.1) as follows

$$A^2T - 2ATA + TA^2 = 0, \quad T \in \operatorname{Alg}\mathcal{N}. \tag{3.2}$$

For $N \in \mathcal{N}$, $x \in N$ and $f \in N_-^\perp$, taking $T = x \otimes f$ in the above equation and then applying this equation to a vector $y \in X$, we get

$$f(y)A^2x - 2f(Ay)Ax + f(A^2y)x = 0. \tag{3.3}$$

We now consider two cases.

Case 1. There is an F in $\mathcal{N}$ with $F \neq (0)$ such that the restriction of A to F is a scalar multiple of the identity operator on F. By translating by a scalar operator, we may assume that the restriction of A to F is zero. Thus, for any N in $\mathcal{N}$ with $(0) < N \leqslant F$ and any $x \in N$, we have that $Ax = 0$. Hence Eq. (3.3) gives that $(A^2)^*f = 0$ for all $f \in N_-^\perp$. Noting that $X^* = \bigvee\{N_-^\perp\colon\ (0) < N \leqslant F\}$, we get $A^2 = 0$. Then Eq. (3.2) becomes $ATA = 0$ for all $T \in \operatorname{Alg}\mathcal{N}$. Hence $A \in \mathfrak{I}(\mathcal{N}, E)$ for some $E \in \mathcal{N}$ by Lemma 2.3.

Case 2. $A|_N$ is not a scalar multiple of the identity operator on N for each non-zero N in $\mathcal{N}$. If we choose $f \in N_-^\perp$ and $y \in X$ such that $f(y) = 1$, from Eq. (3.3) we get

$$A^2x + \mu_N Ax + \gamma_N x = 0$$

for all $x \in N$ with $N_- \neq X$. If $\mu_N \neq \mu_M$ for some $N, M \in \mathcal{N}$, then the restriction of A to $\min\{N, M\}$ is a scalar multiple of the identity operator on $\min\{N, M\}$. So μ_N is independent of N and hence so is γ_N. Consequently, there are scalars μ and γ such that

$$A^2x + \mu Ax + \gamma x = 0$$

for all $x \in \operatorname{span}\{N \in \mathcal{N}\colon\ N_- \neq X\}$. Hence $A^2 + \mu A + \gamma I = 0$. Now translate A by a scalar operator so that $A^2 = \lambda I$. Then Eq. (3.2) yields $ATA = \lambda T$ for all $T \in \operatorname{Alg}\mathcal{N}$. If $\lambda = 0$, then $A \in \mathfrak{I}(\mathcal{N}, E)$ for some $E \in \mathcal{N}$ by Lemma 2.3. If $\lambda \neq 0$, then $AT = \frac{1}{\lambda}(ATA)A = TA$ for all $T \in \operatorname{Alg}\mathcal{N}$ and so A is a scalar multiple of I by Lemma 2.3. $\square$

Theorem 3.4. *We have*:

(1) *Let E be in $\mathcal{N}^0$. Then $\mathbb{F}I + \mathfrak{I}(\mathcal{N}, E)$ is a maximal commutative Lie ideal in* $\operatorname{Alg}\mathcal{N}$.
(2) *Let $\mathfrak{L}$ be a non-trivial commutative Lie ideal in* $\operatorname{Alg}\mathcal{N}$. *Then $\mathfrak{L} \subseteq \mathbb{F}I + \mathfrak{I}(\mathcal{N}, E)$ for some $E \in \mathcal{N}^0$. Hence if $\mathfrak{L}$ is maximal, then $\mathfrak{L} = \mathbb{F}I + \mathfrak{I}(\mathcal{N}, E)$ and such E is unique.*

Proof. (1) Suppose $\mathfrak{K}$ is a commutative Lie ideal in $\operatorname{Alg}\mathcal{N}$ which contains $\mathbb{F}I + \mathfrak{I}(\mathcal{N}, E)$. Let A be in $\mathfrak{K}$. Then for all $x \in E$ and $f \in E^\perp$, we have

$$A(x \otimes f) = (x \otimes f)A. \tag{3.4}$$

From this, we see that there is a scalar λ such that $Ax = \lambda x$ for all $x \in E$. By translating A by a scalar operator we may assume that $\lambda = 0$. Hence Eq. (3.4) gives that $A^* E^\perp = 0$.

(2) Since $\mathfrak{L}$ is a commutative Lie ideal, each operator in $\mathfrak{L}$ satisfies Eq. (3.1). So there are maps $\lambda : \mathfrak{L} \to \mathbb{F}$ and $\eta : \mathfrak{L} \to \bigcup\{\mathfrak{I}(\mathcal{N}, F)\colon\ F \in \mathcal{N}^0\}$ such that

$$A = \lambda(A)I + \eta(A), \quad A \in \mathfrak{L}.$$

Furthermore, for $A, B \in \mathfrak{L}$ and $T \in \operatorname{Alg}\mathcal{N}$,

$$\big[\eta(A), \big[\eta(B), T\big]\big] = \big[A, [B, T]\big] = 0,$$

that is,

$$\eta(A)\eta(B)T - \eta(A)T\eta(B) - \eta(B)T\eta(A) + T\eta(B)\eta(A) = 0. \tag{3.5}$$

Let $E = \bigvee\{F \in \mathcal{N}\colon\ \eta(A)|_F = 0 \text{ for all } A \in \mathfrak{L}\}$. Then $\eta(A)|_E = 0$ for all $A \in \mathfrak{L}$ and hence $E \neq X$ since $\mathfrak{L}$ is non-trivial.

Let A be in $\mathfrak{L}$. For $F > E$, we can choose $x \in F$ and $B \in \mathfrak{L}$ such that $\eta(B)x \neq 0$. Taking $T = x \otimes f$ in Eq. (3.5) for $f \in F_-^\perp$ and noting $\eta(B)^2 = 0$, we get

$$\eta(B)(x \otimes f)\eta(B) = 0$$

and

$$\eta(A)\eta(B)(x \otimes f) - \eta(A)(x \otimes f)\eta(B) - \eta(B)(x \otimes f)\eta(A) + (x \otimes f)\eta(B)\eta(A) = 0.$$

The first equation gives $\eta(B)^* f = 0$ since $\eta(B)x \neq 0$ and then the second equation becomes

$$\eta(A)\eta(B)(x \otimes f) - \eta(B)(x \otimes f)\eta(A) = 0$$

for all $f \in F_-^\perp$. So $\eta(A)^*|_{F_-^\perp}$ is a scalar operator. Hence $\eta(A)^*|_{E^\perp}$ is a scalar operator since $\operatorname{span}\{F_-^\perp\colon\ F > E\}$ is *-weakly dense in $E^\perp$. Now noting that $(\eta(A)^*|_{E^\perp})^2 = (\eta(A)^2)^*|_{E^\perp} = 0$, we get that $\eta(A)^*|_{E^\perp} = 0$.

The proof is complete. $\square$

4. The induced map of ψ on $\mathcal{N}^0$

In this section and foregoing, we assume that $\mathcal{N}$ and $\mathcal{M}$ are nests on X and Y, respectively, and that ψ is a Lie isomorphism from $\operatorname{Alg}\mathcal{N}$ onto $\operatorname{Alg}\mathcal{M}$.

Let E be in $\mathcal{N}^0$. Then $\psi(\mathbb{F}I + \mathfrak{I}(\mathcal{N}, E))$ is a maximal commutative Lie ideal in $\operatorname{Alg}\mathcal{M}$ by Theorem 3.4(1). Hence by Theorem 3.4(2) there is a unique element $F \in \mathcal{M}^0$ such that $\psi(\mathbb{F}I + \mathfrak{I}(\mathcal{N}, E)) = \mathbb{F}I + \mathfrak{I}(\mathcal{M}, F)$. Now we define a map

$$\hat{\psi} : \mathcal{N}^0 \to \mathcal{M}^0$$

by

$$\psi\big(\mathbb{F}I + \mathfrak{I}(\mathcal{N}, E)\big) = \mathbb{F}I + \mathfrak{I}\big(\mathcal{M}, \hat{\psi}(E)\big).$$

Lemma 4.1. *$\hat{\psi}$ is bijective.*

Proof. Let E and F be in $\mathcal{N}^0$ and suppose that $\hat{\psi}(E) = \hat{\psi}(F)$. Then by the definition,

$$\psi\big(\mathbb{F}I + \mathfrak{I}(\mathcal{N}, E)\big) = \psi\big(\mathbb{F}I + \mathfrak{I}(\mathcal{N}, F)\big)$$

and hence by the injectivity of ψ,

$$\mathbb{F}I + \mathfrak{I}(\mathcal{N}, E) = \mathbb{F}I + \mathfrak{I}(\mathcal{N}, F).$$

So $E = F$. This proves the injectivity of $\hat{\psi}$. To show the surjectivity, let M be in $\mathcal{M}^0$. Then $\mathbb{F}I + \Im(\mathcal{M}, M)$ is a maximal commutative Lie ideal and so is $\psi^{-1}(\mathbb{F}I + \Im(\mathcal{M}, M))$. It follows that there is an N in $\mathcal{N}^0$ such that

$$\psi^{-1}\big(\mathbb{F}I + \Im(\mathcal{M}, M)\big) = \mathbb{F}I + \Im(\mathcal{N}, N)$$

and then

$$\psi\big(\mathbb{F}I + \Im(\mathcal{N}, N)\big) = \mathbb{F}I + \Im(\mathcal{M}, M).$$

So $\hat{\psi}(N) = M$ by the definition. $\square$

For $A \in \Im(\mathcal{N}, E)$ with $E \in \mathcal{N}^0$, there is a unique operator $B \in \Im(\mathcal{M}, \hat{\psi}(E))$ such that $\psi(A) - B \in \mathbb{F}I$. We define a map

$$\bar{\psi} : \bigcup\big\{\Im(\mathcal{N}, E)\colon\ E \in \mathcal{N}^0\big\} \to \bigcup\big\{\Im(\mathcal{M}, F)\colon\ F \in \mathcal{M}^0\big\}$$

by

$$\psi(A) - \bar{\psi}(A) \in \mathbb{F}I.$$

Lemma 4.2. *$\bar{\psi}$ is bijective. Furthermore, $\bar{\psi}(\Im(\mathcal{N}, E)) = \Im(\mathcal{M}, \hat{\psi}(E))$ for every $E \in \mathcal{N}^0$.*

Proof. Let A and B be in $\bigcup\{\Im(\mathcal{N}, E)\colon\ E \in \mathcal{N}^0\}$ and suppose that $\bar{\psi}(A) = \bar{\psi}(B)$. Then $\psi(A) - \lambda_A I = \bar{\psi}(A) = \bar{\psi}(B) = \psi(B) - \lambda_B I$ for some $\lambda_A, \lambda_B \in \mathbb{F}$ and hence $\psi(A - B) \in \mathbb{F}I$. So $A - B = \lambda I$ for some $\lambda \in \mathbb{F}$. Since $(A - B)^4 = 0$, we have $\lambda = 0$ and then $A = B$. This proves that $\bar{\psi}$ is injective.

Let E be in $\mathcal{N}^0$. By the definition of $\hat{\psi}$, we know that $\psi(\mathbb{F}I + \Im(\mathcal{N}, E)) = \mathbb{F}I + \Im(\mathcal{M}, \hat{\psi}(E))$. Then for $S \in \Im(\mathcal{M}, \hat{\psi}(E))$, there are a scalar λ and an operator $T \in \Im(\mathcal{N}, E)$ such that $\psi(\lambda I + T) = S$. Hence $\mu I + \psi(T) = S$ for some $\mu \in \mathbb{F}$. Thus $\bar{\psi}(T) = S$ by the definition. So $\bar{\psi}(\Im(\mathcal{N}, E)) = \Im(\mathcal{M}, \hat{\psi}(E))$. This together with the surjectivity of $\hat{\psi}$ shows the surjectivity of $\bar{\psi}$. $\square$

Lemma 4.3. *Let E_1, E_2, E_3 be in $\mathcal{N}^0$ such that $E_1 < E_2 < E_3$. Suppose that one of $\hat{\psi}(E_1) < \hat{\psi}(E_2)$, $\hat{\psi}(E_1) < \hat{\psi}(E_3)$ and $\hat{\psi}(E_2) < \hat{\psi}(E_3)$ holds. Then $\hat{\psi}(E_1) < \hat{\psi}(E_2) < \hat{\psi}(E_3)$.*

Proof. For $i \in \{1, 2, 3\}$, take x_i in E_i and f_i in $E_i^{\perp}$ such that $f_1(x_2) = f_2(x_3) = 1$. Then $x_i \otimes f_i \in \Im(\mathcal{N}, E_i)$ and hence $\bar{\psi}(x_i \otimes f_i) \in \Im(\mathcal{M}, \hat{\psi}(E_i))$. So for each case of $\hat{\psi}(E_1) < \hat{\psi}(E_2)$, $\hat{\psi}(E_1) < \hat{\psi}(E_3)$ and $\hat{\psi}(E_2) < \hat{\psi}(E_3)$, there holds $\bar{\psi}(x_3 \otimes f_3)\bar{\psi}(x_2 \otimes f_2)\bar{\psi}(x_1 \otimes f_1) = 0$. Thus we have

$$\begin{aligned}
\psi(x_1 \otimes f_3) &= \psi\big(\big[x_1 \otimes f_1, [x_2 \otimes f_2, x_3 \otimes f_3]\big]\big) \\
&= \big[\bar{\psi}(x_1 \otimes f_1), \big[\bar{\psi}(x_2 \otimes f_2), \bar{\psi}(x_3 \otimes f_3)\big]\big] \\
&= \bar{\psi}(x_1 \otimes f_1)\bar{\psi}(x_2 \otimes f_2)\bar{\psi}(x_3 \otimes f_3) \\
&\quad - \bar{\psi}(x_1 \otimes f_1)\bar{\psi}(x_3 \otimes f_3)\bar{\psi}(x_2 \otimes f_2) - \bar{\psi}(x_2 \otimes f_2)\bar{\psi}(x_3 \otimes f_3)\bar{\psi}(x_1 \otimes f_1)
\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}
\psi(x_1 \otimes f_3) &= \psi\big(\big[x_3 \otimes f_3, [x_2 \otimes f_2, x_1 \otimes f_1]\big]\big) \\
&= \big[\bar{\psi}(x_3 \otimes f_3), \big[\bar{\psi}(x_2 \otimes f_2), \bar{\psi}(x_1 \otimes f_1)\big]\big] \\
&= \bar{\psi}(x_1 \otimes f_1)\bar{\psi}(x_2 \otimes f_2)\bar{\psi}(x_3 \otimes f_3) \\
&\quad - \bar{\psi}(x_3 \otimes f_3)\bar{\psi}(x_1 \otimes f_1)\bar{\psi}(x_2 \otimes f_2) - \bar{\psi}(x_2 \otimes f_2)\bar{\psi}(x_1 \otimes f_1)\bar{\psi}(x_3 \otimes f_3).
\end{aligned} \tag{4.2}$$

If $\bar{\psi}(x_1 \otimes f_1)\bar{\psi}(x_3 \otimes f_3)\bar{\psi}(x_2 \otimes f_2) \neq 0$, then $\hat{\psi}(E_1) < \hat{\psi}(E_3) < \hat{\psi}(E_2)$ and hence $\psi(x_1 \otimes f_3) = 0$ by Eq. (4.2), a contradiction; If $\bar{\psi}(x_2 \otimes f_2)\bar{\psi}(x_3 \otimes f_3)\bar{\psi}(x_1 \otimes f_1) \neq 0$, then $\hat{\psi}(E_2) < \hat{\psi}(E_3) < \hat{\psi}(E_1)$ and hence $\psi(x_1 \otimes f_3) = 0$ by Eq. (4.2), a contradiction. Consequently, $\bar{\psi}(x_1 \otimes f_1)\bar{\psi}(x_2 \otimes f_2)\bar{\psi}(x_3 \otimes f_3) = \psi(x_1 \otimes f_3) \neq 0$ by Eq. (4.1). This implies that $\hat{\psi}(E_1) < \hat{\psi}(E_2) < \hat{\psi}(E_3)$. $\square$

Proposition 4.4. *$\hat{\psi}$ is either order-preserving or order-reversing.*

Proof. Let E and F be in $\mathcal{N}^0$ with $E < F$ and suppose that $\hat{\psi}(E) < \hat{\psi}(F)$. Let L and N be in $\mathcal{N}^0$ with $L < N$. We shall show that $\hat{\psi}(L) < \hat{\psi}(N)$. Consider twelve possible comparison relations of E, F, L, N:

(1) $E = L < F = N$,
(2) $E < F = L < N$,

(3) $E = L < F < N$,
(4) $E = L < N < F$,
(5) $L < E < N = F$,
(6) $L < E = N < F$,
(7) $E < F < L < N$,
(8) $E < L < F < N$,
(9) $E < L < N < F$,
(10) $L < E < N < F$,
(11) $L < N < E < F$,
(12) $L < E < F < N$.

If the case (1) occurs, then the desired result is clear. If one of cases (2)–(5) occurs, then the desired result immediately follows from Lemma 4.3. If one of cases (7)–(12) occurs, one can get the desired result by using Lemma 4.3 twice. For example, suppose that $E < F < L < N$. Then $\hat{\psi}(E) < \hat{\psi}(F) < \hat{\psi}(L)$ by Lemma 4.3. Hence $\hat{\psi}(F) < \hat{\psi}(L) < \hat{\psi}(N)$ by Lemma 4.3 again. □

5. The behavior on idempotent operators

Recall that idem($\mathcal{N}$) and idem($\mathcal{M}$) denote sets of all idempotent operators in Alg$\mathcal{N}$ and Alg$\mathcal{M}$, respectively. By Lemma 2.2(1), we can define the map

$$\tilde{\psi} : \mathrm{idem}(\mathcal{N}) \to \mathrm{idem}(\mathcal{M})$$

by

$$\psi(P) - \tilde{\psi}(P) \in \mathbb{F}I, \quad P \in \mathrm{idem}(\mathcal{N}).$$

It is easily seen that $\tilde{\psi}$ is bijective.

We shall characterize special elements in idem($\mathcal{N}$) and idem($\mathcal{M}$). To do this, we define, for $E \in \mathcal{N}^0$, that

$$\Omega_1(\mathcal{N}, E) = \{P \in \mathrm{idem}(\mathcal{N}):\ PE = 0,\ P^* E^\perp \neq 0\}$$

and

$$\Omega_2(\mathcal{N}, E) = \{P \in \mathrm{idem}(\mathcal{N}):\ PE \neq 0,\ P^* E^\perp = 0\}.$$

For $F \in \mathcal{M}^0$, the sets $\Omega_1(\mathcal{M}, F)$ and $\Omega_2(\mathcal{M}, F)$ are analogously defined. If $E \in \mathcal{N}^0$ with $E < E_+$, then the typical elements in $\Omega_1(\mathcal{N}, E)$ are rank-one operators of the form $x \otimes f$ with $x \in E_+$, $f \in E^\perp$ and $f(x) = 1$; If $E \in \mathcal{N}^0$ with $E_- < E$, then the typical elements in $\Omega_2(\mathcal{N}, E)$ are rank-one operators of the form $x \otimes f$ with $x \in E$, $f \in E_-^\perp$ and $f(x) = 1$.

For simplicity, we shall say that $\mathcal{N}$ is *incomplemented* if Alg$\mathcal{N}$ has no non-trivial idempotents whose ranges belong to $\mathcal{N}$. Thus by Lemma 2.2, $\mathcal{N}$ is incomplemented if and only if $\mathcal{M}$ is incomplemented.

Remark 5.1. Suppose that $\mathcal{N}$ is incomplemented. Let P be in idem($\mathcal{N}$) and E in $\mathcal{N}^0$.

(1) If $PE = 0$ then $(I - P)^* E^\perp \neq 0$. Otherwise the range of the idempotent operator $I - P$ would be E.
(2) If $P^* E^\perp = 0$ then $(I - P)E \neq 0$. Otherwise the range of the idempotent operator P would be E.

In the following, we write $\hat{E} = \hat{\psi}(E)$ for $E \in \mathcal{N}^0$.

Lemma 5.2. *Suppose that $\mathcal{N}$ is incomplemented. Let E be in $\mathcal{N}^0$. Then $\tilde{\psi}(\Omega_1(\mathcal{N}, E)) \subseteq \Omega_1(\mathcal{M}, \hat{E})$ or $I - \tilde{\psi}(\Omega_1(\mathcal{N}, E)) \subseteq \Omega_2(\mathcal{M}, \hat{E})$.*

Proof. Let P be in $\Omega_1(\mathcal{N}, E)$ and write $\tilde{P} = \tilde{\psi}(P)$. For any $C \in \mathfrak{I}(\mathcal{N}, E)$, we have

$$[P, [P, C]] = [P, -CP] = CP = [C, P].$$

It follows, for any $D \in \mathfrak{I}(\mathcal{M}, \hat{E})$, that

$$[\tilde{P}, [\tilde{P}, D]] = [D, \tilde{P}].$$

So we have that

$$\tilde{P} D \tilde{P} = \tilde{P} D$$

for all $D \in \mathfrak{I}(\mathcal{M}, \hat{E})$. Hence either $\tilde{P}\hat{E} = 0$ or $(I - \tilde{P})^* \hat{E}^\perp = 0$.

If $\tilde{P}\hat{E}=0$, then $\tilde{P}^*\hat{E}^\perp \neq 0$ since $\tilde{P}^2=\tilde{P}\neq 0$. So $\tilde{P}\in \Omega_1(\mathcal{M},\hat{E})$ in this case.

If $(I-\tilde{P})^*\hat{E}^\perp=0$, then $(I-\tilde{P})\hat{E}\neq 0$ since $(I-\tilde{P})^2=I-\tilde{P}\neq 0$. So $I-\tilde{P}\in\Omega_2(\mathcal{M},\hat{E})$ in this case.

Let P and Q be in $\Omega_1(\mathcal{N},E)$, and suppose that $\tilde{P}\in\Omega_1(\mathcal{M},\hat{E})$ and $I-\tilde{Q}\in\Omega_2(\mathcal{M},\hat{E})$. For any $C\in\mathfrak{I}(\mathcal{N},E)$, we have

$$\big[P,[Q,C]\big]=PQC-PCQ-QCP+CQP=CQP=[C,QP].$$

It follows, for any $D\in\mathfrak{I}(\mathcal{M},\hat{E})$, that

$$\big[\tilde{P},[\tilde{Q},D]\big]=\big[D,\psi(QP)\big].$$

Then we have

$$\tilde{P}\tilde{Q}D-\tilde{P}D\tilde{Q}-\tilde{Q}D\tilde{P}+D\tilde{Q}\tilde{P}=DB-BD,$$

where $B=\psi(QP)$. Since $\tilde{P}\hat{E}=0$ and $(I-\tilde{Q})^*\hat{E}^\perp=0$, the above equation becomes

$$(I-\tilde{Q})D\tilde{P}=DB-BD$$

for all $D\in\mathfrak{I}(\mathcal{M},\hat{E})$. Since $(I-\tilde{P})^*\hat{E}^\perp\neq 0$, we can take $f\in\hat{E}^\perp$ and $z\in Y$ such that $f(\tilde{P}z)=1$ and $f(z)=0$. For any $y\in\hat{E}$, we have

$$(I-\tilde{Q})(y\otimes f)\tilde{P}=(y\otimes f)B-By\otimes f.$$

Applying this equation to z, we get a scalar λ such that $(I-\tilde{Q})y=\lambda y$ for all $y\in\hat{E}$. Since $I-\tilde{Q}$ is idempotent and $(I-\tilde{Q})\hat{E}\neq 0$, it follows that $\lambda=1$. Hence $\tilde{Q}\hat{E}=0$, a contradiction by Remark 5.1.

The proof is complete. $\square$

Similarly, we have

Lemma 5.3. *Suppose that $\mathcal{N}$ is incomplemented. Let E be in $\mathcal{N}^0$. Then $\tilde{\psi}(\Omega_2(\mathcal{N},E))\subseteq\Omega_2(\mathcal{M},\hat{E})$ or $I-\tilde{\psi}(\Omega_2(\mathcal{N},E))\subseteq\Omega_1(\mathcal{M},\hat{E})$.*

Lemma 5.4. *Suppose that $\mathcal{N}$ is incomplemented. Let E be in $\mathcal{N}^0$.*

(1) *Suppose that $\Omega_1(\mathcal{N},E)$ is not empty and $\tilde{\psi}(\Omega_1(\mathcal{N},E))\subseteq\Omega_1(\mathcal{M},\hat{E})$. Then $\tilde{\psi}(\Omega_2(\mathcal{N},E))\subseteq\Omega_2(\mathcal{M},\hat{E})$.*
(2) *Suppose that $\Omega_2(\mathcal{N},E)$ is not empty and $\tilde{\psi}(\Omega_2(\mathcal{N},E))\subseteq\Omega_2(\mathcal{M},\hat{E})$. Then $\tilde{\psi}(\Omega_1(\mathcal{N},E))\subseteq\Omega_1(\mathcal{M},\hat{E})$.*

Proof. We only prove (1); the proof of (2) is similar.

Applying Lemma 5.2 to ψ^{-1}, we get $\tilde{\psi}(\Omega_1(\mathcal{N},E))=\Omega_1(\mathcal{M},\hat{E})$. If $\Omega_2(\mathcal{N},E)$ is empty, then the lemma is trivial. Now suppose that $\Omega_2(\mathcal{N},E)$ is not empty. Let Q be in $\Omega_2(\mathcal{N},E)$. If $I-\tilde{Q}\in\Omega_1(\mathcal{M},\hat{E})$, then $I-Q=\tilde{\psi}^{-1}(I-\tilde{Q})\in\Omega_1(\mathcal{N},\hat{E})$. So $(I-Q)E=0$. But $Q^*E^\perp=0$, a contradiction by Remark 5.1. $\square$

Lemma 5.5. *Suppose that $\mathcal{N}$ is incomplemented. Let E and F be in $\mathcal{N}^0$ such that $F<E$. Suppose that $\Omega_1(\mathcal{N},E)$ is not empty and $\tilde{\psi}(\Omega_1(\mathcal{N},E))\subseteq\Omega_1(\mathcal{M},\hat{E})$. Then $\tilde{\psi}(\Omega_2(\mathcal{N},F))\subseteq\Omega_2(\mathcal{M},\hat{F})$.*

Proof. Fix an element P in $\Omega_1(\mathcal{N},E)$. Let Q be in $\Omega_2(\mathcal{N},F)$ and suppose on the contrary that $I-\tilde{Q}\in\Omega_1(\mathcal{M},\hat{F})$. For any $C\in\mathfrak{I}(\mathcal{N},F)\cap\mathfrak{I}(\mathcal{N},E)$ we have

$$\begin{aligned}\psi(QCP)=\psi\big(\big[Q,[C,P]\big]\big)&=\big[\tilde{Q},[D,\tilde{P}]\big]=\tilde{Q}D\tilde{P}-\tilde{Q}\tilde{P}D-D\tilde{P}\tilde{Q}+\tilde{P}D\tilde{Q}=\tilde{Q}D\tilde{P}-D\tilde{P}\tilde{Q}\\&=D\tilde{P}-D\tilde{P}\tilde{Q}=[D,\tilde{P}-\tilde{P}\tilde{Q}],\end{aligned}$$

where $D=\psi(C)$. Applying ψ^{-1} to this equation and writing $A=\psi^{-1}(\tilde{P}-\tilde{P}\tilde{Q})$, we get $QCP=CA-AC$ for all $C\in\mathfrak{I}(\mathcal{N},F)\cap\mathfrak{I}(\mathcal{N},E)$. Thus for all $x\in F$ and $f\in E^\perp$, we have

$$Qx\otimes P^*f=x\otimes A^*f-Ax\otimes f.$$

Hence there exists a scalar λ such that $Qx=\lambda x$ for all $x\in F$ (cf. the proof of Lemma 5.2). Since Q is idempotent, either $\lambda=0$ or $\lambda=1$. If $\lambda=0$, then $QF=0$ and hence $Q=Q^2=0$, a contradiction; If $\lambda=1$, then $I-Q=0$, a contradiction. $\square$

Lemma 5.6. *Suppose that $\mathcal{N}$ is incomplemented and that $\mathcal{N}^0$ has at least two elements and $X_-<X$.*

(1) *If $\tilde{\psi}(\Omega_1(\mathcal{N},X_-))\subseteq\Omega_1(\mathcal{M},\widehat{X_-})$, then $\hat{\psi}$ is order-preserving.*
(2) *If $\tilde{\psi}(\Omega_1(\mathcal{N},X_-))\subseteq I-\Omega_2(\mathcal{M},\widehat{X_-})$, then $\hat{\psi}$ is anti-order-preserving.*

Proof. We only prove (1). The proof of (2) is similar.

For convenience, we write $E_1 = X_-$. Take E_2 in $\mathcal{N}^0$ such that $E_2 < E_1$. Fix an idempotent P in $\Omega_1(\mathcal{N}, E_1)$. Then $\tilde{P} \in \Omega_1(\mathcal{N}, \hat{E}_1)$, where $\tilde{P} = \bar{\psi}(P)$. For any $D_i \in \Im(\mathcal{N}, \hat{E}_i)$, write $C_i = \bar{\psi}^{-1}(D_i)$. Then $C_i \in \Im(\mathcal{N}, E_i)$, $i = 1, 2$.

Suppose on the contrary that $\hat{\psi}$ is anti-order-preserving. Then $\hat{E}_1 < \hat{E}_2$. Thus we have

$$0 = \psi\big(\big[C_1, [C_2, P]\big]\big) = \big[D_1, [D_2, \tilde{P}]\big] = [D_1, D_2\tilde{P} - \tilde{P}D_2] = D_1D_2\tilde{P} - D_1\tilde{P}D_2.$$

Taking $D_i = y_i \otimes g_i$ in the above equation for $y_i \in \hat{E}_i$ and $g_i \in \hat{E}_i^{\perp}$, $i = 1, 2$, we get

$$g_1(y_2)y_1 \otimes \tilde{P}^* g_2 = g_1(\tilde{P}y_2)y_1 \otimes g_2.$$

Choosing $y_2 \in \hat{E}_2$ and $g_1 \in \hat{E}_1^{\perp}$ such that $g_1(y_2) \neq 0$, we see that there is a scalar λ such that $\tilde{P}^* g_2 = \lambda g_2$ for all $g_2 \in \hat{E}_2^{\perp}$. Since $\tilde{P}^*$ is idempotent, either $\lambda = 0$ or $\lambda = 1$.

If $\lambda = 0$, then $g_1(\tilde{P}y_2) = 0$ for all $g_1 \in \hat{E}_1^{\perp}$ and $y_2 \in \hat{E}_2$. This implies that $\tilde{P}^*\hat{E}_1^{\perp} \subseteq \hat{E}_2^{\perp}$. Hence

$$\tilde{P}^*\hat{E}_1^{\perp} = \tilde{P}^*\big(\tilde{P}^*\hat{E}_1^{\perp}\big) \subseteq \tilde{P}^*\hat{E}_2^{\perp} = 0.$$

This conflicts with the fact $\tilde{P} \in \Omega_1(\mathcal{N}, \hat{E}_1)$.

If $\lambda = 1$, then $g_1(\tilde{P}y_2) = g_1(y_2)$ for all $g_1 \in \hat{E}_1^{\perp}$ and $y_2 \in \hat{E}_2$. This implies that $\tilde{(I - P)}^*\hat{E}_1^{\perp} \subseteq \hat{E}_2^{\perp}$. Hence

$$(I - \tilde{P})^*\hat{E}_1^{\perp} = (I - \tilde{P})^*\big((I - \tilde{P})^*\hat{E}_1^{\perp}\big) \subseteq (I - \tilde{P})^*\hat{E}_2^{\perp} = 0.$$

This together with $\tilde{P}\hat{E}_1 = 0$ implies that $\tilde{P}$ is an idempotent operator onto $\hat{E}_1$, a contradiction. □

6. Preserving rank-oneness

Throughout this section we assume that $\mathcal{N}$ is incomplemented and write $\hat{E} = \hat{\psi}(E)$ for $E \in \mathcal{N}^0$. Then $\mathcal{M}$ is incomplemented, either. Moreover, every non-zero subspace in $\mathcal{N}$ and $\mathcal{M}$ is infinite-dimensional.

Proposition 6.1. *Suppose that $(0)_+ = (0)$ and $X_- = X$. Let E be in $\mathcal{N}^0$. Suppose that $x \in E$ and $f \in E^{\perp}$. Then $\bar{\psi}(x \otimes f)$ is of rank one.*

Proof. It follows from Proposition 4.4 that $\mathcal{M}$ is also continuous at (0) and Y. Let $D = \bar{\psi}(x \otimes f)$. Then $D \in \Im(\mathcal{M}, \hat{E})$. Suppose on the contrary that D is of rank greater than one. Then there exist a subspace M in $\mathcal{M}^0$ and two vectors u and v in M such that and Du and Dv are linearly independent. Hence there are a subspace L in $\mathcal{M}^0$ and a functional g in $L^{\perp}$ such that $g(Du) = 0$ and $g(Dv) \neq 0$. Obviously, $L < \hat{E} < M$. Let z be in L and h be in $M^{\perp}$. Let $A = \bar{\psi}^{-1}(z \otimes g)$ and $B = \bar{\psi}^{-1}(u \otimes h)$. Then $A \in \Im(\mathcal{N}, \hat{\psi}^{-1}(L))$ and $B \in \Im(\mathcal{N}, \hat{\psi}^{-1}(M))$.

To get a contradiction, we consider some cases.

Case 1. $\hat{\psi}$ is order-preserving. Then $\hat{\psi}^{-1}(L) < E < \hat{\psi}^{-1}(M)$. So we have

$$\psi\big(A(x \otimes f)B\big) = \psi\big(\big[A, [x \otimes f, B]\big]\big) = \big[z \otimes g, [D, u \otimes h]\big] = (z \otimes g)D(u \otimes h) = 0.$$

It follows from the injectivity of ψ that $A(x \otimes f) = 0$ or $(x \otimes f)B = 0$. If $A(x \otimes f) = 0$, then

$$0 = \psi\big([A, x \otimes f]\big) = [z \otimes g, D] = (z \otimes g)D,$$

a contradiction; If $(x \otimes f)B = 0$, then

$$0 = \psi\big([x \otimes f, B]\big) = [D, u \otimes h] = D(u \otimes h),$$

a contradiction.

Case 2. $\hat{\psi}$ is anti-order-preserving. Then $\hat{\psi}^{-1}(M) < E < \hat{\psi}^{-1}(L)$. So

$$\psi\big(B(x \otimes f)A\big) = \psi\big(\big[B, [x \otimes f, A]\big]\big) = \big[u \otimes h, [D, z \otimes g]\big] = (z \otimes g)D(u \otimes h) = 0.$$

Hence either $B(x \otimes f) = 0$ or $(x \otimes f)A = 0$. If $B(x \otimes f) = 0$, then

$$0 = \psi\big([B, x \otimes f]\big) = [u \otimes h, D] = -D(u \otimes g),$$

a contradiction; If $(x \otimes f)A = 0$, then

$$0 = \psi\big([x \otimes f, A]\big) = [D, z \otimes g] = -(z \otimes g)D,$$

a contradiction. □

Lemma 6.2. *Suppose that $(0) < E = X_- < X$. Suppose that $x \in E$ and $f \in E^{\perp}$. Then $\psi(x \otimes f)$ is of rank one.*

Proof. Let $D = \psi(x \otimes f)$. Then $D \in \mathfrak{I}(\mathcal{M}, \hat{E})$.

Case 1. $\tilde{\psi}(\Omega_1(\mathcal{N}, E)) \subseteq \Omega_1(\mathcal{M}, \hat{E})$.

Suppose on the contrary that D is of rank at least two. Then there are two vectors u and v in Y such that Du and Dv are linearly independent. Moreover u and v are not in $\hat{E}$.

By Lemma 5.6, $\hat{\psi}$ is order-preserving. Then $(0) < Y_- = \hat{E} < Y$. Take h in $\hat{E}^{\perp}$ such that $h(u) = 1$. Then $u \otimes h \in \Omega_1(\mathcal{M}, \hat{E})$. Let $B = \tilde{\psi}^{-1}(u \otimes h)$. Then $B \in \Omega_1(\mathcal{N}, E)$. Let $A = \psi^{-1}(z \otimes g)$. Here z and g are chosen as follows.

If $(0)_+ = (0)$ in $\mathcal{N}$, then $(0)_+ = (0)$ in $\mathcal{M}$. Hence there are M in $\mathcal{M}^0$ and g in $M^{\perp}$ such that $g(Du) = 0$ and $g(Dv) = 1$. Obviously $M < \hat{E}$, and hence $\hat{\psi}^{-1}(M) < E$. Take a non-zero vector z in M. Then $A \in \mathfrak{I}(\mathcal{N}, \hat{\psi}^{-1}(M))$.

If $(0)_+ > (0)$ in $\mathcal{N}$, then $(0)_+ > (0)$ in $\mathcal{M}$. Let $M = \hat{\psi}((0)_+)$. Then $(0) < M = (0)_+ \leqslant \hat{E}$. Since M is infinite-dimensional, there are a vector z in M and a functional g in Y^* such that $g(Du) = 0$ and $g(z) = g(Dv) = 1$. Then $A \in \Omega_2(\mathcal{N}, (0)_+)$.

Note that in both cases we have $A^*\hat{\psi}^{-1}(M)^{\perp} = 0$ and $g(Du) = 0$ and $g(Dv) = 1$. Thus we have

$$\psi\big(A(x \otimes f)B\big) = \psi\big(\big[A, [x \otimes f, B]\big]\big) = \big[z \otimes g, [D, u \otimes h]\big] = (z \otimes g)D(u \otimes h) = 0.$$

So, $A(x \otimes f) = 0$ or $(x \otimes f)B = 0$. If $A(x \otimes f) = 0$, then

$$0 = \psi\big([A, x \otimes f]\big) = [z \otimes g, D] = (z \otimes g)D,$$

a contradiction; If $(x \otimes f)B = 0$, then

$$0 = \psi\big([x \otimes f, B]\big) = [D, u \otimes h] = D(u \otimes h),$$

a contradiction.

Case 2. $\tilde{\psi}(\Omega_1(\mathcal{N}, E)) \subseteq I - \Omega_2(\mathcal{M}, \hat{E})$.

By Lemma 5.6, $\hat{\psi}$ is anti-order-preserving. Then $(0) < (0)_+ = \hat{E} < Y$.

Suppose on the contrary that D is of rank at least two. We shall get a contradiction by considering two subcases.

Case 2.1. $Y = Y_-$. Then there are an element M in $\mathcal{M}^0$ and vectors u and v in M such that Du and Dv are linearly independent. Let h be in $M^{\perp}$ and set $B = \psi^{-1}(u \otimes h)$. Then $B \in \mathfrak{I}(\mathcal{N}, \hat{\psi}^{-1}(M))$. Choose g in Y^* such that $g(Du) = 0$ and $g(Dv) = 1$. Let $z = Dv$. Then $z \in \hat{E}$. Let $A = \tilde{\psi}^{-1}(z \otimes g)$. Then $I - A \in \Omega_1(\mathcal{N}, E)$. Obviously $\hat{E} < M$. It follows that $\hat{\psi}^{-1}(M) < E$. We therefore have

$$\psi\big(B(x \otimes f)(I - A)\big) = \psi\big(\big[B, [x \otimes f, I - A]\big]\big) = \big[u \otimes h, [D, -z \otimes g]\big] = 0.$$

This implies that either $B(x \otimes f) = 0$ or $(x \otimes f)(I - A) = 0$.

If $B(x \otimes f) = 0$, then

$$0 = \psi\big([B, x \otimes f]\big) = [u \otimes h, D] = -D(u \otimes h),$$

a contradiction; If $(x \otimes f)(I - A) = 0$, then

$$0 = \psi\big([x \otimes f, I - A]\big) = [D, -z \otimes g] = (z \otimes g)D,$$

a contradiction.

Case 2.2. $Y_- < Y$. Then there are two vectors u and v such that Du and Dv are linearly independent. If both u and v are in Y_-, we can get a contradiction by an argument similar to that in Case 2.1. Now without loss of generality, we can assume that u is not in Y_-. Choose a functional h in $Y_-^{\perp}$ such that $h(u) = 1$. Set $B = \tilde{\psi}^{-1}(u \otimes h)$. Then $I - B \in \Omega_2(\mathcal{N}, \hat{\psi}(Y_-))$ by Lemma 5.4. Take a functional g in Y^* such that $g(Du) = 0$ and $g(Dv) = 1$. Let $z = Dv$. Then $z \in \hat{E} = (0)_+$. Let $A = \tilde{\psi}^{-1}(z \otimes g)$. Then $I - A \in \Omega_1(\mathcal{N}, E)$. Obviously $\hat{E} \leqslant Y_-$. It follows that $\hat{\psi}^{-1}(Y_-) \leqslant E$. We therefore have

$$\psi\big((I - B)(x \otimes f)(I - A)\big) = \psi\big(\big[I - B, [x \otimes f, I - A]\big]\big) = \big[-u \otimes h, [D, -z \otimes g]\big] = 0.$$

This implies that either $(I - B)(x \otimes f) = 0$ or $(x \otimes f)(I - A) = 0$.

If $(I - B)(x \otimes f) = 0$, then

$$0 = \psi\big([I - B, x \otimes f]\big) = [-u \otimes h, D] = D(u \otimes h),$$

a contradiction; If $(x \otimes f)(I - A) = 0$, then

$$0 = \psi\big([x \otimes f, I - A]\big) = [D, -z \otimes g] = (z \otimes g)D,$$

a contradiction. □

Lemma 6.3. *Suppose that $(0) < X_- < X$. Suppose that P is an idempotent operator of rank one in $\Omega_1(\mathcal{N}, X_-)$. Then $\psi(P)$ is a sum of a scalar operator and a rank-one operator.*

Proof. By Lemma 6.2, ψ maps $X_- \otimes X_-^\perp$ onto $\widehat{X_-} \otimes (\widehat{X_-})^\perp$. Let $\tilde{P} = \tilde{\psi}(P)$.

Case 1. $\tilde{\psi}(\Omega_1(\mathcal{N}, X_-)) \subseteq \Omega_1(\mathcal{M}, \widehat{X_-})$. Then $\widehat{X_-} = Y_-$ by Lemma 5.6. Therefore, ψ maps $\mathbb{F}I + X_- \otimes X_-^\perp$ onto $\mathbb{F}I + Y_- \otimes Y_-^\perp$ by Lemma 6.2. We now consider two subcases according to Proposition 2.4.

Case 1.1. There are a bilinear map $\gamma : X_- \times X_-^\perp \to \mathbb{F}$, and linear bijective maps $C : X_- \to Y_-$ and $D : X_-^\perp \to Y_-^\perp$ such that $\psi(x \otimes f) = \gamma(x, f)I + Cx \otimes Df$ for all $x \in X_-$ and $f \in X_-^\perp$.

Then

$$\psi\big([x \otimes f, P]\big) = \psi\big(x \otimes P^* f\big) = \gamma\big(x, P^* f\big)I + Cx \otimes DP^* f$$

and

$$\psi\big([x \otimes f, P]\big) = [Cx \otimes Df, \tilde{P}] = Cx \otimes \tilde{P}^* Df$$

and so

$$Cx \otimes \tilde{P}^* Df = \gamma\big(x, P^* f\big)I + Cx \otimes DP^* f$$

for all $x \in X_-$ and $f \in X_-^\perp$. Since I is of infinite rank, it follows that $\gamma(x, P^* f) = 0$ and then $Cx \otimes \tilde{P}^* Df = Cx \otimes DP^* f$ for all $x \in X_-$ and $f \in X_-^\perp$. Hence $\tilde{P}^* Df = DP^* f$ for all $f \in X_-^\perp$. Since P is of rank one, it follows that the restriction of $\tilde{P}^*$ to $Y_-^\perp$ is of rank one. Note that $\tilde{P}^* Y^* \subseteq Y_-^\perp$ and hence $\tilde{P}^* Y^* = \tilde{P}^*(\tilde{P}^* Y^*) \subseteq \tilde{P}^* Y_-^\perp$. So $\tilde{P}^*$ is of rank one and so is $\tilde{P}$.

Case 1.2. There are a bilinear map $\gamma : X_- \times X_-^\perp \to \mathbb{F}$, and linear bijective maps $C : X_- \to Y_-^\perp$ and $D : X_-^\perp \to Y_-$ such that $\psi(x \otimes f) = \gamma(x, f)I + Df \otimes Cx$ for all $x \in X_-$ and $f \in X_-^\perp$.

Then

$$\psi\big([x \otimes f, P]\big) = \psi\big(x \otimes P^* f\big) = \gamma\big(x, P^* f\big)I + DP^* f \otimes Cx$$

and

$$\psi\big([x \otimes f, P]\big) = [Df \otimes Cx, \tilde{P}] = Df \otimes \tilde{P}^* Cx$$

and so

$$Df \otimes \tilde{P}^* Cx = \gamma\big(x, P^* f\big)I + DP^* f \otimes Cx$$

for all $x \in X_-$ and $f \in X_-^\perp$. Since I is of infinite rank, it follows that $\gamma(x, P^* f) = 0$ and then $Df \otimes \tilde{P}^* Cx = DP^* f \otimes Cx$ for all $x \in X_-$ and $f \in X_-^\perp$. Since P is of rank one, it follows that D is also of rank one, which conflicts with that fact that $X_-^\perp$ is infinite-dimensional.

Case 2. $\tilde{\psi}(\Omega_1(\mathcal{N}, X_-)) \subseteq I - \Omega_2(\mathcal{M}, \widehat{X_-})$. Then $\widehat{X_-} = (0)_+$ by Lemma 5.6. Therefore, ψ maps $X_- \otimes X_-^\perp$ onto $(0)_+ \otimes ((0)_+)^\perp$. We now consider two subcases according to Proposition 2.4.

Case 2.1. There are a bilinear map $\gamma : X_- \times X_-^\perp \to \mathbb{F}$, and bijective linear maps $C : X_- \to ((0)_+)^\perp$ and $D : X_-^\perp \to (0)_+$ such that $\psi(x \otimes f) = \gamma(x, f)I + Df \otimes Cx$ for all $x \in X_-$ and $f \in X_-^\perp$.

Then

$$\psi\big([x \otimes f, P]\big) = \psi\big(x \otimes P^* f\big) = \gamma\big(x, P^* f\big)I + DP^* f \otimes Cx$$

and

$$\psi\big([x \otimes f, P]\big) = -[Df \otimes Cx, I - \tilde{P}] = (I - \tilde{P})Df \otimes Cx$$

and so

$$(I - \tilde{P})Df \otimes Cx = \gamma\big(x, P^* f\big)I + DP^* f \otimes Cx$$

for all $x \in X_-$ and $f \in X_-^\perp$. Since I is of infinite rank, it follows that $(I - \tilde{P})Df = DP^* f$ for all $f \in X_-^\perp$. Since P is of rank one, it follows that the restriction of $I - \tilde{P}$ to $(0)_+$ is of rank one. Hence $I - \tilde{P}$ is of rank one.

Case 2.2. There are a bilinear map $\gamma : X_- \times X_-^\perp \to \mathbb{F}$, and linear bijective maps $C : X_- \to (0)_+$ and $D : X_-^\perp \to ((0)_+)^\perp$ such that $\psi(x \otimes f) = \gamma(x, f)I + Cx \otimes Df$ for all $x \in X_-$ and $f \in X_-^\perp$.

Then

$$\psi\big([x \otimes f, P]\big) = \psi\big(x \otimes P^* f\big) = \gamma\big(x, P^* f\big)I + Cx \otimes DP^* f$$

and

$$\psi\big([x \otimes f, P]\big) = -[Cx \otimes Df, I - \tilde{P}] = (I - \tilde{P})Cx \otimes Df$$

and so

$$(I-\tilde{P})Cx\otimes Df = Cx\otimes DP^*f$$

for all $x\in X_-$ and $f\in X_-^{\perp}$. Since P is of rank one, it follows that D is also of rank one, which conflicts with the fact that $X_-^{\perp}$ is infinite-dimensional. □

Proposition 6.4. *Suppose that* $(0)<X_-<X$. *Let* x *be in* X *and* f *be in* $X_-^{\perp}$. *Then* $\psi(x\otimes f)$ *is a sum of a scalar operator and a rank-one operator.*

Proof. If $f(x)\neq 0$, the conclusion follows from Lemma 6.3 and the linearity of ψ. Now suppose $f(x)=0$. Take x_1 in X such that $f(x_1)=1$. Let $x_2=x-2x_1$ and $x_3=x-x_1$. Noting $f(x_i)\neq 0$, by Lemma 6.3 we can suppose that

$$\psi(x_i\otimes f)=\lambda_i I+u_i\otimes h_i$$

for $i=1,2,3$. Since

$$\psi(x\otimes f)=\psi\big((x_1+x_3)\otimes f\big)=(\lambda_1+\lambda_3)I+u_1\otimes h_1+u_3\otimes h_3,$$

it is sufficient to show that $u_1\otimes h_1+u_3\otimes h_3$ is of rank one. To this end, we observe that

$$(\lambda_1+\lambda_2)I+u_1\otimes h_1+u_2\otimes h_2=\psi\big((x_1+x_2)\otimes f\big)=\psi(x_3\otimes f)=\lambda_3 I+u_3\otimes h_3.$$

Since I is of infinite rank, it follows that $u_1\otimes h_1+u_2\otimes h_2=u_3\otimes h_3$. This forces that $\{u_1,u_3\}$ or $\{h_1,h_3\}$ is a linearly dependent set. Consequently, $u_1\otimes h_1+u_3\otimes h_3$ is of rank one. □

Similarly we have

Proposition 6.5. *Suppose that* $(0)<(0)_+<X$. *Let* x *be in* $(0)_+$ *and* f *be in* X^*. *Then* $\psi(x\otimes f)$ *is a sum of a scalar operator and a rank-one operator.*

7. Proof of Theorem 1.1

If $\mathcal{N}=\{(0),X\}$. Then $\mathcal{M}=\{(0),Y\}$. So ψ is a Lie isomorphism of $B(X)$ onto $B(Y)$. Now our result follows from [13].

If $\mathcal{N}$ is complemented, so is $\mathcal{M}$. Thus $\mathrm{Alg}\mathcal{N}$ can be written as $\begin{pmatrix}\mathcal{A}_1 & \mathcal{C}_1\\ 0 & \mathcal{B}_1\end{pmatrix}$ with respect to the decomposition $X=E\oplus E'$ for some $E\in\mathcal{N}^0$; and $\mathrm{Alg}\mathcal{M}$ can be written as $\begin{pmatrix}\mathcal{A}_2 & \mathcal{C}_2\\ 0 & \mathcal{B}_2\end{pmatrix}$ with respect to the decomposition $Y=F\oplus F'$ for some $F\in\mathcal{M}^0$. Now our result follows from [2] (in particular, cf. [2, Corollary 4.5]).

In the sequel, we assume that $\mathcal{N}$ is non-trivial and incomplemented. Then $\mathcal{M}$ is non-trivial and incomplemented, either. We distinguish some cases.

Case 1. $(0)=(0)_+$ and $X_-=X$. Then $(0)=(0)_+$ and $Y_-=Y$.

Let E be in $\mathcal{N}^0$. Then by Proposition 6.1, ψ maps $\mathbb{F}I+E\otimes E^{\perp}$ onto $\mathbb{F}I+\hat{\psi}(E)\otimes\hat{\psi}(E)^{\perp}$. Further by Proposition 2.4 and a standard argument (see, for example, [17]), one of the following holds

(a) there exist a bilinear map $\gamma:\mathrm{span}\{E\times E^{\perp}\colon E\in\mathcal{N}^0\}\to\mathbb{F}$, and bijective linear maps $C:\mathrm{span}\{x\in E\colon E\in\mathcal{N}^0\}\to\mathrm{span}\{y\in F\colon F\in\mathcal{M}^0\}$ and $D:\mathrm{span}\{f\in E^{\perp}\colon E\in\mathcal{N}^0\}\to\mathrm{span}\{g\in F^{\perp}\colon F\in\mathcal{M}^0\}$ such that

$$\psi(x\otimes f)=\gamma(x,f)I+Cx\otimes Df \tag{7.1}$$

holds for all $x\in E$ and $f\in E^{\perp}$ with $E\in\mathcal{N}^0$;

(b) there exist a bilinear map $\gamma:\mathrm{span}\{E\times E^{\perp}\colon E\in\mathcal{N}^0\}\to\mathbb{F}$, and bijective linear maps $C:\mathrm{span}\{x\in E\colon E\in\mathcal{N}^0\}\to\mathrm{span}\{g\in F^{\perp}\colon F\in\mathcal{M}^0\}$ and $D:\mathrm{span}\{f\in E^{\perp}\colon E\in\mathcal{N}^0\}\to\mathrm{span}\{y\in F\colon F\in\mathcal{M}^0\}$ such that

$$\psi(x\otimes f)=\gamma(x,f)I+Df\otimes Cx \tag{7.2}$$

holds for all $x\in E$ and $f\in E^{\perp}$ with $E\in\mathcal{N}^0$.

First suppose that (a) holds. For $A\in\mathrm{Alg}\mathcal{N}$, $x\in E$ and $f\in E^{\perp}$ with $E\in\mathcal{N}^0$, from $\psi([A,x\otimes f])=[\psi(A),\psi(x\otimes f)]$ and Eq. (7.1) it follows that

$$\psi(A)Cx\otimes Df-Cx\otimes\psi(A)^*Df=\big(\gamma(Ax,f)-\gamma\big(x,A^*f\big)\big)I+CAx\otimes Df-Cx\otimes DA^*f,$$

and hence

$$CAx \otimes Df - Cx \otimes DA^* f = \psi(A)Cx \otimes Df - Cx \otimes \psi(A)^* Df$$

since I is infinite-rank. So there exists a scalar operator $\tau(A)$ such that

$$\psi(A)Cx = CAx + \tau(A)Cx$$

for all $x \in \operatorname{span}\{x \in E\colon\ E \in \mathcal{N}^0\}$. Evidently, τ is linear. Define $\phi = \psi - \tau$. Then for any $A, B \in \operatorname{Alg}\mathcal{N}$ and $x \in E$ with $E \in \mathcal{N}^0$, we have

$$\phi(AB)Cx = CABx = \phi(A)CBx = \phi(A)\phi(B)Cx.$$

Since $\{Cx\colon\ x \in E,\ E \in \mathcal{N}^0\}$ is dense in Y, we know that ϕ is an isomorphism. Since isomorphisms between nest algebras are spatial, the statement (1) holds.

Now suppose that (b) holds. For $A \in \operatorname{Alg}\mathcal{N}$, $x \in E$ and $f \in E^\perp$ with $E \in \mathcal{N}^0$, from $\psi([A, x \otimes f]) = [\psi(A), \psi(x \otimes f)]$ and Eq. (7.2) it follows that

$$\psi(A)Df \otimes Cx - Df \otimes \psi(A)^* Cx = \big(\gamma(Ax, f) - \gamma\big(x, A^* f\big)\big)I + Df \otimes CAx - DA^* f \otimes Cx,$$

and hence

$$Df \otimes CAx - DA^* f \otimes Cx = \psi(A)Df \otimes Cx - Df \otimes \psi(A)^* Cx.$$

So there exists a scalar operator $\tau(A)$ such that

$$\psi(A)^* Cx = -CAx + \tau(A)^* Cx$$

for all $x \in \operatorname{span}\{x \in E\colon\ E \in \mathcal{N}^0\}$. Evidently, τ is linear. Define $\phi = -\psi + \tau$. Then for any $A, B \in \operatorname{Alg}\mathcal{N}$ and $x \in E$ with $E \in \mathcal{N}^0$, we have

$$\phi(AB)^* Cx = CABx = \phi(A)^* CBx = \phi(A)^* \phi(B)^* Cx.$$

Since $\{Cx\colon\ x \in E,\ E \in \mathcal{N}^0\}$ is dense in Y, we have $\phi(AB)^* = (\phi(B)\phi(A))^*$ and then $\phi(AB) = \phi(B)\phi(A)$. So ϕ is an anti-isomorphism. Hence the statement (2) holds.

Case 2. $(0) < X_- < X$.

By Proposition 6.4, ψ maps $\mathbb{F}I + X \otimes X_-^\perp$ onto $\mathbb{F}I + Y \otimes Y_-^\perp$ or onto $\mathbb{F}I + (0)_+ \otimes Y^*$. Hence by Proposition 2.4 and the argument in the proof of Lemma 6.3, one of the following holds.

(a) For the case $\tilde{\psi}(\Omega_1(\mathcal{N}, X_-)) \subseteq \Omega_1(\mathcal{M}, \widehat{X_-})$, there are linear bijective maps $C : X \to Y$ and $D : X_-^\perp \to Y_-^\perp$ and a function $\gamma : X \times X_-^\perp \to \mathbb{F}I$ such that

$$\psi(x \otimes f) = \gamma(x, f) + Cx \otimes Df$$

holds for all $x \in X$ and $f \in X_-^\perp$.

(b) For the case $\tilde{\psi}(\Omega_1(\mathcal{N}, X_-)) \subseteq I - \Omega_2(\mathcal{M}, \widehat{X_-})$, there are linear bijective maps $C : X \to Y^*$ and $D : X_-^\perp \to (0)_+$ and a function $\gamma : X \times X_-^\perp \to \mathbb{F}I$ such that

$$\psi(x \otimes f) = \gamma(x, f) + Df \otimes Cx$$

holds for all $x \in X$ and $f \in X_-^\perp$.

Now repeating the argument in Case 1, we see that the statement (1) holds if (a) holds, and that the statement (2) holds if (b) holds.

Case 3. $(0) < (0)_+ < X$.

Arguing as that in Case 2.

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