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Bäcklund transformation and soliton interactions for the Zakharov–Kuznetsov equation in plasmas

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ABSTRACT

Symbolically investigated in this paper is the Zakharov–Kuznetsov equation which describes the propagation of the electrostatic excitations in a magnetized, rotating and collisionless three-component plasma. Bilinear form and Bäcklund transformation for the Zakharov–Kuznetsov equation are derived with the Hirota method and symbolic computation. N-soliton solutions in terms of the Wronskian determinant are constructed, and the verification is finished through the direct substitution into bilinear equations. Propagation characteristics and interaction behaviors of the solitons are discussed through a graphical analysis. During the propagation, the one-soliton width and amplitude are both unchanged and not related to the coefficients, while the soliton interactions are elastic.

1. Introduction

Solitons have been studied both theoretically and experimentally in such fields as fluid mechanics, solid state physics, plasmas and nonlinear optics [1–5]. Recently, the higher-dimensional nonlinear evolution equations (NLEEs) have attracted the attention of researchers due to their potential applications in science and engineering [4–7]. For example, the Kadomtsev–Petviashvili equation, one of the integrable NLEEs in (2+1) dimensions, has been derived for plasmas and fluids [8]. In plasma physics, the Davey–Stewartson equation has been shown to explain an electrostatic ion wave propagating perpendicularly to an applied magnetic field [9]. More (2+1)-dimensional NLEEs have been constructed on the basis of the relationship with the hierarchies of the (1+1)-dimensional NLEEs [10].

On the other hand, methods have been proposed for solving the NLEEs [3–5], e.g., the Darboux transformation [11], inverse scattering transformation [12], Bäcklund transformation (BT) [13,14], Painlevé test [15] and Hirota method [16]. Among those methods, the Hirota method is the one that provides a tool for constructing an N-soliton solution which can be expressed in the form of an Nth-order polynomial in N exponentials [16]. The BT can also be used to obtain a nontrivial solution from a seed solution [13,14]. However, it is tedious to write out the derivatives of all the formulations of the N-soliton solution obtained by the Hirota method, so the verification of the solution through the direct substitution into the NLEEs becomes difficult [16]. Another representation of the N-soliton solution, namely, the Wronskian determinant, has been employed to solve such problems [17,18]. Wronskian technique can help to verify the validity of the N-soliton solution, and has been applied to such NLEEs as the KdV, modified KdV and KP equations [17].

In this paper, on the basis of symbolic computation [3–5], we will study the following Zakharov–Kuznetsov (ZK) equation [19]:

$$u_t + \alpha u u_x + \beta u_{xxx} + \gamma u_{xyy} = 0, \tag{1}$$

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where u is a function of the scaled spatial variables x, y and the temporal variable t, and represents the electrostatic wave potential in plasmas, while α, β and γ are the coefficients of the nonlinearity, dispersion, and disturbed wave velocity along the y-direction, respectively. Eq. (1) describes the weakly nonlinear ion-acoustic waves in a strongly-magnetized lossless plasma comprised of cold ions and hot isothermal electrons [19]. In Ref. [20], some solutions have been obtained and the solutions are apparently inelastic. Painlevé analysis has been applied to investigate the integrability of Eq. (1) [21]. Three polynomial conservation laws have been given for Eq. (1) [22]. By employing the multi-dimensional reductive perturbation technique, Ref. [23] has derived Eq. (1) for the evolution of the electric potential perturbation in the electron–positron–ion plasmas, with some solutions given and their characteristics studied. Traveling wave solutions have been obtained via the homotopy perturbation method [24]. A special case of Eq. (1), with $\alpha = 1$, $\beta = \frac{1}{3}$ and $\gamma = \frac{2}{3}$, has also been investigated and some solutions have been obtained [25]. However, to our knowledge, the bilinear BT and N-soliton solutions in terms of the Wronskian determinant for Eq. (1) have not been obtained, which will be the main objectives in the present paper.

This paper will be organized as follows: In Section 2, through the Hirota method, Eq. (1) will be transformed into a bilinear form under a certain parametric constraint. Its N-soliton solutions will be constructed via the formal parameter expansion technique. Explicit one- and two-soliton solutions will be given. In Section 3, the bilinear BT for Eq. (1) will be presented. By means of the BT, a nontrivial solution will be obtained from a trivial solution. In Section 4, we will give the N-soliton solutions in terms of the Wronskian determinant and verify it through direct substitution into the bilinear equations. Section 5 will give our discussion on the soliton interactions. The last section will be our conclusions.

2. The bilinear form and soliton solutions for Eq. (1)

Through the following transformation:

$$u = 2\,(\log f)_{xx} + 2\,(\log f)_{xy},$$

where $f(x, y, t)$ is a real differentiable function, the bilinear form for Eq. (1) can be obtained:

$$\left[\left(D_x + D_y\right)\,\left(D_t + \beta\,D_x^3 + \gamma\,D_x\,D_y^2\right)\right] f\cdot f = 0, \tag{2a}$$

$$(D_x^2 + 3\,D_x\,D_y) f\cdot f = 0, \tag{2b}$$

under the constraint $\alpha - 9\beta - \gamma = 0$. Hereby, D_x, D_y and D_t are the bilinear derivative operators [16,26] defined by

$$D_x^l D_y^m D_t^n(\xi\cdot\zeta) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^l \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n \xi(x, y, t)\,\zeta(x', y', t')|_{x'=x,\, y'=y,\, t'=t}.$$

By virtue of symbolic computation, the N-soliton solutions for Eq. (1) can be derived from bilinear form (2).

Firstly, $f(x, y, t)$ is expanded in powers of a formal parameter ε as

$$f(x, y, t) = 1 + \varepsilon f_1(x, y, t) + \varepsilon^2 f_2(x, y, t) + \cdots, \tag{3}$$

where $f_i(x, y, t)$'s ($i = 1, 2, \ldots$) are all real differentiable functions. Substituting expression (3) into bilinear form (2) and collecting the coefficients of the same power of ε, we have

$$\begin{aligned}
\varepsilon^0 :\ & (D_x + D_y)\,(D_t + \beta\,D_x^3 + \gamma\,D_x\,D_y^2)\,1\cdot 1 = 0,\\
& (D_x^2 + 3\,D_x\,D_y)\,1\cdot 1 = 0,\\
\varepsilon^1 :\ & (D_x + D_y)\,(D_t + \beta\,D_x^3 + \gamma\,D_x\,D_y^2)\,(1\cdot f_1 + f_1\cdot 1) = 0,\\
& (D_x^2 + 3\,D_x\,D_y)\,(1\cdot f_1 + f_1\cdot 1) = 0,\\
\varepsilon^2 :\ & (D_x + D_y)\,(D_t + \beta\,D_x^3 + \gamma\,D_x\,D_y^2)\,(1\cdot f_2 + f_1\cdot f_1 + f_2\cdot 1) = 0,\\
& (D_x^2 + 3\,D_x\,D_y)\,(1\cdot f_2 + f_1\cdot f_1 + f_2\cdot 1) = 0,\\
\varepsilon^3 :\ & (D_x + D_y)\,(D_t + \beta\,D_x^3 + \gamma\,D_x\,D_y^2)\,(1\cdot f_3 + f_1\cdot f_2 + f_2\cdot f_1 + f_3\cdot 1) = 0,\\
& (D_x^2 + 3\,D_x\,D_y)\,(1\cdot f_3 + f_1\cdot f_2 + f_2\cdot f_1 + f_3\cdot 1) = 0,\\
\vdots
\end{aligned}$$

In order to obtain the one-soliton solutions for Eq. (1), let us choose

$$f_1 = \mathrm{e}^{\eta} \quad \text{with } \eta = k\,x + l\,y + w\,t + \eta^0,$$

where k, l, w and η^0 are all constants to be determined. We can truncate the expansion with $f_i(x, y, t) = 0$ ($i = 2,\ 3,\ 4, \ldots$). Consequently, without loss of generality, by setting $\varepsilon = 1$, $f(x, y, t)$ can be reduced to

$$f = 1 + \mathrm{e}^{\eta} \quad \text{with } \eta = k\,x - \frac{1}{3}\,k\,y - \left(\beta + \frac{\gamma}{9}\right)\,k^3\,t + \eta^0.$$

Thus the one-soliton solutions for Eq. (1) can be written as follows:

$$u = \frac{1}{3}\,k^2\,\mathrm{sech}^2\left[\frac{k}{2}\,x - \frac{k}{6}\,y - \left(\frac{\beta}{2} + \frac{\gamma}{18}\right)k^3\,t + \frac{\eta^0}{2}\right]. \tag{4}$$

For constructing the two-soliton solutions, we choose

$$f_1 = \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2}, \qquad f_2 = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}\,\mathrm{e}^{\eta_1+\eta_2},$$

where $\eta_1 = k_1\,x - \frac{1}{3}\,k_1\,y - (\beta + \frac{\gamma}{9})\,k_1^3\,t + \eta_1^0$, $\eta_2 = k_2\,x - \frac{1}{3}\,k_2\,y - (\beta + \frac{\gamma}{9})\,k_2^3\,t + \eta_2^0$ with k_1, k_2, η_1^0 and η_2^0 all as constants. So the two-soliton solutions for Eq. (1) can be written as follows:

$$u = 2\left\{\log\left[1 + \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}\,\mathrm{e}^{\eta_1+\eta_2}\right]\right\}_{xx} + 2\left\{\log\left[1 + \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}\,\mathrm{e}^{\eta_1+\eta_2}\right]\right\}_{xy}. \tag{5}$$

The N-soliton solutions for Eq. (1) can be expressed in the following form:

$$u = 2\,(\log f)_{xx} + 2\,(\log f)_{xy}, \tag{6}$$

$$f = \sum_{\mu=0,1} \exp\left[\sum_{i>j}^{N} A_{ij}\,\mu_i\,\mu_j + \sum_{j=1}^{N} \mu_j\,\eta_j\right], \tag{7}$$

where

$$\eta_j = k_j\,x - \frac{1}{3}\,k_j\,y - \left(\beta + \frac{\gamma}{9}\right)k_j^3\,t + \eta_j^0,$$

while k_j is the parameter characterizing the ith soliton, η_j^0's ($j = 1, 2, \ldots, N$) are all arbitrary constants, $\sum_{\mu=0,1}$ denotes the summation over all possible combinations of $\mu_j = 0, 1$ ($j = 1, 2, \ldots, N$) and $\sum_{i>j}^{N}$ is the summation over all possible pairs chosen from the N elements under the condition $i > j$.

3. The bilinear BT for Eq. (1)

In this section, we will derive the bilinear BT for Eq. (1). Supposing that f and f' are two different solutions for bilinear form (2), we can consider the following expressions:

$$Q_1 = \left[\left(D_x + D_y\right)\left(D_t + \beta\,D_x^3 + \gamma\,D_y^3\right) f'\cdot f'\right] f^2 - \left[\left(D_x + D_y\right)\left(D_t + \beta\,D_x^3 + \gamma\,D_y^3\right) f\cdot f\right] f'^2 = 0, \tag{8a}$$

$$Q_2 = \left[\left(D_x^2 + 3\,D_x\,D_y\right) f'\cdot f'\right] f^2 - \left[\left(D_x^2 + 3\,D_x\,D_y\right) f\cdot f\right] f'^2 = 0. \tag{8b}$$

By using exchange formulas (A.1)–(A.5), the bilinear BT can be given from expressions (8) as follows:

$$\left(D_x + 3\,D_y\right) f'\cdot f = 0, \tag{9a}$$

$$\left(6\,D_t + 4\,\beta\,D_x^3 + 3\,\gamma\,D_x\,D_y^2 + 54\,\beta\,\lambda\,D_x\right) f'\cdot f = 0, \tag{9b}$$

$$\left(2\,D_t - 6\,\gamma\,D_y^3 + \gamma\,D_x^2\,D_y + 18\,\beta\,\lambda\,D_x\right) f'\cdot f = 0, \tag{9c}$$

$$\left[\left(\frac{2}{3}\,\gamma - 6\,\beta\right) D_x\,D_y - \mu\,D_x - 30\,\beta\,\lambda\right] f'\cdot f = 0 \tag{9d}$$

with the constraint $6\beta + \gamma = 0$.

Assuming that $\mu = 0$ and $\lambda = l^2$, substituting the seed solution $f = 1$, i.e., $u = 0$, into Eq. (9), we obtain

$$f' = \mathrm{e}^{\zeta} + \mathrm{e}^{-\zeta}, \tag{10}$$

where $\zeta = -3\,l\,x + l\,y + 36\,\beta\,l^3\,t + \zeta^0$ with ζ^0 an arbitrary constant. Substituting (10) into transformation (6), the one-soliton solutions for Eq. (1) in explicit form can be expressed as

$$u = 12\,l^2\,\mathrm{sech}^2\,\zeta. \tag{11}$$

4. The N-soliton solutions in terms of the Wronskian determinant for Eq. (1)

Assuming that $\mu = 0, \lambda = l_j^2, f = 1$ and $f' = \varphi_j$, where $\phi_j(x, y, t)$ is a real differentiable function, we can get the following results from Eqs. (9):

$$\varphi_{j,x} = -3\,\varphi_{j,y}, \tag{12a}$$

$$\varphi_{j,yy} = l_j^2\,\varphi_j, \tag{12b}$$

$$\varphi_{j,t} = 36\,\beta\,\varphi_{j,yyy}. \tag{12c}$$

Using the Wronskian technique, we assume that Eq. (1) admits the following N-soliton solutions in the Wronskian form:

$$F^{(N)} = W(\varphi_1, \varphi_2, \ldots, \varphi_N) = \begin{vmatrix} \varphi_1 & \varphi_1^{(1)} & \varphi_1^{(2)} & \cdots & \varphi_1^{(N-1)} \\ \varphi_2 & \varphi_2^{(1)} & \varphi_2^{(2)} & \cdots & \varphi_2^{(N-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \varphi_N & \varphi_N^{(1)} & \varphi_N^{(2)} & \cdots & \varphi_N^{(N-1)} \end{vmatrix} = |\widehat{N-1}| \tag{13}$$

with $\varphi_j^i = \frac{\partial^i \varphi_j}{\partial y^i}$, $\varphi_j = e^{\zeta_j} + (-1)^{j+1}\,e^{-\zeta_j}$ and $\zeta_j = -3\,l_j\,x + l_j\,y + 36\,\beta\,l_j^3\,t + \zeta_j^0$ $(i = 1, 2, \ldots, N-1)$. The entries φ_j's $(j = 1, 2, \ldots, N)$ satisfy Eqs. (12). In order to show a brief heuristic description of the Wronskian determinants through using Eq. (13), we take the following example:

$$|\widehat{N-3}, N-1, N| = W(\varphi_1, \varphi_2, \ldots, \varphi_{N-2}, \varphi_N, \varphi_{N+1}) = \begin{vmatrix} \varphi_1 & \varphi_1^{(1)} & \varphi_1^{(2)} & \cdots & \varphi_1^{(N-3)} & \varphi_1^{(N-1)} & \varphi_1^{(N)} \\ \varphi_2 & \varphi_2^{(1)} & \varphi_2^{(2)} & \cdots & \varphi_2^{(N-3)} & \varphi_2^{(N-1)} & \varphi_2^{(N)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \varphi_N & \varphi_N^{(1)} & \varphi_N^{(2)} & \cdots & \varphi_N^{(N-3)} & \varphi_N^{(N-1)} & \varphi_N^{(N)} \end{vmatrix}.$$

Because the Wronskian determinant has the property that each column in the Wronskian determinant is the derivative of the former column, something about the derivative of a determinant can be found by differentiating each column separately and taking the sum of the determinant, and for most terms the determinant is zero since two columns are identical. Therefore, based on the identities of the given φ_j and the property of the Wronskian determinant, the derivatives of $F^{(N)}$ with respect to x, y and t can be computed:

$$\begin{aligned}
F^{(N)} &= |\widehat{N-1}|, \qquad F_y^{(N)} = |\widehat{N-2}, N|, \\
F_{yy}^{(N)} &= |\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|, \\
F_{yyy}^{(N)} &= 2\,|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2| + |\widehat{N-4}, N-2, N-1, N|, \\
F_{yyyy}^{(N)} &= |\widehat{N-2}, N+3| + |\widehat{N-5}, N-3, N-2, N-1, N| + 2\,|\widehat{N-3}, N, N+1| \\
&\quad + 3\,|\widehat{N-3}, N-1, N+2| + 3|\widehat{N-4}, N-2, N-1, N+1|. \\
F_x^{(N)} &= -3\,|\widehat{N-2}, N|, \qquad F_{xy}^{(N)} = -3\,(|\widehat{N-2}, N+1| + |\widehat{N-3}, N-1, N|), \\
F_t^{(N)} &= 36\,\beta\,(|\widehat{N-4}, N-2, N-1, N| - |\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|), \\
F_{yt}^{(N)} &= 36\,\beta\,(|\widehat{N-2}, N+3| - |\widehat{N-3}, N, N+1| + |\widehat{N-5}, N-3, N-2, N-1, N|).
\end{aligned}$$

Substituting the derivatives of $F^{(N)}$ into bilinear form (2), we obtain

$$\begin{aligned}
(D_x + D_y)\,(D_t + \beta\,D_x^3 + \gamma\,D_y^3)\,F^{(N)} \cdot F^{(N)} &= 512\,\beta \begin{vmatrix} \widehat{N-3} & 0 & N-2 & N-1 & N & N+1 \\ 0 & \widehat{N-3} & N-2 & N-1 & N & N+1 \end{vmatrix} \\
&= 0, \\
(D_x^2 + 3\,D_x\,D_y)\,F^{(N)} \cdot F^{(N)} &= 2\left[F_{xx}^{(N)}\,F^{(N)} - \left(F_x^{(N)}\right)^2\right] + 6\,\left(F_{xy}^{(N)}\,F^{(N)} - F_x^{(N)}\,F_y^{(N)}\right) \\
&= 0.
\end{aligned}$$

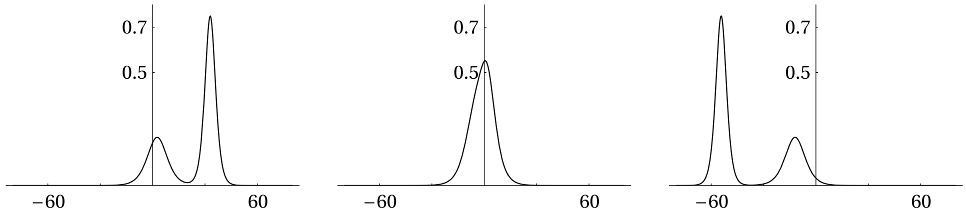
Therefore, the N-soliton solutions in the Wronskian form is verified to be the solutions for Eq. (1).

5. Discussion of the soliton interactions for Eq. (1)

From solutions (4), the width Δ and amplitude A of the soliton can be expressed as

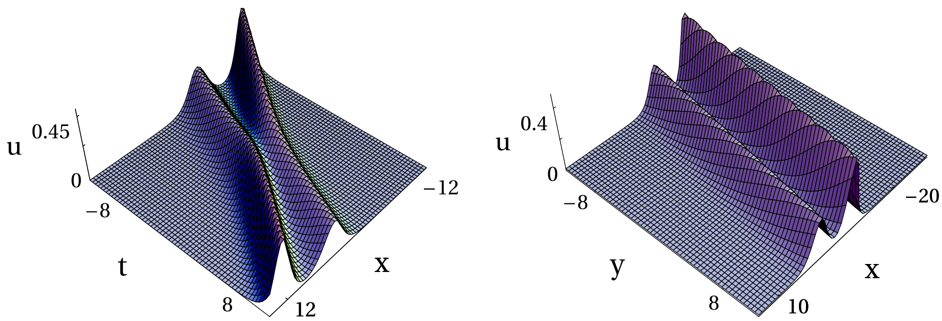
$$A = \frac{1}{3}\,k^2, \qquad \Delta = \frac{3\sqrt{10}}{5|k|}, \tag{14}$$

which are only related to the parameter k from (14).



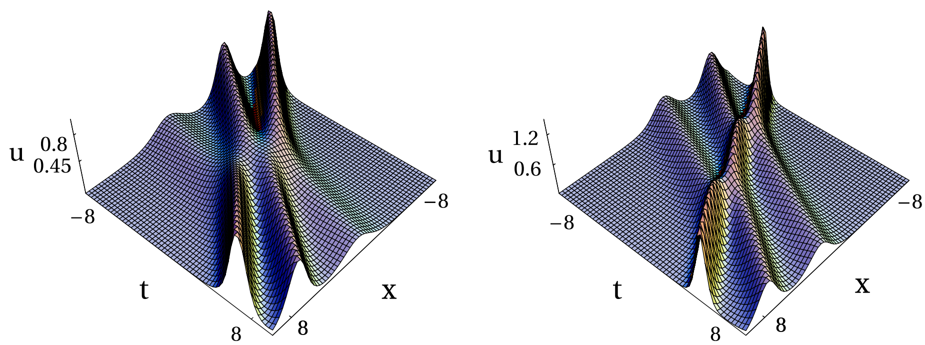
(a) $t = -4$. (b) $t = 0.2$. (c) $t = 7$.

Fig. 1. Interaction of two solitons via solutions (5). The parameters adopted here are: $\alpha = 10, \beta = 1, \gamma = 1, k_1 = 1.5, k_2 = 0.8, \eta_1 = 0, \eta_2 = 0$, at $x = 1$.



(a) $y = 0$. (b) $t = -10$.

Fig. 2. Interaction of the two solitons via solutions (13). The parameters are: $\alpha = 6, \beta = 2, \gamma = -12, l_1 = 0.15, l_2 = -0.2, \zeta_1^0 = 0, \zeta_2^0 = 0$.



(a) $y = 2$. (b) $y = 0$.

Fig. 3. Interaction of the three solitons via solutions (13). The parameters are: $\alpha = 3, \beta = 1, \gamma = -6, \zeta_1^0 = 0, \zeta_2^0 = 0, \zeta_3^0 = 0$. (a) $l_1 = 0.15, l_2 = 0.25, l_3 = 0.3$; (b) $l_1 = 0.35, l_2 = 0.25, l_3 = 0.2$.

Additionally, in order to show the interactions of the solitons, we will analyze the two- and three-soliton solutions through the graphical analysis via solutions (5) and (13) by making some choices of parameters. Fig. 1 illustrates the overtaking interaction of two solitons. After the interaction, the two solitons maintain their original shapes and amplitudes except for the phase shifts. We have also observed that the direction of the propagation can be changed when we adjust the sign of β. More on the solitonic interactions can be seen in [27,28].

Fig. 2(a) illustrates an overtaking elastic interaction of two solitons in the x–t plane which is similar to that plotted in Fig. 1. The soliton with the larger amplitude travels faster, and catches up with the other one with the smaller amplitude. After the interaction, the two solitons retain their original shapes and amplitudes except for the phase shifts. Fig. 2(b) shows the two parallel solitons in the x–y plane at the fixed time t. It can also be seen from solutions (11) that the solitons are parallel at the fixed time t, because they possess the same slope dy/dx in the x–y plane.

With respect to the three solitons, we can also obtain similar behavior of the interaction by modifying the parameters. Fig. 3 displays the elastic interactions of the three solitons with different amplitudes and velocities. The three solitons interact among themselves without exhibiting any change in the properties except for the small phase shifts after the interaction. In Fig. 3(a), the three solitons interact at the same spot in the x–t plane at $y = 2$. However, the interactions among the three solitons at different positions can be shown in Fig. 3(b) with the choices of the parameters. We can depict the analogous interactions in the y–t plane by choosing some parameters.

6. Conclusions

With the aid of symbolic computation, we have investigated Eq. (1) which describes the propagation of the electrostatic excitations in a magnetized, rotating and collisionless three-component plasma. By use of the Hirota method, we have obtained the bilinear form and BT for Eq. (1). Furthermore, N-soliton solutions in terms of the Wronskian determinant has been constructed and proved via the direct substitution into bilinear form (2). In addition, propagation characteristics and interaction behaviors of the solitons have been discussed through the graphical analysis. We have found that the width and amplitude of the one soliton described by solutions (4) are both unchanged and the direction of propagation is related to the sign of β, while the soliton interactions are elastic.

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Appendix. Hirota bilinear operator identities

The following identities are used for (9) in the derivation of the bilinear Bäcklund transformation [16,26]:

$$D_x(D_y f'\cdot f)\cdot(f'f) = D_y(D_x f'\cdot f)\cdot(f'f), \tag{A.1}$$

$$(D_x^4 f'\cdot f')(f)^2 - (D_x^4 f\cdot f)(f')^2 = 2D_x^3(D_x f'\cdot f)\cdot(f'f), \tag{A.2}$$

$$(D_xD_t f'\cdot f')(f)^2 - (D_xD_t f\cdot f)(f')^2 = 2D_x(D_t f'\cdot f)\cdot(f'f), \tag{A.3}$$

$$D_x^3(D_y f'\cdot f)\cdot(f'f) = D_y(D_x^3 f'\cdot f)\cdot(f'f) - 3D_y(D_x^2 f'\cdot f)\cdot(D_x f'\cdot f), \tag{A.4}$$

$$\begin{aligned}(D_x^2D_y^2 f'\cdot f')(f)^2 - (D_x^2D_y^2 f\cdot f)(f')^2 \;=\;& D_x(D_xD_y^2 f'\cdot f)\cdot(f'f) + D_y(D_x^2D_y f'\cdot f)\cdot(f'f)\\ &- 2D_x(D_xD_y f'\cdot f)\cdot(D_y f'\cdot f) - 2D_y(D_xD_y f'\cdot f)\\ &\cdot(D_x f'\cdot f) - D_x(D_y^2 f'\cdot f)\cdot(D_x f'\cdot f) - D_y(D_x^2 f'\cdot f)\cdot(D_y f'\cdot f).\end{aligned} \tag{A.5}$$

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