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A discontinuous Galerkin finite element method for swelling model of polymer gels[☆]

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ABSTRACT

An attractive feature of discontinuous Galerkin (DG) finite element schemes is that this concept offers a unified and versatile discretization platform for various types of partial differential equations. The locality of the trial functions not only supports local mesh refinements but also offers a framework for comfortably varying the order of the discretization. In this paper, we propose and analyze a mixed-DG finite element method for a displacement–pressure model which describes swelling dynamics of polymer gels under mechanical constraints. By introducing a flux variable we first present a reformulation of the governing equations of polymer gels. We then approximate the pressure and flux variables by a mixed finite element space and the displacement by DG finite element method. Existence and uniqueness are proved and error estimates are derived for mixed-DG finite element scheme. Finally, numerical experiments are presented to show the performance of the mixed-DG approximation of polymer gels.

1. Introduction

A gel is a soft material which consists of a solid network and a colloidal solvent. The solid network spans the volume of the solvent medium. The solvent can permeate through the solid network, and the permeation can be controlled by external forces. The phenomenon is important in various industrial processes such as coating and printing [1]. It is also important in the study of gels as actuators and sensors [2,3].

Both by weight and volume, gels are mostly liquid in composition and thus exhibit densities similar to liquids. However, they have the structural coherence of a solid and can be deformed. A gel network can be composed of a wide variety of materials, including particles, polymers, and proteins, which then gives different types of gels such as hydrogels, organogels, and xerogels (cf. [4,5]). Gels have some fascinating properties, in particular, they display thixotropy which means that they become fluid when agitated but resolidify when resting. In general, gels are apparently solid, jelly-like materials, and they exhibit an important state of matter found in a wide variety of biomedical and chemical systems (cf. [6–9] and the references therein).

Although the models of gels have been proposed for a long time, proper theoretical framework to numerically analyze the process is few. Our advisor, Feng and He in [10] presented the Taylor–Hood mixed finite element method to solve the model, which was proposed by Doi in [6] and Yamaue and Doi in [8,9], established optimal error estimates and provided numerical

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experiments. As we all know, discontinuous Galerkin (DG) methods [11–14] for the numerical solution of partial differential equations have enjoyed considerable success because they are both flexible and robust: they allow arbitrary unstructured geometries and easy control of accuracy without compromising simulation stability. And the DG finite element method is also a good method to approximate the displacement–pressure model of kinetics and to alleviate numerical oscillations in the stress field. But, to the best of our knowledge, there are no published results addressing the gel model by the DG finite element method. In this paper, we develop a mixed-DG finite element method for a displacement–pressure model which describes swelling dynamics of polymer gels under mechanical constraints, moreover, numerical results illustrate the fact that the mixed-DG finite element method is better to approximate the gel model than the continuous Galerkin (CG) finite element method [15–17].

Our paper continues in Section 2 with an overview of the swelling model of polymer gels. Section 3 includes a brief description of notions for the DG finite element method, and we develop our mixed-DG finite element method. In Section 4, we examine its existence and uniqueness properties and prove a rigorous proof of convergence results. Finally, Section 5 presents some of our numerical experiments to gauge the performance of the proposed mixed-DG finite element method, and concludes with an examination of numerical results associated to a model that encounters locking when solved with the CG finite element method.

2. The swelling model of polymer gels

First, we explain the displacement–pressure coupling model for polymer gels under mechanical constraints. Let $\Omega \subset \mathbb{R}^d (d = 1, 2, 3)$ be a bounded domain and denote the initial region occupied by the gel. Let $\mathbf{u}(x, t)$ denote the displacement of the gel at the point $x \in \Omega$ in the space and at the time t, let $\mathbf{v}_s(x, t)$ and $p(x, t)$ be the velocity and the pressure, respectively, of the solvent at (x, t). Following [6,10], the governing equations for the swelling dynamics of polymer gels are given by

$$\operatorname{div}(\sigma(\mathbf{u}) - pI) = 0, \tag{2.1}$$

$$\xi(\mathbf{v}_s - \mathbf{u}_t) = -(1-\phi)\nabla p, \tag{2.2}$$

$$\operatorname{div}(\phi\mathbf{u}_t + (1-\phi)\mathbf{v}_s) = 0. \tag{2.3}$$

Here, the first equation stands for the force balance, the second equation represents Darcy's law for the permeation of solvent through the gel network, and the third equation stands for the incompressibility condition. And ξ is the friction constant associated with the motion of the polymer relative to the solvent, ϕ is the volume fraction of the polymer, I denotes the unit tensor defined by the $d \times d$ identity matrix, and $\sigma(\mathbf{u})$ stands for the stress tensor of the gel network, which is given by a constitutive equation. In this paper, we use the following linearized form of the stress tensor:

$$\sigma(\mathbf{u}) \equiv \alpha \operatorname{div} \mathbf{u} I + \beta\varepsilon(\mathbf{u}), \qquad \epsilon(\mathbf{u}) \equiv \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T), \tag{2.4}$$

$$\alpha \equiv K - \frac{2}{3}G, \qquad \beta \equiv 2G, \tag{2.5}$$

where K and G are, respectively, the bulk and shear modulus and α and β are Lamé constants of the gel (cf. [6,10]). In addition, if we introduce the total stress $\widehat{\sigma}(\mathbf{u}, p) \equiv \sigma(\mathbf{u}) - pI$, then (2.1) becomes $\operatorname{div} \widehat{\sigma}(\mathbf{u}, p) = 0$.

Substituting (2.4) into (2.1) and (2.2) into (2.3) yields the following basic equations for swelling dynamics of polymer gels (see [8]):

$$\left(\alpha + \frac{\beta}{2}\right)\nabla \operatorname{div} \mathbf{u} + \frac{\beta}{2}\Delta\mathbf{u} = \nabla p, \tag{2.6a}$$

$$\operatorname{div} \mathbf{u}_t = \kappa \triangle p, \quad \kappa = \frac{(1-\phi)^2}{\xi}, \tag{2.6b}$$

which hold in the space–time domain $\Omega_T = \Omega \times (0, T)$ for some given $T > 0$.

To solve the set of equations, we need to prescribe boundary and initial conditions. Various sets of boundary conditions are possible, and each of them describes a certain type of mechanical condition and solvent permeation condition (cf. [8,9]). In this paper, we consider the following two distinct sets of boundary conditions, one corresponding to the pressure and one corresponding to deformation:

$$p(t) = p_1, \quad \text{on } \Gamma_{p1}, \tag{2.7a}$$

$$\frac{\partial p}{\partial \mathbf{n}} = \nabla p \cdot \mathbf{n} = p_2, \quad \text{on } \Gamma_{p2}, \tag{2.7b}$$

$$\mathbf{u}(t) = \mathbf{c}, \quad \text{on } \Gamma_{\mathbf{u}1}, \tag{2.8a}$$

$$\widehat{\sigma}(\mathbf{u}, p)\mathbf{n} = (\sigma(\mathbf{u}) - pI)\mathbf{n} = \mathbf{f}, \quad \text{on } \Gamma_{\mathbf{u}2}, \tag{2.8b}$$

where $\partial\Omega = \Gamma_{p1} \bigcup \Gamma_{p2} = \Gamma_{\mathbf{u}1} \bigcup \Gamma_{\mathbf{u}2}$, (2.7a) implies that the solvent can permeate freely at the boundary Γ_{p1} which stands for a permeable boundary wall, (2.8b) means that the mechanical force $\mathbf{f}$ is applied on the boundary $\Gamma_{\mathbf{u}2}$ which stands for a deformable boundary, and $\mathbf{n}$ denotes the unit outward normal vector to the boundary $\partial\Omega$.

And Initial conditions

$$p(0) = p_0, \tag{2.9a}$$

$$\mathbf{u}(0) = \mathbf{u}_0. \tag{2.9b}$$

3. The mixed-DG finite element formulation

We first provide some notions important for the development of the mixed DG formulation. Let $\varepsilon_h = \{E_1, E_2, \ldots, E_N\}$ be a nondegenerate subdivision of Ω, where E_j is a triangle or quadrilateral for $d = 2$, or a tetrahedron if $d = 3$, and $h_j = \operatorname{diam}(E_j)$. Here, the nondegeneracy requires the existence of $\rho > 0$ such that E_j contains a ball of radius ρh_j. Moreover, we set $h = \max\{h_j, j = 1, 2, \ldots, N\}$.

We denote the set of interior edges (for $d = 2$, or faces for $d = 3$) of ε_h by Γ_{int}. For each interior edge (or face), $e = E_j \cap E_k (j < k)$, associate a fixed unit normal vector $\mathbf{n}$ pointing from E_j into E_k, and to each boundary edge (or face) $e = E_j \cap \partial\Omega$, let $\mathbf{n}$ be the unit outward pointing normal.

The development of a DG formulation to solve the displacement $\mathbf{u}$, needs the following spaces. For $s \geq 0$, define

$$H^s(\varepsilon_h) \equiv \{v \in L^2(\Omega) : v|_{E_j} \in H^s(E_j), \forall j\},$$

$$\mathbf{H}^s(\varepsilon_h) \equiv [H^s(\varepsilon_h)]^d.$$

And set

$$\mathbf{V}_0^s \equiv \mathbf{H}^s(\varepsilon_h) \cap \{\mathbf{v}; \mathbf{v}|_{\Gamma_{\mathbf{u}1}} = 0\}.$$

For $s \geq \frac{1}{2}$, we define the average and the jump for $v \in H^s(\varepsilon_h)$. Let $e \in \Gamma_{int}$ such that $e = E_j \cap E_k (j < k)$ and set

$$\{v\} \equiv \frac{1}{2}(v|_{E_j})|_e + \frac{1}{2}(v|_{E_k})|_e,$$

$$[v] \equiv (v|_{E_j})|_e - (v|_{E_k})|_e.$$

The usual Sobolev norm of H^n on $E \subset \mathbb{R}^d$ is denoted by $\|\cdot\|_{n,E}$. We thus define the following broken norms for positive integer n:

$$|||v|||_n \equiv \left(\sum_{j=1}^{N} \|v\|_{n,E_j}^2\right)^{\frac{1}{2}}, \quad \forall v \in H^n(\varepsilon_h),$$

$$|||\mathbf{v}|||_n \equiv \left(\sum_{j=1}^{N} \|\mathbf{v}\|_{m,E_j}^2\right)^{\frac{1}{2}}, \quad \forall \mathbf{v} \in \mathbf{H}^n(\varepsilon_h).$$

The finite element space $\mathbf{V}_h$ to approximate displacement $\mathbf{u}$ is the discontinuous piecewise polynomial space

$$\mathbf{V}_h \equiv \{\mathbf{v}; \mathbf{v}|_{E_j} \in [P_r(E_j)]^d, \forall E_j \in \varepsilon_h\},$$

where $P_r(E_j)$ is the set of polynomials of degree less than or equal to r on E_j. And define

$$\mathbf{V}_{0h} \equiv \mathbf{V}_h \cap \{\mathbf{v}; \mathbf{v}|_{\Gamma_{\mathbf{u}1}} = 0\}.$$

By Babuska and Suri [18], for every $\mathbf{v} \in \mathbf{H}^s(\varepsilon_h)$, an interpolant operator $P^I\mathbf{v} \in \mathbf{V}_h$ exists and satisfies the following properties:

$$\|\mathbf{v} - P^I\mathbf{v}\|_{n,E_j} \leq Ch_j^{\nu-n}\|\mathbf{u}\|_{s,E_j}, \quad 0 \leq m \leq s, \tag{3.1}$$

$$\|\mathbf{v} - P^I\mathbf{v}\|_{0,e} \leq Ch_j^{\nu-\frac{1}{2}}\|\mathbf{u}\|_{s,E_j}, \quad s \geq \frac{1}{2}, \tag{3.2}$$

with $\nu = \min\{r + 1, s\}$ and $e \in \partial E_j$ for any element $E_j \in \varepsilon_h$. We can readily obtain a global estimate for (3.1) by summing all elements of ε_h. Additionally, for triangles or tetrahedra, a continuous interpolant satisfying (3.1) and (3.2) can be found.

In order to derive the discontinuous Galerkin variational scheme, plug the solution $\mathbf{u}$ and p into (2.6a) and (2.6b) (assumed to be sufficiently regular) and write the momentum equation in terms of the total stress $\widehat{\sigma}(\mathbf{u}, p)$. Next, multiplying the equation by a test function $\mathbf{v} \in \mathbf{V}_0^{3/2}$ and integrating over a single element E to find

$$\int_E \mathbf{v} \cdot \operatorname{div} \widehat{\sigma}(\mathbf{u}, p) dE = 0. \tag{3.3}$$

Integrating by parts,

$$\int_{\partial E} \mathbf{v} \cdot \widehat{\sigma}(\mathbf{u}, p)\mathbf{n} ds - \int_E \widehat{\sigma}(\mathbf{u}, p) : (\nabla \mathbf{v}) dE = 0. \tag{3.4}$$

Since $\widehat{\sigma}$ is symmetric, we have

$$\int_{\partial E} \mathbf{v} \cdot \widehat{\sigma}(\mathbf{u}, p)\mathbf{n} ds - \int_E \widehat{\sigma}(\mathbf{u}, p) : (\nabla \mathbf{v})^T dE = 0. \tag{3.5}$$

Adding the above two equations together, we get

$$\int_{\partial E} \mathbf{v} \cdot \widehat{\sigma}(\mathbf{u}, p)\mathbf{n} ds - \int_E \widehat{\sigma}(\mathbf{u}, p) : \epsilon(\mathbf{v}) dE = 0. \tag{3.6}$$

Summing over all elements yields

$$\sum_{E \in \varepsilon_h} \int_{\partial E} \mathbf{v} \cdot \widehat{\sigma}(\mathbf{u}, p)\mathbf{n} ds - \sum_{E \in \varepsilon_h} \int_E \widehat{\sigma}(\mathbf{u}, p) : \epsilon(\mathbf{v}) dE = 0. \tag{3.7}$$

Consider that $\mathbf{n}$ is the normal vector from E_i to $E_j (i < j)$, $e = E_i \cap E_j$, and use the equality $[ab] = [a]\{b\} + \{a\}[b]$, then the first item of (3.7) can be written as

$$\begin{aligned}
\sum_{E \in \varepsilon_h} \int_{\partial E} \mathbf{v} \cdot \widehat{\sigma}(\mathbf{u}, p)\mathbf{n} ds &= \sum_{e \in E \bigcap \Gamma_{\mathbf{u}2}} \int_e \mathbf{v} \cdot \mathbf{f} ds + \sum_{e \in \Gamma_{int}} \int_e (\mathbf{v}|_{E_i} \cdot \widehat{\sigma}|_{E_i} \mathbf{n}_i + \mathbf{v}|_{E_j} \cdot \widehat{\sigma}|_{E_j} \mathbf{n}_j) ds \\
&= \sum_{e \in E \bigcap \Gamma_{\mathbf{u}2}} \int_e \mathbf{v} \cdot \mathbf{f} ds + \sum_{e \in \Gamma_{int}} \int_e (\mathbf{v}|_{E_i} \cdot \widehat{\sigma}|_{E_i} \mathbf{n} - \mathbf{v}|_{E_j} \cdot \widehat{\sigma}|_{E_j} \mathbf{n}) ds \\
&= \sum_{e \in E \bigcap \Gamma_{\mathbf{u}2}} \int_e \mathbf{v} \cdot \mathbf{f} ds + \sum_{e \in \Gamma_{int}} \int_e [\mathbf{v} \cdot \widehat{\sigma} \mathbf{n}] ds \\
&= \sum_{e \in E \bigcap \Gamma_{\mathbf{u}2}} \int_e \mathbf{v} \cdot \mathbf{f} ds + \sum_{e \in \Gamma_{int}} \int_e [\mathbf{v}] \cdot \{\widehat{\sigma}(\mathbf{u}, p)\mathbf{n}\} ds;
\end{aligned} \tag{3.8}$$

here, the term $\sum_{e \in \Gamma_{int}} \int_e \{\mathbf{v}\} \cdot [\widehat{\sigma}(\mathbf{u}, p)\mathbf{n}] ds$ is zero by regularity of $\mathbf{u}$ and p.

Taking the above equation into (3.7), we get

$$\sum_{E \in \varepsilon_h} \int_E \widehat{\sigma}(\mathbf{u}, p) : \epsilon(\mathbf{v}) dE - \sum_{e \in \Gamma_{int}} \int_e [\mathbf{v}] \cdot \{\widehat{\sigma}(\mathbf{u}, p)\mathbf{n}\} ds = \sum_{e \in E \bigcap \Gamma_{\mathbf{u}2}} \int_e \mathbf{v} \cdot \mathbf{f} ds. \tag{3.9}$$

Substituting $\widehat{\sigma}(\mathbf{u}, p)$ with $\sigma(\mathbf{u}) - pI$ in (3.9), then

$$\sum_{E \in \varepsilon_h} \int_E \sigma(\mathbf{u}) : \epsilon(\mathbf{v}) dE - \sum_{E \in \varepsilon_h} \int_E p \operatorname{div} \mathbf{v} dE - \sum_{e \in \Gamma_{int}} \int_e [\mathbf{v}] \cdot \{\sigma(\mathbf{u})\mathbf{n}\} ds + \sum_{e \in \Gamma_{int}} \int_e [\mathbf{v}] \cdot \{p\mathbf{n}\} ds = \sum_{e \in E \bigcap \Gamma_{\mathbf{u}2}} \int_e \mathbf{v} \cdot \mathbf{f} ds, \tag{3.10}$$

where $\sum_{E \in \varepsilon_h} \int_E p \operatorname{div}(\mathbf{v}) dE = \sum_{E \in \varepsilon_h} \int_E pI : \epsilon(\mathbf{v}) dE$.

To develop the DG formulation, the following three terms are added to the left-hand side (lhs) of (3.10), which are all equal to zero due to the assumed regularity of $\mathbf{u}$:

$$\epsilon \sum_{e \in \mathbf{\Gamma}_{int}} \int_e [\mathbf{u}] \cdot \{\sigma(\mathbf{v})\mathbf{n}\} ds, \tag{3.11}$$

$$J_0^{\delta,\beta}(\mathbf{u}, \mathbf{v}) \equiv \sum_{e \in \Gamma_{int}} \frac{\delta}{|e|^\beta} \int_e [\mathbf{u}] \cdot [\mathbf{v}] ds, \tag{3.12}$$

$$J_0^{\widehat{\delta},\beta}(\mathbf{u}_t, \mathbf{v}) \equiv \sum_{e \in \Gamma_{int}} \frac{\widehat{\delta}}{|e|^\beta} \int_e [\mathbf{u}_t] \cdot [\mathbf{v}] ds, \tag{3.13}$$

where, the bilinear form in (3.11) characterizes the DG scheme by its value of ε, which affects the overall symmetry of the formulation. The penalty terms in (3.12)–(3.13) contain the parameters δ and $\widehat{\delta}$, which are discrete, positive functions taking a constant value on each edge or face e (with $|e|$ its Lebesgue measure).

Plugging the above items (3.11)–(3.13) into (3.10), we obtain

$$\sum_{E\in\varepsilon_h}\int_E \sigma(\mathbf{u}):\epsilon(\mathbf{v})dE+\epsilon\sum_{e\in\Gamma_{int}}\int_e[\mathbf{u}]\cdot\{\sigma(\mathbf{v})\mathbf{n}\}ds-\sum_{e\in\Gamma_{int}}\int_e[\mathbf{v}]\cdot\{\sigma(\mathbf{u})\mathbf{n}\}ds+J_0^{\delta,\beta}(\mathbf{u},\mathbf{v})+J_0^{\widehat{\delta},\beta}(\mathbf{u}_t,\mathbf{v})$$
$$-\sum_{E\in\varepsilon_h}\int_E p\mathrm{div}\,\mathbf{v}dE+\sum_{e\in\Gamma_{int}}\int_e[\mathbf{v}]\cdot\{p\mathbf{n}\}ds=\sum_{e\in E\bigcap\Gamma_{\mathbf{u}2}}\int_e\mathbf{v}\cdot\mathbf{f}ds. \tag{3.14}$$

The DG variational formulation uses (3.14) to model the momentum conservation equation (2.6a). In order to model the mass conservation equation (2.6b), we use a mixed finite element method, and so we introduce the flux variable $\mathbf{q}=-\nabla p$. Then, (2.6b) can be rewritten as

$$\begin{cases}\dfrac{\partial}{\partial t}(\mathrm{div}\,\mathbf{u})+\kappa\,\mathrm{div}\,\mathbf{q}=0,\\ \mathbf{q}+\nabla p=0.\end{cases} \tag{3.15}$$

Now, we choose $L^2(\Omega)\equiv\{p;\int_\Omega|p|^2dx<+\infty\}$ as the appropriate function space for the pressure p, and $\mathbf{H}(\mathrm{div}\,)\equiv\{\mathbf{s}\in[L^2(\Omega)]^d:\mathrm{div}\,\mathbf{s}\in L^2(\Omega)\}$ as the space used for the flux variable $\mathbf{q}$. For $\mathbf{H}(\mathrm{div})$, we can define the following subset

$$\mathbf{M}_0\equiv\{\mathbf{q}\in\mathbf{H}(\mathrm{div});\,\mathbf{q}\cdot\mathbf{n}|_{\Gamma_{p2}}=0\}.$$

The corresponding finite dimensional approximating spaces are defined as follows. Let $(W_h,\mathbf{M}_h)\subset L^2(\Omega)\times\mathbf{H}(\mathrm{div})$ denote a standard mixed finite element space (see [12,13]) defined on ε_h. Let

$$\mathbf{M}_{0h}\equiv\{\mathbf{q}\in\mathbf{M}_h;\,\mathbf{q}\cdot\mathbf{n}|_{\Gamma_{p2}}=0\},$$

and k refer to the order of this space.

In order to complete our mixed finite element variational scheme, we multiply (3.15) by $(w,\mathbf{s})\in L^2\times\mathbf{M}_0$. The formulation is completed by integrating each of the equations over Ω and integrating by parts when necessary. The essential boundary conditions for the displacement and flux variables are allowed to be inhomogeneous, so for each $t\geq 0$, select some $\bar{\mathbf{u}}(t,\mathbf{x})\in\mathbf{H}^{3/2}(\varepsilon_h)$ such that $\bar{\mathbf{u}}(t,\mathbf{x})|_{\Gamma_{\mathbf{u}1}}=\mathbf{c}$, and choose some $\bar{\mathbf{q}}(t,\mathbf{x})\in H(\mathrm{div})$ such that $\bar{\mathbf{q}}(t,\mathbf{x})\mathbf{n}|_{\Gamma_{\mathbf{p}2}}=p_2$. Then, the mixed-DG variational scheme becomes the following:

Problem I. Find $\mathbf{u}$ satisfying $\mathbf{u}-\bar{\mathbf{u}}\in H^1([0,T];\mathbf{V}_0^{\frac{3}{2}})$, $p\in H^1([0,T];L^2)$, and $\mathbf{q}$ satisfying $\mathbf{q}-\bar{\mathbf{q}}\in L^2([0,T];\mathbf{M}_0)$ such that

$$\begin{aligned}&(a)\ A(\mathbf{u},\mathbf{v})+J_0^{\widehat{\delta},\beta}(\mathbf{u}_t,\mathbf{v})-(p,\mathrm{div}\,\mathbf{v})+\sum_{e\in\Gamma_{int}}\int_e[\mathbf{v}]\cdot\{p\mathbf{n}\}ds=\sum_{e\in E\bigcap\Gamma_{\mathbf{u}2}}\int_e\mathbf{v}\cdot\mathbf{f}ds,\quad\forall\mathbf{v}\in\mathbf{V}_0^{\frac{3}{2}},\\&(b)\ (\mathrm{div}\,\mathbf{u}_t,w)+\kappa(\mathrm{div}\,\mathbf{q},w)=0,\quad\forall w\in L^2,\\&(c)\ (\mathbf{q},\mathbf{s})-(p,\mathrm{div}\,\mathbf{s})=0,\quad\forall\mathbf{s}\in\mathbf{M}_0,\end{aligned} \tag{3.16}$$

hold for every $t\in[0,T]$; here $(\cdot,\cdot)$ represents the inner product in space L^2, and the bilinear form

$$A(\mathbf{u},\mathbf{v})\equiv\sum_{E\in\varepsilon_h}\int_E\sigma(\mathbf{u}):\epsilon(\mathbf{v})dE+\epsilon\sum_{e\in\Gamma_{int}}\int_e[\mathbf{u}]\cdot\{\sigma(\mathbf{v})\mathbf{n}\}ds-\sum_{e\in\Gamma_{int}}\int_e[\mathbf{v}]\cdot\{\sigma(\mathbf{u})\mathbf{n}\}ds+J_0^{\delta,\beta}(\mathbf{u},\mathbf{v}). \tag{3.17}$$

The mixed-DG finite element formulation is based on the above variational form restricted to the finite dimensional spaces $\mathbf{V}_h$, W_h and $\mathbf{M}_h$. Now, for each $t\geq 0$, let $\bar{\mathbf{u}}_h(t,\mathbf{x})=\mathcal{P}^I\bar{\mathbf{u}}(t,\mathbf{x})\in\mathbf{V}_h$ and $\bar{\mathbf{q}}_h(t,\mathbf{x})=\mathcal{Q}_h\bar{\mathbf{q}}(t,\mathbf{x})\in\mathbf{M}_h$. Then the mixed-DG finite element formulation is as follows.

Problem II. Find $\mathbf{u}_h$ satisfying $\mathbf{u}_h-\bar{\mathbf{u}}_h\in H^1([0,T];\mathbf{V}_{0h})$, $p_h\in H^1([0,T];W_h)$, and $\mathbf{q}_h$ satisfying $\mathbf{q}_h-\bar{\mathbf{q}}_h\in L^2([0,T];\mathbf{M}_{0h})$ such that

$$\begin{aligned}&(a)\ A(\mathbf{u}_h,\mathbf{v}_h)+J_0^{\widehat{\delta},\beta}(\mathbf{u}_{ht},\mathbf{v}_h)-(p_h,\mathrm{div}\,\mathbf{v}_h)+\sum_{e\in\Gamma_{int}}\int_e[\mathbf{v}_h]\cdot\{p_h\mathbf{n}\}ds=\sum_{e\in E\bigcap\Gamma_{\mathbf{u}2}}\int_e\mathbf{v}_h\cdot\mathbf{f}ds,\quad\forall\mathbf{v}_h\in\mathbf{V}_{0h},\\&(b)\ (\mathrm{div}\,\mathbf{u}_{ht},w_h)+\kappa(\mathrm{div}\,\mathbf{q}_h,w_h)=0,\quad\forall w_h\in W_h,\\&(c)\ (\mathbf{q}_h,\mathbf{s}_h)-(p_h,\mathrm{div}\,\mathbf{s}_h)=0,\quad\forall\mathbf{s}_h\in\mathbf{M}_{0h},\end{aligned} \tag{3.18}$$

hold for every $t\in[0,T]$. In addition, initial conditions can be given in the following way,

$$A(\mathbf{u}_h|_{t=0},\mathbf{v}_h)=A_{\mathbf{u}}(\mathbf{u}_0,\mathbf{v}_h),\quad\forall\mathbf{v}_h\in\mathbf{V}_{0h}, \tag{3.19}$$
$$(p_h|_{t=0},w_h)=(p_0,w_h),\quad\forall w_h\in W_h. \tag{3.20}$$

4. Convergence analysis

The mixed-DG finite element formulation (Problem II) is a differential algebraic equation (DAE), and we first discuss its existence and uniqueness.

Theorem 4.1. *The mixed-DG finite element formulation (3.18) has a unique solution.*

Proof. For simplicity, we write (3.18) in the matrix form. First, we denote the functions $\mathbf{u}_h(t,\mathbf{x})$, $p_h(t,\mathbf{x})$, and $\mathbf{q}_h(t,\mathbf{x})$ as components in their respective basis functions, which are $\mathbf{\Phi} = [\phi_1, \phi_2, \ldots, \phi_m]$, $\mathbf{\Theta} = [\theta_1, \theta_2, \ldots, \theta_n]$ and $\mathbf{\Psi} = [\psi_1, \psi_2, \ldots, \psi_k]$, then

$$\mathbf{u}_h(t,\mathbf{x}) = \sum_i u_i(t)\phi_i(\mathbf{x}) + \sum_i \bar{u}_i(t)\phi_i(\mathbf{x}) = \mathbf{U}(t)\cdot\mathbf{\Phi}(\mathbf{x}) + \overline{\mathbf{U}}(t)\cdot\mathbf{\Phi}(\mathbf{x}),$$

$$p_h(t,\mathbf{x}) = \sum_i p_i(t)\theta_i(\mathbf{x}) = \mathbf{P}(t)\cdot\mathbf{\Theta}(\mathbf{x}),$$

$$\mathbf{q}_h(t,\mathbf{x}) = \sum_i q_i(t)\psi_i(\mathbf{x}) + \sum_i \bar{q}_i(t)\psi_i(\mathbf{x}) = \mathbf{Q}(t)\cdot\mathbf{\Psi}(\mathbf{x}) + \overline{\mathbf{Q}}(t)\cdot\mathbf{\Psi}(\mathbf{x}),$$

where $\mathbf{U}(t) = [u_1(t), u_2(t), \ldots, u_m(t)]^T$, $\mathbf{P}(t) = [p_1(t), p_2(t), \ldots, p_n(t)]^T$, $\mathbf{Q}(t) = [q_1(t), q_2(t), \ldots, q_k(t)]^T$.$\overline{\mathbf{U}}(t) = [\bar{u}_1(t), \bar{u}_2(t), \ldots, \bar{u}_m(t)]^T$, and $\overline{\mathbf{Q}}(t) = [\bar{q}_1(t), \bar{q}_2(t), \ldots, \bar{q}_k(t)]^T$ are the components of known functions $\bar{\mathbf{u}}_h$ and $\bar{\mathbf{q}}_h$, respectively, which come from the inhomogeneous essential conditions. Taking the representations for $\mathbf{u}_h(t,\mathbf{x})$, $p_h(t,\mathbf{x})$ and $\mathbf{q}_h(t,\mathbf{x})$ into (3.18), and using each basis function as a test function, we get

$$\mathbf{A}\frac{\partial \Gamma(t)}{\partial t} + \mathbf{B}\Gamma(t) = \mathbf{F}(t), \tag{4.1}$$

where

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}^e_{\mathbf{uu}} & 0 & 0 \\ \mathbf{A}^T_{\mathbf{u}p} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} \mathbf{A}_{\mathbf{uu}} & [-\mathbf{A}_{\mathbf{u}p} + \mathbf{A}^e_{\mathbf{u}p}] & 0 \\ 0 & 0 & \mathbf{A}^T_{\mathbf{q}p} \\ 0 & -\mathbf{A}_{\mathbf{q}p} & \mathbf{A}_{\mathbf{qq}} \end{pmatrix},$$

$\Gamma(t) = [\mathbf{U}(t)\mathbf{P}(t)\mathbf{Q}(t)]^T$, $\mathbf{F}(t) = [\mathbf{f}_1(t)\mathbf{f}_2(t)\mathbf{f}_3(t)]^T$ (herein, $\mathbf{f}_1(t), \mathbf{f}_2(t), \mathbf{f}_3(t)$ are suitably modified to include inhomogeneous essential conditions), and the subscript e represents those matrices coming from bilinear forms that are completely defined on edges/faces.

According to theories in [19–21], if $s\mathbf{A} + \mathbf{B}$ is invertible for some $s \neq 0$, then (4.1) is uniquely solvable (there is also a requirement involving sufficient time-differentiability of $\mathbf{F}(t)$). So our goal is to find an $s \neq 0$ such that $(s\mathbf{A} + \mathbf{B})\Lambda = 0$ implies that $\Lambda = 0$.

First, we decompose $\Lambda = [\mathbf{U}_\Lambda \mathbf{P}_\Lambda \mathbf{Q}_\Lambda]$. From the first row of $s\mathbf{A}+\mathbf{B}$, we can get that $(\mathbf{A}_{\mathbf{uu}} + s\mathbf{A}^e_{\mathbf{uu}})\mathbf{U}_\Lambda = [\mathbf{A}_{\mathbf{u}p} - \mathbf{A}^e_{\mathbf{u}p}]\mathbf{P}_\Lambda$, and let $\widehat{\delta} = 0$. Since $\mathbf{A}_{\mathbf{uu}}$ contains a penalty item with parameter δ, for any s, if we make the substitutions $\delta \to \frac{1}{2}\delta$ and $\widehat{\delta} \to \frac{1}{2s}\delta$, then $\mathbf{A}_{\mathbf{uu}} + s\mathbf{A}^e_{\mathbf{uu}} \to \mathbf{A}_{\mathbf{uu}}$, and $\mathbf{A}_{\mathbf{uu}} + s\mathbf{A}^e_{\mathbf{uu}}$ is also invertible and independent of s. we can arrive at

$$\mathbf{U}_\Lambda = (\mathbf{A}_{\mathbf{uu}} + s\mathbf{A}^e_{\mathbf{uu}})^{-1}[\mathbf{A}_{\mathbf{u}p} - \mathbf{A}^e_{\mathbf{u}p}]\mathbf{P}_\Lambda. \tag{4.2}$$

Second, from the third row of $s\mathbf{A}+\mathbf{B}$, since $\mathbf{A}_{\mathbf{qq}}$ is symmetric and positive definite, we can similarly get:

$$\mathbf{Q}_\Lambda = \mathbf{A}^{-1}_{\mathbf{qq}}\mathbf{A}_{\mathbf{q}p}\mathbf{P}_\Lambda. \tag{4.3}$$

Finally, plugging (4.2) and (4.3) into the second row of $s\mathbf{A}+\mathbf{B}$, we deduce

$$(\mathbf{A}^T_{\mathbf{q}p}\mathbf{A}^{-1}_{\mathbf{qq}}\mathbf{A}_{\mathbf{q}p} + s\mathbf{A}^T_{\mathbf{u}p}\mathbf{A}^{-1}_{\mathbf{uu}}[\mathbf{A}_{\mathbf{u}p} - \mathbf{A}^e_{\mathbf{u}p}])\mathbf{P}_\Lambda = 0, \tag{4.4}$$

where $\mathbf{A}^T_{\mathbf{q}p}\mathbf{A}^{-1}_{\mathbf{qq}}\mathbf{A}_{\mathbf{q}p}$ is invertible because $\mathbf{A}_{\mathbf{q}p}$ has full column rank. By the theory in [22], the set of invertible matrices forms an open and dense set, if s is chosen sufficiently small in (4.3), therefore we can get $\mathbf{A}^T_{\mathbf{q}p}\mathbf{A}^{-1}_{\mathbf{qq}}\mathbf{A}_{\mathbf{q}p} + s\mathbf{A}^T_{\mathbf{u}p}\mathbf{A}^{-1}_{\mathbf{uu}}[\mathbf{A}_{\mathbf{u}p} - \mathbf{A}^e_{\mathbf{u}p}]$ is still invertible, that is, $\mathbf{P}_\Lambda = 0$, thus from (4.2) and (4.3), we have $\mathbf{U}_\Lambda = 0$ and $\mathbf{Q}_\Lambda = 0$. Therefore, Eq. (4.1) is uniquely solvable. The proof is completed. □

From now on, we discuss error estimates for the mixed-DG finite element scheme (3.18). First, we introduce some useful lemmas (cf. [23–25]).

Lemma 4.2. *The following component term in* $\mathbf{A}(,)$ *satisfies the properties:*

(a) $\displaystyle\sum_{E\in\varepsilon_h}\int_E \sigma(\mathbf{u}) : \epsilon(\mathbf{v})dE \le C|||\mathbf{u}|||_1|||\mathbf{v}|||_1, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\varepsilon_h),$

(b) $\displaystyle C_K|||\mathbf{u}|||_1^2 \le \sum_{E\in\varepsilon_h}\int_E \sigma(\mathbf{u}) : \epsilon(\mathbf{u})dE + J_0^{\delta,\beta}(\mathbf{u},\mathbf{u}), \quad \forall \mathbf{u} \in \mathbf{H}^1(\varepsilon_h),$

where $C_K > 0$ is a constant, the inequality (b) *is called Korn's inequality, C in this context indicates a positive constant which is possibly different at different occurrences, being independent of the spatial and temporal mesh sizes.*

Lemma 4.3. *For $\forall \mathbf{v} \in P_r(E_j)^d$, $w \in P_r(E_j)$ and any χ in Sobolev spaces, the following trace inequalities hold:*

$$\text{(a) } \|\sigma(\mathbf{u})\mathbf{n}\|^2_{L^2(e)} \le Ch_j^{-1} \int_{E_j} \sigma(\mathbf{v}) : \epsilon(\mathbf{v}) dE,$$

$$\text{(b) } \|w\|_{L^2(e)} \le Ch_j^{-\frac{1}{2}} \|w\|_{L^2(E_j)},$$

$$\text{(c) } \|\chi\|_{L^2(e)} \le C\left(h_j^{-\frac{1}{2}}\|\chi\|_{L^2(E_j)} + h_j^{\frac{1}{2}}\|\chi\|_{H^1(E_j)}\right).$$

Lemma 4.4. *The mixed finite element spaces are required to be endowed with two linear operators, $\mathcal{Q}_h : \mathbf{H}(\mathrm{div}) \to \mathbf{M}_h$ and the L^2 projection $\mathcal{P}_h : L^2 \to W_h$, which satisfy the following properties:*

(a) $(\mathrm{div}(\mathbf{q} - \mathcal{Q}_h\mathbf{q}), w_h) = 0, \quad \forall w_h \in W_h,$

(b) $\|\mathbf{q} - \mathcal{Q}_h\mathbf{q}\|_{L^2} \le Ch^r\|\mathbf{q}\|_{H_r}, \quad 1 \le r \le k+1,$

(c) $\mathrm{div}\, \mathcal{Q}_h = \mathcal{P}_h \mathrm{div},$

(d) $(\mathrm{div}\, \mathbf{q}_h, p - \mathcal{P}_h p) = 0, \quad \forall \mathbf{q}_h \in \mathbf{M}_h,$

(e) $\|p - \mathcal{P}_h p\|_{L^2} \le Ch^r\|p\|_{H_r}, \quad 0 \le r \le k+1.$

Note that not all mixed space operators satisfy each of the above properties, in particular, (b), *where the upper bound for r is sometimes only k.*

For convenience, define the error functions

$$\mathbf{u} - \mathbf{u}_h = (\mathbf{u} - \mathcal{P}^I\mathbf{u}) + (\mathcal{P}^I\mathbf{u} - \mathbf{u}_h) \equiv \eta_\mathbf{u} + \xi_\mathbf{u}, \tag{4.5a}$$

$$p - p_h = (p - \mathcal{P}_h p) + (\mathcal{P}_h p - p_h) \equiv \eta_p + \xi_p, \tag{4.5b}$$

$$\mathbf{q} - \mathbf{q}_h = (\mathbf{q} - \mathcal{Q}_h\mathbf{q}) + (\mathcal{Q}_h\mathbf{q} - \mathbf{q}_h) \equiv \eta_\mathbf{q} + \xi_\mathbf{q}. \tag{4.5c}$$

To obtain the proof of optimal convergence rates for the mixed DG finite element method, it is again useful to prove the following lemma.

Lemma 4.5. *Let $(\mathbf{u}, p, \mathbf{q})$ be the solution to Problem I, and $(\mathbf{u}_h, p_h, \mathbf{q}_h)$ be the solution to Problem II. Then, there is a positive constant $C_\mathbf{q}$ such that, for bounded h, the following inequality holds:*

$$\|\xi_p\|_0 = \|\mathcal{P}_h p - p_h\|_0 \le C_\mathbf{q}\|\mathbf{q} - \mathbf{q}_h\|_0. \tag{4.6}$$

Proof. In (3.16)(c), taking $\mathbf{s} = \mathbf{s}_h \in \mathbf{M}_h$, and subtracting (3.18)(c) from (3.16)(c), we get:

$$(\mathbf{q} - \mathbf{q}_h, \mathbf{s}_h) - (p - p_h, \mathrm{div}\, \mathbf{s}_h) = 0,$$

by (4.5b),

$$(\mathbf{q} - \mathbf{q}_h, \mathbf{s}_h) - (\xi_p, \mathrm{div}\, \mathbf{s}_h) - (\eta_p, \mathrm{div}\, \mathbf{s}_h) = 0,$$

from Lemma 4.4(d), we obtain

$$(\xi_p, \mathrm{div}\, \mathbf{s}_h) = (\mathbf{q} - \mathbf{q}_h, \mathbf{s}_h).$$

We now apply Nitsche technique or duality argument to obtain $\|\xi_p\|_0$. Let $-\Delta\psi = \xi_p$ with boundary conditions compatible with the pressure boundary conditions, and for $\phi \in L^2(\Omega)$, let $\phi = -\nabla\psi$, and there is the regularity estimate $\|\phi\|_1 \le C\|\xi_p\|_0$. Then, by Lemma 4.4(a) and (b), we deduce

$$\begin{aligned}
\|\xi_p\|_0^2 &= (\xi_p, \xi_p) = (\xi_p, -\Delta\psi) \\
&= (\xi_p, \mathrm{div}\, \phi) = (\xi_p, \mathrm{div}\, \mathcal{Q}_h\phi) \\
&= (\mathbf{q} - \mathbf{q}_h, \mathcal{Q}_h\phi) \\
&= (\mathbf{q} - \mathbf{q}_h, \mathcal{Q}_h\phi - \phi) + (\mathbf{q} - \mathbf{q}_h, \phi) \\
&\le \|\mathbf{q} - \mathbf{q}_h\|_0\|\mathcal{Q}_h\phi - \phi\|_0 + \|\mathbf{q} - \mathbf{q}_h\|_0\|\phi\|_0 \\
&\le \|\mathbf{q} - \mathbf{q}_h\|_0\|\mathcal{Q}_h\phi - \phi\|_0 + \|\mathbf{q} - \mathbf{q}_h\|_0\|\phi\|_1 \\
&\le (Ch + 1)\|\mathbf{q} - \mathbf{q}_h\|_0\|\phi\|_1 \\
&\le (Ch + 1)\|\mathbf{q} - \mathbf{q}_h\|_0\|\xi_p\|_0.
\end{aligned}$$

If we assume that h is bounded, and take $C_\mathbf{q} = Ch + 1$, then (4.6) holds. The proof is completed. □

In order to deduce the error estimates, we also need the following Gronwall Lemma.

Lemma 4.6 (*Gronwall Lemma*). *Let $g(t)$ be a positive and integrable function on $[0, T]$, $c > 0$ be a constant. If $\psi(t) \in C^0([0, T])$ satisfies*

$$0 \le \psi(t) \le c + \int_0^t g(s)\psi(s)ds, \quad \forall t \in [0, T],$$

then, $\psi(t)$ also satisfies

$$0 \le \psi(t) \le c \cdot \exp\left(\int_0^t g(s)ds\right), \quad \forall t \in [0, T].$$

Especially, if $c = 0$, then $\psi(t) \equiv 0$.

In order to discuss error estimates, the following minimal regularity requirements are necessary:

$$\mathbf{u} \in L^\infty\left(0, T; \mathbf{H}^{\frac{3}{2}}(\varepsilon_h)\right), \qquad p \in L^2(0, T; H^1),$$

where, the factor $\frac{3}{2}$ is to ensure that the first order derivatives have well-defined traces along boundary edges.

In addition, let s, t, q and r be the largest positive real numbers, such that

$$\begin{aligned} &\mathbf{u} \in L^2(0, T; \mathbf{H}^s(\varepsilon_h)), \qquad \mathbf{u}_t \in L^2(0, T; \mathbf{H}^t(\varepsilon_h)), \\ &p \in L^2(0, T; H^q), \qquad p_t \in L^2(0, T; H^r), \qquad \mathbf{q} \in \mathbf{H}(\mathrm{div}). \end{aligned}$$

For simplicity, we assume the null initial conditions ($\mathbf{u}_0 = \mathbf{0}$ and $p_0 = 0$) and homogeneous essential boundary conditions ($\mathbf{c} = \mathbf{0}$ and $p_2 = 0$) in the following theorem.

Theorem 4.7. *Let $r_1 \ge 0$ be the order of the mixed finite element space $(W_h, \mathbf{M}_h)$, and $r_2 > 0$ be the order of the DG finite element space $\mathbf{V}_h$. Under the above hypotheses, if $\beta = (d-1)^{-1}$, δ and $\widehat{\delta}$ are large enough, and $(\mathbf{u}, p, \mathbf{q})$ and $(\mathbf{u}_h, p_h, \mathbf{q}_h)$ are solutions of Problems I and II, respectively, then*

$$|||\mathbf{u} - \mathbf{u}_h|||_{L^\infty(H^1)} + \|p - p_h\|_{L^2(L^2)} + \|\mathbf{q} - \mathbf{q}_h\|_{L^2(L^2)} \le Ch^R,$$

where C is a constant independent of mesh size h, and $R = \min\{r_1 + 1, r_2, r, q - 1, s - 1, t - 1\}$.

Proof. In (3.16), taking $\mathbf{v} = \mathbf{v}_h \in \mathbf{V}_{0h}$, $p = p_h \in W_h$, $\mathbf{s} = \mathbf{s}_h \in \mathbf{M}_{0h}$, and subtracting (3.18) from (3.16), we get the error equation

$$\begin{aligned} &(\mathrm{a})\ A(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + J_0^{\widehat{\delta},\beta}(\mathbf{u}_t - \mathbf{u}_{ht}, \mathbf{v}_h) - (p - p_h, \mathrm{div}\, \mathbf{v}_h) + \sum_{e \in \Gamma_{int}} \int_e [\mathbf{v}_h] \cdot \{p - p_h\}\mathbf{n} ds = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_{0h}, \\ &(\mathrm{b})\ (\mathrm{div}\, (\mathbf{u} - \mathbf{u}_h)_t, w_h) + \kappa(\mathrm{div}(\mathbf{q} - \mathbf{q}_h), w_h) = 0, \quad \forall w_h \in W_h, \\ &(\mathrm{c})\ (\mathbf{q} - \mathbf{q}_h, \mathbf{s}_h) - (p - p_h, \mathrm{div}\, \mathbf{s}_h) = 0, \quad \forall \mathbf{s}_h \in \mathbf{M}_{0h}. \end{aligned} \tag{4.7}$$

Taking $\mathbf{v}_h = \xi_{\mathbf{u},t}$, $w_h = \xi_p$ and $\mathbf{s}_h = \kappa \xi_{\mathbf{q}}$, then by (4.5), (4.7) can be rewritten as

$$\begin{aligned} &(\mathrm{a})\ A(\eta_{\mathbf{u}}, \xi_{\mathbf{u},t}) + A(\xi_{\mathbf{u}}, \xi_{\mathbf{u},t}) + J_0^{\widehat{\delta},\beta}(\eta_{\mathbf{u},t}, \xi_{\mathbf{u},t}) + J_0^{\widehat{\delta},\beta}(\xi_{\mathbf{u},t}, \xi_{\mathbf{u},t}) - (\eta_p, \mathrm{div}\, \xi_{\mathbf{u},t}) - (\xi_p, \mathrm{div}\, \xi_{\mathbf{u},t}) \\ &\quad + \sum_{e \in \Gamma_{int}} \int_e \{\xi_p + \eta_p\}\mathbf{n} \cdot [\xi_{\mathbf{u},t}] ds = 0 \\ &(\mathrm{b})\ (\mathrm{div}\, \eta_{\mathbf{u},t}, \xi_p) + (\mathrm{div}\, \xi_{\mathbf{u},t}, \xi_p) + \kappa(\mathrm{div}\, \eta_{\mathbf{q}}, \xi_p) + \kappa(\mathrm{div}\, \xi_{\mathbf{q}}, \xi_p) = 0, \\ &(\mathrm{c})\ \kappa(\xi_{\mathbf{q}}, \xi_{\mathbf{q}}) + \kappa(\eta_{\mathbf{q}}, \xi_{\mathbf{q}}) - \kappa(\eta_p, \mathrm{div}\, \xi_{\mathbf{q}}) - \kappa(\xi_p, \mathrm{div}\, \xi_{\mathbf{q}}) = 0. \end{aligned} \tag{4.8}$$

By Lemma 4.4(a) and (d), summing the three equations, we get

$$\begin{aligned} A(\xi_{\mathbf{u}}, \xi_{\mathbf{u},t}) + J_0^{\widehat{\delta},\beta}(\xi_{\mathbf{u},t}, \xi_{\mathbf{u},t}) + \kappa(\xi_{\mathbf{q}}, \xi_{\mathbf{q}}) = &-A_{\mathbf{u}}(\eta_{\mathbf{u}}, \xi_{\mathbf{u},t}) - J_0^{\widehat{\delta},\beta}(\eta_{\mathbf{u},t}, \xi_{\mathbf{u},t}) \\ &+ \sum_{e \in \Gamma_{int}} \int_e \{\xi_p + \eta_p\}\mathbf{n} \cdot [\xi_{\mathbf{u},t}] ds + (\eta_p, \mathrm{div}\, \xi_{\mathbf{u},t}) \\ &- (\mathrm{div}\, \eta_{\mathbf{u},t}, \xi_p) - \kappa(\eta_{\mathbf{q}}, \xi_{\mathbf{q}}). \end{aligned}$$

Integrating the above equation from 0 to T,

$$\int_0^T A(\xi_{\mathbf{u}}, \xi_{\mathbf{u},t})dt + \int_0^T \widehat{J_0^{\delta,\beta}}(\xi_{\mathbf{u},t}, \xi_{\mathbf{u},t})dt + \int_0^T \kappa \|\xi_{\mathbf{q}}\|_0^2 dt$$
$$= -\int_0^T A(\eta_{\mathbf{u}}, \xi_{\mathbf{u},t})dt - \int_0^T \widehat{J_0^{\delta,\beta}}(\eta_{\mathbf{u},t}, \xi_{\mathbf{u},t})dt + \int_0^T \sum_{e\in\Gamma_{int}} \int_e \{\xi_p + \eta_p\}\mathbf{n}\cdot[\xi_{\mathbf{u},t}]dsdt$$
$$+ \int_0^T (\eta_p, \operatorname{div}\xi_{\mathbf{u},t})dt - \int_0^T (\operatorname{div}\eta_{\mathbf{u},t}, \xi_p)dt - \int_0^T \kappa(\eta_{\mathbf{q}}, \xi_{\mathbf{q}})dt.$$

Plugging (3.17) into the left hand side of the above equation, and noting that $\xi_{\mathbf{u}}(0) = 0$ and $\xi_p(0) = 0$, we can obtain

$$\frac{1}{2}\sum_{E\in\varepsilon_h} \int_E (\sigma(\xi_{\mathbf{u}}) : \epsilon(\xi_{\mathbf{u}}))_{t=T} dE + \frac{1}{2} J_0^{\delta,\beta}(\xi_{\mathbf{u}}, \xi_{\mathbf{u}})_{t=T} + \int_0^T \widehat{J_0^{\delta,\beta}}(\xi_{\mathbf{u},t}, \xi_{\mathbf{u},t})dt + \kappa\int_0^T \|\xi_{\mathbf{q}}\|_0^2 dt$$
$$= -\int_0^T A_{\mathbf{u}}(\eta_{\mathbf{u}}, \xi_{\mathbf{u},t})dt - \int_0^T \widehat{J_0^{\delta,\beta}}(\eta_{\mathbf{u},t}, \xi_{\mathbf{u},t})dt + \int_0^T \sum_{e\in\Gamma_{int}} \int_e \{\xi_p + \eta_p\}\mathbf{n}\cdot[\xi_{\mathbf{u},t}]dsdt$$
$$+ \int_0^T (\eta_p, \operatorname{div}\xi_{\mathbf{u},t})dt - \int_0^T (\operatorname{div}\eta_{\mathbf{u},t}, \xi_p)dt - \int_0^T \kappa(\eta_{\mathbf{q}}, \xi_{\mathbf{q}})dt + \int_0^T \sum_{e\in\Gamma_{int}} \int_e [\xi_{\mathbf{u},t}]\cdot\{\sigma(\xi_{\mathbf{u}})\mathbf{n}\}dsdt$$
$$- \epsilon\int_0^T \sum_{e\in\Gamma_{int}} \int_e [\xi_{\mathbf{u}}]\cdot\{\sigma(\xi_{\mathbf{u},t})\mathbf{n}\}dsdt \equiv E_1 + E_2 + \cdots + E_8. \tag{4.9}$$

Integrating by parts in time for E_1,

$$E_1 = -\int_0^T A(\eta_{\mathbf{u}}, \xi_{\mathbf{u},t})dt = \int_0^T A(\eta_{\mathbf{u},t}, \xi_{\mathbf{u}})dt - A(\eta_{\mathbf{u}}, \xi_{\mathbf{u}})_{t=T} \equiv E_{1a} + E_{1b}.$$

For E_{1a} and E_{1b}, by using the projection properties (3.1)–(3.2), Lemmas 4.2(a) and 4.3(a), and the Cauchy–Schwarz and Young inequalities, we get

$$E_{1a} = \int_0^T \sum_{E\in\varepsilon_h} \int_E (\sigma(\eta_{\mathbf{u},t}) : \epsilon(\xi_{\mathbf{u}}))dEdt - \int_0^T \sum_{e\in\Gamma_{int}} \int_e [\xi_{\mathbf{u}}]\cdot\{\sigma(\eta_{\mathbf{u},t})\mathbf{n}\}dsdt$$
$$+ \epsilon\int_0^T \sum_{e\in\Gamma_{int}} \int_e [\eta_{\mathbf{u},t}]\cdot\{\sigma(\xi_{\mathbf{u}})\mathbf{n}\}dsdt + \int_0^T J_0^{\delta,\beta}(\eta_{\mathbf{u},t}, \xi_{\mathbf{u}})dt$$
$$\leq C\int_0^T |||\eta_{\mathbf{u},t}|||_1 |||\xi_{\mathbf{u}}|||_1 dt + \int_0^T \sum_{e\in\Gamma_{int}} \left(\frac{\delta}{|e|^\beta}\right)^{\frac{1}{2}} \|[\xi_{\mathbf{u}}]\|_{0,e} \times \left(\frac{|e|^\beta}{\delta}\right)^{\frac{1}{2}} \|\{\sigma(\eta_{\mathbf{u},t})\mathbf{n}\}\|_{0,e} dt$$
$$+ \int_0^T \sum_{e\in\Gamma_{int}} \left(\frac{\delta}{|e|^\beta}\right)^{\frac{1}{2}} \|[\eta_{\mathbf{u},t}]\|_{0,e} \times \left(\frac{|e|^\beta}{\delta}\right)^{\frac{1}{2}} \|\{\sigma(\xi_{\mathbf{u}})\mathbf{n}\}\|_{0,e} dt$$
$$+ \int_0^T \sum_{e\in\Gamma_{int}} \left(\frac{\delta}{|e|^\beta}\right)^{\frac{1}{2}} \|[\eta_{\mathbf{u},t}]\|_{0,e} \times \left(\frac{\delta}{|e|^\beta}\right)^{\frac{1}{2}} \|[\xi_{\mathbf{u}}]\|_{0,e} dt$$
$$\leq C\int_0^T |||\eta_{\mathbf{u},t}|||_1^2 dt + C\int_0^T |||\xi_{\mathbf{u}}|||_1^2 dt + C\int_0^T \sum_{e\in\Gamma_{int}} \frac{\delta}{h_e}\|[\eta_{\mathbf{u},t}]\|_{0,e}^2 dt$$
$$+ \int_0^T \sum_{e\in\Gamma_{int}} \frac{h_e}{\delta}\|\{\sigma(\eta_{\mathbf{u},t})\mathbf{n}\}\|_{0,e}^2 dt + \frac{1}{8}\int_0^T J_0^{\delta,\beta}(\xi_{\mathbf{u}}, \xi_{\mathbf{u}})dt + \int_0^T \sum_{e\in\Gamma_{int}} \frac{h_e}{\delta}\|\{\sigma(\xi_{\mathbf{u}})\mathbf{n}\}\|_{0,e}^2 dt$$
$$\leq Ch^{\min\{2r_2, 2t-2\}}\int_0^T \|\mathbf{u}_t\|_{\min\{r_2+1,t\}}^2 dt + C\int_0^T |||\xi_{\mathbf{u}}|||_1^2 dt + \frac{1}{8}\int_0^T J_0^{\delta,\beta}(\xi_{\mathbf{u}}, \xi_{\mathbf{u}})dt,$$

and

$$E_{1b} = -\sum_{E\in\varepsilon_h} \int_E (\sigma(\eta_{\mathbf{u}}) : \epsilon(\xi_{\mathbf{u}}))|_{t=T} dE + \sum_{e\in\Gamma_{int}} \int_e [\xi_{\mathbf{u}}]\cdot\{\sigma(\eta_{\mathbf{u}})\mathbf{n}\}|_{t=T} ds$$
$$- \epsilon\sum_{e\in\Gamma_{int}} \int_e [\eta_{\mathbf{u}}]\cdot\{\sigma(\xi_{\mathbf{u}})\mathbf{n}\}|_{t=T} ds - J_0^{\delta,\beta}(\eta_{\mathbf{u}}, \xi_{\mathbf{u}})|_{t=T}$$

$$
\begin{aligned}
&\le \|\!|\eta_{\mathbf{u}}(T)|\!\|_1 \|\!|\xi_{\mathbf{u}}(T)|\!\|_1 + \sum_{e\in\Gamma_{int}} \left(\frac{\delta}{|e|^{\beta}}\right)^{\frac{1}{2}} \|[\xi_{\mathbf{u}}(T)]\|_{0,e} \times \left(\frac{|e|^{\beta}}{\delta}\right)^{\frac{1}{2}} \|\{\sigma(\eta_{\mathbf{u}}(T))\mathbf{n}\}\|_{0,e} \\
&\quad + \sum_{e\in\Gamma_{int}} \left(\frac{\delta}{|e|^{\beta}}\right)^{\frac{1}{2}} \|[\eta_{\mathbf{u}}(T)]\|_{0,e} \times \left(\frac{|e|^{\beta}}{\delta}\right)^{\frac{1}{2}} \|\{\sigma(\xi_{\mathbf{u}}(T))\mathbf{n}\}\|_{0,e} \\
&\quad + \sum_{e\in\Gamma_{int}} \left(\frac{\delta}{|e|^{\beta}}\right)^{\frac{1}{2}} \|[\eta_{\mathbf{u}}(T)]\|_{0,e} \times \left(\frac{\delta}{|e|^{\beta}}\right)^{\frac{1}{2}} \|[\xi_{\mathbf{u}}(T)]\|_{0,e} \\
&\le C\|\!|\eta_{\mathbf{u}}(T)|\!\|_1^2 + \epsilon\|\!|\xi_{\mathbf{u}}(T)|\!\|_1^2 + C\sum_{e\in\Gamma_{int}} \frac{\delta}{h_e}\|[\eta_{\mathbf{u}}(T)]\|_{0,e}^2 \\
&\quad + \sum_{e\in\Gamma_{int}} \frac{h_e}{\delta}\|\{\sigma(\eta_{\mathbf{u}}(T))\mathbf{n}\}\|_{0,e}^2 + \frac{1}{8}J_0^{\delta,\beta}(\xi_{\mathbf{u}}(T),\xi_{\mathbf{u}}(T)) + \epsilon\sum_{e\in\Gamma_{int}} \frac{h_e}{\delta}\|\{\sigma(\xi_{\mathbf{u}}(T))\mathbf{n}\}\|_{0,e}^2 \\
&\le Ch^{\min\{2r_2,2t-2\}}\|\mathbf{u}(T)\|_{\min\{r_2+1,t\}}^2 + \epsilon_1\|\!|\xi_{\mathbf{u}}(T)|\!\|_1^2 + \frac{1}{8}J_0^{\delta,\beta}(\xi_{\mathbf{u}}(T),\xi_{\mathbf{u}}(T)),
\end{aligned}
$$

where $\epsilon_1 > 0$ is an arbitrarily small parameter. For E_2, by virtue of (3.2), (3.13), and the Cauchy–Schwarz and Young inequalities, we deduce

$$
\begin{aligned}
E_2 &= -\int_0^T J_0^{\widehat{\delta},\beta}(\eta_{\mathbf{u},t},\xi_{\mathbf{u},t})dt \\
&= -\int_0^T \sum_{e\in\Gamma_{int}} \int_e \left(\frac{\widehat{\delta}}{|e|^{\beta}}\right)^{\frac{1}{2}} [\eta_{\mathbf{u},t}] \cdot \left(\frac{\widehat{\delta}}{|e|^{\beta}}\right)^{\frac{1}{2}} [\xi_{\mathbf{u},t}]dsdt \\
&\le \int_0^T \sum_{e\in\Gamma_{int}} \left(\frac{\widehat{\delta}}{|e|^{\beta}}\right)^{\frac{1}{2}} \|[\eta_{\mathbf{u},t}]\|_{0,e} \cdot \left(\frac{\widehat{\delta}}{|e|^{\beta}}\right)^{\frac{1}{2}} \|[\xi_{\mathbf{u},t}]\|_{0,e}dt \\
&\le C\int_0^T \sum_{e\in\Gamma_{int}} \frac{\widehat{\delta}}{|e|^{\beta}}\|[\eta_{\mathbf{u},t}]\|_{0,e}^2 dt + \frac{1}{4}\int_0^T J_0^{\widehat{\delta},\beta}(\xi_{\mathbf{u},t},\xi_{\mathbf{u},t})dt \\
&\le Ch^{\min\{2r_2,2t-2\}}\int_0^T \|\mathbf{u}_t\|_{\min\{r_2+1,t\}}^2 dt + \frac{1}{4}\int_0^T J_0^{\widehat{\delta},\beta}(\xi_{\mathbf{u},t},\xi_{\mathbf{u},t})dt.
\end{aligned}
$$

E_3 can be written as

$$
\begin{aligned}
E_3 &= \int_0^T \sum_{e\in\Gamma_{int}} \int_e \{\xi_p + \eta_p\}\mathbf{n}\cdot[\xi_{\mathbf{u},t}]dsdt \\
&= \int_0^T \sum_{e\in\Gamma_{int}} \int_e \{\xi_p\}\mathbf{n}\cdot[\xi_{\mathbf{u},t}]dsdt + \int_0^T \sum_{e\in\Gamma_{int}} \int_e \{\eta_p\}\mathbf{n}\cdot[\xi_{\mathbf{u},t}]dsdt \\
&\equiv E_{3a} + E_{3b}.
\end{aligned}
$$

For E_{3a}, using Lemmas 4.5 and 4.3(b),

$$
\begin{aligned}
E_{3a} &\le \int_0^T \sum_{e\in\Gamma_{int}} \int_e |\{\xi_p\}|\cdot|[\xi_{\mathbf{u},t}]|dsdt \\
&\le \int_0^T \sum_{e\in\Gamma_{int}} \left(\frac{|e|^{\beta}}{\widehat{\delta}}\right)^{\frac{1}{2}} \|\{\xi_p\}\|_{0,e} \left(\frac{\widehat{\delta}}{|e|^{\beta}}\right)^{\frac{1}{2}} \|[\xi_{\mathbf{u},t}]\|_{0,e}dt \\
&\le C\int_0^T \sum_{e\in\Gamma_{int}} \frac{|e|^{\beta}}{\widehat{\delta}}\|\{\xi_p\}\|_{0,e}^2 dt + \frac{1}{4}\int_0^T J_0^{\widehat{\delta},\beta}(\xi_{\mathbf{u},t},\xi_{\mathbf{u},t})dt \\
&\le C\int_0^T \sum_{e\in\Gamma_{int}} \frac{|e|^{\beta}}{\widehat{\delta}}\left(\|\xi_p|_{E_i^e}\|_{0,e}^2 + \|\xi_p|_{E_j^e}\|_{0,e}^2\right)dt + \frac{1}{4}\int_0^T J_0^{\widehat{\delta},\beta}(\xi_{\mathbf{u},t},\xi_{\mathbf{u},t})dt \\
&\le C\int_0^T \sum_{e\in\Gamma_{int}} \frac{|e|^{\beta}}{\widehat{\delta}}\left(h_{ie}^{-1}\|\xi_p|_{E_i^e}\|_{0,E_i^e}^2 + h_{je}^{-1}\|\xi_p|_{E_j^e}\|_{0,E_j^e}^2\right)dt + \frac{1}{4}\int_0^T J_0^{\widehat{\delta},\beta}(\xi_{\mathbf{u},t},\xi_{\mathbf{u},t})dt
\end{aligned}
$$

$$\le \frac{C}{\widehat{\delta}_{\min}}\int_0^T \|\xi_p\|_0^2 dt + \frac{1}{4}\int_0^T J_0^{\widehat{\delta},\beta}(\xi_{\mathbf{u},t},\xi_{\mathbf{u},t})dt$$
$$\le \frac{CC_{\mathbf{q}}^2}{\widehat{\delta}_{\min}}\int_0^T (\|\eta_{\mathbf{q}}\|_0^2 + \|\xi_{\mathbf{q}}\|_0^2)dt + \frac{1}{4}\int_0^T J_0^{\widehat{\delta},\beta}(\xi_{\mathbf{u},t},\xi_{\mathbf{u},t})dt$$
$$\le Ch^{\min\{2r_1+2,2q-2\}}\int_0^T \|\mathbf{q}\|^2_{\min\{r_1+1,q-1\}}dt + \frac{CC_{\mathbf{q}}^2}{\widehat{\delta}_{\min}}\int_0^T \|\xi_{\mathbf{q}}\|_0^2 dt + \frac{1}{4}\int_0^T J_0^{\widehat{\delta},\beta}(\xi_{\mathbf{u},t},\xi_{\mathbf{u},t})dt.$$

By Lemma 4.3(c), similar to the deduction of E_{3a}, we can get

$$E_{3b} \le C\int_0^T \sum_{e\in\Gamma_{int}} \frac{|e|^\beta}{\widehat{\delta}}\left(\|\eta_p|_{E_i^e}\|_{0,e}^2 + \|\eta_p|_{E_j^e}\|_{0,e}^2\right)dt + \frac{1}{4}\int_0^T J_0^{\widehat{\delta},\beta}(\xi_{\mathbf{u},t},\xi_{\mathbf{u},t})dt$$
$$\le C\int_0^T \sum_{e\in\Gamma_{int}} \frac{|e|^\beta}{\widehat{\delta}}\left(h_{ie}^{-1}\|\eta_p\|_{0,E_i^e}^2 + h_{ie}\|\eta_p\|_{1,E_i^e}^2 + h_{je}^{-1}\|\eta_p\|_{0,E_j^e}^2 + h_{je}\|\eta_p|\|_{1,E_j^e}^2\right)dt$$
$$+\frac{1}{4}\int_0^T J_0^{\widehat{\delta},\beta}(\xi_{\mathbf{u},t},\xi_{\mathbf{u},t})dt$$
$$\le \frac{C}{\widehat{\delta}_{\min}}\int_0^T \sum_{E\in\varepsilon_h}\left(\|\eta_p\|_{0,E}^2 + h_e^2\|\eta_p\|_{1,E}^2\right)dt + \frac{1}{4}\int_0^T J_0^{\widehat{\delta},\beta}(\xi_{\mathbf{u},t},\xi_{\mathbf{u},t})dt$$
$$\le \frac{C}{\widehat{\delta}_{\min}}h^{\min\{2r_1+2,2q\}}\int_0^T \|p\|^2_{\min\{r_1+1,q\}}dt + \frac{1}{4}\int_0^T J_0^{\widehat{\delta},\beta}(\xi_{\mathbf{u},t},\xi_{\mathbf{u},t})dt.$$

The bound for E_4 follows by integrating by parts and using the initial time assumption $\operatorname{div}\xi_{\mathbf{u}}|_{t=0} = 0$,

$$E_4 = \int_0^T (\eta_p, \operatorname{div}\xi_{\mathbf{u},t})dt = -\int_0^T (\eta_{p,t}, \operatorname{div}\xi_{\mathbf{u}})dt + (\eta_p, \operatorname{div}\xi_{\mathbf{u}})|_{t=T}$$
$$\le \int_0^T \sum_{E\in\varepsilon_h}\left(\|\eta_{p,t}\|_{0,E}\|\operatorname{div}\xi_{\mathbf{u}}\|_{0,E} + \|\eta_p(T)\|_{0,E}\|\operatorname{div}\xi_{\mathbf{u}}(T)\|_{0,E}\right)dt$$
$$\le C\int_0^T \|\eta_{p,t}\|_0^2 dt + \frac{1}{2}\int_0^T |||\xi_{\mathbf{u}}|||_1^2 dt + C\|\eta_p(T)\|_0^2 + \epsilon_2|||\xi_{\mathbf{u}}(T)|||_1^2$$
$$\le Ch^{\min\{2r_1+1,2r\}}\int_0^T \|p_t\|^2_{\min\{r_1+1,r\}}dt + \frac{1}{2}\int_0^T \|\xi_{\mathbf{u}}\|_1^2 dt$$
$$+ Ch^{\min\{2r_1+1,2q\}}\|p(T)\|^2_{\min\{r_1+1,q\}} + \epsilon_2|||\xi_{\mathbf{u}}(T)|||_1^2,$$

where $\epsilon_2 > 0$ is an arbitrarily small parameter. For E_5, by Lemma 4.5 and the Young inequality with $\epsilon = \frac{\kappa}{4C_{\mathbf{q}}^2}$, we get

$$E_5 = -\int_0^T (\operatorname{div}\eta_{\mathbf{u},t}, \xi_p)dt \le \int_0^T \sum_{E\in\varepsilon_h}\|\xi_p\|_{0,E}\|\operatorname{div}\eta_{\mathbf{u},t}\|_{0,E}dt$$
$$\le C\int_0^T |||\eta_{\mathbf{u},t}|||_1^2 dt + \frac{\kappa}{4C_{\mathbf{q}}^2}\int_0^T \|\xi_p\|_0^2 dt$$
$$\le C\int_0^T |||\eta_{\mathbf{u},t}|||_1^2 dt + \frac{\kappa}{4}\int_0^T (\|\eta_{\mathbf{q}}\|_0^2 + \|\xi_{\mathbf{q}}\|_0^2)dt$$
$$\le Ch^{\min\{2r_1+2,2q-2\}}\int_0^T \|\mathbf{q}\|^2_{\min\{r_1+1,q-1\}} + \frac{\kappa}{4}\int_0^T \|\xi_{\mathbf{q}}\|_0^2 dt + Ch^{\min\{2r_2,2t-2\}}\int_0^T \|\mathbf{u}_t\|^2_{\min\{r_2+1,t\}}.$$

To bound E_6, applying the Cauchy–Schwarz and Young inequalities

$$E_6 = -\int_0^T \kappa(\eta_{\mathbf{q}}, \xi_{\mathbf{q}})dt \le \kappa\int_0^T \|\xi_{\mathbf{q}}\|_0\|\eta_{\mathbf{q}}\|_0 dt$$
$$\le C\int_0^T \|\eta_{\mathbf{q}}\|_0^2 dt + \frac{\kappa}{4}\int_0^T \|\xi_{\mathbf{q}}\|_0^2 dt$$
$$\le Ch^{\min\{2r_1+2,2q-2\}}\int_0^T \|\mathbf{q}\|^2_{\min\{r_1+1,q-1\}}dt + \frac{\kappa}{4}\int_0^T \|\xi_{\mathbf{q}}\|_0^2 dt.$$

Integrate by parts in time for E_8, then we get

$$\begin{aligned}
E_7 + E_8 &= \int_0^T \sum_{e\in\Gamma_{int}} \int_e [\xi_{\mathbf{u},t}] \cdot \{\sigma(\xi_{\mathbf{u}})\mathbf{n}\} ds dt - \epsilon \int_0^T \sum_{e\in\Gamma_{int}} \int_e [\xi_{\mathbf{u}}] \cdot \{\sigma(\xi_{\mathbf{u},t})\mathbf{n}\} ds dt \\
&= (1+\epsilon) \int_0^T \sum_{e\in\Gamma_{int}} \int_e \{\sigma(\xi_{\mathbf{u}})\mathbf{n}\} \cdot [\xi_{\mathbf{u},t}] ds dt - \epsilon \sum_{e\in\Gamma_{int}} \int_e \{\sigma(\xi_{\mathbf{u}})\mathbf{n}\} \cdot [\xi_{\mathbf{u}}]_{t=T} ds \\
&\equiv E_{7'} + E_{8'}.
\end{aligned}$$

The bound for $E_{7'}$ uses Lemma 4.3(a), the Cauchy–Schwarz and Young inequalities

$$\begin{aligned}
E_{7'} &= (1+\epsilon) \int_0^T \sum_{e\in\Gamma_{int}} \left(\frac{|e|^\beta}{\widehat{\delta}}\right)^{\frac{1}{2}} \{\sigma(\xi_{\mathbf{u}})\mathbf{n}\} \cdot \left(\frac{\widehat{\delta}}{|e|^\beta}\right)^{\frac{1}{2}} [\xi_{\mathbf{u},t}] dt \\
&\le C \int_0^T \sum_{e\in\Gamma_{int}} \left(\frac{|e|^\beta}{\widehat{\delta}}\right)^{\frac{1}{2}} \|\{\sigma(\xi_{\mathbf{u}})\mathbf{n}\}\|_{0,e} \cdot \left(\frac{\widehat{\delta}}{|e|^\beta}\right)^{\frac{1}{2}} \|[\xi_{\mathbf{u},t}]\|_{0,e} dt \\
&\le Ch^{\beta(d-1)} \int_0^T \sum_{e\in\Gamma_{int}} \|\{\sigma(\xi_{\mathbf{u}})\mathbf{n}\}\|_{0,e}^2 dt + \frac{1}{4} \int_0^T \sum_{e\in\Gamma_{int}} \frac{\widehat{\delta}}{|e|^\beta} \|[\xi_{\mathbf{u},t}]\|_{0,e}^2 dt \\
&\le Ch^{\beta(d-1)-1} \int_0^T \sum_{e\in\Gamma_{int}} \int_{E^e} (\sigma(\xi_{\mathbf{u}}) : \epsilon(\xi_{\mathbf{u}})) dt + \frac{1}{4} \int_0^T J_0^{\widehat{\delta},\beta}(\xi_{\mathbf{u},t}, \xi_{\mathbf{u},t}) dt \\
&\le C \int_0^T |||\xi_{\mathbf{u}}|||_1^2 dt + \frac{1}{4} \int_0^T J_0^{\widehat{\delta},\beta}(\xi_{\mathbf{u},t}, \xi_{\mathbf{u},t}) dt.
\end{aligned}$$

Similarly, we can get

$$\begin{aligned}
E_{8'} &\le C \sum_{e\in\Gamma_{int}} \int_e \left(\frac{|e|^\beta}{\delta}\right)^{\frac{1}{2}} \{\sigma(\xi_{\mathbf{u}})\mathbf{n}\}_{t=T} \cdot \left(\frac{\delta}{|e|^\beta}\right)^{\frac{1}{2}} [\xi_{\mathbf{u}}]_{t=T} ds \\
&\le C \sum_{e\in\Gamma_{int}} \left(\frac{|e|^\beta}{\delta}\right)^{\frac{1}{2}} \|\{\sigma(\xi_{\mathbf{u}})\mathbf{n}\}_{t=T}\|_{0,e} \cdot \left(\frac{\delta}{|e|^\beta}\right)^{\frac{1}{2}} \|[\xi_{\mathbf{u}}(T)]\|_{0,e} \\
&\le \frac{C}{\delta_{\min}} h^{\beta(d-1)} \sum_{e\in\Gamma_{int}} \|\{\sigma(\xi_{\mathbf{u}})\mathbf{n}\}_{t=T}\|_{0,e}^2 + \frac{1}{8} \sum_{e\in\Gamma_{int}} \frac{\delta}{|e|^\beta} \|[\xi_{\mathbf{u}}(T)]\|_{0,e}^2 \\
&\le \frac{C}{\delta_{\min}} \sum_{e\in\Gamma_{int}} \int_{E^e} (\sigma(\xi_{\mathbf{u}}) : \epsilon(\xi_{\mathbf{u}}))_{t=T} dE + \frac{1}{8} J_0^{\delta,\beta}(\xi_{\mathbf{u}}, \xi_{\mathbf{u}})_{t=T} \\
&\le \frac{C}{\delta_{\min}} \sum_{E\in\varepsilon_h} \int_E (\sigma(\xi_{\mathbf{u}}) : \epsilon(\xi_{\mathbf{u}}))_{t=T} dE + \frac{1}{8} J_0^{\delta,\beta}(\xi_{\mathbf{u}}(T), \xi_{\mathbf{u}}(T)).
\end{aligned}$$

Combining the estimates of E_1 through E_8 with (4.9), we obtain that

$$\begin{aligned}
&\left(\frac{1}{2} - \frac{C}{\delta_{\min}}\right) \sum_{E\in\varepsilon_h} \int_E (\sigma(\xi_{\mathbf{u}}) : \epsilon(\xi_{\mathbf{u}}))_{t=T} dE + \left(\frac{1}{2} - \frac{1}{8} - \frac{1}{8}\right) J_0^{\delta,\beta}(\xi_{\mathbf{u}}(T), \xi_{\mathbf{u}}(T)) \\
&\quad + \left(\kappa - \frac{\kappa}{4} - \frac{\kappa}{4} - \frac{CC_{\mathbf{q}}^2}{\widehat{\delta}_{\min}}\right) \int_0^T \|\xi_{\mathbf{q}}\|_0^2 dt - (\epsilon_1 + \epsilon_2) |||\xi_{\mathbf{u}}(T)|||_1^2 \\
&\le C\Bigg(h^{\min\{2r_2,2t-2\}} \int_0^T \|\mathbf{u}_t\|_{\min\{r_2+1,t\}}^2 dt + h^{\min\{2r_2,2t-2\}} \|\mathbf{u}(T)\|_{\min\{r_2+1,t\}}^2 \\
&\quad + h^{\min\{2r_1+2,2q-2\}} \int_0^T \|\mathbf{q}\|_{\min\{r_1+1,q-1\}}^2 dt + h^{\min\{2r_1+2,2q\}} \int_0^T \|p\|_{\min\{r_1+1,q\}}^2 dt \\
&\quad + h^{\min\{2r_1+1,2r\}} \int_0^T \|p_t\|_{\min\{r_1+1,r\}}^2 dt + h^{\min\{2r_1+1,2q\}} \|p(T)\|_{\min\{r_1+1,q\}}^2 \Bigg) \\
&\quad + C\left(\int_0^T |||\xi_{\mathbf{u}}|||_1^2 dt + \int_0^T J_0^{\delta,\beta}(\xi_{\mathbf{u}}, \xi_{\mathbf{u}}) dt\right). \qquad (4.10)
\end{aligned}$$

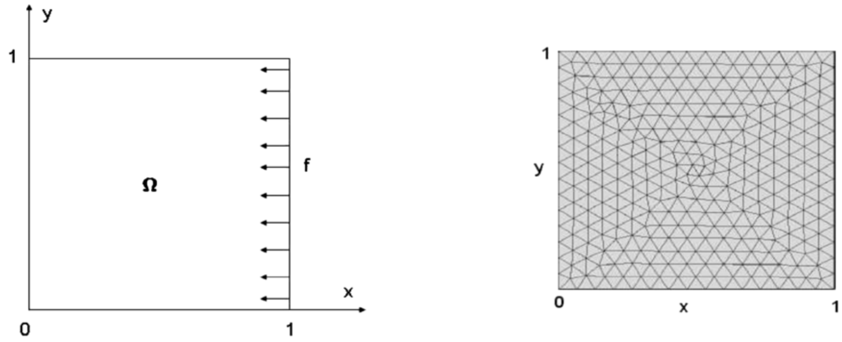


Fig. 1. Arrow plot of the force function and the mesh.

Let δ and $\widehat{\delta}$ be chosen so that $\frac{1}{2} - \frac{C}{\delta_{\min}} = \frac{1}{8}$ and $\frac{\kappa}{2} - \frac{CC_{\mathbf{q}}^2}{\widehat{\delta}_{\min}} > 0$; then by Lemma 4.2(b), the left-hand side of (4.10) can be written as

$$\begin{aligned} \text{lhs} &\geq \left(\frac{1}{8}C_K - \epsilon_1 - \epsilon_2\right) \|\!|\xi_{\mathbf{u}}(T)|\!\|_1^2 + \frac{1}{8}J_0^{\delta,\beta}(\xi_{\mathbf{u}}(T), \xi_{\mathbf{u}}(T)) + \left(\frac{\kappa}{2} - \frac{CC_{\mathbf{q}}^2}{\widehat{\delta}_{\min}}\right)\int_0^T \|\xi_{\mathbf{q}}\|_0^2 dt \\ &\geq C'\left(\|\!|\xi_{\mathbf{u}}(T)|\!\|_1^2 + J_0^{\delta,\beta}(\xi_{\mathbf{u}}(T), \xi_{\mathbf{u}}(T)) + \int_0^T \|\xi_{\mathbf{q}}\|_0^2 dt\right), \end{aligned} \tag{4.11}$$

where C' is the positive minimum coefficient of the three terms of (4.11). By virtue of (4.10) and (4.11), we can first eliminate the last two terms from the right-hand side of (4.10) using Gronwall Lemma 4.6, then drop the inessential positive term $J_0^{\delta,\beta}(\xi_{\mathbf{u}}(T), \xi_{\mathbf{u}}(T))$, so we can get

$$\|\!|\xi_{\mathbf{u}}|\!\|_{L^\infty(\mathbf{H}^1)}^2 + \|\xi_{\mathbf{q}}\|_{L^2(L^2)}^2 \leq Ch^{2R}. \tag{4.12}$$

The theorem readily follows from (4.12), (3.2), Lemma 4.4(b) and (d), and Lemma 4.5. The proof is completed. $\square$

5. Numerical examples

In this section, one two-dimensional numerical experiment of swelling model is conducted to gauge the performance of the mixed-DG finite element method developed in this paper. The gel used in the test is the Poly hydrogel (cf. [10,19] and the references therein). The material constants, which were reported in [10,26], are given:

Young's modulus $E = 6000$, Poisson's ratio $\nu = 0.43$, bulk modulus $K = \frac{E}{3(1-2\nu)} = 14285.7$, shear modulus $G = \frac{E}{2(1+\nu)} = 2097.9$, friction constant $\xi = 100$, Lamé constants $\alpha = K - \frac{2G}{3} = 12887.1$ and $\beta = 2G = 4195.8$, volume fraction $\phi = 0.15$, and $\kappa = \frac{(1-\phi)^2}{\xi} = 7.225 \times 10^{-3}$.

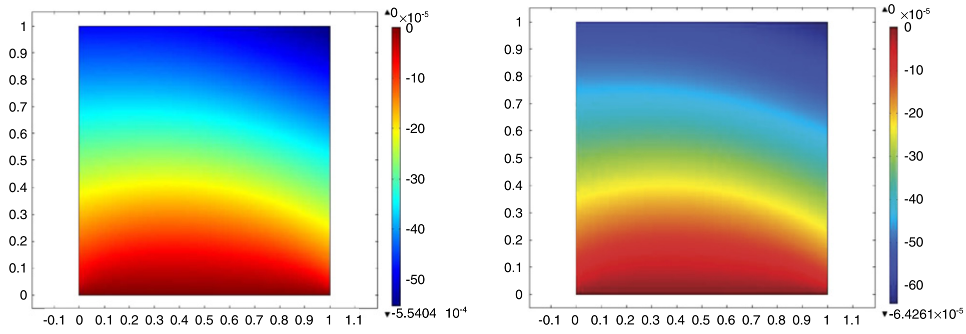
Let $\Omega = [0, 1] \times [0, 1]$. The external force $\mathbf{f} = (f_1, f_2)$ on the boundary is taken as follows for the test:

$$f_1(x, y) = \begin{cases} -1, & x = 1, \\ 0, & \text{others}, \end{cases} \qquad f_2(x, y) = 0.$$

Fig. 1 shows the computational domain, the arrow plot of the force function $\mathbf{f}$ on $\partial\Omega$, and the mesh on which the numerical solution is computed. $\Delta t = 0.0001$ is used in the test.

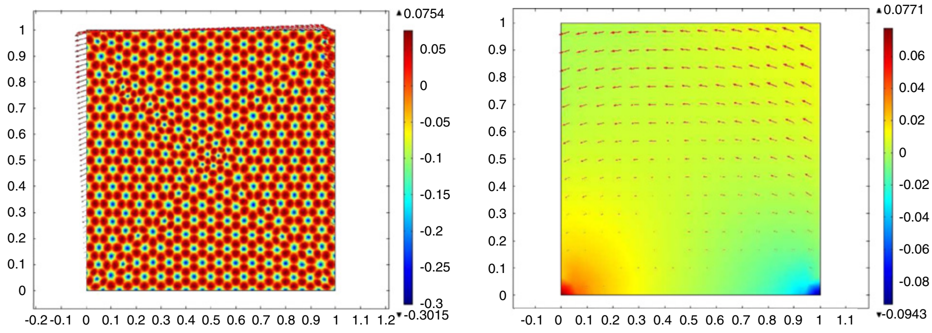
Figs. 2 and 3 display snapshots of the computed solution at time $t = 0.001$. Each graph in Fig. 2 contains the color plots of the displacement, and each graph in Fig. 3 contains not only the color plots of the pressure, but also the arrow plots of the displacement, which show the deformation of the square gel under the mechanical force $\mathbf{f}$ on the boundary. As expected, the gel is moved to the left a little when the right-hand-side force is applied.

At the same time, Figs. 2–4 show a comparison of the pressure and displacement produced by the CG and mixed DG algorithms, respectively. The three graphs denoted by (a) are computed by the CG finite element method, while three graphs denoted by (b) are the corresponding plots by the mixed DG finite element method. Figs. 2–4 show that the scheme using continuous elements for displacements clearly suffers from nonphysical pressure oscillations, whereas the one using discontinuous elements has virtually eliminated the problem.



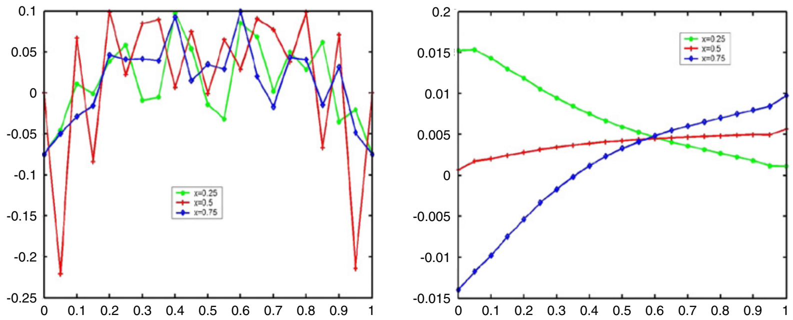
(a) CG. (b) Mixed-DG.

Fig. 2. The profile of x-axis displacement $\mathbf{u}$ at time $t = 0.001$.



(a) CG. (b) Mixed DG.

Fig. 3. The profile of pressure p and displacement $\mathbf{u}$ at time $t = 0.001$.



(a) CG. (b) Mixed DG.

Fig. 4. The profile of pressure $p(x = 0.2, 0.5, 0.8)$ at time $t = 0.001$.

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