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Equivalent monotone versions of PRV functions

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ABSTRACT

Let f_* denote the *lower function* of a real function f. Assuming that $f_*(c) \geq 1$ for all $c > 1$ and $f_*(c_0) > 1$ for some $c_0 > 1$ we prove that, if f is a *pseudo-regularly varying* function, then f has an equivalent monotone version.

1. Introduction

The early developments in the theory of regularly varying functions were often related to situations, in which monotonicity was assumed. Such a preliminary assumption allows one to develop key aspects of the theory, particularly characterizations, either more comprehensively or in a much simpler manner (see, e.g., Seneta [14], Section 1.8).

The simplest generalization of the monotonicity property is that the function possesses an (asymptotically) *equivalent monotone version*. This weaker condition, nevertheless, leads to almost the same simplifications in the proofs of a number of results for regularly varying functions. Thus a natural question to ask is, for a given function f (not necessarily regularly varying), about conditions for f to have an equivalent monotone version. We give a partial answer to this question for the so-called *pseudo-regularly varying* (PRV) functions (see the definitions below). The well-known statement that a regularly varying function of nonzero index has an equivalent monotone version (see Seneta [14], Section 1.5) is a particular case of our result. In contrast, a regularly varying function of zero index (a so-called slowly varying function) does not always have an equivalent monotone version. We show this by constructing an example of such a function below (cf. Theorem 3.2).

The monotonicity of functions under consideration plays a crucial role in various problems of probability theory. To be more specific we mention the following example. Let $\{\xi_n\}_{n\geq 1}$ be a sequence of nondegenerate, nonnegative, independent, identically distributed random variables and let $\{\zeta_n\}_{n\geq 1}$, $\zeta_0 = 0$, be their partial sums. Note that the sequence $\{\zeta_n\}$ is nondecreasing with respect to n, since the summands ξ_n are nonnegative. Denote by $\{N(t)\}_{t\geq 0}$ the corresponding renewal counting process defined by

$$N(t) = \max\{n : \zeta_n \leq t\}, \quad t \geq 0.$$

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Note that $\zeta_n \to \infty$ almost surely as $n \to \infty$, since the distribution of ξ_1 is non-degenerate, whence $N(t) < \infty$ almost surely for all $t \geq 0$. Assuming that

$$\lim_{n\to\infty} \frac{\zeta_n}{b_n} = 1 \quad \text{almost surely}$$

for a deterministic sequence of positive numbers $\{b_n\}_{n\geq 1}$, the problem studied in Klesov, Rychlik, and Steinebach [12] is to find a normalizing function $\{c(t)\}_{t\geq 0}$ such that

$$\lim_{t\to\infty} \frac{N(t)}{c(t)} = 1 \quad \text{almost surely.}$$

This problem has been solved in [12] if $\{b_n\}$ is an increasing sequence. It turned out that the function $\{c(t)\}$ can be constructed from the sequence $\{b_n\}$ by applying the concept of *generalized inverse functions*. Moreover, the conclusion holds if the sequence $\{b_n\}$ is not necessarily monotone but (asymptotically) equivalent to a monotone sequence and if a certain additional restriction is imposed.

Karamata's ideas about the notion of regular variation turned out to be very fruitful in many fields of mathematics and have soon been generalized in various ways, keeping the main properties, however, in one way or another. Among those generalizations is the notion of *O-regularity*, denoted here by OR, which, for a positive function f, means that

$$f^*(c) = \limsup_{t\to\infty} \frac{f(ct)}{f(t)} < \infty \quad \text{for all } c > 0$$

(see Avakumović [1] and Karamata [11]). The function f^* is called the *upper function* for f, while

$$f_*(c) = \liminf_{t\to\infty} \frac{f(ct)}{f(t)}, \quad c > 0, \tag{1.1}$$

is called its *lower function*. Karamata's regularly varying functions are characterized by the property that $f_*(c) = f^*(c)$ for all $c > 0$, in which case automatically the common value of $f_*(c)$ and $f^*(c)$ equals c^ρ for some real ρ and all $c > 0$.

OR functions have many properties, which are similar to those of regularly varying functions, for example, they exhibit similar uniform convergence properties, integral representations and characterizations, etc. Again, results for OR functions f become simpler, if f is monotone or has an equivalent monotone version. Here are some selected examples of such situations.

Independently of Avakumović [1] and Karamata [11], Bari and Stechkin in their memoir [2] also introduced the ORV property for monotone functions. They used such a property to describe a relationship between the modulus of continuity of a periodic function and its best approximation by trigonometric polynomials. In fact, their results also hold if just the existence of an equivalent monotone version is assumed instead of monotonicity.

Feller [9] observed that some of Karamata's [11] results do not require the full concept of regular variation. Instead he introduced the notion of *dominated variation* for monotone functions and successfully applied it to some local limit theorems in probability theory and to the tail behavior of infinitely divisible distribution functions. All his results on dominatedly varying functions can easily be extended to the non-monotone case by assuming that the corresponding functions have an equivalent monotone version.

De Haan and Stadtmüller [8] studied Abelian and Tauberian theorems for Laplace transforms under conditions of dominated variation and related concepts. Again, their results only require the existence of an equivalent monotone version.

Several subclasses of OR functions have been studied in the literature. A key role therein sets the question of whether, given two functions f and g such that $f \sim g$, it is true that $f^{-1} \sim g^{-1}$. (Examples from probability theory can be found in Gut, Klesov, and Steinebach [10], Klesov, Rychlik and Steinebach [12] and Buldygin, Klesov and Steinebach [4–6].) Here f^{-1} and g^{-1} denote the inverse functions to f and g, respectively.

A more general problem is to obtain a similar conclusion if the inverse functions do not exist and are substituted by so-called *quasi-inverse* or *asymptotically quasi-inverse functions* denoted by f^{inv}, g^{inv}, etc. A natural substitution for the inverse functions suggested by the construction of the renewal process given above are the so-called *generalized inverse functions* denoted by $f^{\leftarrow}$, $g^{\leftarrow}$, etc. An even larger class of functions, for which one can prove the above assertion, is the class of POV functions. Here, a function f is defined to be of *positive order of variation*, denoted by POV, if

$$f_*(c) > 1 \quad \text{for all } c > 1 \tag{1.2}$$

and

$$\liminf_{c\downarrow 1} f_*(c) = 1. \tag{1.3}$$

The latter property characterizes the so-called *pseudo-regularly varying* functions, denoted by PRV. For general OR, PRV or POV functions, the questions about the existence of equivalent monotone versions are not so obvious. For POV functions, for example, the existence of an equivalent monotone version has been proved by Buldygin, Klesov and Steinebach [5].

Our goal in this paper is to solve the problem on the monotonicity for a much larger class of functions. A related problem, studying whether a function has an infinite limit, has been presented by Buldygin, Klesov, and Steinebach [7], where also a motivation is given.

2. Some assumptions

Throughout the paper we assume that

$$f(t) > 0 \quad \text{for all } t \geq 0. \tag{2.1}$$

Definition 2.1. Let f be a real function. We say that f has an *equivalent monotone version* if there exists a nondecreasing function $f^{\uparrow}$ such that

$$f \sim f^{\uparrow} \quad (t \to \infty), \quad \text{i.e.,} \lim_{t\to\infty} \frac{f(t)}{f^{\uparrow}(t)} = 1.$$

A necessary assumption for a function f to have an equivalent monotone version is that

$$f_*(c) \geq 1 \quad \text{for all } c \geq 1, \tag{2.2}$$

see Theorem 3.1 below. Moreover, the condition

$$f_*(c_0) > 1 \quad \text{for some } c_0 > 1 \tag{2.3}$$

is also necessary (in the sense that there is a function f for which (2.3) does not hold and f has no equivalent monotone version, see Theorem 3.2).

We sometimes use a stronger condition than (2.2), namely

$$\liminf_{n\to\infty} \frac{f(c_n t_n)}{f(t_n)} \geq 1 \quad \text{for all sequences } c_n \downarrow 1 \text{ and } t_n \to \infty. \tag{2.4}$$

First we study the question under which condition an equivalent monotone version does not exist.

3. Condition under which no equivalent monotone version exists

Theorem 3.1. *Let*

$$f_*(c_0) < 1 \quad \textit{for some } c_0 > 1. \tag{3.1}$$

Then f has no equivalent monotone version.

Proof. Assume the converse, i.e. that there is a monotone version $f^{\uparrow}$. Condition (3.1) implies that there are some $q \in (0, 1)$ and a sequence $\{t_n\}$ such that $t_n \to \infty$ and

$$\frac{f(c_0 t_n)}{f(t_n)} \leq q \quad \text{for all } n \geq 1.$$

Thus

$$\frac{f^{\uparrow}(t_n)}{f(t_n)} \leq \frac{f^{\uparrow}(c_0 t_n)}{f(t_n)} \leq \frac{f^{\uparrow}(c_0 t_n)}{f(c_0 t_n)} \cdot \frac{f(c_0 t_n)}{f(t_n)} \leq q \frac{f^{\uparrow}(c_0 t_n)}{f(c_0 t_n)}.$$

Letting $n \to \infty$, we obtain the contradiction $1 \leq q$, which proves (3.1). □

Thus when finding conditions for the existence of an equivalent monotone version, one necessarily has to assume (2.2) and also (2.3) as a companion. If (2.3) does not hold, then a monotone version may not exist. An example, where (2.2) holds but (2.3) does not, is given by a *slowly varying* function for which $f_*(c) = f^*(c) = 1$ for all $c > 0$.

Theorem 3.2. *There is a slowly varying function f such that*

$$\limsup_{t\to\infty} f(t) = \infty \quad \textit{and} \quad \liminf_{t\to\infty} f(t) = 0.$$

It will be obvious from the construction of the function f in Theorem 3.2 that it does not have an equivalent monotone version. The proof of Theorem 3.2 is postponed to Section 6.

4. Sufficient conditions for equivalent monotone versions to exist

Theorem 4.1. *Let f be a function satisfying (2.1). Then, under conditions (2.2)–(2.4), there exists an equivalent monotone version $f^{\uparrow}$.*

The proof of this result is based on the two lemmas below. Moreover, in what follows we deal with a special equivalent monotone version, that is with

$$f^{\uparrow}(t) = \inf_{s\geq t} f(s). \tag{4.1}$$

Lemma 4.1. *Let f be a function satisfying (2.1). Then, under conditions (2.2) and (2.3), there is a $t_0 > 0$ such that*

$$f^{\uparrow}(t) \geq \inf_{t\leq s\leq c_0 t} f(s) \quad \text{for all } t \geq t_0. \tag{4.2}$$

Remark 4.1. Inequality (4.2), in fact, is an equality, since the upper bound is obvious.

Proof of Lemma 4.1. Due to conditions (2.2)–(2.3), there are numbers $r > 1$ and $t_0 > 0$ such that

$$f(c_0 t) \geq r f(t) \quad \text{for all } t \geq t_0.$$

Moreover, for $t \geq t_0$,

$$\begin{aligned} f^{\uparrow}(t) &= \min\left\{\inf_{t\leq s\leq c_0 t} f(s), \inf_{s\geq c_0 t} f(s)\right\} = \min\left\{\inf_{t\leq s\leq c_0 t} f(s), \inf_{s\geq t} f(c_0 s)\right\} \\ &= \min\left\{\inf_{t\leq s\leq c_0 t} f(s), \inf_{s\geq t} f(s)\frac{f(c_0 s)}{f(s)}\right\} \geq \min\left\{\inf_{t\leq s\leq c_0 t} f(s), r\inf_{s\geq t} f(s)\right\} \\ &= \min\left\{\inf_{t\leq s\leq c_0 t} f(s), r f^{\uparrow}(t)\right\} = \inf_{t\leq s\leq c_0 t} f(s). \end{aligned}$$

Therefore,

$$f^{\uparrow}(t) \geq \inf_{t\leq s\leq c_0 t} f(s) \quad \text{for all } t \geq t_0,$$

which completes the proof. □

Lemma 4.2. *Let f be a function satisfying (2.1). Then, under conditions (2.2) and (2.4),*

$$\liminf_{t\to\infty} \inf_{1\leq c\leq C} \frac{f(ct)}{f(t)} \geq 1 \tag{4.3}$$

for all $C \geq 1$.

Proof of Lemma 4.2. If the required property does not hold, then there are a constant $C \geq 1$ and two sequences $\{c_n\}$ and $\{t_n\}$ such that $1 \leq c_n \leq C$, $t_n \to \infty$, and

$$\lim_{n\to\infty} \frac{f(c_n t_n)}{f(t_n)} < 1. \tag{4.4}$$

Choose a monotone convergent subsequence of $\{c_n\}$ and denote its limit by c'. Without loss of generality we assume that the whole sequence $\{c_n\}$ converges to c'. Then

$$\liminf_{n\to\infty} \frac{f(c_n t_n)}{f(t_n)} \geq \liminf_{n\to\infty} \frac{f(c_n t_n)}{f(c' t_n)} \cdot \liminf_{n\to\infty} \frac{f(c' t_n)}{f(t_n)} \geq \liminf_{n\to\infty} \frac{f(c_n t_n)}{f(c' t_n)}$$

by condition (2.2). On setting $\tau_n = c' t_n$ and $d_n = c_n/c'$, we have $\tau_n \to \infty$ and $d_n \downarrow 1$. Thus condition (2.4) implies that

$$\liminf_{n\to\infty} \frac{f(c_n t_n)}{f(t_n)} \geq \liminf_{n\to\infty} \frac{f(d_n \tau_n)}{f(\tau_n)} \geq 1.$$

This contradicts (4.4) and completes the proof. □

Proof of Theorem 4.1. By Lemma 4.1, there is a $t_0 > 0$ such that

$$\frac{f^{\uparrow}(t)}{f(t)} \geq \inf_{t\leq s\leq c_0 t} \frac{f(s)}{f(t)} = \inf_{1\leq c\leq c_0} \frac{f(ct)}{f(t)} \quad \text{for all } t \geq t_0,$$

where $f^{\uparrow}$ is defined by (4.1). Passing to the lim inf as $t \to \infty$ and then applying Lemma 4.2 we complete the proof, since the upper bound

$$\limsup_{t\to\infty} \frac{f^{\uparrow}(t)}{f(t)} \leq 1$$

is obvious. □

Corollary 4.1. *Let f be a PRV function, that is, let (1.2) and (1.3) hold. Then an equivalent monotone version $f^{\uparrow}$ exists.*

Corollary 4.2. *Let f be a regularly varying function of positive index. Then an equivalent monotone version $f^{\uparrow}$ exists.*

5. Equivalent monotone versions

In this section, we study the role of an asymptotic form of (4.3) for the existence of an equivalent monotone version to a given positive function f. Let

$$\chi_f = \lim_{c\downarrow 1} \liminf_{t\to\infty} \inf_{1\leq\lambda\leq c} \frac{f(\lambda t)}{f(t)}. \tag{5.1}$$

We want to decide about the existence of an equivalent monotone version just via the single number χ_f. Note that $\chi_f \leq 1$ for any positive function f and

$$\chi_f = 1 \tag{5.2}$$

if f is increasing. Moreover, (5.2) is necessary for f to have an equivalent monotone version.

Theorem 5.1. *If $\chi_f < 1$, then there is no equivalent monotone version $f^{\uparrow}$.*

Proof of Theorem 5.1. Since $\chi_f < 1$, there are a number $q \in (0, 1)$ and two sequences $\{\lambda_n\}$ and $\{t_n\}$ such that $\lambda_n \downarrow 1$, $t_n \to \infty$, and

$$\frac{f(\lambda_n t_n)}{f(t_n)} \leq q \quad \text{for all } n \geq 1.$$

If a monotone version $f^{\uparrow}$ exists, then, in particular,

$$f(t_n) \sim f^{\uparrow}(t_n) \quad \text{and} \quad f(\lambda_n t_n) \sim f^{\uparrow}(\lambda_n t_n).$$

On the other hand,

$$\frac{f^{\uparrow}(\lambda_n t_n)}{f(\lambda_n t_n)} = \frac{f^{\uparrow}(\lambda_n t_n)}{f(t_n)} \cdot \frac{f(t_n)}{f(\lambda_n t_n)} \geq \frac{1}{q} \cdot \frac{f^{\uparrow}(\lambda_n t_n)}{f(t_n)} \geq \frac{1}{q} \cdot \frac{f^{\uparrow}(t_n)}{f(t_n)} \quad \text{for all } n \geq 1$$

in view of the monotonicity of $f^{\uparrow}$. Letting $n \to \infty$, we have a contradiction, which completes the proof. □

It turns out that one can drop condition (2.4) if f is measurable and the (necessary) condition (5.2) holds.

Theorem 5.2. *Assume that f is a measurable and positive real function satisfying conditions (2.2) and (2.3). Then, if $\chi_f = 1$, there exists an equivalent monotone version $f^{\uparrow}$. Moreover, one can choose*

$$f^{\uparrow}(t) = \inf_{s\geq t} f(s).$$

For the proof of Theorem 5.2 we need the following result whose proof is postponed to Section 7.

Lemma 5.1. *If f is measurable and (2.2) holds, then, for all $1 < a < b < \infty$,*

$$\liminf_{t\to\infty} \inf_{a\leq c\leq b} \frac{f(ct)}{f(t)} \geq 1. \tag{5.3}$$

Proof of Theorem 5.2. By Lemma 4.1, there is a $t_0 > 0$ such that

$$f^{\uparrow}(t) \geq \inf_{t \leq s \leq c_0 t} f(s) \quad \text{for all } t \geq t_0.$$

Thus, for all $1 < c < c_0$ and $t \geq t_0$,

$$f^{\uparrow}(t) \geq \min\left\{ \inf_{t \leq s \leq ct} f(s), \inf_{ct \leq s \leq c_0 t} f(s) \right\}.$$

Hence, by Lemma 5.1,

$$\liminf_{t\to\infty} \frac{f^{\uparrow}(t)}{f(t)} \geq \min\left\{ \liminf_{t\to\infty} \inf_{1\leq\lambda\leq c} \frac{f(\lambda t)}{f(t)}, 1 \right\}.$$

Letting $c \downarrow 1$, we get

$$\liminf_{t\to\infty} \frac{f^{\uparrow}(t)}{f(t)} \geq \min\left\{ \chi_f, 1 \right\} = 1,$$

since $\chi_f = 1$. On the other hand, by the definition of $f^{\uparrow}$,

$$\limsup_{t\to\infty} \frac{f^{\uparrow}(t)}{f(t)} \leq 1,$$

which completes the proof. □

6. Proof of Theorem 3.2

Let $t_1 = 1$ and set, for $k \geq 1$,

$$t_{k+1} = e^{2k^2} t_k,$$
$$\tau_k = e^{k^2} t_k.$$

Define the function β on $(0, \infty)$ as follows: for $0 < s < 1$, β is such that

$$\int_0^1 \frac{\beta(s)}{s}\, ds = 1,$$

e.g., $\beta(s) = s$ on $(0, 1)$. Otherwise the function β is defined by

$$\beta(s) = \begin{cases} \beta_k^{(1)}, & \text{for } t_k \leq s < \tau_k, \\ \beta_k^{(2)}, & \text{for } \tau_k \leq s < t_{k+1}, \end{cases} \qquad k \geq 1,$$

where the sequences $\{\beta_k^{(1)}\}$ and $\{\beta_k^{(2)}\}$ are determined by the conditions

$$\int_0^{t_k} \frac{\beta(s)}{s}\, ds = k, \qquad \int_0^{\tau_k} \frac{\beta(s)}{s}\, ds = -k, \quad k \geq 2.$$

This immediately implies that

$$\int_{t_k}^{\tau_k} \frac{\beta(s)}{s}\, ds = -2k, \qquad \int_{\tau_k}^{t_{k+1}} \frac{\beta(s)}{s}\, ds = 2k+1,$$

or, equivalently,

$$\beta_k^{(1)} \ln \frac{\tau_k}{t_k} = -2k, \qquad \beta_k^{(2)} \ln \frac{t_{k+1}}{\tau_k} = 2k+1,$$

whence

$$\beta_k^{(1)} = -\frac{2}{k}, \qquad \beta_k^{(2)} = \frac{2k+1}{k^2}.$$

Thus

$$\lim_{s\to\infty} \beta(s) = 0.$$

Finally, set

$$f(t) = \exp\left\{ \int_0^t \frac{\beta(s)}{s}\, ds \right\}, \quad t \geq t_1.$$

By the integral representation theorem for slowly varying functions, f is slowly varying (see Theorem 1.2 in Seneta [14]). Moreover,

$$\lim_{k\to\infty} f(t_k) = \infty \quad \text{and} \quad \lim_{k\to\infty} f(\tau_k) = 0,$$

but obviously f does not have an equivalent monotone version, since, by monotonicity, one would necessarily have that $\lim_{k\to\infty} f^{\uparrow}(\tau_k) = \infty$, so there can be no equivalence with f for this subsequence. □

7. Proof of Lemma 5.1

In order to prove Lemma 5.1 we need a more general result, i.e. Lemma 7.1 below. In the modern theory of regular variation, analogs of Lemma 7.1 are used in the proofs of two fundamental results of this theory, namely for the proof of the uniform convergence theorem and the representation theorem. The classical variant of Lemma 7.1 deals with the case where the limit of the corresponding ratio exists, while we require only a lower bound for the lim inf. Several proofs of the classical result are known (see, e.g., Bingham, Goldie, and Teugels [3], Section 1.2). We follow an idea of an earlier proof due to Korevaar, Ardenne-Ehrenfest and de Bruijn [13] and apply it to a more general case.

Lemma 7.1. *Let f be a measurable function. Assume that, for some $0 < \ell < \infty$,*

$$\liminf_{t\to\infty} \frac{f(ct)}{f(t)} \ge \ell \quad \textit{for all } c > 1. \tag{7.1}$$

Then

$$\liminf_{n\to\infty} \frac{f(c_n t_n)}{f(t_n)} \ge \ell^2 \tag{7.2}$$

for every sequence $\{t_n\}$ such that $t_n \to \infty$ and every sequence $\{c_n\}$ such that $\liminf_{n\to\infty} c_n > 1$ and $\limsup_{n\to\infty} c_n < \infty$.

Moreover, if

$$\liminf_{t\to\infty} \frac{f(ct)}{f(t)} = \infty \quad \textit{for all} \quad c > 1, \tag{7.3}$$

then

$$\liminf_{t\to\infty} \frac{f(c_n t_n)}{f(t_n)} = \infty \tag{7.4}$$

for all sequences $\{t_n\}$ and $\{c_n\}$ possessing the same properties as in the case of $\ell < \infty$. (Of course, if (7.3) holds, then “lim inf” *can be replaced by* “lim” *in both equations (7.3) and (7.4)).*

Remark 7.1. If $\ell > 1$ in (7.1), then $f_*(c) = \infty$ for all $c > 1$. Indeed, (7.1) can be rewritten as $f_*(c) \ge \ell$ for all $c > 1$. Lemma 7.1 implies (7.2) and one can use ℓ^2 instead of ℓ on the right-hand side of (7.1), i.e. $f_*(c) \ge \ell^2$ for all $c > 1$. Repeating this argument we arrive at the conclusion that $f_*(c) = \infty$.

Proof of Lemma 7.1. Put $\nu = \inf_{c>1} f_*(c)$. Note that either $\nu = \infty$ or $0 \le \nu \le 1$. Indeed, assume that $0 \le \nu < \infty$. Since $f_*(c_1)f_*(c_2) \le f_*(c_1c_2)$ for all $c_1 > 1$ and $c_2 > 1$, we have $\nu^2 \le \nu$, whence $\nu \le 1$.

First we consider the case of $0 < \ell < \infty$. In what follows, we switch to an “additive” notation instead of the “multiplicative” one above. Namely we set $h(t) = \ln f(e^t)$ and deal with h rather than f in the following. The assumptions of Lemma 7.1 can be reformulated as follows: h is measurable and

$$\liminf_{x\to\infty} \big(h(x+u) - h(x)\big) \ge \ln \ell \quad \text{for all } u > 0.$$

We want to prove that

$$\liminf_{n\to\infty} \big(h(x_n+u_n) - h(x_n)\big) \ge 2\ln \ell$$

for every sequence $\{x_n\}$ such that $x_n \to \infty$ and every sequence $\{u_n\}$ such that $\liminf_{n\to\infty} u_n > 0$ and $\limsup_{n\to\infty} u_n < \infty$.

Fix sequences $\{x_n\}$ and $\{u_n\}$ with the above properties. Note that there are numbers $\delta > 0$ and $\Delta < \infty$ for which

$$\delta \le u_n \le \Delta \quad \text{for all sufficiently large } n.$$

Without loss of generality we assume that this holds for all $n \ge 1$. Let $0 < \varkappa < \delta$ and

$$\delta_1 = \delta - \varkappa, \qquad \delta_2 = \Delta - \frac{\varkappa}{2}, \qquad \varepsilon_1 = \varkappa, \qquad \varepsilon_2 = \frac{\varkappa}{2}.$$

Obviously $\delta_1 < \delta_2$ and $\varepsilon_1 > \varepsilon_2$. Fix $-\infty < \gamma < \ln \ell$ and define the sets

$$A_n = \{u \in [\delta_1, \delta_2] : h(x_k + u) - h(x_k) \geq \gamma \text{ for all } k \geq n\},$$
$$B_n = \{v \in [-\varepsilon_1, -\varepsilon_2] : h(x_k + u_k) - h(x_k + u_k + v) \geq \gamma \text{ for all } k \geq n\}.$$

It is obvious that $A_n \subseteq A_{n+1}$ and $B_n \subseteq B_{n+1}$. Moreover

$$\bigcup_{n=1}^{\infty} A_n = [\delta_1, \delta_2], \qquad \bigcup_{n=1}^{\infty} B_n = [-\varepsilon_1, -\varepsilon_2],$$

whence

$$|A_n| \to \delta_2 - \delta_1, \qquad |B_n| \to \varepsilon_1 - \varepsilon_2$$

as $n \to \infty$, where $|\cdot|$ stands for the Lebesgue measure. Now we introduce the sets $B_n' = B_n + u_n$ and note that $|B_n'| = |B_n|$ for each $n \geq 1$. Moreover,

$$A_n \subseteq [\delta_1, \delta_2], \qquad B_n' \subseteq [-\varepsilon_1 + \delta, -\varepsilon_2 + \Delta],$$

so that

$$A_n \cup B_n' \subseteq [\delta_1, \delta_2].$$

Since

$$|A_n| + |B_n'| \to \delta_2 - \delta_1 + \varepsilon_1 - \varepsilon_2 > \delta_2 - \delta_1,$$

we get that $A_n \cap B_n' \neq \varnothing$ for sufficiently large n. For such an integer n, let $u_0 \in A_n \cap B_n'$. According to the definition of the set B_n', this means that there is a $v_0 \in B_n$ such that $v_0 = u_0 - u_n$ for some $u_0 \in A_n$. Therefore, with this n,

$$h(x_n + u_0) - h(x_n) \geq \gamma, \qquad h(x_n + u_n) - h(x_n + u_n + v_0) \geq \gamma,$$

or, equivalently,

$$h(x_n + u_0) - h(x_n) \geq \gamma, \qquad h(x_n + u_n) - h(x_n + u_0) \geq \gamma,$$

whence

$$h(x_n + u_n) - h(x_n) \geq 2\gamma$$

and

$$\liminf_{n\to\infty} \big(h(x_n + u_n) - h(x_n)\big) \geq 2\gamma.$$

Since $\gamma < \ln \ell$ is arbitrary, the lemma is proved for the case of $0 < \ell < \infty$. To prove that (7.3) implies (7.4), we note that (7.1) holds for every $\ell > 1$, which results in (7.2) by the first part of the proof. So, since ℓ is arbitrary, (7.4) follows from (7.3). $\square$

Proof of Lemma 5.1. If (5.3) does not hold for some $1 < a < b < \infty$, then there exist sequences $t_n \uparrow \infty$ and $a \leq c_n \leq b$ such that

$$\liminf_{t\to\infty} \frac{f(c_n t_n)}{f(t_n)} < 1,$$

which contradicts Lemma 7.1. $\square$

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