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Existence theorems for the initial value problem of the cometary flow equation with an external force☆

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ABSTRACT

The initial value problem of the cometary flow equation with a given external force is investigated. By assuming that the initial microscopic density has finite mass and finite momentum and belongs to L^p for some $p > 1$, three existence results of weak solutions with mass conservation and local estimates for the kinetic energy are established for different external forces, each of which is assumed to be divergence free with respect to particle velocities. The first result deals with a bounded smooth force and a Lorentz force with bounded smooth electric and magnetic intensities, and the second one concerns a force belonging to L^q with $\frac{1}{p} + \frac{1}{q} = 1$. In the third theorem, we discuss a force that can be divided into two parts: one is in L^q and the other is linearly growing at infinity; in this case we need to assume further that the initial density has finite first order spatial moment.

1. Introduction

In this paper, we consider the Cauchy problem for a kinetic equation of the form

$$\begin{cases} \partial_t f + \xi \cdot \nabla_x f + F(t,x,\xi) \cdot \nabla_\xi f = Q_{u_f}(f), \\ f(0,x,\xi) = f_0(x,\xi) \geq 0. \end{cases} \tag{1.1}$$

This nonlinear partial differential equation is an important mathematical model derived from the theory of astrophysical plasmas, and provides a statistical description of a cometary flow in light of its microscopic density $f(t,x,\xi)$ depending upon position $x \in \mathbb{R}^3$, velocity $\xi \in \mathbb{R}^3$ and time $t \geq 0$ (see, e.g., [12,22–24]). In this connection, (1.1) is commonly known as the cometary flow equation.

Here $F(t,x,\xi)$ stands for an external force imposed on the cometary flow, and the collision operator $Q_{u_f}(f)$ describing wave–particle interactions in the cometary flow, is nonlinearly related to the unknown f through its velocity moments, namely

$$Q_{u_f}(f) = P_{u_f}(f)(t,x,\xi) - f(t,x,\xi),$$

$$P_{u_f}(f) = \begin{cases} \dfrac{1}{4\pi} \displaystyle\int_{\mathbb{S}^2} f(t,x,u_f + |\xi - u_f|\omega)d\omega, & \rho_f \neq 0, \\ 0, & \rho_f = 0, \end{cases}$$

where $\mathbb{S}^2$ designates the unit sphere of $\mathbb{R}^3$, and where the macroscopic density $\rho_f(t,x)$ and the bulk velocity $u_f(t,x)$

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corresponding to f are defined by

$$\begin{pmatrix} \rho_f \\ \rho_f u_f \end{pmatrix}(t,x) = \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ \xi \end{pmatrix} f(t,x,\xi)d\xi, \quad t \geq 0,\ x \in \mathbb{R}^3$$

respectively. It is not to hard to see that the nonlinear part $P_{u_f}(f)$ of the collision operator is in fact a nonlinear projection, so the mathematical structure of Eq. (1.1) is quite similar to that of the Boltzmann–BGK equation which is widely used in gas dynamics (see, e.g., [2,20,26]). Nevertheless, the nonlinearity of $Q_{u_f}(f)$ is obviously much stronger than that of the classical Boltzmann–BGK collision operator. Again, there is a significant difference between the wave–particle interaction $Q_{u_f}(f)$ and the Boltzmann–BGK collision operator: the former has infinitely many collision invariants (see, e.g., [7,5]), while the latter possesses exactly five [2]. This means that they have different sets of equilibria which possibly lead to different large time asymptotic behaviors. As for fundamental properties of the wave–particle interaction operator $Q_{u_f}(f)$ and their proofs, we refer the readers to [7,5,6,14,13,15] (see also Section 3).

Due to the significance of this kinetic equation in the theory of astrophysics, some researchers have been devoted themselves to establishing its rigorous theory. In the absence of an external field (i.e., $F(t,x,\xi) \equiv 0$), qualitative treatments in terms of nonnegative weak solutions have been received a great deal of attention both for Cauchy problems and for initial boundary value problems, various results including global existence with conservation laws and entropy dissipation, propagation of higher order moments, perturbation theory of global equilibria and large time behavior, were well established (see, e.g., [6,14,13,15]).

Recently, the Cauchy problem (1.1) with a force field satisfying certain integrability conditions was discussed in Ref. [3]. Specifically, assuming that the external force $F(t,x,\xi)$ belongs to $L^q((0,\ T) \times \mathbb{R}^3_x \times \mathbb{R}^3_\xi)$ and is divergence free with respect to the velocity variable ξ, assuming further that the initial density verifies $(1+|\xi|^2)f_0(x,\xi) \in L^1(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$ and $f_0(x,\xi) \in L^p(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)$, finally assuming that the integrability exponents verify $p,\ q > 1$ and $\frac{1}{p} + \frac{1}{q} < 1$, then it was shown in [3] that the Cauchy problem (1.1) has a nonnegative weak solution $f(t,x,\xi)$ such that $\|f(t)\|_1 = \|f_0\|_1$ and $\|f(t)\|_p \leq \|f_0\|_p$ for $t \in [0,\ T]$. An important case was also treated in Ref. [3] namely $F(t,x,\xi) = E(t,x) + \xi \times B(t,x)$ being a Lorentz field, where $E(t,x)$ and $B(t,x)$ are given electric and magnetic intensities respectively. Then, (1.1) can be written as

$$\begin{cases} \partial_t f + \xi \cdot \nabla_x f + [E(t,x) + \xi \times B(t,x)] \cdot \nabla_\xi f = Q_{u_f}(f), \\ f(0,x,\xi) = f_0(x,\xi) \geq 0. \end{cases} \tag{1.2}$$

It was shown that if $E(t,x) \in L^q((0,\ T) \times \mathbb{R}^3)$, $B(t,x) \in L^{p'}((0,\ T) \times \mathbb{R}^3)$ and $f_0 \in L^p(\mathbb{R}^3_x \times \mathbb{R}^3_x)$ with $p > 1$ and $q > 3 + p'$ and if $(1+|\xi|^2)f_0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$, then there exists a nonnegative weak solution to (1.2) in the function space $L^\infty((0,\ T);\ L^p(\mathbb{R}^3_x \times \mathbb{R}^3_\xi))$. Here and in the following of this paper, we denote the conjugate exponent of p by p', i.e., $p' = \frac{p}{p-1}$ for $1 < p \leq \infty$ and $p' = \infty$ for $p = 1$.

Scrutinizing those results, one can find from a mathematical point of view that they are far from optimal and may be improved at least in two ways. First, the definition of the equation do not require finiteness of second order velocity moment, as a matter of fact, the finiteness of velocity moment of order one is sufficient for the description of the wave–particle collision operator $Q_{u_f}(f)$. Second, to define the nonlinear terms $F(t,x,\xi) \cdot \nabla_\xi f$ and $[E(t,x)+\xi \times B(t,x)] \cdot \nabla_\xi f$ in distributional sense, it is sufficient, for example, to ask for $f(t,x,\xi) \in L^\infty_{\text{loc}}((0,\ T);\ L^p_{\text{loc}}(\mathbb{R}^3_x \times \mathbb{R}^3_\xi))$ and $F(t,x,\xi) \in L^{p'}_{\text{loc}}((0,\ T) \times \mathbb{R}^3_x \times \mathbb{R}^3_\xi)$ or $E(t,x),\ B(t,x) \in L^{p'}_{\text{loc}}((0,\ T) \times \mathbb{R}^3)$. These observations directly motivate our present discussion. In fact, we shall establish in this paper three global existence results, each of which only requires that the first order velocity (space) moment of the initial density is finite. The first one concerns a general smooth force and a smooth Lorentz field (Theorem 2.1), the rest two are aimed at improving global existence results concerning integrable force fields described above (see also Theorems 3.2 and 3.3 in Ref. [3]), especially, we shall prove the critical case $\frac{1}{p} + \frac{1}{q} = 1$ and also consider force fields growing linearly at infinity (Theorems 2.2 and 2.3). As a consequence, we obtain a new existence theorem for the Cauchy problem (1.2) concerning a different kind of Lorentz field (Corollary 2.4).

Concerning the cometary flow equation (1.1), there is another kind of interesting problem, namely the Cauchy or the initial boundary value problem with a self-consistent force field $F(t,x,\xi)$. The Cauchy problem with a self-generated electrostatic field and the Cauchy problem with a self-consistent Lorentz field were studied carefully in [25,3] and in [4] respectively. Nevertheless, the corresponding initial boundary value problem has not been investigated so far. Before going further, we first present the definition of weak solutions.

Definition 1.1. Let T be a given positive constant, a nonnegative function $f(t,x,\xi) \in L^\infty((0,\ T);\ L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ is said to be a weak solution on $[0,\ T]$ to (1.1) if the product fF is well-defined in distributional sense, e.g., $fF \in L^1_{\text{loc}}((0,\ T) \times \mathbb{R}^3 \times \mathbb{R}^3)$, and if $f(t,x,\xi)$ verifies

$$\int_0^T dt \int_{\mathbb{R}^3 \times \mathbb{R}^3} f\left[\partial_t \phi + \xi \cdot \nabla_x \phi + F \cdot \nabla_\xi \phi\right] dxd\xi + \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_0 \phi|_{t=0} dxd\xi = -\int_0^T dt \int_{\mathbb{R}^3 \times \mathbb{R}^3} Q_{u_f}(f)\phi dxd\xi \tag{1.3}$$

for any test function $\phi(t,x,\xi) \in C^1_c([0,\ T) \times \mathbb{R}^3 \times \mathbb{R}^3)$.

Weak solutions to the Cauchy problem (1.2) can be defined in the same way. As explained above our goal in this paper is to establish various existence results of weak solutions for the Cauchy problems (1.1) and (1.2). Nevertheless, whether there is a classical solution to the Cauchy problem (1.1) or (1.2) has not been known so far, even if the initial density and the force field are sufficiently smooth.

The outline of the remainder of this paper is as follows. In Section 2, we present the main results of this paper and prove a corollary concerning global existence of weak solutions to the Cauchy problem (1.2). In Section 3, we summarize main properties of the collision operator $Q_{u_f}(f)$ and some auxiliary tools including a L^1 velocity averaging lemma, all of which are standard and can be found in literature. Then, we discuss a linear problem and a nonlinear approximate problem, the main results are global existence of weak solutions to the approximate problem and their desired estimates. Section 4 contains the proof of the main Theorems.

2. Main results

In the rest of this paper, we shall denote by $C_b^1(\mathbb{R}^N)$ the set consisting of all functions having bounded continuous derivatives on $\mathbb{R}^N$ up to order one, the symbol $K_x \subset\subset \mathbb{R}_x^3$ means that K_x is a compact subset of $\mathbb{R}_x^3$. Again, in order to simplify our presentation, we shall use the following shorthand notations for various norms:

$$\|f(t)\|_p = \|f(t,\cdot,\cdot)\|_{L^p(\mathbb{R}_x^3\times\mathbb{R}_\xi^3)}, \qquad \|f_0\|_p = \|f_0(\cdot,\cdot)\|_{L^p(\mathbb{R}_x^3\times\mathbb{R}_\xi^3)},$$
$$\|F(t)\|_p = \|F(t,\cdot,\cdot)\|_{L^q(\mathbb{R}_x^3\times\mathbb{R}_\xi^3)}, \qquad \|F\|_q = \|F(\cdot,\cdot,\cdot)\|_{L^q((0,\ T)\times\mathbb{R}_x^3\times\mathbb{R}_x^3)},$$
$$\|E(t)\|_q = \|E(t,\cdot)\|_{L^q(\mathbb{R}_x^3)}, \qquad \|E\|_q = \|E(\cdot,\cdot)\|_{L^q((0,\ T)\times\mathbb{R}_x^3)},$$
$$\|B(t)\|_q = \|B(t,\cdot)\|_{L^q(\mathbb{R}_x^3)}, \qquad \|B\|_q = \|B(\cdot,\cdot)\|_{L^q((0,\ T)\times\mathbb{R}_x^3)},$$

where, e.g., $f(t,x,\xi) \in L^\infty((0,\ T);\ L^p(\mathbb{R}_x^3\times\mathbb{R}_\xi^3))\ (1 \le p \le \infty),\ F(t,x,\xi) \in L^q((0,\ T)\times\mathbb{R}_x^3\times\mathbb{R}_\xi^3)$ and $E(t,x),\ B(t,x) \in L^q((0,\ T)\times\mathbb{R}_x^3)\ (1 \le q \le \infty)$.

Our first result concerns a smooth force field and a smooth Lorentz field, namely $F(t,x,\xi) \in C([0,\ T];\ C_b^1(\mathbb{R}_x^3\times\mathbb{R}_\xi^3))$ and $E(t,x), B(t,x) \in C([0,\ T];\ C_b^1(\mathbb{R}^3))$, which is a generalization of Lemma 3.1 in Ref. [3] in the sense that we assume that the initial datum only has finite velocity moment of order one, rather than order two. This result is not only meaningful in itself but also the cornerstone of our next two theorems.

Theorem 2.1. *Let the initial microscopic density $f_0(x,\xi) \ge 0$ verify*

$$f_0 \in L^1\cap L^p(\mathbb{R}^3\times\mathbb{R}^3) \quad (p > 1), \qquad \int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi|f_0 dx d\xi < \infty. \tag{2.1}$$

(1) *If $F(t,x,\xi) \in C([0,\ T];\ C_b^1(\mathbb{R}_x^3\times\mathbb{R}_\xi^3))$ and $\nabla_\xi\cdot F(t,x,\xi) = 0$, then there exists a nonnegative weak solution $f(t,x,\xi)$ to the Cauchy problem (1.1) such that*

$$\|f(t)\|_1 = \|f_0\|_1, \qquad \|f(t)\|_p \le \|f_0\|_p, \quad 0 \le t \le T. \tag{2.2}$$

Furthermore, if $K_x \subset\subset \mathbb{R}_x^3$, then there is a positive constant C depending continuously upon the parameters diamK_x, $\|f_0\|_1$, $\||\xi|f_0\|_1$, T *and* $\|F\|_\infty$ *such that*

$$\int_0^T dt \int_{K_x\times\mathbb{R}^3} |\xi|^2 f dx d\xi \le C. \tag{2.3}$$

(2) *If $E(t,x) \in C([0,\ T];\ C_b^1(\mathbb{R}^3))$ and $B(t,x) \in C([0,\ T];\ C_b^1(\mathbb{R}^3))$, then there exists a nonnegative weak solution $f(t,x,\xi)$ to the Cauchy problem (1.2) such that*

$$\|f(t)\|_1 = \|f_0\|_1, \qquad \|f(t)\|_p \le \|f_0\|_p, \quad 0 \le t \le T. \tag{2.4}$$

Further, for any $K_x \subset\subset \mathbb{R}_x^3$, there is a positive constant C depending continuously upon the parameters diamK_x, $\|f_0\|_1$, $\||\xi|f_0\|_1$, T, $\|E\|$ *and* $\|B\|_\infty$ *such that*

$$\int_0^T dt \int_{K_x\times\mathbb{R}^3} |\xi|^2 f dx d\xi \le C. \tag{2.5}$$

Remark 2.1. (1) By the proofs given in the following sections, it is clear that the assumption $\nabla_\xi\cdot F = 0$ in conclusion (1) is not necessary for the existence of a nonnegative solution to (1.1). If we remove this assumption, the estimate (2.2) of the solution f should be replaced by

$$\|f(t)\|_1 \le \|f_0\|_1\cdot\exp\left(\int_0^t \|\nabla_\xi\cdot F(\tau)\|_\infty d\tau\right), \quad 0 \le t \le T,$$

and

$$\|f(t)\|_p \le \|f_0\|_p \cdot \exp\left(\frac{1}{p}\int_0^t \|\nabla_\xi \cdot F(\tau)\|_\infty d\tau\right), \quad 0 \le t \le T.$$

(2) On the other hand, if we further assume $\int_{\mathbb{R}^3\times\mathbb{R}^3} |x| f_0(x,\xi)dxd\xi < \infty$, then we can show that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} |x| f(t,x,\xi)dxd\xi \le \tilde{C}, \quad 0 \le t \le T,$$

where the constant $\tilde{C}$ depends only upon $\|f_0\|_1$, $\||\xi|f_0\|_1$, $\||x|f_0\|_1$, T and $\|F\|_\infty$, $\|\nabla_\xi \cdot F\|_\infty$ (or $\|E\|_\infty$, $\|B\|_\infty$). For details, see Remark 3.3.

The second result solves the global existence problem involving critical integrability exponents, namely, $f_0 \in L^p$, $F \in L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$. Again, we only assume that the initial density just has finite velocity moment of order one. Thus, it is a substantial improvement of Theorem 3.2 in Ref. [3]. We describe it as the following theorem.

Theorem 2.2. *Assume that the initial microscopic density $f_0(x,\xi) \ge 0$ verifies*

$$f_0 \in L^1 \cap L^p(\mathbb{R}^3\times\mathbb{R}^3) \quad (1 < p \le \infty), \qquad \int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi| f_0 dxd\xi < \infty. \tag{2.6}$$

Assume further that $F(t,x,\xi) \in L^{p'}((0,T)\times\mathbb{R}^3\times\mathbb{R}^3)$ such that $\nabla_\xi \cdot F = 0$ in distributional sense. Then there exists a nonnegative weak solution $f(t,x,\xi)$ to the Cauchy problem (1.1) such that

$$\|f(t)\|_1 = \|f_0\|_1, \qquad \|f(t)\|_p \le \|f_0\|_p, \quad 0 \le t \le T. \tag{2.7}$$

Moreover, for any given $K_x \subset\subset \mathbb{R}^3$, there is a positive constant C depending only upon T, $\||\xi|f_0\|_1$, $\|f_0\|_p$, $\|F\|_{p'}$ and diamK_x *such that*

$$\int_0^T dt \int_{K_x\times\mathbb{R}^3} |\xi|^2 f dxd\xi \le C. \tag{2.8}$$

Next, we deal with force fields that are not integrable at infinity. We suppose that the force field can be decomposed into two parts: the first part has the same integrability as the force field in Theorem 2.2, the second part is linearly growing at infinity. To be specific, we establish the following theorem.

Theorem 2.3. *Assume that the initial microscopic density $f_0(x,\xi) \ge 0$ verifies*

$$f_0 \in L^1 \cap L^p(\mathbb{R}^3\times\mathbb{R}^3) \quad (1 < p < \infty), \qquad \int_{\mathbb{R}^3\times\mathbb{R}^3} (|x| + |\xi|) f_0 dxd\xi < \infty. \tag{2.9}$$

Let $\nabla_\xi \cdot F(t,x,\xi) = 0$ in distributional sense and let $F(t,x,\xi) = F_1 + F_2$, where $F_1 \in L^{p'}((0,T)\times\mathbb{R}^3\times\mathbb{R}^3)$, $\frac{F_2}{1+|x|+|\xi|} = M(t,x,\xi) \in L^\infty((0,T); L^\infty(\mathbb{R}^3\times\mathbb{R}^3))$. Then there exists a nonnegative weak solution $f(t,x,\xi)$ to the Cauchy problem (1.1) such that

$$\|f(t)\|_1 = \|f_0\|_1, \qquad \|f(t)\|_p \le \|f_0\|_p, \quad 0 \le t \le T. \tag{2.10}$$

Moreover, for any given $K_x \subset\subset \mathbb{R}^3$, there is a positive constant C depending only upon T, f_0, F_1, M and diamK_x *such that*

$$\int_0^T dt \int_{K_x\times\mathbb{R}^3} |\xi|^2 f dxd\xi \le C. \tag{2.11}$$

Notice that in Theorem 2.3 we assume that the initial density not only has finite velocity moment of order one but also has finite space moment of the same order. From Theorem 2.3, we can deduce a global existence result of the Cauchy problem (1.2) with a Lorentz field.

Corollary 2.4. *Suppose that the initial microscopic density $f_0(x,\xi) \ge 0$ verifies*

$$f_0 \in L^1 \cap L^p(\mathbb{R}^3\times\mathbb{R}^3) \quad (p > 1), \qquad \int_{\mathbb{R}^3\times\mathbb{R}^3} (|x| + |\xi|) f_0 dxd\xi < \infty.$$

Suppose also that $\frac{E(t,x)}{1+|x|} \in L^\infty((0,T); L^\infty(\mathbb{R}^3))$, $B(t,x) \in L^\infty((0,T); L^\infty(\mathbb{R}^3))$. Then there exists a nonnegative weak solution $f(t,x,\xi)$ to the Cauchy problem (1.2) such that

$$\|f(t)\|_1 = \|f_0\|_1, \qquad \|f(t)\|_p \le \|f_0\|_p, \quad 0 \le t \le T.$$

Conclusions in Corollary 2.4 are the same as those obtained by Theorem 3.3 in Ref. [3] where the authors assumed that the initial density f_0 has finite second order velocity moment and that the magnetic and electric intensities satisfy $B(t,x) \in L^{p'}([0,\ T] \times \mathbb{R}^3)$ and $E(t,x) \in L^q([0,\ T] \times \mathbb{R}^3)$ for $q > 3 + p'$ respectively. Here, we assume that f_0 has finite first order moment both in velocity variable and in space variable and that the electric intensity $E(t,x)$ can be linearly growing at infinity. Therefore, Corollary 2.4 can be served as a counterpart of Theorem 3.3 in Ref. [3].

Proof of Corollary 2.4. It is sufficient to take $F_1(t,x,\xi) \equiv 0,\ F_2(t,x,\xi) = E(t,x) + \xi \times B(t,x)$. Then

$$\begin{aligned}\frac{|F_2(t,x,\xi)|}{1+|x|+|\xi|} &\le \frac{|E(t,x)| + |\xi \times B(t,x)|}{1+|x|+|\xi|}\\ &\le \frac{|E(t,x)|}{1+|x|} + |B(t,x)| \in L^\infty([0,\ T];\ L^\infty(\mathbb{R}^3)).\end{aligned}$$

Consequently, $F_1 \in L^{p'}((0,\ T) \times \mathbb{R}^3 \times \mathbb{R}^3),\ \frac{F_2}{1+|x|+|\xi|} \in L^\infty((0,\ T);\ L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$. $\square$

3. Preliminary results

3.1. About the collision operator and the velocity averaging lemma

First, we summarize main properties of the collision operator $Q_{u_f}(f)$. Define the linear collision operator $Q_u(f)$ for a given $u \in \mathbb{R}^3$ as follows: for any function $f(\xi) \in L^1(\mathbb{R}^3)$,

$$Q_u(f)(\xi) = P_u(f)(\xi) - f(\xi), \qquad P_u(f)(\xi) = \frac{1}{4\pi}\int_{\mathbb{S}^2} f(u + |\xi - u|\omega)d\omega.$$

Then, we have (for details, see, e.g., [3,4,7,5,6,14])

Lemma 3.1. *Let $f(\xi),\ g(\xi) \in L^1(\mathbb{R}^3)$ be nonnegative functions and ψ be a measurable function on $(0,\ \infty)$, all of which are assumed to be regular enough to ensure the existence of the following integrals. Then*

(1) *$P_u(f)$ is a projector: $P_u^2(f) = P_u(f)$.*
(2) *$Q_u(f)$ is symmetric: $\int_{\mathbb{R}^3} Q_u(f)gd\xi = \int_{\mathbb{R}^3} Q_u(g)fd\xi = -\int_{\mathbb{R}^3} Q_u(f)Q_u(g)d\xi$.*
(3) *Collision invariants: $\int_{\mathbb{R}^3} \xi Q_u(f)(\xi)d\xi = \int_{\mathbb{R}^3} \psi(|\xi - u|)Q_u(f)(\xi)d\xi = 0$.*
(4) *$Q_u(f) = 0$ if and only if there exist $u \in \mathbb{R}^3$ and a function F defined on $[0,\ \infty)$ such that $f(\xi) = F(|\xi - u|^2)$.*
(5) *H-theorem: $\int_{\mathbb{R}^3} Q_u(f)fd\xi = -\int_{\mathbb{R}^3} Q_u(f)Q_u(f)d\xi \le 0$.*

Lemma 3.2. (1) *Let $u(t,x),\ u_n(t,x)\ :\ (0,\ T) \times \mathbb{R}^3 \to \mathbb{R}^3$ be locally integrable functions such that $\lim_{n\to\infty} u_n = u$ in $L^1_{\mathrm{loc}}((0,\ T) \times \mathbb{R}^3)$, and let $f(t,x,\xi) \in L^q((0,\ T);\ L^p(\mathbb{R}^3 \times \mathbb{R}^3))$. Then, we have for $1 \le p,q \le \infty$,*

$$\|P_u(f)\|_{L^q((0,\ T);\ L^p(\mathbb{R}^3\times\mathbb{R}^3))} \le \|f\|_{L^q((0,\ T);\ L^p(\mathbb{R}^3\times\mathbb{R}^3))},$$

and for $1 \le p,q < \infty$,

$$\lim_{n\to\infty} \|P_{u_n}(f) - P_u(f)\|_{L^q((0,\ T);\ L^p(\mathbb{R}^3\times\mathbb{R}^3))} = 0.$$

(2) *Given $r \in [1,\ \infty)$ and a nonnegative function f such that $(1+|\xi|^r)f \in L^1(\mathbb{R}^3)$, then there exists a positive constant C_r such that*

$$\rho_f|u_f|^r \le \int_{\mathbb{R}^3} |\xi|^r fd\xi, \qquad \int_{\mathbb{R}^3} |\xi|^r P_{u_f}(f)d\xi \le C_r \int_{\mathbb{R}^3} |\xi|^r fd\xi.$$

Second, we discuss a very useful tool called velocity averaging lemma (see, e.g., [18,17,6,8–11,20,3,4]). We directly cite the following one which was recently proved in [3].

Lemma 3.3. *Let the sequence $\{F_n(t,x,\xi)\ :\ n = 1,2,\ldots\} \subset C([0,\ T];\ C_b^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3))$ be bounded in $L^q_{\mathrm{loc}}((0,\ T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ for a fixed $q > 1$ and satisfy $\nabla_\xi \cdot F_n = 0 (n = 1,2,\ldots)$. Suppose that the sequences $\{f_n\ :\ n = 1,2,\ldots\}$ is weakly compact in $L^1((0,\ T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ and $\{g_n\ :\ n = 1,2,\ldots\}$ is weakly compact in $L^1_{\mathrm{loc}}((0,\ T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ such that*

$$\partial_t f_n + \xi \cdot \nabla_x f_n + F_n(t,x,\xi) \cdot \nabla_\xi f_n = g_n \tag{3.1}$$

in distributional sense. Then for any bounded sequence $\{\psi_n(t,x,\xi)\ :\ n = 1,2,\ldots\} \subset L^\infty((0,\ T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ that converges almost everywhere to $\psi(t,x,\xi)$, the sequence

$$\int_{\mathbb{R}^3} f_n(t,x,\xi)\psi_n(t,x,\xi)d\xi, \quad n = 1,2,\ldots$$

is relatively compact in $L^1((0,\ T)\times\mathbb{R}^3)$. *Furthermore, if* $f_n(t,x,\xi)\to f(t,x,\xi)$ $(n\to\infty)$ *weakly in* $L^1((0,\ T)\times\mathbb{R}^3\times\mathbb{R}^3)$, *then up to a subsequence if necessary*

$$\int_{\mathbb{R}^3} f_n(t,x,\xi)\psi_n(t,x,\xi)d\xi \to \int_{\mathbb{R}^3} f(t,x,\xi)\psi(t,x,\xi)d\xi,\quad n\to\infty$$

strongly in $L^1((0,\ T)\times\mathbb{R}^3)$.

3.2. A linear Cauchy problem

In order to construct suitably approximate solutions for the Cauchy problems (1.1) and (1.2), we begin with a discussion of the following linear Cauchy problems.

$$\begin{cases}\partial_t f+\xi\cdot\nabla_x f+F(t,x,\xi)\cdot\nabla_\xi f=Q_u(f),\\ f(0,x,\xi)=f_0(x,\xi)\end{cases}\tag{3.2}$$

and

$$\begin{cases}\partial_t f+\xi\cdot\nabla_x f+[E(t,x)+\xi\times B(t,x)]\cdot\nabla_\xi f=Q_u(f),\\ f(0,x,\xi)=f_0(x,\xi),\end{cases}\tag{3.3}$$

where $u\in L^\infty((0,\ T)\times\mathbb{R}^3_x)$ is a given velocity field. The existence and uniqueness of a solution to (3.2) or (3.3) is a consequence of the Banach's fixed point theorem, just as the procedure given in Ref. [6]. Here, the major difficulty is to obtain a suitable uniform bound of the second order velocity moment. To this end, we improve the duality method (see, e.g., [16]) through selecting specific test functions.

Lemma 3.4. *Suppose that* T *is a fixed positive constant and that the initial microscopic density* $f_0(x,\xi)\ge 0$ *verifies*

$$f_0\in L^1\cap L^p(\mathbb{R}^3\times\mathbb{R}^3)\quad(p>1),\qquad \int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|f_0dxd\xi<\infty\tag{3.4}$$

and there exist constants $a,\ b>0$ *such that*

$$f_0(x,\xi)\ge a\exp(-b(x^2+\xi^2)),\quad x,\ \xi\in\mathbb{R}^3.\tag{3.5}$$

Furthermore, let $u\in L^\infty((0,\ T)\times\mathbb{R}^3_x)$ *and denote* $J=\|u\|_{L^\infty((0,\ T)\times\mathbb{R}^3_x)}$.

(1) *If* $F(t,x,\xi)\in C([0,\ T];\ C^1_b(\mathbb{R}^3_x\times\mathbb{R}^3_\xi))$ *and* $\nabla_\xi\cdot F(t,x,\xi)=0$, *then* (3.2) *possesses a unique nonnegative weak solution* $f\in L^\infty((0,\ T);\ L^1\cap L^p(\mathbb{R}^3\times\mathbb{R}^3))$ *such that*

$$\|f(t)\|_1=\|f_0\|_1,\qquad \|f(t)\|_p\le\|f_0\|_p,\quad 0\le t\le T.\tag{3.6}$$

Moreover, for any $K_x\subset\subset\mathbb{R}^3$, *there is a positive constant* C *depending continuously upon the parameters* $\mathrm{diam}K_x,\ \|f_0\|_1,\ \||\xi|f_0\|_1,\ T,\ \|F\|_\infty$ *and* J *such that*

$$\int_0^T dt\int_{K_x\times\mathbb{R}^3}|\xi|^2fdxd\xi\le C,\quad u_f\in L^\infty([0,\ T];\ L^1_{\mathrm{loc}}(\mathbb{R}^3)).\tag{3.7}$$

(2) *If* $E(t,x),\ B(t,x)\in C([0,\ T];\ C^1_b(\mathbb{R}^3))$, *then the Cauchy problem* (3.3) *possesses a unique nonnegative weak solution* $f\in L^\infty((0,\ T);\ L^1\cap L^p(\mathbb{R}^3\times\mathbb{R}^3))$ *such that*

$$\|f(t)\|_1=\|f_0\|_1,\qquad \|f(t)\|_p\le\|f_0\|_p,\quad 0\le t\le T.\tag{3.8}$$

Moreover, for any fixed $K_x\subset\subset\mathbb{R}^3$, *there is a positive constant* C *depending continuously upon the parameters* $\mathrm{diam}K_x,\ \|f_0\|_1,\ \||\xi|f_0\|_1,\ T,\ J,\ \|E\|_\infty$ *and* $\|B\|_\infty$ *such that*

$$\int_0^T dt\int_{K_x\times\mathbb{R}^3}|\xi|^2fdxd\xi\le C,\quad u_f\in L^\infty([0,\ T];\ L^1_{\mathrm{loc}}(\mathbb{R}^3)).\tag{3.9}$$

Proof. We give a detail proof of conclusion (1) and a sketchy proof of conclusion (2).

Step 1. *Existence and uniqueness.* For any given $(t,x,\xi)\in[0,\ T]\times\mathbb{R}^3\times\mathbb{R}^3$, it follows from $F(t,x,\xi)\in C([0,\ T];\ C^1_b(\mathbb{R}^3_x\times\mathbb{R}^3_\xi))$ and the theory of ordinary differential equations (see, e.g., [1]) that for any fixed $(t,x,\xi)\in[0,\ T]\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi$, the characteristic equation

$$\begin{cases}\dot X(s)=\Xi(s), & X(t)=x;\\ \dot\Xi(s)=F(s,X(s),\Xi(s)), & \Xi(t)=\xi,\end{cases}$$

corresponding (3.2) has a unique solution $Z(s, t, x, \xi) = (X(s), \ \Xi(s)) = (X(s, t), \ \Xi(s, t)) = (X(s, t, x, \xi), \ \Xi(s, t, x, \xi))$ defined on $[0, \ T]$. The characteristic flow verifies that for any fixed $s, t \in [0, \ T]$, $Z(s, t, x, \xi)$ belongs to $C^1([0, \ T] \times [0, \ T] \times \mathbb{R}^3 \times \mathbb{R}^3; \ \mathbb{R}^3 \times \mathbb{R}^3)$ and is a measure preserving and homeomorphic mapping from $\mathbb{R}^3 \times \mathbb{R}^3$ onto itself (see, e.g., [1,19]). Hence, we can write $f(t, x, \xi)$ as follows

$$f(t, x, \xi) = \exp(-t)f_0(X(0), \ \Xi(0)) + \int_0^t \exp(s-t)P_u(f)(s, X(s), \ \Xi(s))ds, \tag{3.10}$$

or equivalently

$$f(t, x, \xi) = f_0(X(0), \ \Xi(0)) + \int_0^t Q_u(f)(s, X(s), \ \Xi(s))ds.$$

In the following, we denote

$$Y = \{f \in L^\infty((0, \ T); \ L^1 \cap L^p(\mathbb{R}^3 \times \mathbb{R}^3)) : f(t, x, \xi) \geq 0, \|f(t)\|_1 = \|f_0\|_1, \ \|f(t)\|_p \leq \|f_0\|_p, \ \forall t \in [0, T]\}.$$

Then, Y is a closed subset of the Banach space $L^\infty([0, \ T]; \ L^1 \cap L^p(\mathbb{R}^3 \times \mathbb{R}^3))$. Define the operator G on Y as follows

$$G(f) = \exp(-t)f_0(X(0), \ \Xi(0)) + \int_0^t \exp(s-t)P_u(f)(s, X(s), \ \Xi(s))ds.$$

Then, Lemma 3.1 implies that $\int_{\mathbb{R}^3} P_u(f)d\xi = \int_{\mathbb{R}^3} fd\xi$, so $\|P_u(f)\|_1 = \|f\|_1$. Integrating the above equation against $x, \ \xi$, due to the measure preserving of the characteristic flow we obtain through integration by substitution

$$\begin{aligned} \|G(f)(t)\|_1 &= \exp(-t)\|f_0\|_1 + \int_0^t \exp(\tau - t)\|P_u(f)(\tau)\|_1 d\tau \\ &= \exp(-t)\|f_0\|_1 + \int_0^t \exp(\tau - t)\|f_0\|_1 d\tau = \|f_0\|_1. \end{aligned}$$

On the other hand, it follows from Lemma 3.2 that $\|P_u(f)\|_p \leq \|f_0\|_p$. Consequently, we obtain by the same reason that

$$\begin{aligned} \|G(f)(t)\|_p &= \exp(-t)\|f_0\|_p + \int_0^t \exp(\tau - t)\|P_u(f)(\tau)\|_p d\tau \\ &\leq \exp(-t)\|f_0\|_p + \int_0^t \exp(\tau - t)\|f\|_p d\tau \\ &\leq \exp(-t)\|f_0\|_p + [1 - \exp(-t)]\|f_0\|_p = \|f_0\|_p. \end{aligned}$$

Hence, G is a mapping from Y into itself.

Let $f_1, f_2 \in Y$, similarly we get (notice that $\|P_u(f)\|_1 = \|f_0\|_1$, $\|P_u(f)\|_p \leq \|f_0\|_p$)

$$\begin{aligned} \|G(f_1) - G(f_2)\|_{L^\infty([0, \ T]; \ L^1(\mathbb{R}^3\times\mathbb{R}^3))} &\leq \sup_{0<t<T} \int_0^t \exp(\tau - t)\|P_u(f_1 - f_2)(\tau)\|_1 d\tau \\ &\leq (1 - \exp(-T))\|f_1 - f_2\|_{L^\infty([0, \ T]; \ L^1(\mathbb{R}^3\times\mathbb{R}^3))} \end{aligned}$$

and

$$\begin{aligned} \|G(f_1) - G(f_2)\|_{L^\infty([0, \ T]; \ L^p(\mathbb{R}^3\times\mathbb{R}^3))} &\leq \sup_{0<t<T} \int_0^t \exp(\tau - t)\|P_u(f_1 - f_2)(\tau)\|_p d\tau \\ &\leq (1 - \exp(-T))\|f_1 - f_2\|_{L^\infty([0, \ T]; \ L^p(\mathbb{R}^3\times\mathbb{R}^3))}. \end{aligned}$$

It follows from $(1 - \exp(-T)) < 1$ that G is a contraction. So, G has a unique fixed point f in Y, namely $f = G(f)$. Obviously, f is the unique solution to (3.2) and verifies (3.6).

Step 2. *Proof of the first inequality in* (3.7). Multiplying both sides of (3.2) by $|\xi|$ and then integrating against x and ξ, we have

$$\left| \frac{d}{dt} \int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi| f dx d\xi \right| \leq \left| \int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi| Q_u(f) dx d\xi \right| + \left| \int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi| F \cdot \nabla_\xi f dx d\xi \right|.$$

By Lemma 3.1 (2), we get

$$\left| \int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi| Q_u(f) dx d\xi \right| = \left| \int_{\mathbb{R}^3\times\mathbb{R}^3} f \cdot Q_u(|\xi|) dx d\xi \right| = \left| \int_{\mathbb{R}^3\times\mathbb{R}^3} f \cdot (P_u(|\xi|) - |\xi|) dx d\xi \right|.$$

Since

$$P_u(|\xi|) = \frac{1}{4\pi}\int_{\mathbb{S}^2} |u + |\xi - u|\omega|\, d\omega \le \frac{1}{4\pi}\int_{\mathbb{S}^2} (|u| + |\xi - u|)d\omega \\ \le \frac{1}{4\pi}\int_{\mathbb{S}^2} (|\xi| + 2|u|)d\omega = |\xi| + 2|u|,$$

we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi| Q_u(f) dxd\xi\right| &\le \left|\int_{\mathbb{R}^3\times\mathbb{R}^3} f\cdot(|\xi| + 2|u| + |\xi|)dxd\xi\right| \\ &= 2\int_{\mathbb{R}^3\times\mathbb{R}^3} f|u|dxd\xi + 2\int_{\mathbb{R}^3\times\mathbb{R}^3} f|\xi|dxd\xi \\ &\le 2\|u\|_\infty \int_{\mathbb{R}^3\times\mathbb{R}^3} fdxd\xi + 2\int_{\mathbb{R}^3\times\mathbb{R}^3} f|\xi|dxd\xi \\ &\le 2J\|f_0\|_1 + 2\int_{\mathbb{R}^3\times\mathbb{R}^3} f|\xi|dxd\xi.\end{aligned}$$

On the other hand,

$$\left|\int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi| F\cdot\nabla_\xi fdxd\xi\right| = \left|\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\xi}{|\xi|}\cdot Ffdxd\xi\right| \le \|F\|_\infty \|f_0\|_1.$$

Hence,

$$\left|\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi| fdxd\xi\right| \le (2J + \|F\|_\infty)\|f_0\|_1 + 2\int_{\mathbb{R}^3\times\mathbb{R}^3} f|\xi|dxd\xi.$$

By Gronwall's inequality, we get

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi| fdxd\xi \le \exp(2t)\int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi| f_0dxd\xi + \left(J + \frac{\|F\|_\infty}{2}\right)\|f_0\|_1(\exp(2t) - 1) \le C_T, \quad 0 \le t \le T,$$

namely, $\int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi| fdxd\xi$ is uniformly bounded.

Next, we are going to improve the order of the velocity moment by duality (see, e.g., [16]). Multiplying both sides of (3.2) by $\phi(x,\xi) = \frac{(x-x_0)\cdot\xi}{(1+|x-x_0|^2)^{\frac{1}{2}}}$, where $x_0 \in \mathbb{R}^3$ is given, and then integrating against x and ξ, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3} \phi(x,\xi)fdxd\xi = \int_{\mathbb{R}^3\times\mathbb{R}^3} \phi(x,\xi)Q_u(f)dxd\xi - \int_{\mathbb{R}^3\times\mathbb{R}^3} \phi(x,\xi)\xi\cdot\nabla_x fdxd\xi - \int_{\mathbb{R}^3\times\mathbb{R}^3} \phi(x,\xi)F\cdot\nabla_\xi fdxd\xi.$$

For the sake of simplicity, we denote the above equation by $I_1 = I_4 - I_2 - I_3$. We estimate each term in this equation separately as follows.

Estimate of I_1:

$$\begin{aligned}\left|\int_0^T I_1 dt\right| &= \left|\int_0^T \left(\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3} \phi(x,\xi)fdxd\xi\right)dt\right| \\ &\le \int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi|(f(T,x,\xi) + f_0(x,\xi))dxd\xi \le 2C_T.\end{aligned}$$

Estimate of I_4:

$$\begin{aligned}\left|\int_0^T I_4 dt\right| &= \left|\int_0^T dt\int_{\mathbb{R}^3\times\mathbb{R}^3} \phi(x,\xi)Q_u(f)dxd\xi\right| \\ &\le \int_0^T dt\int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi||Q_u(f)|dxd\xi \\ &\le 2J\|f_0\|_1 T + 2\int_0^T\int_{\mathbb{R}^3\times\mathbb{R}^3} f|\xi|dtdxd\xi \le 2J\|f_0\|_1 T + 2TC_T.\end{aligned}$$

Estimate of I_3:

$$\left|\int_0^T I_3 dt\right| = \left|\int_0^T dt \int_{\mathbb{R}^3\times\mathbb{R}^3} \phi(x,\xi)\cdot\nabla_\xi F f dx d\xi\right|$$
$$= \left|\int_0^T dt \int_{\mathbb{R}^3\times\mathbb{R}^3} \nabla_\xi \left(\frac{(x-x_0)\cdot\xi}{(1+|x-x_0|^2)^{\frac{1}{2}}}\right)\cdot F f dx d\xi\right| \le T\|F\|_\infty \|f_0\|_1.$$

Estimate of I_2: For any $K_x \subset \mathbb{R}^3$, if $x_0 \in K_x$ then

$$|I_2| = \left|\int_{\mathbb{R}^3\times\mathbb{R}^3} \phi(x,\xi)\xi\cdot\nabla_x f dx d\xi\right|$$
$$= \left|\int_{\mathbb{R}^3\times\mathbb{R}^3} \nabla_x \left(\frac{(x-x_0)\cdot\xi}{(1+|x-x_0|^2)^{\frac{1}{2}}}\right)\cdot\xi f dx d\xi\right|$$
$$= \left|\int_{\mathbb{R}^3\times\mathbb{R}^3} \left[\frac{\xi}{(1+|x-x_0|^2)^{\frac{1}{2}}} - \frac{1}{2}\cdot\frac{(x-x_0)\cdot\xi}{(1+|x-x_0|^2)^{\frac{3}{2}}} 2(x-x_0)\right]\cdot\xi f dx d\xi\right|$$
$$\ge \int_{K_x\times\mathbb{R}^3} \frac{|\xi|^2}{(1+|x-x_0|^2)^{\frac{3}{2}}} f dx d\xi \ge \frac{1}{(1+(\mathrm{diam}K_x)^2)^{\frac{3}{2}}}\int_{K_x\times\mathbb{R}^3}|\xi|^2 f dx d\xi.$$

It follows that

$$\left|\int_0^T I_2 dt\right| \ge \frac{1}{(1+(\mathrm{diam}K_x)^2)^{\frac{3}{2}}}\int_0^T dt\int_{K_x\times\mathbb{R}^3}|\xi|^2 f dx d\xi.$$

Combine the above estimates, we get

$$\int_0^T dt\int_{K_x\times\mathbb{R}^3}|\xi|^2 f dx d\xi \le (1+(\mathrm{diam}K_x)^2)^{\frac{3}{2}}(2C_T + 2J\|f_0\|_1 T + 2TC_T + T\|F\|_\infty\|f_0\|_1)$$
$$\le C(\mathrm{diam}K_x, f_0, T, \|F\|_\infty, J).$$

Step 3. *Proof of the integrability of* u_f. First, we demonstrate that u_f is well defined by establishing a positive lower bound of ρ_f. In fact, by (3.10) we obtain

$$f(t,x,\xi) \ge \exp(-t) f_0(X(0), \qquad \Xi(0)) \ge a\exp(-b(|X(0)|^2 + |\Xi(0)|^2))\exp(-t),$$

on the other hand, it follows from [21] that

$$|X(0)| \le |x| + T|\xi| + T^2\|F\|_\infty, \qquad |\Xi(0)| \le |\xi| + T\|F\|_\infty.$$

So,

$$f(t,x,\xi) \ge A\exp(-B(|x|^2+|\xi|^2)),$$

where A, B are positive constants depending on a, b, T and $\|F\|_\infty$. Integrating the above inequality against ξ, we have

$$\rho_f = \int_{\mathbb{R}^3} f(t,x,\xi)d\xi \ge A'\exp(-B|x|^2) > 0, \quad A' = A\int_{\mathbb{R}^3}\exp(-B|\xi|^2)d\xi > 0.$$

Consequently, by the uniform bound of $\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi| f dx d\xi$ proved above we know that

$$|u_f| = \frac{|m_f|}{\rho_f} \le \frac{1}{A'}\exp(B|x|^2)|m_f| \in L^\infty([0,\ T];\ L^1_{\mathrm{loc}}(\mathbb{R}^3)),$$

where $m_f(t,x) = \rho_f u_f(t,x) = \int_{\mathbb{R}^3\times\mathbb{R}^3}\xi f(t,x,\xi)dx d\xi$ is the momentum density of the system.

Step 4. *Sketch of the proof of conclusion* (2). The characteristic equation

$$\begin{cases}\dot{X}(s) = \Xi(s), & X(t) = x;\\ \dot{\Xi}(s) = E(s,X(s)) + \xi\times B(s,X(s)), & \Xi(t) = \xi,\end{cases}$$

of the Cauchy problem (3.3) has a unique solution $(X(s),\ \Xi(s))$. Similar to the first step and the third step of the proof of conclusion (1), we can show that the Cauchy problem (3.3) has a unique positive solution

$$f \in L^\infty((0,\ T);\ L^1(\mathbb{R}^3\times\mathbb{R}^3)\cap L^p(\mathbb{R}^3\times\mathbb{R}^3))$$

verifying

$$\|f(t)\|_1 = \|f_0\|_1, \qquad \|f(t)\|_p \le \|f_0\|_p, \quad 0 \le t \le T.$$

Furthermore,

$$|X(0)| \le |x| + T|\xi| + T^2(\|E\|_\infty + |\xi|\|B\|_\infty),$$
$$|\Xi(0)| \le |\xi| + T(\|E\|_\infty + |\xi|\|B\|_\infty),$$

it follows that $u_f \in L^\infty([0,\ T];\ L^1_{\mathrm{loc}}(\mathbb{R}^3))$.

In order to finish the proof of conclusion (2), it is sufficient to show the first inequality in (3.9). Multiplying both sides of (3.3) by $|\xi|$ and then integrating against x and ξ, it follows that

$$\begin{aligned}
\left|\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|fdxd\xi\right| &\le \left|\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|E\cdot\nabla_\xi fdxd\xi\right| + \left|\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|(\xi\times B)\cdot\nabla_\xi fdxd\xi\right| + \left|\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|Q_u(f)dxd\xi\right| \\
&\le \|E\|_\infty\|f_0\|_1 + \int_{\mathbb{R}^3\times\mathbb{R}^3}|(\xi\times B)|fdxd\xi + \int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|Q_u(f)dxd\xi \\
&\le \|E\|_\infty\|f_0\|_1 + \int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi\|B|fdxd\xi + 2J\|f_0||_1 + 2\int_{\mathbb{R}^3\times\mathbb{R}^3}f|\xi|dxd\xi \\
&\le \|E\|_\infty\|f_0\|_1 + \|B\|_\infty\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|fdxd\xi + 2J\|f_0\|_1 + 2\int_{\mathbb{R}^3\times\mathbb{R}^3}f|\xi|dxd\xi \\
&\le (\|E\|_\infty + 2J)\|f_0\|_1 + (\|B\|_\infty + 2)\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|fdxd\xi,
\end{aligned}$$

then Gronwall's lemma implies that

$$\begin{aligned}
\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|fdxd\xi &\le \exp((\|B\|_\infty+2)t)\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|f_0dxd\xi + \left(J + \frac{\|E\|_\infty}{2}\right)\|f_0\|_1(\exp((\|B\|_\infty+2)t) - 1) \\
&\le C_T, \quad 0 \le t \le T.
\end{aligned}$$

Multiplying both sides of (3.3) by $\phi(x,\xi) = \frac{(x-x_0)\cdot\xi}{(1+|x-x_0|^2)^{\frac{1}{2}}}$ and then integrating against x and ξ, we obtain

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3}\phi(x,\xi)fdxd\xi &= \int_{\mathbb{R}^3\times\mathbb{R}^3}\phi(x,\xi)Q_u(f)dxd\xi \\
&\quad - \int_{\mathbb{R}^3\times\mathbb{R}^3}\phi(x,\xi)\xi\cdot\nabla_x fdxd\xi - \int_{\mathbb{R}^3\times\mathbb{R}^3}\phi(x,\xi)(E+\xi\times B)\cdot\nabla_\xi fdxd\xi.
\end{aligned}$$

Similarly, we denote the above equation by $I_1 = I_4 - I_2 - I_3$. We only need to estimate I_3 since estimates of the other terms are just similar to above.

$$\begin{aligned}
|I_3| &= \left|\int_{\mathbb{R}^3\times\mathbb{R}^3}\phi(x,\xi)(E+\xi\times B)\cdot\nabla_\xi fdxd\xi\right| \\
&\le \left|\int_{\mathbb{R}^3\times\mathbb{R}^3}\phi(x,\xi)E\cdot\nabla_\xi fdxd\xi\right| + \left|\int_{\mathbb{R}^3\times\mathbb{R}^3}\phi(x,\xi)(\xi\times B)\cdot\nabla_\xi fdxd\xi\right| \\
&\le \|E\|_\infty\|f_0\|_1 + \left|\int_{\mathbb{R}^3\times\mathbb{R}^3}\phi(x,\xi)(\xi\times B)\cdot\nabla_\xi fdxd\xi\right| \\
&\le \|E\|_\infty\|f_0\|_1 + \int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi||B|fdxd\xi \\
&\le \|E\|_\infty\|f_0\|_1 + \|B\|_\infty\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|fdxd\xi \\
&\le \|E\|_\infty\|f_0\|_1 + \|B\|_\infty C_T.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int_0^T dt\int_{K_x\times\mathbb{R}^3}|\xi|^2fdxd\xi &\le (1+(\mathrm{diam}K_x)^2)^{\frac{3}{2}}(2C_T + 2J\|f_0\|_1T + 2TC_T + T\|E\|_\infty\|f_0\|_1 + T\|B\|_\infty C_T) \\
&\le C(\mathrm{diam}K_x, f_0, T, \|E\|_\infty, \|B\|_\infty, J).
\end{aligned}$$

This completes the proof. $\square$

Remark 3.1. The assumption $\nabla_\xi \cdot F(t,x,\xi) = 0$ is not necessary for the existence of a nonnegative solution to (3.2), it only ensures (3.6), i.e., mass conservation of the system and non-increasing of the L^p norm of the solution. Specifically, if we drop this condition, the existence of a nonnegative solution $f(t,x,\xi)$ to (3.2) is also valid. Nevertheless, the estimate (3.6) should be modified as follows:

$$\|f(t)\|_1 \le \exp\left(\int_0^t \|\nabla_\xi \cdot F(\tau)\|_\infty d\tau\right)\|f_0\|_1, \quad 0 \le t \le T$$

and

$$\|f(t)\|_p \le \exp\left(\frac{1}{p}\int_0^t \|\nabla_\xi \cdot F(\tau)\|_\infty d\tau\right)\|f_0\|_p, \quad 0 \le t \le T.$$

Furthermore, the constant C in the estimate (3.7) also depends continuously on $\|\nabla_\xi \cdot F\|_\infty$. In fact, it follows from Liouville's theorem (see, e.g., [1]) that for any fixed $s, t \in [0, T]$, the Jacobian of the transformation $Z(s,t,\cdot,\cdot) : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ verifies

$$\det\left(\frac{\partial Z(s,t,x,\xi)}{\partial(x,\xi)}\right) = \exp\left(\int_t^s \nabla_\xi \cdot F(\tau, Z(\tau,t,x,\xi))d\tau\right).$$

Hence, for any integrable function $g(t,x,\xi)$, we have

$$\begin{aligned}\int_{\mathbb{R}^3\times\mathbb{R}^3} |g(s,Z(s,t,x,\xi))|dxd\xi &= \int_{\mathbb{R}^3\times\mathbb{R}^3} |g(s,x,\xi)| \exp\left(\int_s^t \nabla_\xi \cdot F(\tau, Z(\tau,s,x,\xi))d\tau\right)dxd\xi \\ &\le \exp\left(\int_s^t \|\nabla_\xi \cdot F(\tau)\|_\infty d\tau\right)\int_{\mathbb{R}^3\times\mathbb{R}^3} |g(s,x,\xi)|dxd\xi.\end{aligned}$$

Due to this estimate, we can obtain the above results by a little modification of the previous proof. On the other hand, that the constant C depends continuously upon $\|\nabla_\xi \cdot F\|_\infty$ results obviously from estimates of

$$\left|\int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi| F \cdot \nabla_\xi f dxd\xi\right|$$

and I_3.

Remark 3.2. If we assume further $\int_{\mathbb{R}^3\times\mathbb{R}^3} |x| f_0(x,\xi)dxd\xi < \infty$, then there is a positive constant $\tilde{C}$ depending continuously upon $\|f_0\|_1$, $\||\xi| f_0\|_1$, $\||x| f_0\|_1$, T, J and $\|F\|_\infty$, $\|\nabla_\xi \cdot F\|_\infty$ (or $\|E\|_\infty$, $\|B\|_\infty$) such that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} |x| f(t,x,\xi)dxd\xi \le \tilde{C}, \quad 0 \le t \le T.$$

Actually, multiplying both sides of (3.2) by $|x|$ and integrating against x, ξ, we obtain by a direct calculation that (notice that for (3.3), the last term of the following inequality disappears)

$$\left|\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3} |x| f(t,x,\xi)dxd\xi\right| \le \int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi| f(t,x,\xi)dxd\xi + \|\nabla_\xi \cdot F\|_\infty \int_{\mathbb{R}^3\times\mathbb{R}^3} |x| f(t,x,\xi)dxd\xi.$$

Then, the desired result is obtained by the Gronwall's trick and the estimate of

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi| f(t,x,\xi)dxd\xi.$$

3.3. *On approximate solutions*

Now, we are in a position to construct approximate solutions to the Cauchy problem (1.1) and the Cauchy problem (1.2). Following the method used in [6,14], for any given velocity field $u(t,x)$ and any $n \in \mathbb{N}$, we define the cutoff velocity field $\varphi_n(u)(t,x)$ as follows

$$\varphi_n(u)(t,x) = \begin{cases} u(t,x), & |x| < n, |u(t,x)| \le n, \\ n\dfrac{u(t,x)}{|u(t,x)|}, & |x| < n, |u(t,x)| > n, \\ 0, & |x| > n. \end{cases} \tag{3.11}$$

Then, the approximate equations for Eqs. (1.1) and (1.2) are respectively defined by

$$\begin{cases} \partial_t f^n + \xi \cdot \nabla_x f^n + F(t,x,\xi) \cdot \nabla_\xi f^n = Q_{\varphi_n(u_{f^n})}(f^n), \\ f^n(0,x,\xi) = f_0(x,\xi) \end{cases} \tag{3.12}$$

and

$$\begin{cases}\partial_t f^n + \xi \cdot \nabla_x f^n + [E(t,x) + \xi \times B(t,x)] \cdot \nabla_\xi f^n = Q_{\varphi_n(u_{f^n})}(f^n),\\ f^n(0,x,\xi) = f_0(x,\xi).\end{cases} \tag{3.13}$$

Concerning approximate equation (3.12), we are going to prove the following

Proposition 3.5. *Suppose that the initial microscopic density $f_0(x,\xi) \geq 0$ verifies*

$$f_0 \in L^1 \cap L^p(\mathbb{R}^3 \times \mathbb{R}^3) \quad (p > 1), \qquad \int_{\mathbb{R}^3 \times \mathbb{R}^3} |\xi| f_0 dx d\xi < \infty \tag{3.14}$$

and there exist constants $a,\ b > 0$ such that

$$f_0(x,\xi) \geq a \exp(-b(x^2 + \xi^2)), \quad x,\ \xi \in \mathbb{R}^3. \tag{3.15}$$

If $F(t,x,\xi) \in C([0,\ T];\ C_b^1(\mathbb{R}_x^3 \times \mathbb{R}_\xi^3))$ and $\nabla_\xi \cdot F(t,x,\xi) = 0$, then for any $n \in \mathbb{N}$ there exists a nonnegative weak solution $f^n \in L^\infty((0,\ T);\ L^1 \cap L^p(\mathbb{R}^3 \times \mathbb{R}^3))$ to the Cauchy problem (3.12) such that

$$\|f^n(t)\|_1 = \|f_0\|_1, \qquad \|f^n(t)\|_p \leq \|f_0\|_p, \quad 0 \leq t \leq T. \tag{3.16}$$

Moreover, for any $K_x \subset\subset \mathbb{R}^3$, there is a positive constant $C = C(\mathrm{diam}K_x, f_0, T, \|F\|_\infty)$ independent of n and continuously depending upon the parameters $\mathrm{diam}K_x$, $\|f_0\|_1$, $\||\xi| f_0\|_1$, T and $\|F\|_\infty$ such that

$$\int_0^T dt \int_{K_x \times \mathbb{R}^3} |\xi|^2 f^n dx d\xi \leq C, \quad u_{f^n} \in L^\infty([0,\ T];\ L^1_{\mathrm{loc}}(\mathbb{R}^3)). \tag{3.17}$$

Proof. First, we show that for any given $n \in \mathbb{N}$ there exists a nonnegative solution f^n to Eq. (3.12). Inspired by the method used in [6,14], let $B_n = \{x \in \mathbb{R}^3 :\ |x| < n\}$ and

$$\begin{aligned} S_n = \{&u \in L^1((0,T) \times \mathbb{R}^3)^3 :\ |u(t,x)| \leq n \text{ for almost all } (t,x) \in (0,T) \times B_n \\ &\text{and } u(t,x) = 0 \text{ for almost all } (t,x) \in (0,T) \times (\mathbb{R}^3 \setminus B_n)\}, \end{aligned}$$

then S_n is a bounded and closed convex subset of $L^1((0,T) \times \mathbb{R}^3)^3$. By the definition of $\varphi_n(u)$, we know that φ_n maps $L^1((0,T) \times \mathbb{R}^3)^3$ onto S_n. Define the operator $T_1 :\ S_n \to L^1((0,T) \times \mathbb{R}^3)^3$ as follows: for any $u \in S_n$, let f be the unique nonnegative solution to Eq. (3.2) corresponding to the fixed velocity field u (see Lemma 3.4), we set $T_1(u) = \chi_{(0,T)\times B_n} \cdot u_f$, hereafter χ_A is designated for the characteristic function of the set A. Further, let $T : S_n \to S_n;\ T(u) = \varphi_n(T_1(u))$. Obviously, a fixed point u of the operator T determines a nonnegative solution to Eq. (3.12) (denoted by f^n) and it follows from Lemma 3.4 that the solution f^n satisfies (3.16) and (3.17) (notice that for the time being the constant C in (3.17) is not claimed to be independent of n). Since $\varphi_n : L^1((0,T) \times \mathbb{R}^3)^3 \to S_n$ is continuous, the continuity and compactness of T_1 would imply that the operator T has a fixed point in S_n due to Schauder's theorem. Consequently, to prove the existence of a nonnegative solution to Eq. (3.12), it is sufficient to show the compactness and continuity of T_1.

Proof of the compactness of T_1. For any given $u \in S_n$, let f be the unique nonnegative solution to Eq. (3.2) corresponding to the velocity field u, then it verifies (3.6) and (3.7). Suppose that $\phi(x)$ is any C_c^∞ function such that $\phi|_{B_n} \equiv 1$, let $\tilde{f} = \phi f,\ \tilde{g} = \phi Q_u(f) + (\xi \cdot \nabla_x \phi) f$. Then

$$\partial_t \tilde{f} + \xi \cdot \nabla_x \tilde{f} + F(t,x,\xi) \cdot \nabla_\xi \tilde{f} = \tilde{g}$$

in distributional sense. It follows from (3.6) and (3.7), Dunford–Pettis theorem and boundedness of the linear operator Q_u that $\{\tilde{f} :\ u \in S_n\}$ and $\{\tilde{g} :\ u \in S_n\}$ are relatively compact in the weak topology of $L^1((0,T)\times\mathbb{R}^3\times\mathbb{R}^3)$ and $L^1_{\mathrm{loc}}((0,T)\times\mathbb{R}^3\times\mathbb{R}^3)$ respectively. Then, Lemma 3.3 implies that for any $\psi \in L^\infty(\mathbb{R}_\xi^3)$ with support contained in $K_\xi = \{\xi \in \mathbb{R}^3 :\ |\xi| \leq R\}$ $(R > 0)$, the subset

$$\left\{\int_{K_\xi} \psi(\xi) \tilde{f}(t,x,\xi) d\xi,\ u \in S_n\right\}$$

is relatively compact in the strong topology of $L^1((0,T) \times \mathbb{R}^3)$. Letting $\phi(x) \to \chi_{B_n}(x)$, we obtain that

$$\left\{\int_{K_\xi} \psi(\xi) f(t,x,\xi) d\xi,\ u \in S_n\right\}$$

is relatively compact in the strong topology of $L^1((0,T)\times\mathbb{R}^3)$. On the other hand, by Lemma 3.4 (especially inequality (3.7)) we know that $\int_0^T dt\int_{B_n\times\mathbb{R}^3}|\xi|^2 f dxd\xi$ is uniformly bounded, it follows that for any fixed $0\le r<2$

$$\sup_{u\in S_n}\int_{(0,T)\times B_n}\left(\int_{\mathbb{R}^3}|\xi|^r f(t,x,\xi)d\xi-\int_{K_\xi}|\xi|^r f(t,x,\xi)d\xi\right)dtdx=\sup_{u\in S_n}\int_{(0,T)\times B_n}\left(\int_{K_\xi^c}|\xi|^r f(t,x,\xi)d\xi\right)dtdx$$
$$\le\frac{1}{R^{2-r}}\sup_{u\in S_n}\int_{(0,T)\times B_n}\left(\int_{K_\xi^c}|\xi|^2 f(t,x,\xi)d\xi\right)dtdx\to 0,\quad R\to\infty.$$

Hence, for any given $\varepsilon>0$, if $R>0$ is large enough, then $\left\{\int_{K_\xi}|\xi|^r f(t,x,\xi)d\xi\,:\,u\in S_n\right\}$ is a sequentially compact ε-net of $\left\{\int_{\mathbb{R}^3}|\xi|^r f(t,x,\xi)d\xi\,:\,u\in S_n\right\}\subset L^1((0,T)\times B_n)$. This implies that for any fixed $0\le r<2$, $\left\{\int_{\mathbb{R}^3}|\xi|^r f(t,x,\xi)d\xi\,:\,u\in S_n\right\}$, and therefore ρ_f and m_f are relatively compact in the strong topology of $L^1((0,T)\times B_n)$. Since $\rho_f\ge A'\exp(-B|x|^2)>0$ for any $(t,x)\in(0,T)\times B_n$, we know that

$$T_1S_n=\left\{\chi_{(0,T)\times B_n}\cdot u_f=\chi_{(0,T)\times B_n}\cdot\frac{m_f}{\rho_f}\,:\,u\in S_n\right\}$$

is relatively compact in the strong topology of $L^1((0,T)\times B_n)$. Hence, T_1 is compact.

Proof of the continuity of T_1. Let $u_k\in S_n$ and $u_k\to u\in S_n$ $(k\to\infty)$, and denote by f_k and f the unique nonnegative solution to Eq. (3.2) corresponding to the velocity field u_k and u, respectively. Then, Lemma 3.4 (especially inequality (3.6)) implies that there is a subsequence of f_k (still denoted by f_k for the sake of simplicity) that converges to some $\tilde f$ in the weak topology of $L^p((0,T)\times\mathbb{R}^3\times\mathbb{R}^3)$. By Lemma 3.2 (1), we can pass to the limit $k\to\infty$ in the sense of distributions in Eq. (3.2) corresponding to the velocity field u_k and thus obtain that

$$\begin{cases}\partial_t\tilde f+\xi\cdot\nabla_x\tilde f+F(t,x,\xi)\cdot\nabla_\xi\tilde f=Q_u(\tilde f),\\ \tilde f(0,x,\xi)=f_0(x,\xi).\end{cases}$$

So, the uniqueness of the linear problem (3.2) gives $\tilde f=f$. This implies that the full sequence f_k is also convergence. Again, using velocity averaging lemma we get that $\rho_{f_k}\to\rho_f$, $m_{f_k}\to m_f$ in the strong topology of $L^1((0,T)\times B_n)$. Furthermore, by the lower bound estimates for ρ_{f_k} obtained above we have $u_{f_k}\to u_f$ $(k\to\infty)$. Hence, we have proved that if $u_k\to u$ $(k\to\infty)$, then $\chi_{(0,T)\times B_n}\cdot u_{f_k}\to\chi_{(0,T)\times B_n}\cdot u_f$ $(k\to\infty)$, i.e., T_1 is a continuous operator.

To finish the proof, we have to show that the constant C in (3.17) is independent of $n\in\mathbb{N}$. First, we notice

$$\begin{aligned}|Q_{\varphi_n(u_{f^n})}(|\xi|)|&=|P_{\varphi_n(u_{f^n})}(|\xi|)-|\xi||\\&=\left|\frac{1}{4\pi}\int_{\mathbb{S}^2}(|\varphi_n(u_{f^n})+|\xi-\varphi_n(u_{f^n})|\omega|)d\omega-|\xi|\right|\\&\le 2|\varphi_n(u_{f^n})|+2|\xi|\le 2|u_{f^n}|+2|\xi|,\end{aligned}$$

which implies

$$\begin{aligned}\left|\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|Q_{\varphi_n(u_{f^n})}(f^n)dxd\xi\right|&=\left|\int_{\mathbb{R}^3\times\mathbb{R}^3}f^nQ_{\varphi_n(u_{f^n})}(|\xi|)dxd\xi\right|\\&\le\int_{\mathbb{R}^3\times\mathbb{R}^3}f^n[2|u_{f^n}|+2|\xi|]dxd\xi\\&\le 2\int_{\mathbb{R}^3}\rho_{f^n}\cdot|u_{f^n}|dx+2\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|f^ndxd\xi\\&=4\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|f^ndxd\xi.\end{aligned}$$

Multiplying both sides of (3.12) by $|\xi|$ and then integrating against x and ξ, by the above estimate of the collision operator and similar to the proof of Lemma 3.4 we obtain

$$\left|\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|f^ndxd\xi\right|\le\|F\|_\infty\|f_0\|_1+4\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|f^ndxd\xi.$$

By Gronwall's inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|f^ndxd\xi&\le\exp(4t)\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|f_0dxd\xi+\frac14\|F\|_\infty\|f_0\|_1(\exp(4t)-1)\\&\le\exp(4T)\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|f_0dxd\xi+\frac14\|F\|_\infty\|f_0\|_1(\exp(4T)-1)=:C_T,\quad 0\le t\le T.\end{aligned}$$

Multiplying both sides of (3.12) by $\phi(x,\xi) = \frac{(x-x_0)\cdot\xi}{(1+|x-x_0|^2)^{\frac{1}{2}}}$ and then integrating against x and ξ, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3}\phi(x,\xi)f^n dxd\xi = \int_{\mathbb{R}^3\times\mathbb{R}^3}\phi(x,\xi)Q_{\varphi_n(u_{f^n})}(f^n)dxd\xi - \int_{\mathbb{R}^3\times\mathbb{R}^3}\phi(x,\xi)\xi\cdot\nabla_x f^n dxd\xi - \int_{\mathbb{R}^3\times\mathbb{R}^3}\phi(x,\xi)F\cdot\nabla_\xi f^n dxd\xi.$$

Similar to the proof of Lemma 3.4, we denote the above equation by $I_1 = I_4 - I_2 - I_3$. Again, similar to the proof of Lemma 3.4, we can show

$$\left|\int_0^T I_1 dt\right| \le 2C_T, \qquad \left|\int_0^T I_3 dt\right| \le T\|F\|_\infty\|f_0\|_1, \qquad \left|\int_0^T I_4 dt\right| \le 4TC_T,$$

furthermore, for any $K_x \subset\subset \mathbb{R}^3$ such that $x_0 \in K_x$, we have

$$\left|\int_0^T I_2 dt\right| \ge \frac{1}{(1+(\mathrm{diam}K_x)^2)^{\frac{3}{2}}}\int_0^T dt\int_{K_x\times\mathbb{R}^3}|\xi|^2 f dxd\xi.$$

Consequently, we obtain

$$\int_0^T dt\int_{K_x\times\mathbb{R}^3}|\xi|^2 f dxd\xi \le (1+(\mathrm{diam}K_x)^2)^{\frac{3}{2}}\left[2C_T + 4TC_T + T\|F\|_\infty\|f_0\|_1\right].$$

Taking $C = (1+(\mathrm{diam}K_x)^2)^{\frac{3}{2}}[2C_T + 4TC_T + T\|F\|_\infty\|f_0\|_1]$, obviously C is independent of n and continuously depends upon the parameters $\mathrm{diam}K_x$, $\|f_0\|_1$, $\||\xi|f_0\|_1$, T and $\|F\|_\infty$. This completes the proof. □

Remark 3.3. Due to Remarks 3.1, 3.2 and the procedure given above, similar results in Proposition 3.5 are still valid when $\nabla_\xi\cdot F(t,x,\xi) \not\equiv 0$. But (3.16) should be replaced by

$$\|f^n(t)\|_1 \le \exp\left(\int_0^t \|\nabla_\xi\cdot F(\tau)\|_\infty d\tau\right)\|f_0\|_1, \quad 0\le t\le T$$

and

$$\|f^n(t)\|_p \le \exp\left(\frac{1}{p}\int_0^t \|\nabla_\xi\cdot F(\tau)\|_\infty d\tau\right)\|f_0\|_p, \quad 0\le t\le T.$$

Notice that in this case, the constant C in (3.17) should also depend upon $\|\nabla_\xi\cdot F\|_\infty$.

Furthermore, if $\int_{\mathbb{R}^3\times\mathbb{R}^3}|x|f_0(x,\xi)dxd\xi < \infty$, then

$$\int_{\mathbb{R}^3\times\mathbb{R}^3}|x|f^n(t,x,\xi)dxd\xi \le \tilde{C}, \quad 0\le t\le T,$$

where the constant $\tilde{C}$ depends upon $\|f_0\|_1$, $\||\xi|f_0\|_1$, $\||x|f_0\|_1$, T, $\|F\|_\infty$ and $\|\nabla_\xi\cdot F\|_\infty$, but not upon n. This estimate also applies to the following proposition.

Similarly, we can establish the following result for the approximate problem (3.13).

Proposition 3.6. *Suppose the initial microscopic density $f_0(x,\xi) \ge 0$ verifies (3.14) and (3.15). If $E(t,x)$, $B(t,x) \in C([0,T]; C_b^1(\mathbb{R}^3))$, then for any $n\in\mathbb{N}$ there exists a nonnegative solution $f^n \in L^\infty((0,T); L^1\cap L^p(\mathbb{R}^3\times\mathbb{R}^3))$ to the Cauchy problem (3.13) such that*

$$\|f^n(t)\|_1 = \|f_0\|_1, \qquad \|f^n(t)\|_p \le \|f_0\|_p, \quad 0\le t\le T. \tag{3.18}$$

Further, for any $K_x \subset\subset \mathbb{R}^3$, there is a positive constant C independent of n and continuously depending upon the parameters $\mathrm{diam}K_x$, $\|f_0\|_1$, $\||\xi|f_0\|_1$, T, $\|E\|_\infty$ *and* $\|B\|_\infty$ *such that*

$$\int_0^T dt\int_{K_x\times\mathbb{R}^3}|\xi|^2 f^n dxd\xi \le C, \quad u_{f^n}\in L^\infty([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^3)). \tag{3.19}$$

4. Proofs of the main theorems

In this section, we shall finish the proofs of the main results in this paper. First, we give a short proof of Theorem 2.1 based upon Propositions 3.5 and 3.6.

Proof of Theorem 2.1. Let $f_0^n(x,\xi) = f_0(x,\xi) + \frac{1}{n}\exp(-(|x|^2+|\xi|^2))$ $(n = 1, 2, \ldots)$, then f_0^n satisfies (3.14) and (3.15) since f_0 verifies (2.1). It follows from Proposition 3.5 that there is a nonnegative solution f^n to (3.12) verifying $f^n|_{t=0} = f_0^n$, (3.16) and (3.17). Especially, (3.16) implies

$$\|f^n(t)\|_1 = \|f_0^n\|_1 \le \|f_0\|_1 + \pi^3/n, \qquad \|f^n(t)\|_p \le \|f_0^n\|_p \le \|f_0\|_p + \pi^3/n, \quad 0 \le t \le T. \tag{4.1}$$

It follows that up to a subsequence, for any $1 < q < \infty$

$$f^n \to f\ (n \to \infty) \quad \text{weakly in } L^q((0,\ T);\ L^p(\mathbb{R}^3 \times \mathbb{R}^3)), \tag{4.2}$$

where $f \in L^\infty((0,\ T);\ L^1 \cap L^p(\mathbb{R}^3 \times \mathbb{R}^3))$ and $f \ge 0$. On the other hand, by (4.1), (3.17) and the velocity averaging lemma (Lemma 3.3), similar to the proof of Proposition 3.5 we can show that $\left\{\int_{\mathbb{R}^3} |\xi|^r f^n(t,x,\xi)d\xi \,:\, n = 1, 2, \ldots\right\}$ is relatively compact in $L^1((0,\ T) \times K_x)$ for any $0 \le r < 2$ and $K_x \subset\subset \mathbb{R}^3_x$. This obviously implies that $\{\rho_{f^n} \,:\, n = 1, 2, \ldots\}$ and $\{\rho_{f^n} u_{f^n} \,:\, n = 1, 2, \ldots\}$ are relatively compact in $L^1((0,\ T) \times K_x)$ for any $K_x \subset\subset \mathbb{R}^3_x$. Without loss of generality and in consideration of (4.2), we may assume that as $n \to \infty$,

$$\rho_{f^n} \to \rho_f, \qquad \rho_{f^n} u_{f^n} \to \rho_f u_f \quad \text{strongly in } L^1((0,\ T) \times K_x) \tag{4.3}$$

for any $K_x \subset\subset \mathbb{R}^3_x$. Due to (4.2), (4.3) and the proof of Theorem 2 in [14], we obtain

$$Q_{\varphi_n(u_{f^n})}(f^n) \to Q_{u_f}(f) \quad \text{in } \mathcal{D}'((0,\ T) \times \mathbb{R}^3 \times \mathbb{R}^3), \quad \text{as } n \to \infty. \tag{4.4}$$

Since f^n is a solution of (3.12) with initial datum f_0^n, we have by Definition 1.1 that for any test function $\phi(t,x,\xi) \in C_c^1([0,\ T) \times \mathbb{R}^3 \times \mathbb{R}^3)$

$$\begin{aligned} &\int_0^T dt \int_{\mathbb{R}^3\times\mathbb{R}^3} f^n \left[\partial_t \phi + \xi \cdot \nabla_x \phi + F \cdot \nabla_\xi \phi\right] dx d\xi + \int_{\mathbb{R}^3\times\mathbb{R}^3} f_0^n \phi|_{t=0} dx d\xi \\ &\quad = -\int_0^T dt \int_{\mathbb{R}^3\times\mathbb{R}^3} Q_{\varphi_n(u_{f^n})}(f^n)\phi dx d\xi. \end{aligned} \tag{4.5}$$

Due to (4.2), (4.4) and $\|f_0^n - f_0\|_1 \to 0 (n \to \infty)$, we can pass to limit $n \to \infty$ in (4.5) and obtain that for any test function $\phi(t,x,\xi) \in C_c^1([0,\ T) \times \mathbb{R}^3 \times \mathbb{R}^3)$

$$\int_0^T dt \int_{\mathbb{R}^3\times\mathbb{R}^3} f\left[\partial_t \phi + \xi \cdot \nabla_x \phi + F \cdot \nabla_\xi \phi\right] dx d\xi + \int_{\mathbb{R}^3\times\mathbb{R}^3} f_0 \phi|_{t=0} dx d\xi = -\int_0^T dt \int_{\mathbb{R}^3\times\mathbb{R}^3} Q_{u_f}(f)\phi dx d\xi,$$

namely f is a nonnegative solution of (1.1). Furthermore, passing to limits $n \to \infty$ in (4.1) and (3.17) we obtain the desired estimates (2.2) and (2.3). This completes the proof of part (1).

Using almost the same method and starting from Proposition 3.6, we can show the second part of this theorem. □

Proof of Theorem 2.2. Let $F^\varepsilon = F(t,x,\xi) * \eta_\varepsilon(t,x,\xi)$, where η_ε $(\varepsilon > 0)$ is the standard mollifier. Then, we have $\nabla_\xi \cdot F^\varepsilon = 0$ and $F^\varepsilon(t,x,\xi) \in C([0,\ T];\ C_b^1(\mathbb{R}^3_x \times \mathbb{R}^3_\xi))$, furthermore,

$$\|F^\varepsilon\|_{L^{p'}([0,T]\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi)} \le \|F\|_{L^{p'}([0,T]\times\mathbb{R}^3_x\times\mathbb{R}^3_\xi)} \tag{4.6}$$

and

$$F^\varepsilon \to F, \text{ in } L^{p'}([0,T] \times \mathbb{R}^3_x \times \mathbb{R}^3_\xi), \quad \text{as } \varepsilon \to 0. \tag{4.7}$$

It follows from Theorem 2.1 that for each $\varepsilon > 0$, the Cauchy problem

$$\begin{cases} \partial_t f^\varepsilon + \xi \cdot \nabla_x f^\varepsilon + F^\varepsilon \cdot \nabla_\xi f^\varepsilon = Q_{u_{f^\varepsilon}}(f^\varepsilon), \\ f^\varepsilon(0,x,\xi) = f_0(x,\xi) \end{cases} \tag{4.8}$$

has a nonnegative solution f^ε such that

$$\|f^\varepsilon(t)\|_1 = \|f_0\|_1, \qquad \|f^\varepsilon(t)\|_p \le \|f_0\|_p, \quad 0 \le t \le T \tag{4.9}$$

and for any $K_x \subset\subset \mathbb{R}^3_x$

$$\int_0^T dt \int_{K_x\times\mathbb{R}^3} |\xi|^2 f^\varepsilon dx d\xi < \infty. \tag{4.10}$$

Multiplying both sides of (4.8) by $|\xi|$ and then integrating against x and ξ, we obtain by (4.6), (4.9), Lemma 3.2 (2) and Hölder's inequality that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|f^\varepsilon dxd\xi &= \int_{\mathbb{R}^3\times\mathbb{R}^3}\nabla_\xi|\xi|\cdot F^\varepsilon f^\varepsilon dxd\xi + \int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|Q_{u_{f^\varepsilon}}(f^\varepsilon)dxd\xi\\ &\le \int_{\mathbb{R}^3\times\mathbb{R}^3}|F^\varepsilon|f^\varepsilon dxd\xi + (C_1+1)\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|f^\varepsilon dxd\xi\\ &\le \|f^\varepsilon(t)\|_p\|F\|_{p'} + (C_1+1)\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|f^\varepsilon dxd\xi\\ &\le \|f_0\|_p\|F\|_{p'} + (C_1+1)\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|f^\varepsilon dxd\xi,\end{aligned}$$

where C_1 is the positive constant from Lemma 3.2 (2). Then, Gronwall's inequality implies that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|f^\varepsilon dxd\xi \le C_T,\quad 0\le t\le T,\ \varepsilon>0, \tag{4.11}$$

where the constant C_T is independent of ε, i.e., $\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|f^\varepsilon dxd\xi$ is uniformly bounded.

Again, we use duality method to obtain uniform estimate of the second velocity moment. We have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3}\phi(x,\xi)f^\varepsilon dxd\xi &= \int_{\mathbb{R}^3\times\mathbb{R}^3}\phi(x,\xi)Q_{u_{f^\varepsilon}}(f^\varepsilon)dxd\xi\\ &\quad - \int_{\mathbb{R}^3\times\mathbb{R}^3}\phi(x,\xi)\xi\cdot\nabla_x f^\varepsilon dxd\xi - \int_{\mathbb{R}^3\times\mathbb{R}^3}\phi(x,\xi)F^\varepsilon\cdot\nabla_\xi f^\varepsilon dxd\xi,\end{aligned}$$

where $\phi(x,\xi)=\frac{(x-x_0)\cdot\xi}{(1+|x-x_0|^2)^{\frac12}}$. For the sake of simplicity, we denote the last equation by $I_1=I_4-I_2-I_3$. On one hand, by (4.11) we get the estimate of I_1 as follows:

$$\begin{aligned}\left|\int_0^T I_1dt\right| &= \left|\int_0^T\int_{\mathbb{R}^3\times\mathbb{R}^3}\phi(x,\xi)\partial_t f^\varepsilon dtdxd\xi\right|\\ &\le \int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|(f^\varepsilon(T,x,\xi)+f_0^\varepsilon(x,\xi))dxd\xi \le 2C_T.\end{aligned}$$

On the other hand, for any $t\in[0,\ T]$ we obtain due to Lemma 3.2 (2), (4.6), (4.9) and (4.11)

$$\begin{aligned}|I_4| &= \left|\int_{\mathbb{R}^3\times\mathbb{R}^3}\phi(x,\xi)Q_{u_{f^\varepsilon}}(f^\varepsilon)dxd\xi\right|\\ &\le \int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|Q_{u_{f^\varepsilon}}(f^\varepsilon)dxd\xi\\ &\le (C_1+1)\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|f^\varepsilon dxd\xi \le (C_1+1)\cdot C_T\end{aligned}$$

and

$$\begin{aligned}|I_3| &= \left|\int_{\mathbb{R}^3\times\mathbb{R}^3}\phi(x,\xi)\cdot\nabla_\xi F^\varepsilon f^\varepsilon dxd\xi\right|\\ &= \left|\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{\nabla_\xi((x-x_0)\cdot\xi)}{(1+|x-x_0|^2)^{\frac12}}\cdot F^\varepsilon f^\varepsilon dxd\xi\right|\\ &\le \|F^\varepsilon(t)\|_{p'}\|f^\varepsilon(t)\|_p \le \|F(t)\|_{p'}\|f_0\|_p.\end{aligned}$$

Lastly, for $K_x\subset\subset\mathbb{R}^3$, we choose $x_0\in K_x$, then

$$|I_2| \ge \frac{1}{(1+(\mathrm{diam}K_x)^2)^{\frac32}}\int_{K_x\times\mathbb{R}^3}|\xi|^2f^\varepsilon dxd\xi.$$

Integrating the equation $I_1=I_4-I_2-I_3$ with respect to t from 0 to T and using the above estimates, we obtain

$$\int_0^T dt\int_{K_x\times\mathbb{R}^3}|\xi|^2f^\varepsilon dxd\xi \le (1+(\mathrm{diam}K_x)^2)^{\frac32}(2C_T+\|F\|_{p'}\|f_0\|_p+TC_T(C_1+1)).$$

Namely, for any given $K_x \subset\subset \mathbb{R}^3$, there is a positive constant C depending only upon $\||\xi|f_0\|_1$, $\|f_0\|_p$, $\|F\|_{p'}$ and diamK_x such that

$$\int_0^T dt \int_{K_x\times\mathbb{R}^3} |\xi|^2 f^\varepsilon dx d\xi \le C. \tag{4.12}$$

From (4.9), (4.12) and Dunford–Pettis theorem, we know that for any given $R > 0$, the sequence f^ε is relatively compact in the weak topology of $L^1((0,\ T)\times B_R\times\mathbb{R}^3_\xi)$. It follows from Lemma 3.3 that for any $\varphi\in C_c^\infty(\mathbb{R}^3)$, we have (extracting a subsequence if necessary)

$$\int_{\mathbb{R}^3} f^\varepsilon\varphi(\xi)d\xi \to \int_{\mathbb{R}^3} f\varphi(\xi)d\xi, \quad \text{as } \varepsilon\to 0 \tag{4.13}$$

in the strong topology of $L^1_{\mathrm{loc}}([0,\ T]\times\mathbb{R}^3)$, where $f\ge 0$ is the weak limit of $f^\varepsilon \in L^1((0,\ T)\times B_R\times\mathbb{R}^3_\xi)$ $(\varepsilon>0)$. From (4.12) and (4.13), we obtain

$$\rho_{f^\varepsilon}\to\rho_f, \qquad \rho_{f^\varepsilon}u_{f^\varepsilon}\to\rho_f u_f, \quad \text{as } \varepsilon\to 0$$

in the strong topology of $L^1_{\mathrm{loc}}([0,\ T]\times\mathbb{R}^3)$.

On the other hand, if $1<p<\infty$, then for any $1<s\le p$ we have $f^\varepsilon\to f$ (as $\varepsilon\to 0$) in the weak topology of $L^s((0,\ T)\times\mathbb{R}^3\times\mathbb{R}^3)$, if $p=\infty$ then $f^\varepsilon\to f$ (as $\varepsilon\to 0$) in the weak $*$ topology of $L^\infty((0,\ T)\times\mathbb{R}^3\times\mathbb{R}^3)$. Consequently, we finally get (see the proof of (4.4)),

$$Q_{u_{f^\varepsilon}}(f^\varepsilon)\to Q_{u_f}(f) \text{ in } \mathcal{D}'((0,\ T)\times\mathbb{R}^3\times\mathbb{R}^3), \quad \text{as } \varepsilon\to 0.$$

Furthermore, (4.7) and $f^\varepsilon\to f$ (as $\varepsilon\to 0$) in the weak or weak $*$ topology of $L^p((0,\ T)\times\mathbb{R}^3\times\mathbb{R}^3)$ implies that

$$F^\varepsilon f^\varepsilon \to Ff \text{ in } \mathcal{D}'((0,\ T)\times\mathbb{R}^3\times\mathbb{R}^3), \quad \text{as } \varepsilon\to 0.$$

Due to these results we can pass to limit $\varepsilon\to 0$ in the weak form of (4.8), then we obtain that $f(t,x,\xi)$ is a nonnegative weak solution to the Cauchy problem (1.1). Finally, estimates (2.7) and (2.8) follow from (4.9) and (4.12). $\square$

Proof of Theorem 2.3. Let $F^\varepsilon = (F(t,x,\xi)\cdot\chi_\varepsilon(t,x,\xi)) * \eta_\varepsilon(t,x,\xi)$, where η_ε $(0<\varepsilon<1)$ is the standard mollifier and the convolution is in t,x,ξ, and where $\chi_\varepsilon\in C_c^\infty\left((\varepsilon,T-\varepsilon)\times B\left(0,\frac{2}{\varepsilon^r}\right)\times B\left(0,\frac{2}{\varepsilon^r}\right)\right)$ is a cutoff function such that $\chi_\varepsilon\equiv 1$ on $[2\varepsilon,T-2\varepsilon]\times B\left(0,\frac{1}{\varepsilon^r}\right)\times B\left(0,\frac{1}{\varepsilon^r}\right)$, and $0\le\chi_\varepsilon\le 1$ on $(\varepsilon,T-\varepsilon)\times B\left(0,\frac{2}{\varepsilon^r}\right)\times B\left(0,\frac{2}{\varepsilon^r}\right)\triangleq H_\varepsilon$ (here we set $r=7(p-1)$). Then, $F^\varepsilon(t,x,\xi)\in C([0,\ T];\ C_b^1(\mathbb{R}^3_x\times\mathbb{R}^3_\xi))$ and

$$\begin{aligned}\nabla_\xi\cdot F^\varepsilon &= \nabla_\xi\cdot[(F\chi_\varepsilon)*\eta_\varepsilon]\\ &= [\nabla_\xi\cdot(F\chi_\varepsilon)]*\eta_\varepsilon = (F\cdot\nabla_\xi\chi_\varepsilon)*\eta_\varepsilon\\ &= (F_1\cdot\nabla_\xi\chi_\varepsilon)*\eta_\varepsilon + (F_2\cdot\nabla_\xi\chi_\varepsilon)*\eta_\varepsilon.\end{aligned}$$

Consequently,

$$\begin{aligned}\|\nabla_\xi\cdot F^\varepsilon(t)\|_\infty &\le \|F_1\|_{p'}\cdot\|\eta_\varepsilon\|_p\cdot\|\nabla_\xi\chi_\varepsilon\|_\infty + \sup_{(t,x,\xi)\in H_\varepsilon}(1+|x|+|\xi|)\left\|\frac{|F_2|}{1+|x|+|\xi|}\right\|_\infty\|\nabla_\xi\chi_\varepsilon\|_\infty\\ &\le\left[\|F_1\|_{p'}\|\eta_\varepsilon\|_p+\left(1+\frac{2}{\varepsilon^r}+\frac{2}{\varepsilon^r}\right)\|M\|_\infty\right]\cdot C\cdot\varepsilon^r\\ &= C\left[\varepsilon^r\|F_1\|_{p'}\|\eta_\varepsilon\|_p+(4+\varepsilon^r)\|M\|_\infty\right].\end{aligned}$$

So, we get

$$\int_0^T\|\nabla_\xi\cdot F^\varepsilon(t)\|_\infty dt\le C'\left[\|F_1\|_{p'}+T\|M\|_\infty\right], \tag{4.14}$$

where the positive constant C' is independent of ε. Hence, $\nabla_\xi\cdot F^\varepsilon\in L^1([0,\ T];\ L^\infty(\mathbb{R}^3\times\mathbb{R}^3))$. It follows from Remark 2.1 that the Cauchy problem

$$\begin{cases}\partial_t f^\varepsilon+\xi\cdot\nabla_x f^\varepsilon+F^\varepsilon(t,x,\xi)\cdot\nabla_\xi f^\varepsilon = Q_{u_{f^\varepsilon}}(f^\varepsilon),\\ f^\varepsilon(0,x,\xi)=f_0(x,\xi)\end{cases} \tag{4.15}$$

has a nonnegative solution f^ε such that

$$\|f^\varepsilon(t)\|_1\le\|f_0\|_1\cdot\exp\left(\int_0^t\|\nabla_\xi\cdot F^\varepsilon(\tau)\|_\infty d\tau\right), \quad 0\le t\le T$$

and

$$\|f^\varepsilon(t)\|_p \le \|f_0\|_p \cdot \exp\left(\frac{1}{p}\int_0^t \|\nabla_\xi \cdot F(^\varepsilon \tau)\|_\infty d\tau\right), \quad 0 \le t \le T.$$

Combining these estimates with (4.14), we get as in the proof of Theorem 2.2

$$\|f^\varepsilon(t)\|_1 \le \|f_0\|_1 \cdot \exp\left(C'\left[\|F_1\|_{p'} + T\|M\|_\infty\right]\right), \quad 0 \le t \le T \tag{4.16}$$

and

$$\|f^\varepsilon(t)\|_p \le \|f_0\|_p \cdot \exp\left(\frac{C'}{p}\left[\|F_1\|_{p'} + T\|M\|_\infty\right]\right), \quad 0 \le t \le T. \tag{4.17}$$

On the other hand, multiplying both sides of the approximate problem (4.15) by ξ and then integrating against x and ξ, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi| f^\varepsilon dxd\xi &= \int_{\mathbb{R}^3\times\mathbb{R}^3} f^\varepsilon \nabla_\xi \cdot (|\xi| F^\varepsilon) dxd\xi + \int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi| Q_{u_{f^\varepsilon}}(f^\varepsilon) dxd\xi \\ &\le \|\nabla_\xi \cdot F^\varepsilon(t)\|_\infty \cdot \int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi| f^\varepsilon dxd\xi + \int_{\mathbb{R}^3\times\mathbb{R}^3} |F^\varepsilon| f^\varepsilon dxd\xi + (C_1+1)\int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi| f^\varepsilon dxd\xi,\end{aligned}$$

in which the term $\int_{\mathbb{R}^3\times\mathbb{R}^3} |F^\varepsilon| f^\varepsilon dxd\xi$ can be estimated as follows

$$\begin{aligned}\int_{\mathbb{R}^3\times\mathbb{R}^3} |F^\varepsilon| f^\varepsilon dxd\xi &\le \int_{\mathbb{R}^3\times\mathbb{R}^3} |F_1^\varepsilon| f^\varepsilon dxd\xi + \int_{\mathbb{R}^3\times\mathbb{R}^3} |F_2^\varepsilon| f^\varepsilon dxd\xi \\ &\le \|F_1^\varepsilon(t)\|_{p'}\|f^\varepsilon(t)\|_p + \|M(t)\|_\infty \int_{\mathbb{R}^3\times\mathbb{R}^3} (1+|x|+|\xi|) f^\varepsilon dxd\xi.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi| f^\varepsilon dxd\xi &\le \|\nabla_\xi \cdot F^\varepsilon(t)\|_\infty \cdot \int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi| f^\varepsilon dxd\xi + \|F_1^\varepsilon(t)\|_{p'}\|f^\varepsilon(t)\|_p \\ &\quad + \|M(t)\|_\infty \int_{\mathbb{R}^3\times\mathbb{R}^3} (1+|x|+|\xi|) f^\varepsilon dxd\xi + (C_1+1)\int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi| f^\varepsilon dxd\xi.\end{aligned}$$

Similarly, multiplying both sides of the approximate problem by x and then integrating against x and ξ, we have

$$\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3} |x| f^\varepsilon dxd\xi \le \int_{\mathbb{R}^3\times\mathbb{R}^3} |\xi| f^\varepsilon dxd\xi + \|\nabla_\xi \cdot F^\varepsilon(t)\|_\infty \int_{\mathbb{R}^3\times\mathbb{R}^3} |x| f^\varepsilon dxd\xi.$$

Combining the above two differential inequality, we discover

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3} (|x|+|\xi|) f^\varepsilon dxd\xi &\le (\|\nabla_\xi \cdot F^\varepsilon(t)\|_\infty + \|M(t)\|_\infty + C_1 + 2)\int_{\mathbb{R}^3\times\mathbb{R}^3} (|x|+|\xi|) f^\varepsilon dxd\xi \\ &\quad + \|F_1(t)\|_{p'}\|f_0\|_p + \|M(t)\|_\infty \|f_0\|_1.\end{aligned}$$

It follows from (4.14) and Gronwall's inequality that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} (|x|+|\xi|) f^\varepsilon(t,x,\xi) dxd\xi \le C_T, \quad 0 \le t \le T, \tag{4.18}$$

where the positive constant C_T is independent of ε.

Similar to the proof of Theorem 2.2, multiplying both sides of (4.15) by $\phi(x,\xi) = \frac{(x-x_0)\cdot\xi}{(1+|x-x_0|^2)^{\frac{1}{2}}}$ and then integrating against x and ξ, we obtain

$$\begin{aligned}\int_{\mathbb{R}^3\times\mathbb{R}^3} \phi(x,\xi)\partial_t f^\varepsilon dxd\xi &= \int_{\mathbb{R}^3\times\mathbb{R}^3} \phi(x,\xi) Q_{u_{f^\varepsilon}}(f^\varepsilon) dxd\xi \\ &\quad - \int_{\mathbb{R}^3\times\mathbb{R}^3} \phi(x,\xi)\xi \cdot \nabla_x f^\varepsilon dxd\xi - \int_{\mathbb{R}^3\times\mathbb{R}^3} \phi(x,\xi) F^\varepsilon \cdot \nabla_\xi f^\varepsilon dxd\xi,\end{aligned}$$

which is denoted by $I_1 = I_4 - I_2 - I_3$. The integrals $I_1,\ I_2$ and I_4 have been estimated in the proof of Theorem 2.2, and the integral I_3 can be estimated as follows. First,

$$\begin{aligned}\left|\int_0^T I_3 dt\right| &= \left|\int_0^T dt\int_{\mathbb{R}^3\times\mathbb{R}^3}\phi(x,\xi)F^\varepsilon\cdot\nabla_\xi f^\varepsilon dxd\xi\right|\\ &\le \left|\int_0^T dt\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(x-x_0)\cdot\xi}{(1+|x-x_0|^2)^{\frac12}}(\nabla_\xi\cdot F^\varepsilon)f^\varepsilon dxd\xi\right| + \left|\int_0^T dt\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{(x-x_0)}{(1+|x-x_0|^2)^{\frac12}}\cdot F^\varepsilon f^\varepsilon dxd\xi\right|\\ &\le \int_0^T dt\|\nabla_\xi\cdot F^\varepsilon(t)\|_\infty\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|f^\varepsilon dxd\xi + \int_0^T dt\int_{\mathbb{R}^3\times\mathbb{R}^3}|F^\varepsilon|f^\varepsilon dxd\xi.\end{aligned}$$

Second, by (4.14) and (4.18), we get

$$\int_0^T dt\|\nabla_\xi\cdot F^\varepsilon(t)\|_\infty\int_{\mathbb{R}^3\times\mathbb{R}^3}|\xi|f^\varepsilon dxd\xi \le C_T C'\left[\|F_1\|_{p'} + T\|M\|_\infty\right].$$

Third, since

$$\begin{aligned}|[(1+|x|+|\xi|)\chi_\varepsilon]*\eta_\varepsilon| &\le |(1+|x|+|\xi|)*\eta_\varepsilon|\\ &\le 1+\int_{-\infty}^{\infty}dt\int_{\mathbb{R}^3\times\mathbb{R}^3}(|x-y|+|\xi-\eta|)\cdot\eta_\varepsilon(t,y,\eta)dyd\eta\\ &\le 1+\int_{-\infty}^{\infty}dt\int_{\mathbb{R}^3\times\mathbb{R}^3}(|x|+|y|+|\xi|+|\eta|)\cdot\eta_\varepsilon(t,y,\eta)dyd\eta\\ &\le 1+|x|+|\xi|+\int_{-\infty}^{\infty}dt\int_{\mathbb{R}^3\times\mathbb{R}^3}(|y|+|\eta|)\cdot\eta_\varepsilon(t,y,\eta)dyd\eta\\ &\le 1+|x|+|\xi|+2\varepsilon,\end{aligned}$$

in consideration of (4.16)–(4.18) we have

$$\begin{aligned}\int_0^T dt\int_{\mathbb{R}^3\times\mathbb{R}^3}|F^\varepsilon|f^\varepsilon dxd\xi &= \int_0^T dt\int_{\mathbb{R}^3\times\mathbb{R}^3}|[(F_1+F_2)\chi_\varepsilon]*\eta_\varepsilon|f^\varepsilon dxd\xi\\ &= \int_0^T dt\int_{\mathbb{R}^3\times\mathbb{R}^3}|(F_1\chi_\varepsilon)*\eta_\varepsilon|f^\varepsilon dxd\xi + \int_0^T dt\int_{\mathbb{R}^3\times\mathbb{R}^3}|(F_2\chi_\varepsilon)*\eta_\varepsilon|f^\varepsilon dxd\xi\\ &\le \int_0^T \|F_1\chi_\varepsilon(t)\|_{p'}\|f^\varepsilon(t)\|_p dt + \int_0^T dt\int_{\mathbb{R}^3\times\mathbb{R}^3}|[M(1+|x|+|\xi|)\chi_\varepsilon]*\eta_\varepsilon|f^\varepsilon dxd\xi\\ &\le \int_0^T \|F_1(t)\|_{p'}\|f^\varepsilon(t)\|_p dt + \|M\|_\infty\int_0^T dt\int_{\mathbb{R}^3\times\mathbb{R}^3}|[(1+|x|+|\xi|)\chi_\varepsilon]*\eta_\varepsilon|f^\varepsilon dxd\xi\\ &\le \int_0^T \|F_1(t)\|_{p'}\|f^\varepsilon(t)\|_p dt + \|M\|_\infty\int_0^T dt\int_{\mathbb{R}^3\times\mathbb{R}^3}(1+2\varepsilon+|x|+|\xi|)f^\varepsilon dxd\xi\\ &\le T^{1/p}\|F_1\|_{p'}\|f_0\|_p\cdot\exp\left(\frac{C'}{p}\left[\|F_1\|_{p'}+T\|M\|_\infty\right]\right) + T\Big[3\|f_0\|_1\cdot\exp\left(C'\left[\|F_1\|_{p'}+T\|M\|_\infty\right]\right)+C_T\Big]\|M\|_\infty.\end{aligned}$$

To sum up, we obtain

$$\left|\int_0^T I_3 dt\right| \le C_T',$$

where

$$\begin{aligned}C_T' &= C_T C'\left[\|F_1\|_{p'}+T\|M\|_\infty\right] + T^{1/p}\|F_1\|_{p'}\|f_0\|_p\cdot\exp\left(\frac{C'}{p}\left[\|F_1\|_{p'}+T\|M\|_\infty\right]\right)\\ &\quad + T\Big[3\|f_0\|_1\cdot\exp\left(C'\left[\|F_1\|_{p'}+T\|M\|_\infty\right]\right)+C_T\Big]\|M\|_\infty.\end{aligned}$$

Using the above estimates for I_k ($k = 1, 2, 3, 4$), similar to the proof of (4.12) we can show that for any fixed $K_x \subset\subset \mathbb{R}^3$,

$$\int_0^T dt\int_{K_x\times\mathbb{R}^3}|\xi|^2 f^\varepsilon dxd\xi \le C, \tag{4.19}$$

where the positive constant C depends continuously on $\mathrm{diam}K_x,\ f_0,\ T$ and F, but is independent of ε.

Again, similar to the proof of Theorem 2.2, we can show by (4.16), (4.17), (4.19) and Lemma 3.3 that for any $1 < s \le p$ we have $f^\varepsilon \to f$ (as $\varepsilon \to 0$) in the weak topology of $L^s((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$, and $\rho_{f^\varepsilon} \to \rho_f$, $\rho_{f^\varepsilon} u_{f^\varepsilon} \to \rho_f u_f$ (as $\varepsilon \to 0$) in the strong topology of $L^1_{\mathrm{loc}}([0, T] \times \mathbb{R}^3)$. Here $f \ge 0$ is the weak limit of $\{f^\varepsilon : \varepsilon > 0\}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^3_x; L^1((0, T) \times \mathbb{R}^3_\xi))$. As a consequence, we have

$$Q_{u_{f^\varepsilon}}(f^\varepsilon) \to Q_{u_f}(f) \text{ in } \mathcal{D}'((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3), \quad \text{as } \varepsilon \to 0. \tag{4.20}$$

Next, we show that

$$F^\varepsilon f^\varepsilon \to Ff \text{ in } \mathcal{D}'((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3), \quad \text{as } \varepsilon \to 0. \tag{4.21}$$

Actually, by the fact that $F_1 \in L^q([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3)$ and the proof of Theorem 2.2 it is obvious that $F_1^\varepsilon f^\varepsilon \to F_1 f$ (as $\varepsilon \to 0$) in $\mathcal{D}'((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$. So, in order to finish the proof of (4.21) we only need to show $F_2^\varepsilon f^\varepsilon \to F_2 f$ (as $\varepsilon \to 0$) in $\mathcal{D}'((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$. In fact, the fact that $\frac{F_2}{1+|x|+|\xi|} \in L^\infty([0, T]; L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ implies that $F_2 \in L^\infty([0, T]; L^\infty_{\mathrm{loc}}(\mathbb{R}^3 \times \mathbb{R}^3))$. Hence, $F_2^\varepsilon \to F_2$ (as $\varepsilon \to 0$) strongly in $L^{p'}_{\mathrm{loc}}([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3)$, which and $f^\varepsilon \to f$ (as $\varepsilon \to 0$) in the weak topology of $L^p((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ obviously imply that $F_2^\varepsilon f^\varepsilon \to F_2 f$ (as $\varepsilon \to 0$) in the sense of distributions.

Combining (4.20), (4.21) and $f^\varepsilon \to f$ weakly as $\varepsilon \to 0$, we can pass to limit $\varepsilon \to 0$ in (4.15) and obtain that $f(t, x, \xi)$ is a nonnegative solution to (1.1).

In order to finish the proof, we have to show the estimates claimed by this theorem. Actually, (2.11) is obviously implied by (4.19) through a limit process, and the mass conservation in (2.11) can be easily proved. Now, we show the second part in (2.11). Multiplying both sides of (1.1) by pf^{p-1} and then integrating against x and ξ, we get

$$\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3} f^p dx d\xi + \int_{\mathbb{R}^3\times\mathbb{R}^3} [\xi \cdot \nabla_x f^p + F \cdot \nabla_\xi f^p] dx d\xi = p\int_{\mathbb{R}^3\times\mathbb{R}^3} f^{p-1} Q_{u_f}(f) dx d\xi.$$

Since $\nabla_\xi \cdot F = 0$, the second term in the right hand side disappears by integration by parts. Consequently,

$$\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3} f^p dx d\xi = p\int_{\mathbb{R}^3\times\mathbb{R}^3} f^{p-1} Q_{u_f}(f) dx d\xi.$$

On the other hand, it follows from Lemma 3.2 (1) and Hölder's inequality that

$$\begin{aligned}
\int_{\mathbb{R}^3\times\mathbb{R}^3} f^{p-1} Q_{u_f}(f) dx d\xi &= \int_{\mathbb{R}^3\times\mathbb{R}^3} f^{p-1} P_{u_f}(f) dx d\xi - \int_{\mathbb{R}^3\times\mathbb{R}^3} f^p dx d\xi \\
&\le \left(\int_{\mathbb{R}^3\times\mathbb{R}^3} f^p dx d\xi\right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^3\times\mathbb{R}^3} [P_{u_f}(f)]^p dx d\xi\right)^{\frac{1}{p}} - \int_{\mathbb{R}^3\times\mathbb{R}^3} f^p dx d\xi \\
&\le \left(\int_{\mathbb{R}^3\times\mathbb{R}^3} f^p dx d\xi\right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^3\times\mathbb{R}^3} f^p dx d\xi\right)^{\frac{1}{p}} - \int_{\mathbb{R}^3\times\mathbb{R}^3} f^p dx d\xi \\
&= 0.
\end{aligned}$$

So, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3} f^p dx d\xi \le 0, \quad 0 \le t \le T,$$

which obviously implies that $\|f(t)\|_p \le \|f_0\|_p$ $(0 \le t \le T)$. The proof is completed. □

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