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Mean separations in Banach spaces under abstract interpolation and extrapolation

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ABSTRACT

We introduce the mean separation for bounded sequences in Banach spaces and the related seminorm for bounded linear operators. The introduced quantities are closely related to the geometric characterizations of the Banach–Saks property and the alternate signs Banach–Saks property. We investigate the behavior of the mean separations for a class of operators between vector-valued Banach sequence spaces $E(X_\nu)$, providing that a Banach sequence lattice E has the Banach–Saks property. We estimate the mean separations for operators under abstract interpolation and extrapolation methods. In particular, we obtain quantitative and qualitative results on the heredity of the Banach–Saks properties under these methods.

1. Introduction

The distances between various combinations of sequence elements are used in the geometric descriptions of several properties of Banach spaces and operators. For instance, the close packing of equal spheres in Banach spaces can be characterized by Kottman's constant [23] which employs the separated elements of sequences in the unit ball. A similar idea applied to bounded subsets of a Banach space leads to the separation measure of noncompactness and the related seminorm for operators (see [1,2]). Reflexivity of spaces and weak compactness of operators are geometrically described by James' condition [20] on the separated convex hulls of sequence elements (see also [27]). The mean separations, which we consider in this paper, appear in the context of the Banach–Saks property and its variant with alternate signs. Recall that a bounded linear operator T acting between Banach spaces X and Y has the alternate signs Banach–Saks (ABS) property, if every bounded sequence (x_n) in X contains a subsequence (x_n') such that the Cesàro means of $((-1)^n Tx_n')$ converge in Y. If in this definition $((-1)^n Tx_n')$ is replaced by (Tx_n'), then T is said to have the Banach–Saks (BS) property. The ABS and BS properties of a Banach space X are defined by the respective property of the identity operator on X.

A naturally arising question is the heredity of properties of a Banach space X by the spaces built on X or by operators between such spaces. The problem for the BS property was widely studied. Partington [38] proved that direct sums of Banach spaces preserve the BS property, if the sums are built on a Banach space with a hyperorthogonal basis and the BS property. In particular, the BS property passes from X to $l_p(X)$ with $1 < p < \infty$. Bourgain [16] constructed a Banach space X with the BS property such that $L_2(X)$ does not have the BS property (see also [39]). Bourgain proved also that $L_p(X)$ with $1 < p < \infty$ has the BS property if and only if X has the Komlós property. If E is a reflexive Köthe function space with the subsequence splitting property and X has the Komlós property, then the Köthe–Bochner function space $E(X)$ has the BS property [28, Theorem 5.6.7].

In this paper, we define the mean separations for bounded sequences in Banach spaces. The notion we introduce is flexible and can be used as a separation of a mean from zero or between successive means. In particular, the mean separations applied

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to operators give quantities which include the measures of deviation from the ABS and BS properties considered in [24,25]. We investigate the behavior of these quantities for a special class of operators between $E(X_\nu)$ sequence spaces over a Banach sequence lattice E with the BS property, where (X_ν) is a sequence of Banach spaces indexed by integers. Other properties of this type of spaces, namely, of the Köthe–Bochner sequence spaces $E(X)$, were investigated in [19,22]. The class of operators we consider is characteristic for interpolation and extrapolation methods. Our main result states that the means in $E(X_\nu)$ cannot be separated by an operator more than in any space X_ν. This fact is decisive for the heredity of the properties we study.

The result obtained for operators between $E(X_\nu)$ sequence spaces is applied to the abstract interpolation method of Nilsson [36]. Recall that the basic facts on the behavior of the ABS and BS properties under the real interpolation method of Lions and Peetre [29], together with factorization theorems for these properties, were established by Beauzamy [3,4]. Heinrich [18] proved that the BS property of the embedding $I: A_0 \cap A_1 \to A_0 + A_1$ implies the BS property of the real interpolation space $(A_0, A_1)_{\theta,p}$ for all $0 < \theta < 1$ and $1 < p < \infty$. The interpolation J- and K-methods presented in [36] and developed also in [5] are generalizations of the classical methods of Lions and Peetre [29]. The norm of a weighted space L_p, used in the real interpolation spaces $(A_0, A_1)_{\theta,p}$, is replaced here by a more general lattice norm. In this way, the abstract J- and K-spaces cover also another extension of $(A_0, A_1)_{\theta,p}$ in which t^θ is replaced by a certain function parameter. Another application of the result for $E(X_\nu)$ sequence spaces concerns extrapolation of spaces and operators. The extrapolation methods, which are strongly connected with interpolation, were investigated by Jawerth and Milman [21]. The extrapolation spaces were also studied as the logarithmic spaces by Edmunds and Triebel [11]. In the case of extrapolation, we investigate whether a property is transmitted from a family of spaces or operators to extrapolation spaces or operators between such spaces.

The approach we apply in this paper, besides quantitative and qualitative results in a general setting, provides a uniform view on the heredity of the ABS and BS properties under interpolation and extrapolation methods.

By $\mathcal{L}(X, Y)$ we will denote the space of all bounded linear operators acting between Banach spaces X and Y. The open unit ball of X will be denoted by $\mathsf{B}(X)$. The number of elements of a finite subset $A \subset \mathbb{N} = \{1, 2, 3, \ldots\}$ will be denoted by $|A|$.

2. Mean separations for sequences in Banach spaces

The mean separations we are going to introduce in this section are motivated by the following result of Beauzamy [3]: a Banach space X does not have the ABS property if and only if there exist $\delta > 0$ and a bounded sequence (x_n) in X such that $\left\|\sum_{n\in A} \epsilon_n x_n\right\| \geq \delta\,|A|$ for all finite subsets $A \subset \mathbb{N}$ and all sequences of signs (ϵ_n), $\epsilon_n = \pm 1$. The result was applied in a quantitative approach to the real interpolation of the ABS and BS properties in [24,25]. A key fact for the interpolation estimates is that the quantity $\left\|\sum_{n\in A} \epsilon_n x_n / |A|\right\|$ can be stabilized by passing from (x_n) to a certain sequence of means built on successive equipollent blocks of (x_n) (see Proposition 2.3 of [24]). This can be done with the use of the spreading models of Brunel and Sucheston [6]. Beauzamy's results and the techniques worked out in [24,25] will be the framework for a more general approach presented in this paper.

Let X and Y be Banach spaces. By $\mathcal{G}_0$ we denote the set of all sequences $(\epsilon_n)_{n\in G}$ of signs $\epsilon_n = \pm 1$ over all finite subsets $G \subset \mathbb{N}$. Assume that $\mathcal{G} \subset \mathcal{G}_0$ and fix δ equal to 1 or 2.

Definition 1. Let (x_n) be a bounded sequence in a Banach space. The *mean separation* of (x_n) over $\mathcal{G}$ is defined by

$$\phi(x_n) = \inf\left\{\left\|\delta\,|G|^{-1}\sum_{n\in G}\epsilon_n x_n\right\| : (\epsilon_n)_{n\in G} \in \mathcal{G}\right\}$$

and the related quantity for an operator $T \in \mathcal{L}(X, Y)$ by

$$\Phi(T) = \sup\left\{\phi(Tx_n) : (x_n) \subset \mathsf{B}(X)\right\}.$$

Let $\mathcal{H}_0$ denote the set of all sequences $(\zeta_i)_{i\in\bigcup H_n}$ of signs $\zeta_i = \pm 1$ over all subsets $\bigcup_{n\geqslant 1} H_n \subset \mathbb{N}$ such that (H_n) is a sequence of equipollent sets $H_n \subset \mathbb{N}$ with $\max H_n < \min H_{n+1}$. Let $\mathcal{H} \subset \mathcal{H}_0$. For sequences (x_n) and (y_n) in a Banach space, we write $(y_n) \succ_{\mathcal{H}} (x_n)$, if there exists $(\zeta_i)_{i\in\bigcup H_n} \in \mathcal{H}$ such that $y_n = m^{-1}\sum_{i\in H_n}\zeta_i x_i$ with $m = |H_n|$ for all n.

We will deal only with those subsets $\mathcal{G} \subset \mathcal{G}_0$, for which there exists $\mathcal{H} \subset \mathcal{H}_0$ such that ϕ over $\mathcal{G}$ satisfies a stability condition with respect to the relation $\succ_{\mathcal{H}}$. A pair $(\mathcal{G}, \mathcal{H})$ will be called *stable* for ϕ in a Banach space X, if the following conditions hold:

(s1) $\mathcal{H}$ contains all $(\zeta_i)_{i\in\bigcup H_n} \in \mathcal{H}_0$ such that $\zeta_i = 1$ for all $i \in \bigcup_{n\geqslant 1} H_n$.

(s2) If $(\epsilon_n)_{n\in G} \in \mathcal{G}$, $(\zeta_i)_{i\in\bigcup H_n} \in \mathcal{H}$, $G' = \bigcup_{n\in G} H_n$ and $\zeta'_i = \epsilon_n\zeta_i$ for $n \in G$ and $i \in H_n$, then $(\zeta'_i)_{i\in G'} \in \mathcal{G}$.

(s3) For every bounded sequence (x_n) in X, there is $(y_n) \succ_{\mathcal{H}} (x_n)$ such that for all $(\epsilon_n)_{n\in G} \in \mathcal{G}$,

$$\left\|\delta\,|G|^{-1}\sum_{n\in G}\epsilon_n y_n\right\| \leqslant \phi(y_n) + \varepsilon.$$

The above conditions imply several properties which will be frequently used in the proofs. In particular, by (s1), if (x'_n) is a subsequence of (x_n), then $(x'_n) \succ_{\mathcal{H}} (x_n)$. By (s2), if $(y_n) \succ_{\mathcal{H}} (x_n)$, then $\phi(y_n) \geqslant \phi(x_n)$. Combining (s2) and (s3), we conclude that multiple averaging with the use of the relation $\succ_{\mathcal{H}}$ keeps the stability of ϕ (note that the transitivity of $\succ_{\mathcal{H}}$ is not assumed).

Corollary 2. *Let $(\mathcal{G}, \mathcal{H})$ be a stable pair for ϕ in X. If $(y_n) \succ_{\mathcal{H}} (x_n)$ satisfies condition (s3) and $(y_n^i) \succ_{\mathcal{H}} (y_n^{i-1})$ for $i = 1, 2, \ldots, k$, where $k \in \mathbb{N}$ and $(y_n^0) = (y_n)$, then for all $(\epsilon_n)_{n\in G} \in \mathcal{G}$,*

$$\left\| \delta\, |G|^{-1} \sum_{n\in G} \epsilon_n y_n^i \right\| \leqslant \phi(y_n) + \varepsilon, \quad i = 1, 2, \ldots, k.$$

If $(\mathcal{G}, \mathcal{H})$ is stable for ϕ in Y, applying Corollary 2 we can prove that $\Phi(S + T) \leqslant \Phi(S) + \Phi(T)$ for all $S, T \in \mathcal{L}(X, Y)$ (see the proof of Proposition 2.5 in [24]). Clearly, $\Phi(\lambda T) = |\lambda|\, \Phi(T)$ for every scalar λ. Thus Φ is a seminorm in $\mathcal{L}(X, Y)$.

From now on, we consider only those pairs $(\mathcal{G}, \mathcal{H})$, which are stable for ϕ in every Banach space. The examples of such pairs are the following:

1. $(\mathcal{G}_0, \mathcal{H}_0)$.
2. $(\mathcal{G}_1, \mathcal{H}_1)$, where $\mathcal{G}_1$ and $\mathcal{H}_1$ consist of all respectively $(\epsilon_n)_{n\in G} \in \mathcal{G}_0$ and $(\zeta_i)_{i\in\bigcup H_n} \in \mathcal{H}_0$ with $\epsilon_n = 1$ for all $n \in G$ and $\zeta_i = 1$ for all $i \in \bigcup_{n\geqslant 1} H_n$.
3. $(\mathcal{G}_2, \mathcal{H}_2)$, where $\mathcal{H}_2 = \mathcal{H}_1$ and $\mathcal{G}_2$ consists of all $(\epsilon_n)_{n\in G} \in \mathcal{G}_0$ such that $G = E \cup F$ with $|E| = |F|$ and $\max E < \min F$, $\epsilon_n = 1$ for $n \in E$, $\epsilon_n = -1$ for $n \in F$.

Indeed, (s1) and (s2) are obviously satisfied. The proofs of (s3) for $(\mathcal{G}_i, \mathcal{H}_i)$, $i = 0, 1, 2$, are not trivial and depend on the Ramsey theorem (see Proposition 2.3 of [24] for $i = 0, 1$ and Proposition 3.1 of [25] for $i = 2$).

The pair $(\mathcal{G}_0, \mathcal{H}_0)$ corresponds to the ABS property. By the aforesaid Beauzamy's result of [3], considering ϕ over $\mathcal{G}_0$ with $\delta = 1$, we obtain that a Banach space X has the ABS property if and only if $\phi(x_n) = 0$ for every bounded sequence (x_n) in X. By Proposition 2.5 of [24], Φ is a seminorm in $\mathcal{L}(X, Y)$ such that $T \in \mathcal{L}(X, Y)$ has the ABS property if and only if $\Phi(T) = 0$.

The BS property is related to $(\mathcal{G}_2, \mathcal{H}_2)$. The mean separation $\phi(x_n)$ over $\mathcal{G}_2$ with $\delta = 2$ coincides with the arithmetic separation $\operatorname{asep}(x_n)$ introduced in [25], which is a counterpart for the BS property of James' characterization of weak noncompactness based on the separated convex hulls of sequence elements. By Theorem 2.3 of [25], a Banach space X has the BS property if and only if $\phi(x_n) = 0$ for every bounded sequence (x_n) in X. An operator $T \in \mathcal{L}(X, Y)$ has the BS property if and only if $\Phi(T) = 0$. If ϕ is taken over $\mathcal{G}_2$ with $\delta = 1$, we obtain the index $h(X, (BS_\alpha))$ of a Banach space X introduced by Ostrovskii [37].

3. Mean separations in $E(X_\nu)$ sequence spaces

In this paper, by a *Banach sequence lattice* we mean a Banach space E of real-valued functions on $\mathbb{Z}$ with the natural partial order such that all functions on $\mathbb{Z}$ with finite support belong to E and if $x = (x(\nu)) \in E$ and $|y(\nu)| \leqslant |x(\nu)|$ for every $\nu \in \mathbb{Z}$, then $y = (y(\nu)) \in E$ and $\|y\|_E \leqslant \|x\|_E$.

Let $(X_\nu)_{\nu\in\mathbb{Z}}$ be a sequence of Banach spaces. By $E(X_\nu)$ we mean the Banach space of all $x = (x(\nu))_{\nu\in\mathbb{Z}}$ such that $x(\nu) \in X_\nu$ for every $\nu \in \mathbb{Z}$ and

$$\|x\|_{E(X_\nu)} = \left\| (\|x(\nu)\|_{X_\nu}) \right\|_E < \infty.$$

A Banach space X renormed by $t\|\cdot\|$ with $t > 0$ will be denoted by tX. A scalar-valued l_p space with weight (t_ν) will be denoted by $l_p(t_\nu)$. Given $x = (x(\nu)) \in E(X_\nu)$ and $r \in \mathbb{N}$ let

$$P_r x = (\cdots 0, 0, x(-r), \ldots, x(r), 0, 0, \ldots), \quad Q_r x = x - P_r x.$$

The interpolation and extrapolation results presented in this paper rely on the behavior of Φ on a special class of operators acting between $E(X_\nu)$ spaces. The next theorem is a far generalization of Partington's [38] result concerning the heredity of the BS property for direct sums of Banach spaces. Our result includes also simultaneously Theorems 3.2 of [24] and 4.4 of [25] proved for operators between $l_p(X)$ spaces and the seminorms related respectively to the ABS and BS properties.

The next lemma allows us to reduce considerations to finitely many coordinates of $E(X_\nu)$. This argument was used also in the proof of Theorem 3 of [38] for direct sums of Banach spaces.

Lemma 3. *Let (x_n) be a bounded sequence in $E(X_\nu)$, where E is a Banach sequence lattice with the BS property. Then for every $\varepsilon > 0$ there exist $r \in \mathbb{N}$ and a sequence $(y_n) \succ_{\mathcal{H}_1} (x_n)$ such that for all $(\epsilon_n)_{n\in G} \in \mathcal{G}_0$,*

$$\left\| |G|^{-1} \sum_{n\in G} \epsilon_n Q_r y_n \right\|_{E(X_\nu)} < \varepsilon.$$

Proof. Let $x_n = (x_n(\nu)) \in E(X_\nu)$ and $t_n = (\|x_n(\nu)\|_{X_\nu}) \in E$. Since E has the BS property, by Erdös–Magidor's theorem [13], there exists a subsequence (t_n') of (t_n) such that the Cesàro means of all subsequences of (t_n') converge to the same limit t in E. Clearly, for every sequence $(s_n) \succ_{\mathcal{H}_1} (t_n')$,

$$\inf \left\{ \left\| |G|^{-1} \sum_{n\in G} s_n - t \right\|_E : |G| < \infty \right\} = 0.$$

By Proposition 2.3 of [24], a sequence $(s_n) \succ_{\mathcal{H}_1} (t'_n)$ can be taken so that for every finite $G \subset \mathbb{N}$,

$$\left\| |G|^{-1} \sum_{n \in G} s_n - t \right\|_E < \frac{\varepsilon}{2}.$$

Consequently, for every $r \in \mathbb{N}$,

$$\left\| Q_r \left(|G|^{-1} \sum_{n \in G} s_n \right) \right\|_E < \frac{\varepsilon}{2} + \|Q_r t\|_E .$$

Let (H_n) be a sequence of finite subsets of $\mathbb{N}$ with $|H_n| = m$ and $\max H_n < \min H_{n+1}$ for all n such that $s_n = m^{-1} \sum_{i \in H_n} t'_i$. We extract the subsequence (x'_n) of (x_n) so that $t'_n = \left(\left\| x'_n(\nu) \right\|_{X_\nu} \right)$ and we put $y_n = m^{-1} \sum_{i \in H_n} x'_i$. Since E is reflexive, the norm of E is order continuous (see [40]). Hence there is $r \in \mathbb{N}$ such that $\|Q_r t\|_E < \varepsilon/2$. Since E is a lattice, for every $(\epsilon_n)_{n \in G} \in \mathcal{G}_0$ we get

$$\begin{aligned} \varepsilon &> \left\| Q_r \left(|G|^{-1} \sum_{n \in G} s_n \right) \right\|_E = \left\| Q_r \left(|G|^{-1} \sum_{n \in G} \frac{1}{m} \sum_{i \in H_n} \left\| x'_i(\nu) \right\|_{X_\nu} \right) \right\|_E \\ &\geqslant \left\| Q_r \left(|G|^{-1} \sum_{n \in G} \|y_n(\nu)\|_{X_\nu} \right) \right\|_E \geqslant \left\| Q_r \left(\left\| |G|^{-1} \sum_{n \in G} \epsilon_n y_n(\nu) \right\|_{X_\nu} \right) \right\|_E = \left\| |G|^{-1} \sum_{n \in G} \epsilon_n Q_r y_n \right\|_{E(X_\nu)} . \quad \square \end{aligned}$$

Theorem 4. *Let $(X_\nu)_{\nu \in \mathbb{Z}}$ and $(Y_\nu)_{\nu \in \mathbb{Z}}$ be sequences of Banach spaces and let $(T_\nu)_{\nu \in \mathbb{Z}}$ be a sequence of operators such that $T_\nu \in \mathcal{L}(X_\nu, Y_\nu)$ for every $\nu \in \mathbb{Z}$ and $\sup_{\nu \in \mathbb{Z}} \|T_\nu\| < \infty$. If a Banach sequence lattice E has the BS property and $\overline{T} \in \mathcal{L}(E(X_\nu), E(Y_\nu))$ is given by $\overline{T}x = (T_\nu x(\nu))_{\nu \in \mathbb{Z}}$ for every $x = (x(\nu))_{\nu \in \mathbb{Z}} \in E(X_\nu)$, then $\Phi(\overline{T}) = \sup_{\nu \in \mathbb{Z}} \Phi(T_\nu)$.*

Proof. Since $E(X_\nu)$ and $E(Y_\nu)$ contain isometric copies respectively of X_ν and Y_ν, it is enough to prove that $\Phi(\overline{T}) \leqslant \sup_{\nu \in \mathbb{Z}} \Phi(T_\nu)$.

Fix $\varepsilon > 0$. We choose $(x_n) \subset \mathsf{B}(E(X_\nu))$ so that $\Phi(\overline{T}) - \varepsilon \leqslant \phi(\overline{T}x_n)$. By Lemma 3, there exist $r \in \mathbb{N}$ and a sequence $(x'_n) \succ_{\mathcal{H}_1} (x_n)$ such that for all $(\epsilon_n)_{n \in G} \in \mathcal{G}_0$,

$$\left\| |G|^{-1} \sum_{n \in G} \epsilon_n Q_r \overline{T} x'_n \right\|_{E(Y_\nu)} < \varepsilon. \tag{1}$$

Passing to a subsequence of (x'_n), we may assume that for each coordinate $|\nu| \leqslant r$ the limit $\lambda_\nu = \lim_n \left\| x'_n(\nu) \right\|_{X_\nu}$ exists and $\left\| x'_n(\nu) \right\|_{X_\nu} < \lambda_\nu + \varepsilon / \|P_r e\|_E$ for every n, where $e = (\cdots 1, 1, 1, \ldots)$. Put

$$\beta_\nu = \left(\lambda_\nu + \frac{\varepsilon}{\|P_r e\|_E} \right)^{-1}.$$

Now we stabilize ϕ consecutively on coordinates $k = -r, -r+1, \ldots, r$. Write $x_n^{-r-1} = x'_n$ and $w_n^{-r-1}(\nu) = \beta_\nu T_\nu x_n^{-r-1}(\nu)$. By condition (s3) for $(w_n^{k-1}(k))$, we obtain a sequence $(x_n^k) \succ_{\mathcal{H}} (x_n^{k-1})$ such that for the sequence $(w_n^k(k)) \succ_{\mathcal{H}} (w_n^{k-1}(k))$, where $w_n^k(k) = \beta_k T_k x_n^k(k)$, we have

$$\left\| \delta \, |G|^{-1} \sum_{n \in G} \epsilon_n w_n^k(k) \right\|_{Y_k} \leqslant \phi \left(w_n^k(k) \right) + \varepsilon$$

for all $(\epsilon_n)_{n \in G} \in \mathcal{G}$. Then we put $w_n^k(\nu) = \beta_\nu T_\nu x_n^k(\nu)$ for $\nu \neq k$.

After $2r+1$ steps, all sequences $(w_n^r(\nu))$, $|\nu| \leqslant r$, are built on the common sequence (x_n^r) in such a way that $(x_n^k) \succ_{\mathcal{H}} (x_n^{k-1})$ for $k = -r, -r+1, \ldots, r$. By Corollary 2,

$$\left\| \delta \, |G|^{-1} \sum_{n \in G} \epsilon_n w_n^r(\nu) \right\|_{Y_\nu} \leqslant \phi \left(w_n^\nu(\nu) \right) + \varepsilon \leqslant \phi \left(w_n^r(\nu) \right) + \varepsilon$$

for all $(\epsilon_n)_{n \in G} \in \mathcal{G}$ and every $|\nu| \leqslant r$. Clearly, $\left(w_n^r(\nu) \right) \subset T_\nu(\mathsf{B}(X_\nu))$.

In combination with the lattice properties of E, it follows that

$$\begin{aligned}\left\|\delta\,|G|^{-1}\sum_{n\in G}\epsilon_n P_r\overline{T}x_n^r\right\|_{E(Y_\nu)} &= \left\|P_r\left(\beta_\nu^{-1}\left\|\delta\,|G|^{-1}\sum_{n\in G}\epsilon_n w_n^r(\nu)\right\|_{Y_\nu}\right)\right\|_E\\ &\leqslant \left\|P_r\left(\lambda_\nu+\frac{\varepsilon}{\|P_re\|_E}\right)\right\|_E\max_{|\nu|\leqslant r}\left\|\delta\,|G|^{-1}\sum_{n\in G}\epsilon_n w_n^r(\nu)\right\|_{Y_\nu}\\ &\leqslant (1+\varepsilon)\left(\max_{|\nu|\leqslant r}\phi\left(w_n^r(\nu)\right)+\varepsilon\right).\end{aligned}$$

Assume that $\max_{|\nu|\leqslant r}\phi\left(w_n^r(\nu)\right)$ is attained for j, $|j|\leqslant r$. Since $(\overline{T}x_n')\succ_{\mathcal{H}}(\overline{T}x_n)$ and $(\overline{T}x_n^k)\succ_{\mathcal{H}}(\overline{T}x_n^{k-1})$ for $k=-r,-r+1,\ldots,r$, by (s2) and (1), we have

$$\begin{aligned}\Phi(\overline{T})-\varepsilon &\leqslant \phi(\overline{T}x_n)\leqslant\phi(\overline{T}x_n^r)\leqslant\left\|\delta\,|G|^{-1}\sum_{n\in G}\epsilon_n\overline{T}x_n^r\right\|_{E(Y_\nu)}\\ &\leqslant \left\|\delta\,|G|^{-1}\sum_{n\in G}\epsilon_n P_r\overline{T}x_n^r\right\|_{E(Y_\nu)}+\left\|\delta\,|G|^{-1}\sum_{n\in G}\epsilon_n Q_r\overline{T}x_n^r\right\|_{E(Y_\nu)}\\ &\leqslant (1+\varepsilon)\left(\phi\left(w_n^r(j)\right)+\varepsilon\right)+\delta\varepsilon\leqslant(1+\varepsilon)\left(\Phi(T_j)+\varepsilon\right)+2\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ was chosen arbitrary, $\Phi(\overline{T})\leqslant\sup_{\nu\in\mathbb{Z}}\Phi(T_\nu)$. □

Considering ϕ over $\mathcal{G}_0$ and $\mathcal{G}_2$, we obtain results respectively for the ABS and BS properties. The following corollary includes Corollary 3.3 of [24] for the ABS property of $l_p(X)$ and Theorem 3 of Partington [38] for the BS property of direct sums.

Corollary 5. *The space $E(X_\nu)$ has the ABS property if and only if all X_ν have the ABS property. The same holds for the BS property.*

4. Interpolation results

Interpolation of operator properties by the general real methods was studied both from qualitative and quantitative viewpoints. Weak compactness of operators for the abstract K-method was investigated by Aizenstein and Brudnyi [5, Theorem 4.6.8] and Mastyło [31], who also examined in [32] the Rosenthal property for interpolation spaces. The behavior of compactness under the abstract J- and K-methods of Nilsson [36] was investigated by Cobos, Fernández-Cabrera and Martínez [8]. Other operators which form closed ideals, such as Banach–Saks operators, Asplund operators and Rosenthal operators, were studied in [7,30].

Quantitative studies were undertaken for a measure of noncompactness [9] and for the measures related to closed operator ideals [14]. The aim of a quantitative treatment is to obtain estimates for certain measures of deviation from a given property, which may provide also qualitative results. The measures used in such situations may be designed individually for each property or, as in the case of the ideal measures, several properties are covered by one formula based on some common properties, as surjectivity or injectivity of an operator ideal. In our paper, we examine the behavior of the mean separations for the abstract J- and K-spaces considered by Nilsson [36].

Let $\vec{A}=(A_0,A_1)$ be a Banach couple. Then $\Delta(\vec{A})=A_0\cap A_1$ and $\Sigma(\vec{A})=A_0+A_1$ with norms

$$\|a\|_{\Delta(\vec{A})}=\max\left\{\|a\|_{A_0},\|a\|_{A_1}\right\},\qquad \|a\|_{\Sigma(\vec{A})}=\inf\left\{\|a_0\|_{A_0}+\|a_1\|_{A_1}:a=a_0+a_1\right\}$$

are Banach spaces. The functionals

$$J(t,a)=\max\left\{\|a\|_{A_0},t\,\|a\|_{A_1}\right\},\qquad K(t,a)=\inf\left\{\|a_0\|_{A_0}+t\,\|a_1\|_{A_1}:a=a_0+a_1\right\}$$

with $t>0$ form the families of equivalent norms respectively in $\Delta(\vec{A})$ and $\Sigma(\vec{A})$.

A Banach sequence lattice E is called J-nontrivial, if

$$\sup\left\{\sum_{\nu\in\mathbb{Z}}\min\{1,2^{-\nu}\}\,|\xi(\nu)|:\xi=(\xi(\nu)),\ \|\xi\|_E\leqslant 1\right\}<\infty,$$

and K-nontrivial, if

$$(\min\{1,2^\nu\})\in E.$$

If E is J-nontrivial, the abstract J-space $A_{E;J}$ consists of all sums $\sum_{\nu\in\mathbb{Z}}a(\nu)$ convergent in $\Sigma(\vec{A})$ such that $(a(\nu))\subset\Delta(\vec{A})$ and $(J(2^\nu,a(\nu)))\in E$. If E is K-nontrivial, then the abstract K-space $A_{E;K}$ consists of all $a\in\Sigma(\vec{A})$ such that $(K(2^\nu,a))\in E$. The norms in $A_{E;J}$ and $A_{E;K}$ are given respectively by

$$\|a\|_{A_{E;J}} = \inf\left\{\|(J(2^\nu, a(\nu)))\|_E : a = \sum_{\nu\in\mathbb{Z}} a(\nu)\right\}, \qquad \|a\|_{A_{E;K}} = \|(K(2^\nu, a))\|_E .$$

The space $A_{E;J}$ is continuously embedded in $A_{E;K}$ but need not coincide with it. If $E = l_p(2^{-\nu\theta})$ with $1 \leqslant p \leqslant \infty$ and $0 < \theta < 1$, then $A_{E;J}$ and $A_{E;K}$ agree and they are equal to the real interpolation space $(A_0, A_1)_{\theta,p}$ of Lions and Peetre [29]. They also coincide, if $E = l_p(1/f(2^\nu))$, $1 \leqslant p \leqslant \infty$ and f is a function parameter defined in [17]. Then each of $A_{E;J}$ and $A_{E;K}$ gives the real interpolation space $(A_0, A_1)_{f,p}$ with a function parameter, in particular, $(A_0, A_1)_{\theta,p}$ if $f(t) = t^\theta$. For more examples we refer the reader to [8].

In this paper, we deal with the following equivalent norms in $A_{E;J}$ and $A_{E;K}$:

$$|||a|||_{A_{E;J}} = \inf\max\left\{\left\|(\|a(\nu)\|_{A_0})\right\|_E, \left\|(2^\nu \|a(\nu)\|_{A_1})\right\|_E\right\},$$

the infimum being taken over all representations $(a(\nu)) \subset \Delta(\vec{A})$ of a with $\sum_{\nu\in\mathbb{Z}} a(\nu)$ convergent to a in $\Sigma(\vec{A})$ and $(a(\nu)) \in E(A_0) \cap E(2^\nu A_1)$;

$$|||a|||_{A_{E;K}} = \inf\max\left\{\left\|(\|a_0(\nu)\|_{A_0})\right\|_E, \left\|(2^\nu \|a_1(\nu)\|_{A_1})\right\|_E\right\},$$

the infimum being taken over all decompositions $a = a_0(\nu)+a_1(\nu)$ for all $\nu \in \mathbb{Z}$ with $(a_0(\nu)) \in E(A_0)$ and $(a_1(\nu)) \in E(2^\nu A_1)$.

Let $\vec{A} = (A_0, A_1)$ and $\vec{B} = (B_0, B_1)$ be Banach couples and $T: \Sigma(\vec{A}) \to \Sigma(\vec{B})$ a linear operator. We write $T: \vec{A} \to \vec{B}$, if the restriction $T|A_j$ is a bounded operator into B_j for $j = 0, 1$.

Theorem 6. *Let E be a Banach sequence lattice with the BS property. Let $\vec{A} = (A_0, A_1)$ and $\vec{B} = (B_0, B_1)$ be Banach couples and $T: \vec{A} \to \vec{B}$.*

1. *If E is J-nontrivial and $A_{E;J}$, $B_{E;J}$ are abstract J-spaces respectively for $\vec{A}$, $\vec{B}$, then*
$$\Phi(T: A_{E;J} \to B_{E;J}) \leqslant \max\{\Phi(T: A_0 \to B_0), \Phi(T: A_1 \to B_1)\}.$$
2. *If E is K-nontrivial and $A_{E;K}$, $B_{E;K}$ are abstract K-spaces respectively for $\vec{A}$, $\vec{B}$, then*
$$\Phi(T: A_{E;K} \to B_{E;K}) \leqslant \max\{\Phi(T: A_0 \to B_0), \Phi(T: A_1 \to B_1)\}.$$

Proof. Let $X_j = E(2^{j\nu}A_j)$ and $Y_j = E(2^{j\nu}B_j)$ for $j = 0, 1$. To avoid ambiguity in some cases, we will write $\phi(x_n)_X$ to denote $\phi(x_n)$ in the norm of X. Fix $\varepsilon > 0$.

Let $(a_n) \subset \mathrm{B}(A_{E;J})$ and $b_n = Ta_n$. For each n there exists a representation $u_n = (u_n(\nu))$ of a_n such that $u_n \in \mathrm{B}(X_j)$ for $j = 0, 1$. Write $w_n = (Tu_n(\nu))$. Then w_n is a representation of b_n. By condition (s3), there exists a sequence $(w'_n) \succ_{\mathcal{H}} (w_n)$ such that

$$\left\|\delta\,|G|^{-1}\sum_{n\in G}\epsilon_n w'_n\right\|_{Y_0} \leqslant \phi(w'_n)_{Y_0} + \varepsilon,$$

for all $(\epsilon_n)_{n\in G} \in \mathcal{G}$. Then $w'_n = m^{-1}\sum_{i\in H_n}\zeta_i w_i$ for some $(\zeta_i)_{i\in\bigcup H_n} \in \mathcal{H}$, where $m = |H_n|$ for all n. Put $b'_n = m^{-1}\sum_{i\in H_n}\zeta_i b_i$. Repeating the above procedure for (w'_n) and (b'_n), we obtain the sequences $(w''_n) \succ_{\mathcal{H}} (w'_n)$ and $(b''_n) \succ_{\mathcal{H}} (b'_n)$ such that w''_n is a representation of b''_n and, by Corollary 2,

$$\left\|\delta\,|G|^{-1}\sum_{n\in G}\epsilon_n w''_n\right\|_{Y_j} \leqslant \phi(w''_n)_{Y_j} + \varepsilon,$$

for $j = 0, 1$ and all $(\epsilon_n)_{n\in G} \in \mathcal{G}$. Take now any $(\epsilon_n)_{n\in G} \in \mathcal{G}$. Then

$$\begin{aligned}\phi(b_n) \leqslant \phi(b'_n) \leqslant \phi(b''_n) &\leqslant \left|\left|\left|\delta\,|G|^{-1}\sum_{n\in G}\epsilon_n b''_n\right|\right|\right|_{B_{E;J}} \\ &\leqslant \max_{j=0,1}\left\|\delta\,|G|^{-1}\sum_{n\in G}\epsilon_n w''_n\right\|_{Y_j} \leqslant \max_{j=0,1}\phi(w''_n)_{Y_j} + \varepsilon.\end{aligned}$$

Let $\overline{T}_j: X_j \to Y_j$ be given by $\overline{T}_j x = (T_j x(\nu))$ for every $x = (x(\nu)) \in X_j$, where $T_j = T|A_j$ and $j = 0, 1$. By Theorem 4, $\Phi(\overline{T}_j) = \Phi(T_j)$. Since $(w''_n) \in \overline{T}_j(\mathrm{B}(X_j))$ for $j = 0, 1$, it follows that

$$\phi(b_n) \leqslant \max_{j=0,1}\Phi(\overline{T}_j) + \varepsilon = \max_{j=0,1}\Phi(T_j) + \varepsilon.$$

The proof of the first inequality is complete.

We show the second inequality. Let $(a_n) \subset \mathsf{B}(A_{E;K})$ and $b_n = Ta_n$. For each n there exists a decomposition $a_{j,n} = (a_{j,n}(\nu)) \in \mathsf{B}(X_j), j = 0, 1$ of a_n. Then $b_{j,n} = (Ta_{j,n}(\nu)), j = 0, 1$ is a decomposition of b_n. We stabilize ϕ similarly as in the proof of Theorem 4. By condition (s3), we can find $(b'_{0,n}) \succ_{\mathcal{H}} (b_{0,n})$, where $b'_{0,n} = m^{-1} \sum_{i \in H_n} \zeta_i b_{0,i}$ for some $(\zeta_i)_{i \in \bigcup H_n} \in \mathcal{H}$ with $|H_n| = m$ for all n, such that

$$\left\| \delta \, |G|^{-1} \sum_{n \in G} \epsilon_n b'_{0,n} \right\|_{Y_0} \leqslant \phi(b'_{0,n})_{Y_0} + \varepsilon$$

for all $(\epsilon_n)_{n \in G} \in \mathcal{G}$. We put $b'_{1,n} = m^{-1} \sum_{i \in H_n} \zeta_i b_{1,i}$ and $b'_n = m^{-1} \sum_{i \in H_n} \zeta_i b_i$. Applying again (s3), we choose $(b''_{1,n}) \succ_{\mathcal{H}} (b'_{1,n})$ so that for all $(\epsilon_n)_{n \in G} \in \mathcal{G}$,

$$\left\| \delta \, |G|^{-1} \sum_{n \in G} \epsilon_n b''_{1,n} \right\|_{Y_1} \leqslant \phi(b''_{1,n})_{Y_1} + \varepsilon.$$

Then we construct $(b''_{0,n}) \succ_{\mathcal{H}} (b'_{0,n})$ and $(b''_n) \succ_{\mathcal{H}} (b'_n)$ in the way we did it for $(b'_{1,n})$ and (b'_n), respectively. Take any $(\epsilon_n)_{n \in G} \in \mathcal{G}$. By Corollary 2,

$$\begin{aligned} \phi(b_n) &\leqslant \phi(b'_n) \leqslant \phi(b''_n) \leqslant \left|\left|\left| \delta \, |G|^{-1} \sum_{n \in G} \epsilon_n b''_n \right|\right|\right|_{B_{E;K}} \\ &\leqslant \max_{j=0,1} \left\| \delta \, |G|^{-1} \sum_{n \in G} \epsilon_n b''_{j,n} \right\|_{Y_j} \leqslant \max_{j=0,1} \phi(b''_{j,n})_{Y_j} + \varepsilon. \end{aligned}$$

Applying Theorem 4 as for the operators between abstract J-spaces, we obtain the desired inequality. $\square$

Corollary 7. *If $T: A_0 \to B_0$ and $T: A_1 \to B_1$ have the ABS property, then $T: A_{E;J} \to B_{E;J}$ and $T: A_{E;K} \to B_{E;K}$ have the ABS property. In particular, if A_0 and A_1 have the ABS property, then so have $A_{E;J}$ and $A_{E;K}$. The same holds for the BS property.*

In particular cases, a property may be inherited by an interpolated operator from only one restriction $T: A_0 \to B_0$ or $T: A_1 \to B_1$. It holds, for instance, for operators between the real interpolation spaces of Lions and Peetre [29], since in this case the norms of the interpolation spaces are logarithmically convex up to a constant. In [8,7], by putting additional assumptions on the shift operators in E, the authors show how to obtain results of this type in a more general setting. In the next corollary, if ϕ is taken over $\mathcal{G}_0$ and $\mathcal{G}_2$, we obtain respectively Theorem 4.1 of [24] for the ABS property and Theorem 5.1 of [25] for the BS property.

Corollary 8. *If $E = l_p(2^{-\nu\theta})$ with $1 < p < \infty$ and $0 < \theta < 1$, then*

$$\Phi(T: A_{E;K} \to B_{E;K}) \leqslant 2^{\theta(1-\theta)} \left(\Phi(T: A_0 \to B_0)\right)^{1-\theta} \left(\Phi(T: A_1 \to B_1)\right)^{\theta}.$$

5. Extrapolation results

Let $I \subset [0, 1]$ be an interval. An ordered family $(A_\theta)_{\theta \in I}$ of Banach spaces will be called *compatible* if there exist two Banach spaces A_0 and A_1 such that $A_0 \hookrightarrow A_\theta \hookrightarrow A_\eta \hookrightarrow A_1$ for all $\theta < \eta$ and all the embeddings $\hookrightarrow$ are continuous and uniformly bounded. The abstract extrapolation methods were developed, among others, by Jawerth and Milman in [21] and Edmunds and Triebel in [11]. We follow the approach from a recent paper of Cobos and Kühn [10]. Let $(A_\theta)_{\theta \in I}$ be a compatible family of Banach spaces, $b > 0$ and $1 < p < \infty$ (the construction of extrapolation spaces is valid also for $p = 1$ and $p = \infty$, but our results do not apply for these values). Fix $\theta \in I$ so that $(\theta, \theta + \varepsilon) \subset I$ for some $\varepsilon > 0$. The extrapolation space $A_\theta(\log A)^+_{b,p} = A^+_{\theta,b,p}$ consists of all $a \in \bigcap_{\theta < \eta \leqslant \theta + \varepsilon} A_\eta$ with finite norm given by

$$\|a\|_{A_\theta(\log A)^+_{b,p}} = \left(\int_0^\varepsilon \left(t^b \, \|a\|_{A_{\theta+t}}\right)^p \frac{dt}{t} \right)^{1/p}.$$

Now fix $\theta \in I$ so that $(\theta - \varepsilon, \theta) \subset I$ for some $\varepsilon > 0$. The extrapolation space $A_\theta(\log A)^-_{b,p} = A^-_{\theta,b,p}$ consists of all $a \in A_1$ which can be represented by an integral $a = \int_0^\varepsilon \upsilon(t) \frac{dt}{t}$ convergent in A_1 with $\upsilon(t) \in A_{\theta - t}$ and finite norm given by

$$\|a\|_{A_\theta(\log A)^-_{b,p}} = \inf \left(\int_0^\varepsilon \left(t^{-b} \, \|\upsilon(t)\|_{A_{\theta-t}}\right)^p \frac{dt}{t} \right)^{1/p},$$

the infimum being taken over all representations of a as above. The spaces $A^+_{\theta,b,p}$ and $A^-_{\theta,b,p}$ do not depend on the choice of $\varepsilon > 0$.

The extrapolation spaces $A^+_{\theta,b,p}$ and $A^-_{\theta,b,p}$ can be equivalently renormed by the following discrete norms. For $J \in \mathbb{N}$ such that $2^{-J} < \varepsilon$ and for all $\nu \geqslant J$, we put $\sigma_\nu = \theta + 2^{-\nu}$ and $\lambda_\nu = \theta - 2^{-\nu}$. Let

$$\|a\|_{A^+_{\theta,b,p}} = \left(\sum_{\nu=J}^{\infty} \left(2^{-\nu b}\, \|a\|_{A_{\sigma_\nu}}\right)^p\right)^{1/p}$$

and

$$\|a\|_{A^-_{\theta,b,p}} = \inf \left(\sum_{\nu=J}^{\infty} \left(2^{\nu b}\, \|a(\nu)\|_{A_{\lambda_\nu}}\right)^p\right)^{1/p},$$

the infimum being taken over all representations $a = \sum_{\nu=J}^{\infty} a(\nu)$, $a(\nu) \in A_{\lambda_\nu}$, convergent in A_1. Replacing J by $J' \in \mathbb{N}$ such that $2^{-J'} < \varepsilon$, we generate equivalent norms both in $A^+_{\theta,b,p}$ and $A^-_{\theta,b,p}$.

In particular, if $A_\theta = [A_0, A_1]_\theta$ with $0 < \theta < 1$ are complex interpolation spaces with respect to complex Banach spaces A_0 and A_1 such that A_0 is densely and continuously embedded in A_1, then $A^+_{\theta,b,p}$ and $A^-_{\theta,b,p}$ correspond to the abstract logarithmic spaces $A_\theta(\log A)_{b,p}$ of Edmunds and Triebel [11] with $b \in \mathbb{R} \setminus \{0\}$. In turn, if $A_0 = L_\infty(\Omega)$ and $A_1 = L_1(\Omega)$ for a bounded open subset $\Omega \subset \mathbb{R}^n$, then $A_\theta(\log A)_{b,p}$ coincides with the Zygmund space $L_{1/\theta}(\log L)_b(\Omega)$.

The question of the heredity of properties by extrapolation spaces and operators can be posed analogously as for interpolation. We recall some results based on quantitative tools. In [41], Triebel estimated the entropy numbers for the embeddings of the fractional Besov–Sobolev spaces into Orlicz spaces $L_\infty(\log L)_b(\Omega)$. A similar problem for embeddings into $L_p(\log L)_b(\Omega)$ spaces with $1 < p < \infty$ was investigated by Edmunds and Triebel in [12]. Estimating the modulus of uniform convexity, Nikolova and Zachariades [34] proved that uniform convexity is preserved by the abstract logarithmic spaces. Using Clarkson's inequality, they also examined with Persson [33] the type and cotype of such spaces. A measure of weak noncompactness for operators extrapolated by Δ_p- and Σ_p-methods of Jawerth and Milman [21] with $1 < p < \infty$ was estimated in [26] (see also [35]). A general qualitative approach to extrapolation properties of closed operator ideals was recently presented by Fernández-Cabrera and Martínez in [15].

Let $(A_\theta)_{\theta\in I}$ and $(B_\theta)_{\theta\in I}$ be compatible families of Banach spaces such that $A_0 \hookrightarrow A_\theta \hookrightarrow A_1$ and $B_0 \hookrightarrow B_\theta \hookrightarrow B_1$ for every $\theta \in I$. For a linear operator $T: A_1 \to B_1$, we will write $T: (A_\theta)_{\theta\in I} \to (B_\theta)_{\theta\in I}$, if for every $\theta \in I$ the restriction $T|A_\theta$ is a bounded operator into B_θ and the norms of $T|A_\theta$ are uniformly bounded on I.

Theorem 9. *Let $(A_\theta)_{\theta\in I}$ and $(B_\theta)_{\theta\in I}$ be compatible families of Banach spaces and $1 < p < \infty$. If $T: (A_\theta)_{\theta\in I} \to (B_\theta)_{\theta\in I}$, then*

$$\Phi(T: A^+_{\theta,b,p} \to B^+_{\theta,b,p}) \leqslant \sup_{\theta\in I} \Phi(T: A_\theta \to B_\theta)$$

and

$$\Phi(T: A^-_{\theta,b,p} \to B^-_{\theta,b,p}) \leqslant \sup_{\theta\in I} \Phi(T: A_\theta \to B_\theta).$$

Proof. We prove only the second inequality. The proof of the first one runs similarly. Let (a_n) be a sequence in $\mathsf{B}(A^-_{\theta,b,p})$. Then for each n there exists a representation of a_n such that

$$a_n = \sum_{\nu=J}^{\infty} a_n(\nu), \quad (a_n(\nu))_{\nu\geqslant J} \in \mathsf{B}(l_p(2^{\nu b} A_{\lambda_\nu})).$$

Put $T_\nu = T|A_{\lambda_\nu}$, $b_n = Ta_n$ and $y_n = (T_\nu a_n(\nu))_{\nu\geqslant J}$. Then $y_n \in l_p(2^{\nu b} B_{\lambda_\nu})$ and y_n is a representation of b_n. Define the operator

$$\overline{T} \in \mathcal{L}(l_p(2^{\nu b} A_{\lambda_\nu}), l_p(2^{\nu b} B_{\lambda_\nu})), \quad \overline{T} a = (T_\nu a(\nu))_{\nu\geqslant J}$$

for every $a = (a(\nu))_{\nu\geqslant J} \in l_p(2^{\nu b} A_{\lambda_\nu})$. Thus $y_n \in \overline{T}(\mathsf{B}(l_p(2^{\nu b} A_{\lambda_\nu})))$. Fix $\varepsilon > 0$. By condition (s3), there exists a sequence $(y'_n) \succ_{\mathcal{H}} (y_n)$ such that

$$\left\| \delta\, |G|^{-1} \sum_{n\in G} \epsilon_n y'_n \right\|_{l_p(2^{\nu b} B_{\lambda_\nu})} \leqslant \phi(y'_n) + \varepsilon$$

for all $(\epsilon_n)_{n\in G} \in \mathcal{G}$.

Then $y'_n = m^{-1} \sum_{i\in H_n} \zeta_i y_i$ for some $(\zeta_i)_{i\in\bigcup H_n} \in \mathcal{H}$, where $|H_n| = m$ for all n. Put $b'_n = m^{-1} \sum_{i\in H_n} \zeta_i b_i$. Then $(b'_n) \succ_{\mathcal{H}} (b_n)$ and for any $(\epsilon_n)_{n\in G} \in \mathcal{G}$ we have

$$\phi(b_n) \leqslant \phi(b'_n) \leqslant \left\| \delta\, |G|^{-1} \sum_{n\in G} \epsilon_n b'_n \right\|_{B^-_{\theta,b,p}} \leqslant \left\| \delta\, |G|^{-1} \sum_{n\in G} \epsilon_n y'_n \right\|_{l_p(2^{\nu b} B_{\lambda_\nu})} \leqslant \phi(y'_n) + \varepsilon.$$

Since (y_n') is also a sequence in $\overline{T}(\mathrm{B}(l_p(2^{\nu b}A_{\lambda_\nu})))$, we have

$$\phi(y_n') \leqslant \Phi(\overline{T}: l_p(2^{\nu b}A_{\lambda_\nu}) \to l_p(2^{\nu b}B_{\lambda_\nu})).$$

Substituting $X_\nu = A_{\lambda_\nu}$ and $Y_\nu = B_{\lambda_\nu}$ for $\nu \geqslant J$, $X_\nu = Y_\nu = \{0\}$ for $\nu < J$ and $E = l_p(2^{\nu b})$ in Theorem 4, we obtain

$$\Phi(\overline{T}: l_p(2^{\nu b}A_{\lambda_\nu}) \to l_p(2^{\nu b}B_{\lambda_\nu})) \leqslant \sup_{\nu \geqslant J} \Phi\left(T: A_{\lambda_\nu} \to B_{\lambda_\nu}\right) \leqslant \sup_{\theta \in I} \Phi\left(T: A_\theta \to B_\theta\right).$$

An arbitrary choice of (a_n) and $\varepsilon > 0$ gives the assertion. □

Corollary 10. *If $T: A_\theta \to B_\theta$ has the ABS property for all $\theta \in I$, then $T: A^+_{\theta,b,p} \to B^+_{\theta,b,p}$ and $T: A^-_{\theta,b,p} \to B^-_{\theta,b,p}$ have the ABS property. In particular, if A_θ has the ABS property for all $\theta \in I$, then so have $A^+_{\theta,b,p}$ and $A^-_{\theta,b,p}$. The same holds for the BS property.*

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