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Orthogonal Laurent polynomials. A new algebraic approach

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To Pablo González Vera, our beloved master and friend. In memoriam.

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ABSTRACT

In this paper, some important algebraic aspects in the theory of orthogonal Laurent polynomials, such as the three-term recurrence relation, the Christoffel–Darboux identity or the Liouville–Ostrogradski formula, are revisited from the Riemann–Hilbert window. These topics are considered for general ordered Laurent polynomial sequences, and not only for the usual "balanced" cases. In addition, a connection with Szegö polynomials (orthogonal polynomials in the unit circle) is explored.

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1. Introduction

The first paper devoted to the study of orthogonal Laurent polynomials was [19], although this notion was actually present, without an explicit mention, in the previous [20], where W. Jones et al. posed and solved the strong Stieltjes moment problem. This connection between the strong Stieltjes moment problem and the theory of orthogonal Laurent polynomials (on the sequel, OLP) has remained working until now, allowing to establish since the very beginning, useful links between continued fractions, rational interpolants and quadrature formulas (see [17]), similarly as in the classical polynomial setting. From an algebraic point of view, the development of the theory of OLP has been marked by a series of contributions by Cochran [5], Cooper [6], Hendriksen [15], Njåstad [27] and Sri Ranga [28], among others. It is also worth mentioning some papers devoted to the strong moment problem, but now on the whole real axis, i.e., the so-called Hamburger moment problem (see the survey [16] and references therein). A number of well-known properties in the classical polynomial setting, such as the recurrence relations, the interlacing of zeros, the Christoffel–Darboux identity and the determinantal formula (also called the Liouville–Ostrogradski formula) were then extended for orthogonal Laurent polynomials, though all these starting results were obtained just for two particular situations: the so-called "balanced" cases, associated with the ordered basis $\{1, z^{-1}, z, z^{-2}, z^2, \ldots\}$ and $\{1, z, z^{-1}, z^2, z^{-2}, \ldots\}$. Paralleling, these OLPs became an important tool in the study of the rational approximation of the so-called Stieltjes functions (that is, Cauchy transforms of measures supported on the positive semiaxis). In this sense, it should be pointed out three contributions [21,22,1], where asymptotics of OLP with respect to exponential type weight functions are given, as well as asymptotics of their zeros, making use of the theory of orthogonal polynomials with respect to varying weights. These results, along with the numerical experimentation carried out in [2–4,10] showed the usefulness of OLP in numerical integration when their zeros are taken as quadrature nodes. More recently, some of the most relevant above mentioned algebraic properties were extended to general "non-balanced" cases (see [10,11]).

The aim of this paper is to make a contribution to the theory of OLP, by means of their characterization in terms of a matrix Riemann–Hilbert Problem (RHP). It should be noted that there exists a previous characterization of OLP in terms of a

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different RHP, from which the authors obtained several asymptotics (see [24–26]). However, they restrict themselves to the balanced situations and from their matrix RHP cannot be deduced, for instance, the three-term recurrence relation given in [10]. On the other hand, our characterization also allow us to recover classical results for the so-called Szegö polynomials in the unit circle (see e.g. [23] for a Riemann–Hilbert approach to Szegö polynomials).

The outline of the paper is as follows. In Section 2, the definition and basic properties of OLP are recalled, along with their main algebraic properties, while in Section 3, the characterization of OLP by means of a matrix RHP is studied. Finally, the last section is devoted to the study of some consequences of such a characterization.

2. Orthogonal Laurent polynomials

Let $-\infty \le a < b \le +\infty$ and μ be a real, bounded and nondecreasing function with infinitely many points of increase on (a, b), such that the moments

$$m_k = \int_a^b t^k \, \mathrm{d}\mu(t), \quad k \in \mathbb{Z} \tag{2.1}$$

exist. Under these conditions μ is called a *strong distribution* on (a, b). When $0 \le a < b \le +\infty$, μ will be called a *strong Stieltjes distribution*.

Given two integers p and q such that $p \le q$, by a Laurent polynomial (or L-polynomial in short) we mean a function of the form

$$L(x) = \sum_{j=p}^{q} \alpha_j x^j, \quad \alpha_j \in \mathbb{C}.$$

$\Lambda_{p,q}$ will be the linear space of these functions, i.e.,

$$\Lambda_{p,q} = \operatorname{span}\{x^j : p \le j \le q\},$$

which is a subspace with $\dim(\Lambda_{p,q}) = q - p + 1$ of the linear space of all L-polynomials $\Lambda = \operatorname{span}\{x^k : k \in \mathbb{Z}\}$. Observe that if $\mathbb{P}_k$ denotes the linear space of polynomials of degree at most k, then $\mathbb{P}_k = \Lambda_{0,k}$. Finally, $\mathbb{P}$ will denote the linear space of all algebraic polynomials, and thus, we have $\mathbb{P} = \Lambda_{0,\infty}$.

Let μ be a strong distribution, so that one can define an inner product on Λ (in general it is a non-hermitian product),

$$\langle L, M\rangle_\mu = \int_a^b L(t)\, M(t)\, \mathrm{d}\mu(t), \quad L,\ M \in \Lambda. \tag{2.2}$$

Observe that when we deal with L-polynomials with real coefficients, complex conjugation in (2.2) is not needed, and it is an hermitian inner product.

In order to obtain an appropriate orthogonal system in Λ with respect to (2.2), a certain nested sequence of linear subspaces of Λ, similar to the sequence $\{\mathbb{P}_n\}_{n\ge 0}$ in $\mathbb{P}$, will be required. To do it, let us start from a nondecreasing sequence of nonnegative integers $\{p_n\}_{n\ge 0}$, such that for each n, $0 \le p_n \le n$ and $p_n - p_{n-1} = s_n \in \{0, 1\}$. Now, set

$$\mathcal{L}_n = \Lambda_{-p_n, q_n} = \operatorname{span}\{x^j : -p_n \le j \le q_n\}, \quad p_n + q_n = n,$$

and $\mathcal{L} = \cup_{n\ge 0} \mathcal{L}_n$. Thus, one sees that $\mathcal{L}_n \subset \mathcal{L}_{n+1}$ and $\dim(\mathcal{L}_n) = n + 1$. In this case, it will be said that the sequence $\{p_n\}_{n\ge 0}$ has induced an "ordering" in Λ. Furthermore, $\mathcal{L} = \Lambda$ if and only if $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = \infty$.

Let $R_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$, $R_n(x) = \sum_{j=-p_n}^{q_n} r_{j,n} x^j$, and set

$$\kappa_n = \begin{cases} r_{-p_n,n} & \text{if } s_n = 1, \\ r_{q_n,n} & \text{if } s_n = 0. \end{cases}$$

Then, $\kappa_n \ne 0$ is called the *leading coefficient* of R_n. When $\kappa_n = 1$, for each $n \ge 0$, $\{R_n\}_{n\ge 0}$ is said to be the sequence of monic Laurent polynomials associated with $\{p_n\}_{n\ge 0}$.

On the other hand, by applying the Gram–Schmidt orthogonalization process to the basis $\{x^j : -p_n \le j \le q_n\}$ of $\mathcal{L}_n$, a (monic) orthogonal basis $\{\phi_0, \phi_1, \ldots, \phi_n\}$ can be obtained. When the process is repeated for each $n = 1, 2, \ldots$, an essentially unique sequence $\{\phi_n\}_{n\ge 0}$ is deduced, satisfying

$$\phi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}, \qquad \langle \phi_n, L\rangle_\mu = 0, \quad \forall L \in \mathcal{L}_{n-1},\ n \ge 1 \text{ and } \phi_0 \equiv 1,$$

i.e. $\phi_n \perp \mathcal{L}_{n-1}$. Clearly, for nonnegative integers m and n it holds, $\langle \phi_n, \phi_m\rangle_\mu = \delta_{n,m}\, \gamma_n^2$, with $\gamma_n > 0$ and $\delta_{n,m}$ the Kronecker delta symbol, where $\phi_n(x) = \sum_{k=0}^{n} b_{k,n} x^{-p_n+k}$, with $b_{n(1-s_n),n} = 1$. In addition, we assume that ϕ_n is regular, that is, that its "trailing" coefficient $b_{ns_n,n} \ne 0$. Since in passing from $\mathcal{L}_{n-2}$ to $\mathcal{L}_{n-1}$ the degree of the L-polynomial may increase in powers of z or z^{-1}, depending on the value of s_{n-1}, we need to introduce two new constants. On the one hand, when $s_{n-1} = 1$, set

$$A_{n-1} = -\frac{1}{\gamma_{n-1}^2} \int_a^b \phi_{n-1}(x) x^{q_{n-1}+1} \mathrm{d}\mu(x),$$

and, on the other, if $s_{n-1}=0$, take

$$B_{n-1}=\frac{1}{\gamma_{n-1}^2}\int_a^b \frac{\phi_{n-1}(x)}{x^{p_{n-1}+1}}\mathrm{d}\mu(x).$$

Then, if we denote $\Phi_n(z)=\frac{\phi_n(z)}{\gamma_n}$, for each $n\geq 0$, the sequence $\{\Phi_n\}_{n\geq 0}$ is orthonormal and it is uniquely determined up to a sign. Thus, we can normalize $\{\Phi_n\}_{n=0}^{\infty}$, denoting the corresponding sequence of orthonormal Laurent polynomials, such that $\Phi_n(z)=\frac{B_n(z)}{z^{p_n}}$, where $B_n(z)=b_{0,n}^*+b_{1,n}^*z+\cdots+b_{n,n}^*z^n$, with $b_{k,n}^*=\frac{b_{k,n}}{\gamma_n}$. By construction, we have that $\Phi_n\in\mathcal{L}_n\setminus\mathcal{L}_{n-1}$, $b_{0,n}^*\cdot b_{n,n}^*\neq 0$ and we can suppose that $b_{n,n}^*>0$, for all n. Then, as in the classical polynomial setting, we define (z and y are both different from zero)

$$\mathcal{K}_n(z,y)=\sum_{k=0}^{n}\Phi_k(z)\,\Phi_k(y). \tag{2.3}$$

Thus, if L is an L-polynomial in $\mathcal{L}_n$, one has

$$L(x)=\sum_{k=0}^{n}a_k\,\Phi_k(x)\quad\text{with } a_k=\langle L,\,\Phi_k\rangle_\mu,\ \ k=0,\ 1,\ldots,n.$$

Hence, it follows

$$\langle\mathcal{K}_n(\cdot,y),\ L\rangle_\mu=\int_a^b\mathcal{K}_n(t,y)\,L(t)\,\mathrm{d}\mu(t)=L(y). \tag{2.4}$$

For this reason, $\mathcal{K}_n(z,y)$, given by (2.3), is called the reproducing kernel of $\mathcal{L}_n$. Now, recall the following algebraic properties satisfied by the orthonormal Laurent polynomials, see [10,11] which will be recovered by means of the Riemann–Hilbert approach posed in Section 3. In fact, the last of them, which is formulated in Theorem 3 below, is only mentioned without proof in [11, p. 989] as a consequence of the determinant formula for continued fractions (see e.g. [18]). Thus, in this sense, it is an essentially new result in this context.

Theorem 1 (*Christoffel–Darboux Identity*)**.** *Let $\{\Phi_k\}_{k\in\mathbb{N}}$ be the sequence of orthonormal Laurent polynomials for the strong distribution μ with the ordering induced by a nondecreasing sequence $\{p_n\}_{n\geq 0}$ of nonnegative integers, such that $0\leq p_n\leq n$ and $p_n-p_{n-1}=s_n\in\{0,1\}$ for each n. Then, for z and y real numbers both different from zero, one has*

$$\sum_{k=0}^{n}\Phi_k(z)\,\Phi_k(y)=C_n\left[\frac{z^{s_{n+1}}\Phi_{n+1}(z)\,y^{s_n}\Phi_n(y)-z^{s_n}\Phi_n(z)\,y^{s_{n+1}}\Phi_{n+1}(y)}{z-y}\right], \tag{2.5}$$

where C_n are nonzero real numbers, for $n\geq 1$. Furthermore, (2.5) also holds for $n=0$, with $s_0=0$ and $C_0\neq 0$ conveniently chosen. In addition,

$$C_n=(-1)^{s_n}\left[s_n\frac{b_{0,n}^*}{b_{0,n+1}^*}+(1-s_n)\frac{b_{n,n}^*}{b_{n+1,n+1}^*}\right],\quad n\geq 1. \tag{2.6}$$

Theorem 2 (*Three-Term Recurrence Relation*)**.** *Under the same conditions as in Theorem 1, there exist a sequence of nonzero real numbers $\{\Omega_n\}_{n\geq 1}$, such that for $n\geq 1$,*

$$C_n\,z^{s_{n+1}}\,\Phi_{n+1}(z)=(-1)^{s_n}[z^{1-2s_n}-\Omega_n]\,z^{s_n}\Phi_n(z)-C_{n-1}\,z^{s_{n-1}}\,\Phi_{n-1}(z), \tag{2.7}$$

with $s_0=0$, where $\{C_n\}_{n\geq 0}$ was defined in (2.6). Furthermore, (2.7) also holds for $n=0$ setting $C_0=\frac{1}{\sqrt{m_0}\,\gamma_1}$, $s_{-1}=0$, $\Phi_{-1}\equiv 0$, $\Omega_0=\frac{m_{1-s_1}}{m_{-s_1}}$ and C_{-1} arbitrary (here $\{m_k\}_{k\in\mathbb{Z}}$ are the moments of μ).

To conclude, we will also introduce certain L-polynomials associated with the sequence $\{\Phi_n\}_{n\geq 0}$ (the second kind L-polynomials), namely

$$\Delta_n(x)=\int_a^b\frac{\Phi_n(x)-\Phi_n(t)}{x-t}\,\mathrm{d}\mu(t),\quad n=0,\ 1,\ldots.$$

Theorem 3 (*Liouville–Ostrogradski Identity or Determinantal Formula*)**.** *Under the same conditions as in Theorem 1, it holds,*

$$\Phi_{n+1}(z)\Delta_n(z)-\Phi_n(z)\Delta_{n+1}(z)=-\frac{1}{C_n z^{s_n+s_{n+1}}}, \tag{2.8}$$

where C_n is given by (2.6).

3. Characterization of orthogonal Laurent polynomials by a Riemann–Hilbert problem

In order to state the main result of the paper, throughout this section we suppose that $\mathrm{d}\mu(x) = \omega(x)\,\mathrm{d}x$. Our aim is characterizing the orthogonal Laurent polynomials corresponding to a weight function ω on a closed subset Σ of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ via a boundary value problem for matrix-valued analytic functions. This problem was first formulated for classical orthogonal polynomials in a seminal paper by Fokas, Its and Kitaev in 1992, see [12]. During the last two decades, many works have been devoted to making use of this characterization to derive properties and, specially, to obtain strong asymptotics for orthogonal polynomials (see e.g. [8]) and to apply this approach to several extensions, such as multiple (Hermite–Padé) or matrix orthogonal polynomials (see e.g. [13,14], to cite only a few). Through this section, a Riemann–Hilbert problem characterizing general sequences of OLP will be posed and solved, and in the next section some algebraic properties will be derived following this approach. In a forthcoming paper, this Riemann–Hilbert analysis will be employed to obtain asymptotics for sequences of OLP.

On the sequel, Σ will denote the real line, the positive semiaxis or the unit circle. The Riemann–Hilbert problem for orthogonal Laurent polynomials with respect to a weight function ω on Σ consists of finding a matrix valued function $Y : \mathbb{C}^* \setminus \Sigma \to \mathbb{C}^{2\times 2}$ ($\mathbb{C}^* = \mathbb{C} \setminus \{0\}$) which satisfies the following four conditions:

(RH1) Y is analytic in $\mathbb{C}^* \setminus \Sigma$.

(RH2) $Y_+(x) = Y_-(x) \begin{pmatrix} 1 & \omega(x) \\ 0 & 1 \end{pmatrix}$, $x \in \Sigma$.

(RH3) Y has the following behavior near infinity

$$\begin{aligned} Y(z) &= \left[\begin{pmatrix} b_{n,n}^{s_n} & s_n\tau_n \\ s_{n-1}\widetilde{b}_{n-1,n-1} & A_{n-1}^{s_{n-1}} \end{pmatrix} + \mathcal{O}\left(\frac{1}{z}\right) \right] \begin{pmatrix} z^{q_{n-1}+1} & 0 \\ 0 & \dfrac{1}{z^{q_{n-1}+1}} \end{pmatrix} \\ &= \left[\Gamma_n^{\infty} + \mathcal{O}\left(\frac{1}{z}\right) \right] \begin{pmatrix} z^{q_{n-1}+1} & 0 \\ 0 & \dfrac{1}{z^{q_{n-1}+1}} \end{pmatrix}, \quad z \to \infty, \end{aligned}$$

where $\widetilde{b}_{n-1,n-1} = 2\pi\imath \frac{b_{n-1,n-1}}{\gamma_{n-1}^2}$ when $s_{n-1} = 1$ and $\tau_n = \frac{\gamma_n^2 A_n}{2\pi\imath} \neq 0$ when $s_n = 1$.

(RH4) Y has the following behavior near the origin

$$\begin{aligned} Y(z) &= \left[\begin{pmatrix} b_{0,n}^{1-s_n} & (1-s_n)\upsilon_n \\ (1-s_{n-1})\widetilde{b}_{0,n-1} & B_{n-1}^{1-s_{n-1}} \end{pmatrix} + \mathcal{O}(z) \right] \begin{pmatrix} \dfrac{1}{z^{p_{n-1}}} & 0 \\ 0 & z^{p_{n-1}} \end{pmatrix} \\ &= \left[\Gamma_n^{0} + \mathcal{O}(z) \right] \begin{pmatrix} \dfrac{1}{z^{p_{n-1}}} & 0 \\ 0 & z^{p_{n-1}} \end{pmatrix}, \quad z \to 0, \end{aligned}$$

where $\widetilde{b}_{0,n-1} = -2\pi\imath \frac{b_{0,n-1}}{\gamma_{n-1}^2}$ when $s_{n-1} = 0$ and $\upsilon_n = -\frac{B_n\gamma_n^2}{2\pi\imath} \neq 0$ when $s_n = 0$.

A matrix-valued function Y is analytic in z if each of its components is an analytic function of z. The boundary values Y_+ and Y_- are defined as

$$Y_{\pm}(x) = \lim_{z \to x \pm side} Y(z), \quad x \in \Sigma \text{ (oriented)},$$

where, as usual, the positive side is on the left when we move from left to right in the real case and counterclockwise in the unit circle. The existence of these boundary values is part of the assumptions. Note that when $s_n = 0$, $\forall\, n \in \mathbb{N}$, and we remove the condition (RH4), the classical polynomial setting is recovered (see e.g. [8]); here we can find another difference with the RHP for OLP due to McLaughlin et al. in [24–26].

Now, our main result is announced.

Theorem 4. *Suppose that the weight ω is a Hölder continuous function on Σ such that the Laurent polynomials are absolutely Riemann–Stieltjes integrable with respect to ω, that is,*

$$\int_{\Sigma} |x|^k\, \omega(x)\mathrm{d}x < \infty, \quad k \in \mathbb{Z}.$$

Then, the Riemann–Hilbert problem (RH1)–(RH4) *for Y has a solution given by*

$$Y(z) = \begin{pmatrix} z^{s_n}\phi_n(z) & C_\omega(x^{s_n}\phi_n)(z) \\ c_{n-1}z^{s_{n-1}}\phi_{n-1}(z) & c_{n-1}C_\omega(x^{s_{n-1}}\phi_{n-1})(z) \end{pmatrix}, \tag{3.1}$$

where ϕ_n and ϕ_{n-1} are the monic regular orthogonal Laurent polynomials,

$$c_n = -\frac{(-1)^{s_n}2\pi\imath}{\gamma_n^2}, \tag{3.2}$$

and for a function f, $C_\omega(f)(z)$ denotes its Cauchy transform:

$$C_\omega(f)(z) = \frac{1}{2\pi\imath}\int_\Sigma \frac{f(x)}{x-z}\omega(x)\mathrm{d}x. \tag{3.3}$$

We can see that the solution of the above matrix Riemann–Hilbert problem is obtained in terms of ϕ_n and ϕ_{n-1} and their corresponding Cauchy transforms. Note again that when $s_n = 0, \forall\, n \in \mathbb{N}$, we get the solution of the classical polynomial RHP (see e.g. [8]).

Proof. Let us write the matrix Y by:

$$Y(z) = \begin{pmatrix} Y_{1,1}(z) & Y_{1,2}(z) \\ Y_{2,1}(z) & Y_{2,2}(z) \end{pmatrix}.$$

Conditions on Y imply that all its entries $Y_{i,j}$ are analytic in $\mathbb{C}^*\setminus\Sigma$. In particular, for $Y_{1,1}$ condition (RH2) yields $(Y_{1,1})_+(x) = (Y_{1,1})_-(x)$ for $x \in \Sigma$, and hence, by Morera's theorem $Y_{1,1}$ is analytic in $\mathbb{C}^*$. Now, taking into account (RH3)–(RH4) we obtain that $\lim_{z\to\infty}\frac{Y_{1,1}(z)}{z^{q_{n-1}+1}} = b_{n,n}^{s_n}$ and $\lim_{z\to 0} Y_{1,1}(z)z^{p_{n-1}} = b_{0,n}^{1-s_n}$, which implies that $L_{1,1}(z) = \frac{Y_{1,1}(z)}{z^{s_n}} \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ and, thus, it is regular and monic. Now, consider $Y_{1,2}$. From (RH2), we have

$$(Y_{1,2})_+(x) = (Y_{1,2})_-(x) + \omega(x)x^{s_n}L_{1,1}(x), \quad \forall x \in \Sigma,$$

and thus, by the additive scalar Riemann–Hilbert problem, $Y_{1,2}$ is the Cauchy transform of $Y_{1,1}$, that is,

$$Y_{1,2}(z) = \frac{1}{2\pi\imath}\int_\Sigma \frac{x^{s_n}L_{1,1}(x)}{x-z}\omega(x)\,\mathrm{d}x.$$

Now, from (RH3)–(RH4) we have that $\lim_{z\to\infty} Y_{1,2}(z)z^{q_{n-1}+1} = s_n\tau_n$ and $\lim_{z\to 0}\frac{Y_{1,2}(z)}{z^{p_{n-1}}} = (1-s_n)\upsilon_n$. Thus, taking into account that

$$Y_{1,2}(z) = -\sum_{\nu=0}^{l}\frac{1}{z^{\nu+1}}\frac{1}{2\pi\imath}\int_\Sigma x^{s_n}L_{1,1}(x)x^\nu\omega(x)\mathrm{d}x + \frac{1}{z^{l+1}}\frac{1}{2\pi\imath}\int_\Sigma \frac{x^{s_n}L_{1,1}(x)x^{l+1}\omega(x)}{x-z}\mathrm{d}x$$

and

$$Y_{1,2}(z) = \sum_{k=0}^{l} z^k\frac{1}{2\pi\imath}\int_\Sigma \frac{x^{s_n}L_{1,1}(x)}{x^{k+1}}\omega(x)\mathrm{d}x + z^{l+1}\frac{1}{2\pi\imath}\int_\Sigma \frac{x^{s_n}L_{1,1}(x)\omega(x)}{x^{l+1}(x-z)}\mathrm{d}x,$$

it is immediate to conclude that

$$\int_\Sigma x^{s_n}L_{1,1}(x)x^\nu\omega(x)\mathrm{d}x = 0, \quad \nu = 0, 1, \ldots, q_{n-1}-1,$$

and

$$-\frac{1}{2\pi\imath}\int_\Sigma x^{s_n}L_{1,1}(x)x^{q_{n-1}}\omega(x)\mathrm{d}x = s_n\tau_n. \tag{3.4}$$

In a similar way,

$$\int_\Sigma \frac{x^{s_n}L_{1,1}(x)}{x^k}\omega(x)\mathrm{d}x = 0, \quad k = 1, 2, \ldots, p_{n-1},$$

and

$$\frac{1}{2\pi\imath}\int_\Sigma \frac{x^{s_n}L_{1,1}(x)}{x^{p_{n-1}+1}}\omega(x)\mathrm{d}x = (1-s_n)\upsilon_n. \tag{3.5}$$

Thus, we have that $L_{1,1} \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ is monic and regular and

$$\int_\Sigma L_{1,1}(x)\,L(x)\omega(x)\mathrm{d}x = 0, \quad \forall L \in \mathcal{L}_{n-1}.$$

This shows that $L_{1,1}(z) = \phi_n(z)$. In addition, we have that ϕ_n satisfies

$$-\frac{1}{2\pi\imath}\int_\Sigma \phi_n(x)\left[s_nx^{q_n+1} - \frac{1-s_n}{x^{p_n+1}}\right]\omega(x)\mathrm{d}x = s_n\tau_n + (1-s_n)\upsilon_n.$$

Proceeding in an analogous way with the $(2,1)$-entry of Y, it is easy to see that $L_{2,1}(z)=\frac{Y_{2,1}(z)}{z^{s_{n-1}}}\in\mathcal{L}_{n-1}\setminus\mathcal{L}_{n-2}$ and that $Y_{2,2}$ is the Cauchy transform of $x^{s_{n-1}}L_{2,1}(x)$. Now, from (RH3)–(RH4), we deduce that $L_{2,1}$ satisfies that

$$\begin{aligned}&\int_\Sigma L_{2,1}(x)\,L(x)\omega(x)\mathrm{d}x=0,\quad \forall L\in\mathcal{L}_{n-2},\\ &-\frac{1}{2\pi\imath}\int_\Sigma L_{2,1}(x)\,x^{q_{n-1}+s_{n-1}}\,\omega(x)\mathrm{d}x=A_{n-1}^{s_{n-1}}\end{aligned}\tag{3.6}$$

and

$$\frac{1}{2\pi\imath}\int_\Sigma\frac{L_{2,1}(x)}{x^{p_{n-1}+1-s_{n-1}}}\,\omega(x)\mathrm{d}x=B_{n-1}^{1-s_{n-1}}.\tag{3.7}$$

When $s_{n-1}=0$, setting $L_{2,1}(z)=\sum_{k=0}^{n-1}\tilde{b}_{k,n-1}\,z^{-p_{n-1}+k}$, from (3.6) we deduce that

$$\tilde{b}_{n-1,n-1}\left(-\frac{1}{2\pi\imath}\int_\Sigma\phi_{n-1}(x)\,x^{q_{n-1}}\,\omega(x)\mathrm{d}x\right)=1,$$

and if $s_{n-1}=1$, from (3.7) we obtain

$$\tilde{b}_{0,n-1}\left(\frac{1}{2\pi\imath}\int_\Sigma\frac{\phi_{n-1}(x)}{x^{p_{n-1}}}\,\omega(x)\mathrm{d}x\right)=1.$$

Then, we conclude that $L_{2,1}=\tilde{b}_{(n-1)(1-s_{n-1}),n-1}\phi_{n-1}=c_{n-1}\phi_{n-1}$. □

Corollary 1. *The matrices Γ_n^∞ and Γ_n^0, which characterize the behavior of Y near infinity and near the origin, respectively, are regular and* $\det(\Gamma_n^\infty)=\det(\Gamma_n^0)$.

Proof. First, we will show that

$$\det(\Gamma_n^\infty)=(1-s_{n-1})b_{n,n}^{s_n}+s_{n-1}b_{0,n}^{1-s_n}\neq 0.$$

It will be proved for all the possible values of s_n and s_{n-1}.

1. $s_n=1$. In this case, it should be established that:
$$b_{n,n}A_{n-1}^{s_{n-1}}-s_{n-1}\tilde{b}_{n-1,n-1}\tau_n=(1-s_{n-1})b_{n,n}+s_{n-1}.$$
   - If $s_{n-1}=0$, it is trivial.
   - Suppose $s_{n-1}=1$. It should be shown that
   $$b_{n,n}A_{n-1}-\tilde{b}_{n-1,n-1}\tau_n=1.\tag{3.8}$$
   Indeed, taking into account (3.4)–(3.6) and the orthogonality conditions, and having in mind that $q_{n-2}=q_{n-1}=q_n$, it follows that
   $$\begin{aligned}&A_{n-1}=-\frac{1}{2\pi\imath}\int_\Sigma L_{2,1}(x)x^{q_{n-1}+1}\omega(x)\mathrm{d}x,\qquad b_{n,n}A_{n-1}=-\frac{1}{2\pi\imath}\int_\Sigma L_{2,1}(x)b_{n,n}x^{q_n}x\omega(x)\mathrm{d}x,\\ &\tau_n=-\frac{1}{2\pi\imath}\int_\Sigma\phi_n(x)x^{q_{n-1}+1}\omega(x)\mathrm{d}x,\qquad \tilde{b}_{n-1,n-1}\tau_n=-\frac{1}{2\pi\imath}\int_\Sigma\phi_n(x)L_{2,1}(x)x\omega(x)\mathrm{d}x,\end{aligned}$$
   and therefore,
   $$\begin{aligned}b_{n,n}A_{n-1}-\tilde{b}_{n-1,n-1}\tau_n&=-\frac{1}{2\pi\imath}\int_\Sigma L_{2,1}(x)(b_{n,n}x^{q_n}-\phi_n(x))x\omega(x)\mathrm{d}x\\&=\frac{1}{2\pi\imath}\int_\Sigma\frac{L_{2,1}(x)}{x^{p_{n-1}}}\omega(x)\mathrm{d}x=1.\end{aligned}$$
2. $s_n=0$. We have to prove that $A_{n-1}^{s_{n-1}}=(1-s_{n-1})+s_{n-1}b_{0,n}$.
   - If $s_{n-1}=0$, it is immediate.
   - If $s_{n-1}=1$, we need to check that
   $$A_{n-1}=b_{0,n}\neq 0.\tag{3.9}$$
   But making use of (3.6) and the orthogonality conditions, and taking into account that $q_n=q_{n-1}+1$, we obtain
   $$\begin{aligned}A_{n-1}&=-\frac{1}{2\pi\imath}\int_\Sigma L_{2,1}(x)x^{q_{n-1}+1}\omega(x)\mathrm{d}x\\&=-\frac{1}{2\pi\imath}\int_\Sigma L_{2,1}(x)\left[x^{q_n}+\frac{b_{0,n}}{x^{p_{n-1}}}\right]\omega(x)\mathrm{d}x+\frac{1}{2\pi\imath}\int L_{2,1}(x)\frac{b_{0,n}}{x^{p_{n-1}}}\omega(x)\mathrm{d}x\\&=-\frac{1}{2\pi\imath}\int_\Sigma L_{2,1}(x)\phi_n(x)\omega(x)\mathrm{d}x+b_{0,n}\\&=b_{0,n}.\end{aligned}$$

Now, we will prove that $\det(\Gamma_n^\infty) = \det(\Gamma_n^0)$, that is,

$$b_{n,n}^{s_n} A_{n-1}^{s_{n-1}} - s_n s_{n-1} \tilde{b}_{n-1,n-1}\tau_n = b_{0,n}^{1-s_n} B_{n-1}^{1-s_{n-1}} - (1-s_{n-1})(1-s_n)\tilde{b}_{0,n-1}\upsilon_n.$$

Let us take into account the possible values for s_n and s_{n-1}, as above.

1. $s_n = 0$. We have to prove that

$$A_{n-1}^{s_{n-1}} = b_{0,n} B_{n-1}^{1-s_{n-1}} - (1-s_{n-1})\tilde{b}_{0,n-1}\upsilon_n. \tag{3.10}$$

- $s_{n-1} = 0$. In this case, (3.10) yields $1 = b_{0,n}B_{n-1} - \tilde{b}_{0,n-1}\upsilon_n$, and taking into account that $p_{n-2} = p_{n-1} = p_n$, (3.5)–(3.7) and the orthogonality conditions satisfied by $L_{2,1}$, we have that,

$$B_{n-1} = \frac{1}{2\pi\imath}\int_\Sigma \frac{L_{2,1}(x)}{x^{p_{n-1}+1}}\omega(x)\mathrm{d}x, \qquad b_{0,n}B_{n-1} = \frac{1}{2\pi\imath}\int_\Sigma L_{2,1}(x)\frac{b_{0,n}}{x^{p_n}}\frac{\omega(x)\mathrm{d}x}{x},$$

$$\upsilon_n = \frac{1}{2\pi\imath}\int_\Sigma \frac{\phi_n(x)}{x^{p_{n-1}+1}}\omega(x)\mathrm{d}x, \qquad \tilde{b}_{0,n-1}\upsilon_n = \frac{1}{2\pi\imath}\int_\Sigma \phi_n(x)L_{2,1}(x)\frac{\omega(x)\mathrm{d}x}{x},$$

and thus,

$$\begin{aligned} b_{0,n}B_{n-1} - \tilde{b}_{0,n-1}\upsilon_n &= \frac{1}{2\pi\imath}\int_\Sigma L_{2,1}(x)\left(\frac{b_{0,n}}{x^{p_n}} - \phi_n(x)\right)\frac{\omega(x)\mathrm{d}x}{x} \\ &= -\frac{1}{2\pi\imath}\int_\Sigma L_{2,1}(x)x^{q_{n-1}}\omega(x)\mathrm{d}x = 1. \end{aligned}$$

- $s_{n-1} = 1$. It should be checked that $A_{n-1} = b_{0,n}$, but it was shown in (3.9).

2. $s_n = 1$. Now, we have to prove that

$$b_{n,n}A_{n-1}^{s_{n-1}} - s_{n-1}\tilde{b}_{n-1,n-1}\tau_n = B_{n-1}^{1-s_{n-1}}. \tag{3.11}$$

- $s_{n-1} = 0$. In this case, (3.11) implies that $b_{n,n} - B_{n-1} = 0$, but using the same orthogonality conditions as above and taking into account that now $p_{n-1}+1 = p_n$ and $q_n = q_{n-1}$, and (3.7), we conclude that

$$\begin{aligned} b_{n,n} - B_{n-1} &= -b_{n,n}\int_\Sigma L_{2,1}(x)x^{q_{n-1}}\omega(x)\mathrm{d}x - \int_\Sigma L_{2,1}(x)\frac{1}{x^{p_{n-1}+1}}\omega(x)\mathrm{d}x \\ &= -\frac{1}{2\pi\imath}\int_\Sigma L_{2,1}(x)\phi_n(x)\omega(x)\mathrm{d}x = 0. \end{aligned}$$

- $s_{n-1} = 1$. We need to prove that $b_{n,n}A_{n-1} - \tilde{b}_{n-1,n-1}\tau_n = 1$, but it was established in (3.8). □

Remark 1. Observe that the regularity of ϕ_{n-1} is not needed.

To prove the uniqueness of the solution of our Riemann–Hilbert problem we need the next result.

Corollary 2. *The solution $Y(z)$ of the Riemann–Hilbert problem* (RH1)–(RH4) *is regular for any $z \in \hat{\mathbb{C}}$. Moreover,* $\det(Y(z)) = \det(\Gamma_n^\infty)$.

Proof. Setting $g(z) = \det(Y(z))$, condition (RH1) implies that g is analytic in $\mathbb{C}^* \setminus \Sigma$. But from condition (RH2), we conclude that $g_+(x) = g_-(x)$, $x \in \Sigma$ and, hence, that g is analytic in $\mathbb{C}^*$. Thus, we only need to prove the analyticity at the origin and infinity. However, (RH4) shows that the possible singularity at the origin is removable, which implies that g is an entire function. Finally, (RH3) and Liouville's theorem show that g is constant in $\hat{\mathbb{C}}$. Therefore, $\det(Y(z)) = \det(Y(0)) = \det\Gamma_n^0 \neq 0$, $\forall z \in \hat{\mathbb{C}}$. □

Corollary 3. *The solution $Y(z)$ of the Riemann–Hilbert problem* (RH1)–(RH4) *is unique.*

Proof. Suppose that there exist two solutions Y_1 and Y_2. Since both are regular matrices, it is possible to set $A(z) = Y_1(z)Y_2(z)^{-1}$, which is analytic in $\mathbb{C}^* \setminus \Sigma$. From (RH2), $A_+(x) = Y_{1,+}(x)Y_{2,+}(x)^{-1} = Y_{1,-}(x)Y_{2,-}(x)^{-1} = A_-(x)$, $x \in \Sigma$, and hence, $A(z)$ is analytic in $\mathbb{C}^*$. Moreover, it is easy to see that $A(z)$ agrees with the identity matrix $I = I_2$, both at the origin and at infinity. Thus, Liouville's theorem yields $A(z) = I$ and, hence, $Y_1(z) = Y_2(z)$, $z \in \hat{\mathbb{C}}$. □

4. Some consequences of the characterization

As in the case of classical orthogonal polynomials, in this section some of the most relevant properties of OLP will be derived from their characterization as solutions of the above Riemann–Hilbert problem.

4.1. The three-term recurrence relation

Denote $G_n(z) = Y_n(z)Y_{n-1}^{-1}(z)$, where $Y_n(z)$ is the unique solution of the Riemann–Hilbert problem (RH1)–(RH4). This matrix-valued function satisfies the following conditions.

(A) G_n is analytic in $\mathbb{C}^*$. Indeed, it is immediate that G_n is analytic in $\mathbb{C}^* \setminus \Sigma$ since Y_n, Y_{n-1}^{-1} are analytic there. Now, taking into account that by (RH2) Y_n and Y_{n-1} have the same jump, it follows that $G_n(x)_+ = G_n(x)_-$, $\forall x \in \Sigma$, and thus, G_n is analytic in $\mathbb{C}^*$.

(B) Behavior of $G_n(z)$, when $z \to \infty$.

$$G_n(z) = Y_n(z) \begin{pmatrix} \frac{1}{z^{q_{n-1}+1}} & 0 \\ 0 & z^{q_{n-1}+1} \end{pmatrix} \begin{pmatrix} z^{1-s_{n-1}} & 0 \\ 0 & \frac{1}{z^{1-s_{n-1}}} \end{pmatrix} \begin{pmatrix} z^{q_{n-2}+1} & 0 \\ 0 & \frac{1}{z^{q_{n-2}+1}} \end{pmatrix} Y_{n-1}^{-1}(z),$$

and thus,

$$\frac{G_n(z)}{z^{1-s_{n-1}}} = Y_n(z) \begin{pmatrix} \frac{1}{z^{q_{n-1}+1}} & 0 \\ 0 & z^{q_{n-1}+1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{z^{2(1-s_{n-1})}} \end{pmatrix} \begin{pmatrix} z^{q_{n-2}+1} & 0 \\ 0 & \frac{1}{z^{q_{n-2}+1}} \end{pmatrix} Y_{n-1}^{-1}(z).$$

So, from (RH3) and taking limits when $z \to \infty$, it follows

$$\begin{aligned} \lim_{z\to\infty} \frac{G_n(z)}{z^{1-s_{n-1}}} &= \Gamma_n^\infty \begin{pmatrix} 1 & 0 \\ 0 & s_{n-1} \end{pmatrix} (\Gamma_{n-1}^\infty)^{-1} \\ &= \frac{1}{\det(\Gamma_{n-1}^\infty)} \Gamma_n^\infty \begin{pmatrix} A_{n-2}^{s_{n-2}} & -s_{n-1}\tau_{n-1} \\ -s_{n-1}s_{n-2}\tilde{b}_{n-2,n-2} & s_{n-1}b_{n-1,n-1}^{s_{n-1}} \end{pmatrix}, \end{aligned}$$

that is,

$$G_n(z) = E_n z^{1-s_{n-1}} + \mathcal{O}_\infty(1), \quad z \to \infty,\ E_n \in M_2(\mathbb{C}), \tag{4.1}$$

where $\mathcal{O}_\infty(1)$ means a matrix whose entries are power series in $\frac{1}{z}$, and if we denote $E_n = \frac{1}{\det(\Gamma_{n-1}^\infty)}(E_{i,j}(n))$, it holds

- $E_{1,1}(n) = b_{n,n}^{s_n} A_{n-2}^{s_{n-2}} - s_n s_{n-1} s_{n-2} \tau_n \tilde{b}_{n-2,n-2}$,
- $E_{2,1}(n) = s_{n-1}\tilde{b}_{n-1,n-1} A_{n-2}^{s_{n-2}} - s_{n-1}s_{n-2}\tilde{b}_{n-2,n-2} A_{n-1}^{s_{n-1}}$,
- $E_{1,2}(n) = -s_{n-1} b_{n,n}^{s_n} \tau_{n-1} + s_n s_{n-1} \tau_n b_{n-1,n-1}^{s_{n-1}}$,
- $E_{2,2}(n) = -s_{n-1}^2 \tilde{b}_{n-1,n-1}\tau_{n-1} + s_{n-1} A_{n-1}^{s_{n-1}} b_{n-1,n-1}^{s_{n-1}}$.

(C) Behavior of $G_n(z)$ when $z \to 0$. From (RH4) we have,

$$G_n(z) = Y_n(z) \begin{pmatrix} z^{p_{n-1}} & 0 \\ 0 & \frac{1}{z^{p_{n-1}}} \end{pmatrix} \begin{pmatrix} \frac{1}{z^{s_{n-1}}} & 0 \\ 0 & z^{s_{n-1}} \end{pmatrix} \begin{pmatrix} \frac{1}{z^{p_{n-2}}} & 0 \\ 0 & z^{p_{n-2}} \end{pmatrix} Y_{n-1}(z),$$

thus, proceeding as above,

$$z^{s_{n-1}} G_n(z) = Y_n(z) \begin{pmatrix} z^{p_{n-1}} & 0 \\ 0 & \frac{1}{z^{p_{n-1}}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^{2s_{n-1}} \end{pmatrix} \begin{pmatrix} z^{p_{n-2}} & 0 \\ 0 & \frac{1}{z^{p_{n-2}}} \end{pmatrix} Y_{n-1}(z),$$

and taking limits when $z \to 0$, we get

$$\begin{aligned} \lim_{z\to 0} z^{s_{n-1}} G_n(z) &= \Gamma_n^0 \begin{pmatrix} 1 & 0 \\ 0 & 1-s_{n-1} \end{pmatrix} (\Gamma_{n-1}^0)^{-1} \\ &= \frac{1}{\det(\Gamma_{n-1}^0)} \Gamma_n^\infty \begin{pmatrix} B_{n-2}^{1-s_{n-2}} & -(1-s_{n-1})\upsilon_{n-1} \\ -(1-s_{n-1})(1-s_{n-2})\tilde{b}_{0,n-2} & (1-s_{n-1}) b_{0,n-1}^{1-s_{n-1}} \end{pmatrix}, \end{aligned}$$

that is,

$$G_n(z) = \frac{D_n}{z^{s_{n-1}}} + \mathcal{O}_0(1), \quad z \to 0,\ D_n \in M_2(\mathbb{C}), \tag{4.2}$$

where $\mathcal{O}_0(1)$ denotes a matrix whose entries are power series in z, and setting $D_n = \frac{1}{\det(\Gamma_{n-1}^\infty)}(D_{i,j}(n))$, we have,

- $D_{1,1}(n) = b_{0,n}^{1-s_n} B_{n-2}^{1-s_{n-2}} - (1-s_n)(1-s_{n-1})(1-s_{n-2})\upsilon_n \tilde{b}_{0,n-2}$,
- $D_{2,1}(n) = (1-s_{n-1})\tilde{b}_{0,n-1} B_{n-2}^{1-s_{n-2}} - (1-s_{n-1})^2(1-s_{n-2})\tilde{b}_{0,n-2} B_{n-1}^{1-s_{n-1}}$,

- $D_{1,2}(n) = -(1-s_{n-1})\upsilon_{n-1}b_{0,n}^{1-s_n} + (1-s_n)(1-s_{n-1})\upsilon_n b_{0,n-1}^{1-s_{n-1}}$,
- $D_{2,2}(n) = -(1-s_{n-1})^2\tilde{b}_{0,n-1}\upsilon_{n-1} - (1-s_{n-1})b_{0,n-1}^{1-s_{n-1}}B_{n-1}^{1-s_{n-1}}$.

Therefore, by (4.1) and (4.2), G_n is analytic in $\mathbb{C}^*$ and it satisfies

$$G_n(z) = E_n z^{1-s_{n-1}} + \frac{D_n}{z^{s_{n-1}}}, \quad z \in \mathbb{C}^*.$$

Denoting by $G_{i,j}(n)$ the entries of G_n, it follows

$$G_{1,1}(n)(z) = \frac{1}{\det(\Gamma_{n-1}^{\infty})}\left[E_{1,1}(n)z^{1-s_{n-1}} + \frac{D_{1,1}(n)}{z^{s_{n-1}}}\right],$$

and it is easy to see that $G_{12}(n), G_{21}(n), G_{22}(n) \in \mathbb{R}$. Therefore,

$$G_n(z) = \begin{pmatrix} \dfrac{1}{\det(\Gamma_{n-1}^{\infty})}\left[E_{1,1}(n)z^{1-s_{n-1}} + \dfrac{D_{1,1}(n)}{z^{s_{n-1}}}\right] & G_{1,2}(n) \\ G_{2,1}(n) & G_{2,2}(n) \end{pmatrix},$$

or, what is the same,

$$Y_n(z) = \begin{pmatrix} \dfrac{1}{\det(\Gamma_{n-1}^{\infty})}\left[E_{1,1}(n)z^{1-s_{n-1}} + \dfrac{D_{1,1}(n)}{z^{s_{n-1}}}\right] & G_{1,2}(n) \\ G_{2,1}(n) & G_{2,2}(n) \end{pmatrix} Y_{n-1}(z).$$

From this last expression, we obtain that

$$Y_n(z) = \begin{pmatrix} \dfrac{1}{\det(\Gamma_{n-1}^{\infty})}\left[E_{1,1}(n)z^{1-s_{n-1}} + \dfrac{D_{1,1}(n)}{z^{s_{n-1}}}\right] & G_{1,2}(n) \\ c_{n-1} & 0 \end{pmatrix} Y_{n-1}(z).$$

Now, the fact that $\det(Y_n(z)) = \det(\Gamma_n^{\infty})$ implies that

$$G_{1,2}(n) = -\frac{\det(\Gamma_n^{\infty})}{c_{n-1}\det(\Gamma_{n-1}^{\infty})},$$

and therefore,

$$z^{s_n}\phi_n(z) = \frac{1}{\det(\Gamma_{n-1}^{\infty})}\left[E_{1,1}(n)z^{1-s_{n-1}} + \frac{D_{1,1}(n)}{z^{s_{n-1}}}\right]z^{s_{n-1}}\phi_{n-1}(z) - \frac{c_{n-2}\det(\Gamma_n^{\infty})}{c_{n-1}\det(\Gamma_{n-1}^{\infty})}z^{s_{n-2}}\phi_{n-2}(z).$$

Thus, taking into account that $E_{1,1}(n) \neq 0$ and $D_{1,1}(n) \neq 0$, when $s_{n-1} = 0$ and $s_{n-1} = 1$, respectively, it follows,

$$\begin{aligned} z^{s_n}\phi_n(z) = {} & (-1)^{s_{n-1}}\frac{E_{1,1}(n)^{1-s_{n-1}}(-D_{1,1}(n))^{s_{n-1}}}{\det(\Gamma_{n-1}^{\infty})}\left[z^{1-2s_{n-1}} - \left(\frac{-D_{1,1}(n)}{E_{1,1}(n)}\right)^{1-2s_{n-1}}\right]z^{s_{n-1}}\phi_{n-1}(z) \\ & - \frac{c_{n-2}\,\det(\Gamma_n^{\infty})}{c_{n-1}\det(\Gamma_{n-1}^{\infty})}z^{s_{n-2}}\phi_{n-2}(z). \end{aligned} \tag{4.3}$$

It is not hard to conclude that $\frac{E_{1,1}(n)^{1-s_{n-1}}(-D_{1,1}(n))^{s_{n-1}}}{\det(\Gamma_{n-1}^{\infty})} = (-1)^{s_{n-1}}\det(\Gamma_n^{\infty})$. Hence, denoting $\Omega_{n-1} = \left(\frac{-D_{1,1}(n)}{E_{1,1}(n)}\right)^{1-2s_{n-1}}$, (4.3) may be rewritten in terms of the orthonormal Laurent polynomials Φ_n, Φ_{n-1}, Φ_{n-2}, where $\Phi_n(z) = \frac{\operatorname{sgn}(b_{n,n})^{s_n}}{\gamma_n}\phi_n(z)$, $\operatorname{sgn}(x)$ denotes the sign of x and $\gamma_n = \langle\phi_n, \phi_n\rangle^{\frac{1}{2}}$. Namely,

$$\begin{aligned} \frac{(-1)^{s_{n-1}}\gamma_n(\operatorname{sgn}(b_{n,n}))^{s_n}}{\det(\Gamma_n^{\infty})\gamma_{n-1}(\operatorname{sgn}(b_{n-1,n-1}))^{s_{n-1}}}z^{s_n}\Phi_n(z) = {} & (-1)^{s_{n-1}}[z^{1-2s_{n-1}} - \Omega_{n-1}]z^{s_{n-1}}\Phi_{n-1}(z) \\ & - \frac{(-1)^{s_{n-1}}c_{n-2}\gamma_{n-2}(\operatorname{sgn}(b_{n-2,n-2}))^{s_{n-2}}}{c_{n-1}\det(\Gamma_{n-1}^{\infty})\gamma_{n-1}(\operatorname{sgn}(b_{n-1,n-1}))^{s_{n-1}}}z^{s_{n-2}}\Phi_{n-2}(z). \end{aligned}$$

Finally, denoting $C_{n-1} = (-1)^{s_{n-1}}\frac{\gamma_n(\operatorname{sgn}(b_{n,n}))^{s_n}}{\det(\Gamma_n^{\infty})\gamma_{n-1}(\operatorname{sgn}(b_{n-1,n-1}))^{s_{n-1}}}$, we have

$$C_{n-1}z^{s_n}\Phi_n(z) = (-1)^{s_{n-1}}[z^{1-2s_{n-1}} - \Omega_{n-1}]z^{s_{n-1}}\Phi_{n-1}(z) - C_{n-2}z^{s_{n-2}}\Phi_{n-2}(z),$$

and the three-term recurrence relation (2.7) is finally recovered.

4.2. Christoffel–Darboux identity

In order to prove the Christoffel–Darboux identity it is equivalent to show that the following identity holds:

$$\mathcal{K}_{n-1}(z,t)=\frac{1}{2\pi\imath}\frac{1}{t-z}\,(0,1)\,Y_n^{-1}(z)Y_n(t)\,(1,0)^T, \tag{4.4}$$

where $\mathcal{K}_{n-1}(z,t)=\sum_{k=0}^{n-1}\Phi_k(z)\Phi_k(t)$.

Indeed, we have

$$Y_n(z)\,(1,0)^T=(z^{s_n}\phi_n(z),c_{n-1}z^{s_{n-1}}\phi_{n-1}(z))^T,\qquad (0,1)\,Y_n^{-1}(z)=\frac{1}{\det(\Gamma_n^\infty)}(-c_{n-1}z^{s_{n-1}}\phi_{n-1}(z),z^{s_n}\phi_n(z)),$$

which implies that

$$\frac{1}{2\pi\imath}\frac{1}{t-z}\,(0,1)\,Y_n^{-1}(z)Y_n(t)\,(1,0)^T=-\frac{c_{n-1}}{2\pi\imath\det(\Gamma_n^\infty)}\frac{t^{s_n}\phi_n(t)z^{s_{n-1}}\phi_{n-1}(z)-t^{s_{n-1}}\phi_{n-1}(t)z^{s_n}\phi_n(z)}{t-z}.$$

Observe that $\frac{1}{t-z}\,(0,1)\,Y_n^{-1}(z)Y_n(t)\,(1,0)^T\in\mathcal{L}_{n-1}$. Now, it is easy to check, on the one hand, that

$$\frac{1}{2\pi\imath}\int_\Sigma\frac{1}{t-z}\,(0,1)\,Y_n^{-1}(z)Y_n(t)\,(1,0)^T\,\omega(t)\mathrm{d}t=\frac{\det(Y_n(z))}{\det(\Gamma_n^\infty)}=1,$$

or, what is the same,

$$\frac{1}{2\pi\imath}\int_\Sigma\frac{1}{t-z}\,(0,1)\,Y_n^{-1}(z)Y_n(t)\,(1,0)^T\Phi_0(t)\omega(t)\mathrm{d}t=\Phi_0(z),$$

and, on the other, by the orthogonality conditions,

$$\frac{1}{2\pi\imath}\int_\Sigma(0,1)\,Y_n^{-1}(z)Y_n(t)\,(1,0)^T\left(\frac{\Phi_k(t)-\Phi_k(z)}{t-z}\right)\omega(t)\mathrm{d}t=0,\quad k=1,\dots,n-1,$$

or, equivalently,

$$\frac{1}{2\pi\imath}\int_\Sigma\frac{1}{t-z}\,(0,1)\,Y_n^{-1}(z)Y_n(t)\,(1,0)^T\,\Phi_k(t)\omega(t)\mathrm{d}t=\Phi_k(z),\quad k=1,\dots,n-1.$$

Finally, by linearity we have that

$$\frac{1}{2\pi\imath}\int_\Sigma\frac{1}{t-z}\,(0,1)\,Y_n^{-1}(z)Y_n(t)\,(1,0)^T\,L(t)\,\omega(t)\mathrm{d}t=L(z),\quad\forall L\in\mathcal{L}_{n-1}.$$

This shows that property (2.4) holds, and it renders the proof of (4.4).

4.3. Liouville–Ostrogradski formula

To prove the Liouville–Ostrogradski formula, we will show first that

$$\phi_n(z)\delta_{n-1}(z)-\delta_n(z)\phi_{n-1}(z)=-\frac{(-1)^{s_{n-1}}\gamma_{n-1}^2\det(\Gamma_n^\infty)}{z^{s_n+s_{n-1}}}, \tag{4.5}$$

where $\delta_n(z)=\int_\Sigma\frac{\phi_n(t)-\phi_n(z)}{t-z}\omega(t)\mathrm{d}t$. Indeed, it is easy to see that

$$\int_\Sigma\frac{t^{s_n}\phi_n(t)}{t-z}\omega(t)\mathrm{d}t=z^{s_n}\int_\Sigma\frac{\phi_n(t)}{t-z}\omega(t)\mathrm{d}t.$$

Now, we have that

$$\begin{aligned}\det(\Gamma_n^\infty)&=\det(Y_n(z))\\&=c_{n-1}\left[z^{s_n}\phi_n(z)\frac{1}{2\pi\imath}\int_\Sigma\frac{t^{s_{n-1}}\phi_{n-1}(t)}{t-z}\omega(t)\,\mathrm{d}t-z^{s_{n-1}}\phi_{n-1}(z)\frac{1}{2\pi\imath}\int_\Sigma\frac{t^{s_n}\phi_n(t)}{t-z}\omega(t)\,\mathrm{d}t\right]\\&=(-1)^{1-s_{n-1}}\frac{1}{\gamma_{n-1}^2}\left[z^{s_n}\phi_n(z)z^{s_{n-1}}\int_\Sigma\frac{\phi_{n-1}(t)}{t-z}\omega(t)\,\mathrm{d}t-z^{s_n}\phi_n(z)z^{s_{n-1}}\phi_{n-1}(z)\int_\Sigma\frac{\omega(t)}{t-z}\mathrm{d}t\right.\\&\quad\left.+z^{s_n}\phi_n(z)z^{s_{n-1}}\phi_{n-1}(z)\int_\Sigma\frac{\omega(t)}{t-z}\mathrm{d}t-z^{s_{n-1}}\phi_{n-1}(z)z^{s_n}\int\frac{\phi_n(t)}{t-z}\omega(t)\,\mathrm{d}t\right]\\&=(-1)^{1-s_{n-1}}\frac{z^{s_n+s_{n-1}}}{\gamma_{n-1}^2}\left[\phi_n(z)\delta_{n-1}(z)-\phi_{n-1}(z)\delta_n(z)\right],\end{aligned}$$

which implies (4.5).

Rewriting (4.5) in terms of the orthonormal Laurent polynomials, it follows that

$$\Phi_n(z)\Delta_{n-1}(z) - \Phi_{n-1}(z)\Delta_n(z) = -\frac{(-1)^{s_{n-1}}\det(\Gamma_n^\infty)\gamma_{n-1}}{z^{s_n+s_{n-1}}(\operatorname{sgn}(b_{n,n}))^{s_n}(\operatorname{sgn}(b_{n-1,n-1}))^{s_{n-1}}\gamma_n},$$

and thus, a comparison with (2.6) yields

$$C_{n-1} = (-1)^{s_{n-1}}\frac{\gamma_n(\operatorname{sgn}(b_{n,n}))^{s_n}}{\gamma_{n-1}(\operatorname{sgn}(b_{n-1,n-1}))^{s_{n-1}}}\frac{1}{\det(\Gamma_n^\infty)}.$$

4.4. Unit circle: Szegö polynomials

First, let us to show the connection with Szegö polynomials (see e.g. [29]), which will allow us to derive some results in this theory, some of them are well-known but others are new (see for instance Corollary 5). Let μ be now a positive measure such that $\operatorname{supp}\mu \subset \mathbb{T}$, the unit circle, and assume that $\mu'(\theta) > 0$ a.e. on $\mathbb{T}$ with respect to the Lebesgue measure (that is, the usual regularity condition on measures supported in the unit circle). In this situation, the expression of the moments (2.1) is replaced by

$$m_k = \int_0^{2\pi} e^{\imath\theta k}\mu'(\theta)\mathrm{d}\theta, \quad k \in \mathbb{Z}.$$

Since in this case the inner product is non-hermitian, to ensure that the Gram–Schmidt process is feasible, as well as the regularity of the corresponding orthogonal Laurent polynomials, it must be assumed that certain Hankel type determinants do not vanish (see [9] for details).

Through this subsection we only deal with one of the so-called balanced cases, that is, that related to the sequence $p_n = [\frac{n+1}{2}]$ or, what is the same, $s_n = \frac{1-(-1)^n}{2}$, where $[x]$ denotes the integer part of x (see also [7]). As we see below, the fact that the corresponding Hankel determinants do not vanish will be immediate. Remark that in this situation, $s_n + s_{n-1} = 1$, $n \in \mathbb{N}$.

Let $\{\rho_n\}_{n=0}^\infty$ be the sequence of monic orthogonal polynomials w.r.t. μ (the so-called Szegő polynomials). For each $n \geq 1$, ρ_n is a monic polynomial of exact degree n. Denote by $\{\rho_n^*\}_{n=0}^\infty$ the sequence of reversed (or reciprocal) polynomials, that is, $\rho_n^*(z) = z^n\overline{\rho_n(1/\bar{z})} \in \mathbb{P}_n$, with $\rho_0^*(z) = \rho_0(z) \equiv 1$. Taking into account their orthogonality properties, we have that

$$\int_{\mathbb{T}} \rho_n(z)\,\bar{z}^k\mu'(\theta)\mathrm{d}\theta = \int_{\mathbb{T}} \rho_n^*(z)\,\bar{z}^{k+1}\mu'(\theta)\mathrm{d}\theta = 0, \quad k = 0, 1, \ldots, n-1,\ z = e^{\imath\theta}.$$

Now, to show the link with orthogonal Laurent polynomials, set $\omega(z)\mathrm{d}z = \imath z\mu'(\theta)\mathrm{d}\theta$, $z = e^{\imath\theta}$. Then, we have

$$\begin{aligned}\int_{\mathbb{T}} \rho_n(z)\,\bar{z}^k\mu'(\theta)\mathrm{d}\theta &= \frac{1}{\imath}\int_{\mathbb{T}} \rho_n(z)\,z^{n-1-k}\,\frac{\omega(z)}{z^n}\mathrm{d}z = 0, \quad k = 0, 1, \ldots, n-1\\ &= \frac{1}{\imath}\int_{\mathbb{T}} \rho_n(z)\,z^j\,\frac{\omega(z)}{z^n}\mathrm{d}z = 0, \quad j = 0, 1, \ldots, n-1,\end{aligned} \tag{4.6}$$

and then, assuming that $\rho_n(0) \neq 0$ when $s_n = 1$,

$$\frac{1}{\imath}\int_{\mathbb{T}} \frac{\rho_n(z)}{z^{p_n}}\,L(z)\omega(z)\mathrm{d}z = \frac{(\rho_n(0))^{s_n}}{\imath}\int_{\mathbb{T}} \phi_n(z)\,L(z)\omega(z)\mathrm{d}z = 0, \quad \forall L \in \mathcal{L}_{n-1}, \qquad p_n = \left[\frac{n+1}{2}\right].$$

Therefore, when we deal with $p_n = [\frac{n+1}{2}]$, it is easy to conclude that $\phi_n(z) = \frac{\rho_n(z)}{(\rho_n(0))^{s_n}z^{p_n}}$ is the monic regular orthogonal Laurent polynomial with respect to the weight function ω, with $b_{ns_n,n} = \det(\Gamma_n^\infty) = (\rho_n(0))^{1-2s_n}$. Then, making use of the above Riemann–Hilbert problem (RH1)–(RH4), some well-known algebraic properties of Szegö polynomials may be deduced; first, the three-term recurrence relation will be obtained, with the coefficients given in terms of the entries of the matrices which describe the behavior near the origin and near infinity in the corresponding Riemann–Hilbert problem.

Corollary 4. *Let $\{\rho_n\}_{n=0}^\infty$ the sequence of monic orthogonal polynomials w.r.t. μ, where μ is a positive measure with $\operatorname{supp}\mu \subset \mathbb{T}$ (unit circle) and assume that $\mu'(\theta) > 0$ a.e. on $\mathbb{T}$ with respect to the Lebesgue measure. If $\rho_n(0) \neq 0$, $n \in \mathbb{N}$, then*

$$\rho_n(z) = \left(z + \frac{\rho_n(0)}{\rho_{n-1}(0)}\right)\rho_{n-1}(z) + \left[\frac{\gamma_{n-1}}{\gamma_{n-2}}\right]^2 (\rho_{n-1}(0))^{2s_{n-1}-1}(\rho_{n-2}(0))^{s_{n-1}-1}(\rho_n(0))^{s_{n-1}}z\rho_{n-2}(z),$$

where $\gamma_n^2 = \frac{1}{(\rho_n(0))^{2s_n}}\frac{1}{\imath}\int_{\mathbb{T}}\left(\frac{\rho_n(x)}{x^{p_n}}\right)^2\omega(x)\mathrm{d}x$.

Now, as a consequence of the Liouville–Ostrogradski formula, an expression relating first and second kind Szegö polynomials is obtained, which, as far as we know, is a new result.

Corollary 5. *Let $\{\rho_n\}_{n=0}^{\infty}$ the sequence of monic orthogonal polynomials w.r.t. μ, where μ is a positive measure with* $\operatorname{supp}\mu \subset \mathbb{T}$, *and assume that $\mu'(\theta) > 0$ a.e. on $\mathbb{T}$ with respect to the Lebesgue measure. Denote by $\{\xi_n\}_{n=0}^{\infty}$, the associated sequence of second kind polynomials. If $\rho_n(0) \neq 0$, $n \in \mathbb{N}$, then*

$$\rho_n(z)\xi_{n-1}(z) - \rho_{n-1}(z)\xi_n(z) = -2(-1)^{s_{n-1}} z^{n-1} \gamma_{n-1}^2 (\rho_n(0))^{1-s_n} (\rho_{n-1}(0))^{s_{n-1}}.$$

Proof. (4.5) yields

$$\phi_n(z)\delta_{n-1}(z) - \delta_n(z)\phi_{n-1}(z) = -(-1)^{s_{n-1}} \frac{\gamma_{n-1}^2 \det(\Gamma_n^{\infty})}{z^{s_n+s_{n-1}}},$$

and taking into account that $\phi_n(z) = \frac{\rho_n(z)}{(\rho_n(0))^{s_n} z^{p_n}}$, we have that $\delta_n(z) = \frac{1}{(\rho_n(0))^{s_n}} \int_{\mathbb{T}} \frac{\rho_n(t)}{t-z} \frac{\omega(t)}{t^{p_n}} \mathrm{d}t - \phi_n(z) \int_{\mathbb{T}} \frac{\omega(t)}{t-z} \mathrm{d}t$. Now, by the non-hermitian orthogonality relations (4.6), it is easy to see that

$$\int_{\mathbb{T}} \rho_n(t) \frac{t^{p_n} - z^{p_n}}{t-z} \frac{\omega(t)}{t^{p_n}} \mathrm{d}t = 0,$$

and therefore,

$$\delta_n(z) = \frac{1}{(\rho_n(0))^{s_n} z^{p_n}} \left[\int_{\mathbb{T}} \frac{\rho_n(t) - \rho_n(z)}{t-z} \omega(t) \mathrm{d}t \right].$$

On the other hand, we know that

$$\xi_n(z) = \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} \left[\rho_n(e^{i\theta}) - \rho_n(z) \right] \mu'(\theta) \mathrm{d}\theta,$$

and after some calculations, it yields

$$\int_{\mathbb{T}} \frac{\rho_n(t) - \rho_n(z)}{t-z} \omega(t) \mathrm{d}t = \frac{1}{2} \left(\xi_n(z) - m_0 \rho_n(z) \right),$$

from which it follows that

$$\delta_n(z) = \frac{1}{2(\rho_n(0))^{s_n} z^{p_n}} \left(\xi_n(z) - m_0 \rho_n(z) \right).$$

Then,

$$\begin{aligned} \rho_n(z)\xi_{n-1}(z) - \xi_n(z)\rho_{n-1}(z) &= -2(-1)^{s_{n-1}} z^{p_{n-1}+p_{n-2}} \gamma_{n-1}^2 (\rho_n(0))^{1-s_n} (\rho_{n-1}(0))^{s_{n-1}} \\ &= -2(-1)^{s_{n-1}} z^{n-1} \gamma_{n-1}^2 (\rho_n(0))^{1-s_n} (\rho_{n-1}(0))^{s_{n-1}}. \quad \square \end{aligned}$$

Next, the usual recurrence relation for Szegö polynomials will be derived. To do this, let us concentrate on the reversed polynomials. Proceeding in an analogous way as in (4.6), we have that

$$\begin{aligned} \int_{\mathbb{T}} \rho_n^*(z)\, \bar{z}^k \mu'(\theta) \mathrm{d}\theta &= \int_{\mathbb{T}} \rho_n^*(z)\, z^{n-k} \frac{\mu'(\theta)}{z^n} \mathrm{d}\theta, \quad k = 1, 2, \ldots, n \\ &= \int_{\mathbb{T}} \rho_n^*(z)\, z^j \frac{\omega(z)}{z^{n+1}} \mathrm{d}z = 0, \quad j = 0, 1, \ldots, n-1, \end{aligned}$$

and then,

$$\int_{\mathbb{T}} \frac{\rho_n^*(z)}{z^{p_n+1}} L(z)\, \omega(z)\, \mathrm{d}z = \int_{\mathbb{T}} \varphi_n(z)\, L(z)\, \omega(z)\, \mathrm{d}z, \quad \forall L \in \mathcal{L}_{n-1}.$$

But it is easy to see that in general $\varphi_n \notin \mathcal{L}_{n+1}$ ($s_{n+1} = 0$), and thus, we need to define the following subspace

$$\tilde{\mathcal{L}}_{n+1} = \mathcal{L}_n \oplus \operatorname{span}\left\{ \frac{1}{x^{p_n+1}} \right\},$$

so that

$$\varphi_n \in \tilde{\mathcal{L}}_{n+1}. \tag{4.7}$$

Let us now consider the finite sequence of nested subspaces

$$\mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_n \subset \tilde{\mathcal{L}}_{n+1},$$

and let $\{\hat{\Phi}_k\}_{k=0}^{n+1}$ be an orthonormal basis for $\tilde{\mathcal{L}}_{n+1}$. Then, it holds that $\hat{\Phi}_k = \Phi_k,\ k = 0, 1, \ldots, n$ and $\hat{\Phi}_{n+1} \in \tilde{\mathcal{L}}_{n+1} \setminus \mathcal{L}_n$. Hence, we can write

$$\varphi_n = \frac{1}{\hat{b}_{0,n+1}}\hat{\Phi}_{n+1} + \sum_{k=0}^{n} \tilde{c}_k \Phi_k,$$

where $\hat{\Phi}_{n+1}(z) = \sum_{k=0}^{n+1} \hat{b}_{k,n+1} z^{k-p_n-1}$. Thus, taking into account that φ_n is orthogonal to $\mathcal{L}_{n-1}$, then

$$\varphi_n(z) = \frac{1}{\hat{b}_{0,n+1}}\hat{\Phi}_{n+1}(z) + \tilde{c}_n \Phi_n(z).$$

We have that $\left(\frac{\rho_{n+1}(0)}{\hat{b}_{0,n+1}}\hat{\Phi}_{n+1}(z) - z^{-1}\phi_{n+1}(z)\right) \in \mathcal{L}_n$ is orthogonal to $\mathcal{L}_{n-1}$, and therefore,

$$\rho_{n+1}(0)\varphi_n(z) = z^{-1}\phi_{n+1}(z) + c_n^* \Phi_n(z),$$

which yields,

$$z\rho_{n+1}(0)\varphi_n(z) = \phi_{n+1}(z) + c_n z\phi_n(z).$$

Then,

$$\rho_{n+1}(0)\rho_n^*(z) = \rho_{n+1}(z) + c_n z\rho_n(z),$$

or, equivalently,

$$\rho_{n+1}(z) = c_n z\rho_n(z) + \rho_{n+1}(0)\rho_n^*(z),$$

which allows to conclude that $c_n = 1$. Finally, we get

$$\rho_{n+1}(z) = z\rho_n(z) + \rho_{n+1}(0)\rho_n^*(z), \tag{4.8}$$

that is, the classical relation between the Szegö polynomials and their reversed ones is recovered. Note that the assumption on the Verblunsky coefficient made in Corollary 5 is not needed in this case. Now, if $s_{n+1} = 1$, $\varphi_n \in \mathcal{L}_{n+1}$ then we can write

$$\varphi_n = \Phi_{n+1} + \sum_{k=0}^{n} \tilde{c}_k \Phi_k,$$

and taking into account that φ_n is orthogonal to $\mathcal{L}_{n-1}$, then

$$\varphi_n(z) = \Phi_{n+1}(z) + \tilde{c}_n \Phi_n(z).$$

Observe that we assume $\rho_{n+1}(0) \neq 0$, because $\Phi_{n+1}(z) = \frac{\rho_{n+1}(z)}{\rho_{n+1}(0)z^{p_{n+1}}}$. Then,

$$\rho_n^*(z) = \frac{\rho_{n+1}(z)}{\rho_{n+1}(0)} + c_n^* z\rho_n(z),$$

therefore,

$$\rho_{n+1}(0)\rho_n^*(z) = \rho_{n+1}(z) + c_n z\rho_n(z),$$

or, equivalently,

$$\rho_{n+1}(z) = c_n z\rho_n(z) + \rho_{n+1}(0)\rho_n^*(z),$$

which allows to conclude that $c_n = 1$. Finally, (4.8) is established, but now assuming that the Verblunsky coefficient is nonzero.

To end this subsection devoted to the connection with Szegő polynomials, let us show the relation between our RHP for OLP and the corresponding for Szegő polynomials [23]. Recall that we consider the particular setting when $p_n = \left[\frac{n+1}{2}\right]$, so that $p_n + p_{n-1} = n$ and $s_{n-1} + s_n = 1$, for each $n \in \mathbb{N}$.

Indeed, it is shown in [23] that the unique solution of the Riemann–Hilbert Problem

(RHI) Z is analytic in $\mathbb{C} \setminus \mathbb{T}$.

(RHII) $Z_+(z) = Z_-(z) \begin{pmatrix} 1 & \omega(z)/z^n \\ 0 & 1 \end{pmatrix},\ z \in \mathbb{T}$.

(RHIII) $\lim_{z\to\infty} Z(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = I$

is given by

$$Z(z)=\begin{pmatrix}\rho_n(z) & C_\omega\left(\dfrac{\rho_n}{x^n}\right)(z)\\ -2\pi\kappa_{n-1}^2\rho_{n-1}^*(z) & -2\pi\kappa_{n-1}^2\,C_\omega\left(\dfrac{\rho_{n-1}^*}{x^n}\right)(z)\end{pmatrix},$$

where $C_\omega(\cdot)$ has the same meaning as in (3.3) and κ_n denotes, as usual, the leading coefficient of the orthonormal Szegő polynomial of degree n.

First, take into account that from the orthogonality relations, the solution of our Riemann–Hilbert problem (RH1)–(RH4) (3.1) can be written in the form

$$Y(z)=\begin{pmatrix}\dfrac{\rho_n(z)}{V_n z^{p_{n-1}}} & \dfrac{z^{p_n}}{V_n}C_\omega\left(\dfrac{\rho_n}{x^n}\right)(z)\\ \dfrac{c_{n-1}\rho_{n-1}(z)}{V_{n-1}z^{p_{n-2}}} & \dfrac{c_{n-1}z^{p_{n-1}}}{V_{n-1}}C_\omega\left(\dfrac{\rho_{n-1}}{x^{n-1}}\right)(z)\end{pmatrix},\tag{4.9}$$

where $V_n=(\rho_n(0))^{s_n}$ and, by (3.2), $c_n=-\frac{(-1)^{s_n}2\pi\imath}{\gamma_n^2}=2\pi\frac{V_n}{V_{n+1}}\kappa_n^2$. Thus, defining the matrices

$$A(z)=\begin{pmatrix}V_n & 0\\ -\dfrac{c_{n-1}V_n}{V_{n-1}} & z^{1-s_{n-1}}\end{pmatrix},\qquad B(z)=\begin{pmatrix}z^{p_{n-1}} & 0\\ 0 & z^{-p_n}\end{pmatrix},$$

it is not hard to see that the solutions of our RHP and the RHP for Szegő polynomials in [23] are linked by the relation

$$Z(z)=A(z)Y(z)B(z).$$

In addition, if we denote by $J_Y(z)$ and $J_Z(z)$ the jump matrices in (RH2) and (RHII), respectively, we have that they are also linked in the suitable way, that is,

$$J_Y=B(z)^{-1}J_Z(z)B(z).$$

Thus, it only remains to check the relation between the behaviors at infinity (RH3)–(RHIII). In order to do it, it is enough to prove that as $z\to\infty$,

$$A(z)\left[\Gamma_n^\infty+\mathcal{O}\left(\frac{1}{z}\right)\right]\begin{pmatrix}1 & 0\\ 0 & z^{-s_n}\end{pmatrix}=I+\mathcal{O}\left(\frac{1}{z}\right),$$

but it readily follows when we analyze separately the two possible cases for which $s_n+s_{n-1}=1$.

Therefore, in a certain sense we have that for the particular choice $p_n=\left[\frac{n+1}{2}\right]$, the first three conditions of our RHP for OLP allows to recover the RHP for Szegő polynomials in [23].

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