



An attractor for the singularly perturbed Kirchhoff equation with supercritical nonlinearity



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ARTICLE INFO

Article history:

Received 22 April 2013

Available online 23 September 2013

Submitted by X. Zhang

Keywords:

Kirchhoff equation

Global attractor

ABSTRACT

We consider a quasilinear wave equation of Kirchhoff type

$$\epsilon u_{tt} - (1 + \|\nabla u\|^2) \Delta u + u_t + f(u) = g(x),$$

where $\epsilon > 0$ is a small parameter. Without any growth restrictions on the nonlinearity $f(u)$, we prove the existence of a finite-dimensional global attractor on an appropriate (bounded) phase space. The key step is the estimate of the difference between the solutions of a quasilinear dissipative hyperbolic equation of Kirchhoff type and the corresponding quasilinear parabolic equation.

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1. Introduction

In this article, we investigate the longtime behavior of the solutions to the following problem

$$\epsilon u_{tt} - (1 + \|\nabla u\|^2) \Delta u + u_t + f(u) = g(x), \quad (1.1)$$

on $\Omega \times (0, +\infty)$, Ω being a bounded smooth subset of \mathbb{R}^3 , $0 < \epsilon < 1$, endowed with initial and boundary conditions

$$u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1, \quad (1.2)$$

$$u|_{\partial\Omega} = 0. \quad (1.3)$$

In one dimension space, Eqs. (1.1)–(1.3) without dissipation and source terms were introduced by G. Kirchhoff [8] as a model for the small transversal vibrations of an elastic string with fixed endpoints. The nonlinear term $f(u)$ appears commonly to describe various self-interactions in evolution equation.

The singular perturbation problem for Kirchhoff equations has generated a lot of impressive literatures in the last 30 years. We refer to [2] and the references therein. [3,4,7] obtain the decay-error estimates for the difference between a solution of the hyperbolic problem and the solution of the corresponding parabolic problem. However the main attention of [3,4,7] is paid the case when the term $f(u)$ is absent.

On the other hand, the Cauchy problem (for the case $\Omega = \mathbb{R}^n$) and the boundary value problem for Eq. (1.1) ($\epsilon = 1$) with dissipation u_t or $-\Delta u_t$ or $h(u_t)$ have been studied in many papers (see [1,9–16] and the references therein). It is well known that the problem (1.1)–(1.3) ($\epsilon = 1$) admits a unique global solution only as the norms of the initial data in the usual Sobolev spaces of appropriate order are suitably small, so we could not discuss the global attractor. Recently, Nakao [10]

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constructs a local attractor in a small neighborhood of zero for the problem (1.1)–(1.3) ($\epsilon = 1$), under the assumption of the criticality of $f(u)$.

We see that the smallness assumption for the initial data in [10] can be replaced by the smallness of ϵ . The main purpose of the present work is, under the assumption that ϵ is small enough, to study the well-posedness and the longtime dynamics of the problem (1.1)–(1.3) on an approximation phase space. Here $f(u)$ is of supercritical growth. This article extends and generalizes the local stability results of [10] in some extent.

More precisely, following the idea from [5], we firstly prove the existence and uniqueness of the weak solution if $t \in \mathbb{R}$ is small enough. These results are obtained without the assumption that $\epsilon > 0$ is small enough. Secondly, under the assumptions that the initial data is bounded by a constant R_ϵ which blows up as ϵ tends to 0, we prove the global well-posedness and the dissipation of the solution. This result is based on the comparison of solutions between the hyperbolic equation and the limiting ($\epsilon = 0$) parabolic equation. We finally obtain the existence of a global attractor with finite fractal dimension.

The paper is organized as follows. In Section 2, we formulate an abstract version of problem (1.1)–(1.3), by means of a general operator defined on a suitable Hilbert space. The basic assumptions are stated. In Section 3, the existence of a bounded absorbing set, along with the characterization of the system is carried out. In Section 4, the existence of a global attractor with finite fractal dimension is shown.

2. Preliminaries

We first introduce some notation that we will use throughout this paper, then state the basic mathematical assumptions for considering the longtime behaviors of Eqs. (1.1)–(1.3).

We denote the inner product and the norm on $H = L^2(\Omega)$ by (\cdot, \cdot) and $\|\cdot\|$, respectively. Let

$$A = -\Delta \quad \text{with domain } \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega),$$

$\{e_i\}_{i=1}^\infty$ be the orthonormal system of eigenvectors of the operator A and $\{\lambda_i\}_{i=1}^\infty$ be the corresponding eigenvalue:

$$Ae_i = \lambda_i e_i, \quad e_i|_{\partial\Omega} = 0, \quad \lambda_{i+1} \geq \lambda_i.$$

We consider a family of Hilbert spaces

$$H_s = \mathcal{D}(A^{s/2}), \quad s \in \mathbb{R},$$

whose inner products and norms are given by

$$(x, z)_s = (A^{s/2}x, A^{s/2}z) \quad \text{and} \quad \|\cdot\|_s = \|A^{s/2}\cdot\|.$$

Obviously,

$$H_s \hookrightarrow H_r, \quad \text{and} \quad \|x\|_r \leq \lambda_1^{(r-s)/2} \|x\|_s, \quad \forall s > r,$$

where λ_1 is the first eigenvalue of the operator A .

$$H_2 = H^2(\Omega) \cap H_0^1(\Omega) \Subset H_1 = H_0^1(\Omega) \Subset H = L^2(\Omega),$$

and $H'_1 := H^{-1}(\Omega)$ is the topological dual of H_1 . The duality between H_1 and H'_1 will be noted by $\langle \cdot, \cdot \rangle$.

We define the product spaces

$$\mathcal{H}_s = H_s \times H_{s-1},$$

endowed with the norm

$$\|(x, z)\|_{\mathcal{H}_s}^2 = \|x\|_s^2 + \epsilon \|z\|_{s-1}^2.$$

As before, $\mathcal{H}_{s+1} \subset \mathcal{H}_s$.

General assumptions. We assume that $g(x) \in H_1$, and the nonlinear function $f(u)$ satisfies the following assumptions:

$$f \in C^2(\mathbb{R}, \mathbb{R}), \quad \text{with } f(0) = 0, \tag{2.1}$$

$$|f''(v)| \leq C(1 + |v|^p), \tag{2.2}$$

$$f'(v) \geq -K + \delta|v|^{p+1}, \tag{2.3}$$

where $p > 0$, $C > 0$, $K > 0$ and $\delta > 0$ are given constants. We now recall (see [17]) that assumption (2.3) implies that

$$f(v)v \geq |v|^2 \left(-K + \frac{\delta}{p+2} |v|^{p+1} \right), \quad (2.4)$$

$$F(v) \geq |v|^2 \left(-\frac{K}{2} + \frac{\delta}{(p+2)(p+3)} |v|^{p+1} \right), \quad (2.5)$$

$$F(v) \leq f(v)v - |v|^2 \left(-\frac{K}{2} + \frac{\delta}{(p+3)} |v|^{p+1} \right), \quad (2.6)$$

where $F(v) := \int_0^v f(\tau) d\tau$. Moreover, assumption (2.2) implies that

$$|f(v)| \leq C|v|(1 + |v|^{p+1}), \quad F(v) \leq C|v|^2(1 + |v|^{p+1}). \quad (2.7)$$

Throughout the paper, C_i ($i = 1, 2, \dots$) and Q_i ($i = 1, 2, \dots$) stand for a generic positive constant and a generic positive increasing function, respectively, independent of ϵ . Moreover, for any function $u(t)$, we will write for short

$$\xi_u = (u, u_t).$$

Rewriting Eq. (1.1) in the form of an operator equation, we get the problem which is equivalent to (1.1)–(1.3):

$$\epsilon u_{tt} + (1 + \|\nabla u\|^2) Au + u_t + f(u) = g, \quad (2.8)$$

$$u(0) = u_0, \quad u_t(0) = u_1. \quad (2.9)$$

3. The absorbing set

We begin the section with the following lemma which will be needed later.

Lemma 3.1. Let Φ be an absolutely continuous positive function on R^+ , which satisfies for some $\eta > 0$ the differential inequality

$$\frac{d}{dt} \Phi(t) + 2\eta \Phi(t) \leq g(t)\Phi(t) + h(t), \quad t > 0,$$

where $h \in L^1_{loc}(R^+)$ and

$$\int_{\tau}^t g(y) dy \leq \eta(t - \tau) + m, \quad \text{for } t \geq \tau \geq 0,$$

with some $m > 0$. Then

$$\Phi(t) \leq e^m \left(\Phi(0)e^{-\eta t} + \int_0^t |h(\tau)| e^{-\eta(t-\tau)} d\tau \right), \quad \forall t > 0.$$

Definition 3.1. A function $u(t)$ is said to be a weak solution of problem (1.1)–(1.3) on an interval $(0, T)$ if

$$\xi_u \in L^\infty(0, T; \mathcal{H}_2), \quad u_{tt} \in L^\infty(0, T; H),$$

and

$$\epsilon(u_{tt}, v) + (1 + \|\nabla u\|^2)(Au, v) + (u_t, v) + (f(u), v) = (g, v), \quad v \in H, \text{ for a.e. } t \in (0, T).$$

Theorem 3.2. For any $\epsilon \in (0, 1]$, $g(x) \in H_1$, let assumptions (2.1)–(2.3) hold. If

$$\xi_u(0) = (u_0, u_1) \in \mathcal{H}_2.$$

Then, there exist a time $T = T(\|\xi_u(0)\|_{\mathcal{H}_2}, \|g\|_{H_1})$ and a constant

$$Q(\|\xi_u(0)\|_{\mathcal{H}_2}, \|g\|_{H_1}) > 0,$$

both independent of ϵ , such that problem (1.1)–(1.3) has a weak solution

$$u \in W^{2,\infty}(0, T; H) \cap W^{1,\infty}(0, T; H_1) \cap L^\infty(0, T; H_2),$$

which satisfies

$$\|\xi_u(t)\|_{\mathcal{H}_1} + \int_0^t \|u_t(s)\|^2 ds \leq Q(\|\xi_u(0)\|_{\mathcal{H}_2})e^{-\kappa t} + Q(\|g\|_{H_1}), \quad \forall t > 0, \quad (3.1)$$

$$\|\xi_u(t)\|_{\mathcal{H}_2} + \int_0^t \|\nabla u_t(s)\|^2 ds \leq Q(\|\xi_u(0)\|_{\mathcal{H}_2}) + Q(\|g\|_{H_1}), \quad \forall t \in [0, T], \quad (3.2)$$

where κ is a positive constant and Q is a positive increasing function both independent of ϵ .

Proof. Since the results of the theorem is more or less standard, we give below only formal derivation of estimates (3.1) and (3.2), which can be easily justified using the Galerkin approximation.

Taking H -inner product by u_t in (2.8), so that

$$\frac{d}{dt} \left[\frac{\epsilon}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 + \frac{1}{4} \|\nabla u\|^4 + (F(u), 1) - (g, u) \right] + \|u_t\|^2 = 0. \quad (3.3)$$

Taking H -inner product by ηu in (2.8), where $\eta > 0$ is a small parameter will be fixed later. By adding the resulting equality to (3.3), we get

$$\frac{d}{dt} \phi_1 + (1 - \epsilon\eta) \|u_t\|^2 + \eta \|\nabla u\|^2 + \eta \|\nabla u\|^4 + \eta(f(u), u) - \eta(g, u) = 0, \quad (3.4)$$

where

$$\phi_1 = \frac{\epsilon}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 + \frac{1}{4} \|\nabla u\|^4 + (F(u), 1) - (g, u) + \epsilon\eta(u_t, u) + \frac{\eta}{2} \|u\|^2.$$

Choosing η small enough that

$$\begin{aligned} C_1(\|\nabla u\|^4 + \|u\|_{L^{p+3}}^{p+3} + \epsilon\|u_t\|^2) - C(1 + \|g\|_{H_1}^2) \\ \leq \phi_1 \leq C(\|\nabla u\|^4 + \|u\|_{L^{p+3}}^{p+3} + \epsilon\|u_t\|^2) + C(1 + \|g\|_{H_1}^2), \end{aligned} \quad (3.5)$$

where (2.5) and (2.7) have been used. Then, the last inequality, (2.6) and (3.4) give

$$\frac{d}{dt} \phi_1 + \eta\phi_1 + \kappa\|u_t\|^2 \leq C\eta, \quad (3.6)$$

for some positive constants $\kappa > 0$ and C both independent of ϵ . Applying the Gronwall inequality to this estimate and using (3.5), we arrive at

$$\|\nabla u\|^2 + \|u\|_{L^{p+3}}^{p+3} + \epsilon\|u_t\|^2 \leq Q(\|\xi_u(0)\|_{\mathcal{H}_2})e^{-\kappa t} + Q(\|g\|_{H_1}). \quad (3.7)$$

Integrating (3.3) over $[0, t]$, letting $t \rightarrow \infty$, using (2.7) and (3.7), we get

$$\int_0^\infty \|u_t(s)\|^2 ds \leq Q(\|\xi_u(0)\|_{\mathcal{H}_2})e^{-\kappa t} + Q(\|g\|_{H_1}). \quad (3.8)$$

Again, taking H -inner product by $Au_t + \eta Au$ in (2.8), we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \phi_2 + (1 - \epsilon\eta) \|\nabla u_t\|^2 + \eta \|Au\|^2 + \eta \|\nabla u\|^2 \|Au\|^2 + \eta(f'(u)\nabla u, \nabla u) - \eta(g, Au) \\ = (\nabla u_t, \nabla u) \|Au\|^2 + \frac{1}{2} (f''(u)|\nabla u|^2, u_t), \end{aligned} \quad (3.9)$$

where

$$\phi_2 = \epsilon \|\nabla u_t\|^2 + 2\epsilon\eta(\nabla u_t, \nabla u) + \|Au\|^2 + \|\nabla u\|^2 \|Au\|^2 + \eta \|\nabla u\|^2 + (f'(u)\nabla u, \nabla u) - 2(g, Au).$$

In light of (2.3), (3.7), $f \in C^2(\mathbb{R})$, the embedding $H_2 \rightarrow C^0(\bar{\Omega})$, and the Young inequality, we have (for η small enough)

$$\begin{aligned} \phi_2(t) &\geq C(\|Au\|^2 + \epsilon\|\nabla u_t\|^2) - K\|\nabla u\|^2 - C\|g\|_{H_1}^2 \\ &\geq C(\|Au\|^2 + \epsilon\|\nabla u_t\|^2) - Q(\|\xi_u(0)\|_{\mathcal{H}_2})e^{-\kappa t} - Q(\|g\|_{H_1}), \end{aligned} \quad (3.10)$$

$$\phi_2(0) \leq Q(\|\xi_u(0)\|_{\mathcal{H}_2}) + Q(\|g\|_{H_1}). \quad (3.11)$$

It follows that

$$\frac{d}{dt} \phi_2 + \kappa \phi_2 + \kappa \|\nabla u_t\|^2 \leq Q_1(\phi_2), \quad \kappa > 0, \quad (3.12)$$

where the inequalities

$$(\nabla u_t, \nabla u) \|Au\|^2 \leq \frac{1}{4} \|\nabla u_t\|^2 + \|\nabla u\|^2 \|Au\|^4, \quad (3.13)$$

and

$$\frac{1}{2} (f''(u) |\nabla u|^2, u_t) \leq \frac{1}{4} \|u_t\|^2 + \frac{1}{4} \|f''(u)\|_{L^\infty(\Omega)}^2 \|\nabla u\|_{L^4(\Omega)}^4, \quad (3.14)$$

have been used.

Let y be the solution to the Cauchy problem

$$\frac{d}{dt} y(t) + \kappa y(t) = Q_1(y(t)), \quad y(0) = \phi_2(0). \quad (3.15)$$

Then, the comparison principle gives

$$\phi_2(t) \leq y(t), \quad (3.16)$$

for every $t \in [0, T]$, where $T = T(\|\xi_u(0)\|_{\mathcal{H}_2}, \|g\|_{H_1})$ is some positive time which is strictly smaller than both the life span of y and of the Galerkin solution (i.e. T^*). Thus,

$$C(\|Au\|^2 + \epsilon \|\nabla u_t\|^2) - Q(\|\xi_u(0)\|_{\mathcal{H}_2}) - Q(\|g\|_{H_1}) \leq \phi_2(t) \leq Q(\|\xi_u(0)\|_{\mathcal{H}_2}) + Q(\|g\|_{H_1}), \quad t \in [0, T]. \quad (3.17)$$

Obviously, the estimates (3.16) and (3.17) yield

$$\|Au\|^2 + \epsilon \|\nabla u_t\|^2 \leq C, \quad (3.18)$$

for a positive constant C independent of ϵ but, of course, depending on $\xi_u(0)$.

Taking H -inner product by u_{tt} in (2.8), we have

$$u_{tt} \in L^\infty(0, T; H).$$

Theorem 3.2 is proven. \square

Remark 3.3. The results of **Theorem 3.2** certainly hold for the case $g \in L^2(\Omega)$.

Lemma 3.4. Let the assumptions of **Theorem 3.2** hold. Then, the weak solution of problem (1.1)–(1.3) is unique and satisfies the time continuity property

$$\xi_u \in C^0([0, T]; \mathcal{H}_2).$$

Proof. To prove uniqueness, let $\xi_u(t)$ and $\xi_v(t)$ be two weak solutions of problem (1.1)–(1.3) on $(0, T)$, with the same initial data $\xi_u(0)$. Set $w = u - v$, so that

$$\epsilon w_{tt} - \Delta w - \|\nabla u\|^2 \Delta u + \|\nabla v\|^2 \Delta v + w_t + f(u) - f(v) = 0. \quad (3.19)$$

Multiplying (3.19) by $w_t + \eta w$, we get

$$\begin{aligned} & \frac{d}{dt} Y + (1 - \epsilon \eta) \|w_t\|^2 + \eta \|\nabla w\|^2 - \eta (\|\nabla u\|^2 \Delta u - \|\nabla v\|^2 \Delta v, w) \\ & = -(\Delta u, u_t) \|\nabla w\|^2 + (\|\nabla u\|^2 - \|\nabla v\|^2) (\Delta v, w_t) - (f(u) - f(v), w_t + \eta w), \end{aligned} \quad (3.20)$$

where

$$Y(t) = \frac{1}{2} [\epsilon \|w_t\|^2 + 2\epsilon \eta (w_t, w) + \|\nabla w\|^2 + \|\nabla u\|^2 \|\nabla w\|^2 + \eta \|w\|^2].$$

Consequently, for $\eta > 0$ suitably small,

$$\frac{1}{4} (\epsilon \|w_t\|^2 + \|\nabla w\|^2) \leq Y(t) \leq C(\epsilon \|w_t\|^2 + \|\nabla w\|^2). \quad (3.21)$$

It is not hard to show that

$$-(\|\nabla u\|^2 \Delta u - \|\nabla v\|^2 \Delta v, w) \geq 0, \quad (3.22)$$

$$\begin{aligned} & |-(\Delta u, u_t)\|\nabla w\|^2 + (\|\nabla u\|^2 - \|\nabla v\|^2)(\Delta v, w_t)| \\ & \leq \|\Delta u\|\|u_t\|\|\nabla w\|^2 + (\|\nabla u\| + \|\nabla v\|)\|\nabla w\|\|\Delta v\|\|w_t\| \\ & \leq C\|u_t\|Y + \frac{1}{4}\|w_t\|^2 + C\|\nabla w\|^2 \\ & \leq \frac{1}{4}\|w_t\|^2 + C(\|u_t\| + 1)Y, \end{aligned} \quad (3.23)$$

$$\begin{aligned} |(f(u) - f(v), w_t + \eta w)| & \leq \|f(u) - f(v)\|(\|w_t\| + \|w\|) \\ & \leq C(1 + \|\Delta u\|^{p+1} + \|\Delta v\|^{p+1})\|w\|(\|w_t\| + \|w\|) \\ & \leq \frac{1}{4}\|w_t\|^2 + CY. \end{aligned} \quad (3.24)$$

Substituting (3.22)–(3.24) into (3.20) turns out

$$\frac{d}{dt}Y \leq C(\|u_t\| + 1)Y. \quad (3.25)$$

The Gronwall lemma together with (3.1) gives

$$Y(t) \leq Y(0)e^{C(t+t^{1/2})},$$

the uniqueness follows from (3.21) and $Y(0) = 0$.

Now, we prove the time continuity property of the weak solution. Let us rewrite Eq. (1.1) as

$$\epsilon u_{tt} - (1 + \|\nabla u\|^2)\Delta u + u_t = g(x) - f(u) := h. \quad (3.26)$$

Actually, the estimate (3.2) gives

$$h \in L^2(0, T; H_1).$$

Given some $T > 0$, we can take sequences such that

$$\begin{aligned} \{h^N\} & \subset C^0([0, T]; H_2), \quad h^N \rightarrow h \text{ strongly in } L^2(0, T; H_1), \\ \xi_{u^N}(0) & \subset \mathcal{H}_2, \quad \xi_{u^N}(0) \rightarrow \xi_u(0) \text{ strongly in } \mathcal{H}_2. \end{aligned}$$

Now, we consider the solution ξ_{u^N} to

$$\epsilon u_{tt}^N - (1 + \|\nabla u\|^2)\Delta u^N + u_t^N = h^N, \quad (3.27)$$

with the initial data $\xi_{u^N}(0)$. The linear theory yields

$$u^N \in C^1([0, T]; H_1) \cap C^0([0, T]; H_2).$$

Writing (3.27) for the couple of indexes N, M , taking the difference $w := u^N - u^M$, testing by Aw_t , and integrating over $(0, t)$ for $t \leq T$, we have

$$\begin{aligned} & \epsilon \|\nabla w_t(t)\|^2 + (1 + \|\nabla u(t)\|^2)\|Aw(t)\|^2 + \int_0^t \|\nabla w\|^2 d\tau \\ & \leq \epsilon \|\nabla w_t(0)\|^2 + (1 + \|\nabla u(0)\|^2)\|Aw(0)\|^2 + \int_0^t \|\nabla(h^N - h^M)\|^2 d\tau + 2 \int_0^t \|\nabla u\|\|\nabla u_t\|\|Aw\|^2 d\tau \\ & \leq \epsilon \|\nabla w_t(0)\|^2 + (1 + \|\nabla u(0)\|^2)\|Aw(0)\|^2 + \int_0^t \|\nabla(h^N - h^M)\|^2 d\tau \\ & \quad + 2 \int_0^t \|\nabla u\|\|\nabla u_t\|[\epsilon \|\nabla w_t\|^2 + (1 + \|\nabla u\|^2)\|Aw\|^2] d\tau. \end{aligned}$$

The Gronwall lemma gives

$$\begin{aligned} & \epsilon \|\nabla w_t(t)\|^2 + (1 + \|\nabla u(t)\|^2) \|Aw(t)\|^2 \\ & \leq \left(\epsilon \|\nabla w_t(0)\|^2 + (1 + \|\nabla u(0)\|^2) \|Aw(0)\|^2 + \int_0^t \|\nabla(h^N - h^M)\|^2 d\tau \right) \exp \left\{ \int_0^t \|\nabla u\| \|\nabla u_t\| d\tau \right\}. \end{aligned}$$

Thus, taking the supremum with respect to $t \in [0, T]$, we deduce that $\{\xi_{u^N}\}$ is a Cauchy sequence in $C^0([0, T]; \mathcal{H}_2)$ from the estimate (3.2). This entails that $\xi_u \in C^0([0, T]; \mathcal{H}_2)$. \square

Now, we construct a solution of the parabolic problem

$$v_t - (1 + \|\nabla v\|^2) \Delta v + f(v) + \lambda v = g + \lambda u, \quad \text{in } H, \quad 0 < t < T, \quad (3.28)$$

$$v(0) = u_0, \quad \text{in } \Omega, \quad (3.29)$$

where

$$\lambda > K, \quad (3.30)$$

the constants K, T are given by (2.3) and Theorem 3.2, respectively. Here, the initial datum u_0 in (3.29) is the first component of the initial data of (1.2).

Lemma 3.5. Let $g(x) \in H_1$, and assumptions (2.1)–(2.3) hold. Let v be a solution of problem (3.28)–(3.29). Then, the following estimate holds

$$\|Av(t)\|^2 + \|v_t(t)\|^2 \leq Q(\|\xi_u(0)\|_{\mathcal{H}_2}) e^{-\kappa t} + Q(\|g\|_{H_1}), \quad t \in (0, T), \quad (3.31)$$

for some positive constant κ and some positive increasing function Q both independent of ϵ and T .

Proof. Taking H -inner product by $v_t + \eta v$ in (3.28), we have

$$\frac{d}{dt} \phi_3 + \|v_t\|^2 + \eta \|\nabla v\|^2 + \eta \|\nabla v\|^4 + \eta (\beta(v), v) - \eta (g, v) = \lambda (u, v_t + \eta v), \quad (3.32)$$

where

$$\begin{aligned} \phi_3 &= \frac{\eta}{2} \|v\|^2 + \frac{1}{2} \|\nabla v\|^2 + \frac{1}{4} \|\nabla v\|^4 + (B(v), 1) - (g, v), \\ B(r) &:= F(r) + \frac{\lambda}{2} r^2, \quad \beta(r) = f(r) + \lambda r. \end{aligned} \quad (3.33)$$

Using (2.5) and (3.30), and choosing $\eta > 0$ small enough, we have

$$\frac{1}{4} (\|\nabla v\|^4 + \|v\|_{L^{p+3}}^{p+3}) - C(1 + \|g\|_{H_1}^2) \leq \phi_3 \leq C(\|\nabla v\|^4 + \|v\|_{L^{p+3}}^{p+3}) + C(1 + \|g\|_{H_1}^2). \quad (3.34)$$

Then, the last two inequalities give that

$$\frac{d}{dt} \phi_3 + \kappa \phi_3 + \kappa \|v_t\|^2 \leq C \|u\|^2 + C, \quad \kappa > 0. \quad (3.35)$$

Applying the Gronwall lemma to (3.35), using (3.1) and (3.34), we get

$$\|\nabla v(t)\|^2 + \|v(t)\|_{L^{p+3}}^{p+3} + \int_0^t e^{-\kappa(t-s)} \|v_t(s)\|^2 ds \leq Q(\|\xi_v(0)\|_{\mathcal{H}_2}) e^{-\kappa t} + Q(\|g\|_{H_1}). \quad (3.36)$$

Taking H -inner product by $A^2 v$ in (3.28), we have

$$\begin{aligned} \frac{d}{dt} \|Av\|^2 + \|A^{\frac{3}{2}} v\|^2 + \|\nabla v\|^2 \|A^{\frac{3}{2}} v\|^2 & \leq C(\|\nabla g\|^2 + \|\nabla u\|^2 + \|\nabla \beta(v)\|^2) \\ & \leq C(\|\nabla g\|^2 + \|\nabla u\|^2 + Q(\|Av\|^2)), \end{aligned} \quad (3.37)$$

whence, recalling (3.1) and arguing as in (3.15) and (3.16), it follows that there exists a time $T_1 (\leq T)$, depending on $\|\xi_u(0)\|_{\mathcal{H}_2}$ and $\|g\|_{H_1}$ such that

$$\|Av(t)\|^2 \leq Q(\|\xi_u(0)\|_{\mathcal{H}_2}) + Q(\|g\|_{H_1}), \quad \forall t \in [0, T_1]. \quad (3.38)$$

Taking H -inner product by v_t in (3.28), we have

$$\|v_t\|^2 = -(1 + \|\nabla v\|^2)(Av, v_t) - (f(v), v_t) - \lambda(v, v_t) + (g + \lambda u, v_t).$$

Integrating this equality on $(0, T_1)$ and using (3.38), we have

$$\int_0^{T_1} \|v_t(s)\|^2 ds \leq CT_1(Q(\|\xi_u(0)\|_{\mathcal{H}_2}) + Q(\|g\|_{H_1})). \quad (3.39)$$

Differentiating (3.28), testing the resulting equation by tv_t and v_t , respectively, we obtain

$$\frac{d}{dt}(t\|v_t\|^2) + t(1 + \|\nabla v\|^2)\|\nabla v_t\|^2 + 4t(\nabla v, \nabla v_t)^2 + 2t(f'(v) + \lambda, |v_t|^2) \leq \|v_t\|^2 + Ct\|u_t\|^2, \quad (3.40)$$

$$\frac{1}{2} \frac{d}{dt}\|v_t\|^2 + (1 + \|\nabla v\|^2)\|\nabla v_t\|^2 + 2(\nabla v, \nabla v_t)^2 + (f'(v) + \lambda, |v_t|^2) = \lambda(u_t, v_t). \quad (3.41)$$

Integrating (3.40) on $(0, T_1)$ and making use of (2.3), (3.2), (3.30) and (3.39) yield

$$\|v_t(T_1)\|^2 \leq Q(\|\xi_u(0)\|_{\mathcal{H}_2}) + Q(\|g\|_{H_1}). \quad (3.42)$$

From (3.41), we get

$$\frac{d}{dt}\|v_t\|^2 + \kappa\|v_t\|^2 \leq C\|u_t\|^2, \quad \kappa > 0. \quad (3.43)$$

Integrating this inequality on $[T_1, t]$ and using (3.1) and (3.42), we have

$$\begin{aligned} \|v_t(t)\|^2 &\leq Ce^{-\kappa(t-T_1)}\|v_t(T_1)\|^2 + C \int_{T_1}^t e^{-\kappa(t-s)}\|u_t(s)\|^2 ds \\ &\leq Q(\|\xi_u(0)\|_{\mathcal{H}_2})e^{-\kappa t} + Q(\|g\|_{H_1}), \quad \forall t \geq T_1. \end{aligned} \quad (3.44)$$

The standard elliptic regularity entails that

$$\|Av(t)\|^2 \leq Q(\|\xi_u(0)\|_{\mathcal{H}_2})e^{-\kappa t} + Q(\|g\|_{H_1}), \quad \forall t \geq T_1, \quad (3.45)$$

which concludes the proof of Lemma 3.5. \square

Lemma 3.6. Let $g(x) \in H_1$, and assumptions (2.1)–(2.3) hold. Let u and v be the corresponding (unique) solution to problem (1.1)–(1.3) and (3.28)–(3.29), respectively. Then, the following estimate holds

$$\|v(t) - u(t)\|^2 + \int_0^t e^{-(t-s)} \|\nabla(v(s) - u(s))\|^2 ds \leq \epsilon [Q(\|\xi_u(0)\|_{\mathcal{H}_2})e^{-\kappa t} + Q(\|g\|_{H_1})], \quad \forall t \in (0, T), \quad (3.46)$$

where Q, κ are independent both of ϵ and T .

Proof. Setting $\bar{u} := u - v$, then the difference \bar{u} solves the equation

$$\bar{u}_t - \Delta \bar{u} - \|\nabla u\|^2 \Delta u + \|\nabla v\|^2 \Delta v + \beta(u) - \beta(v) = -\epsilon u_{tt}, \quad (3.47)$$

where $\beta(r)$ is given by (3.33). Multiplying (3.47) by \bar{u} , we get

$$\begin{aligned} \frac{d}{dt} \left[\frac{1}{2} \|\bar{u}\|^2 + \epsilon(u_t, \bar{u}) \right] + \|\nabla \bar{u}\|^2 + (\beta(u) - \beta(v), \bar{u}) + (-\|\nabla u\|^2 \Delta u + \|\nabla v\|^2 \Delta v, \bar{u}) \\ = \epsilon(\bar{u}_t, u_t) \leq \epsilon(\|\bar{u}_t\|^2 + \|u_t\|^2). \end{aligned} \quad (3.48)$$

By (2.3), (3.30), (3.33), we have

$$(\beta(u) - \beta(v), \bar{u}) > 0.$$

Substituting the above inequality and

$$(-\|\nabla u\|^2 \Delta u + \|\nabla v\|^2 \Delta v, \bar{u}) \geq 0,$$

into (3.48), we get

$$\frac{d}{dt} \left[\frac{1}{2} \|\bar{u}\|^2 + \epsilon(u_t, \bar{u}) \right] + \kappa \left[\frac{1}{2} \|\bar{u}\|^2 + \epsilon(u_t, \bar{u}) \right] + \frac{1}{2} \|\nabla \bar{u}\|^2 \leq \epsilon C (\|u_t\|^2 + \|v_t\|^2). \quad (3.49)$$

In light of (3.1) and (3.31), we have

$$\int_t^{t+1} (\|u_t(s)\|^2 + \|v_t(s)\|^2) ds \leq Q(\|\xi_u(0)\|_{\mathcal{H}_2}) e^{-\kappa t} + Q(\|g\|_{H_1}).$$

Hence, the comparison principle, $\bar{u}(0) = 0$ and (3.1) ensure that

$$\|\bar{u}\|^2 + \int_0^t e^{-(t-s)} \|\nabla \bar{u}(s)\|^2 ds \leq \epsilon [Q(\|\xi_u(0)\|_{\mathcal{H}_2}) e^{-\kappa t} + Q(\|g\|_{H_1})]. \quad (3.50)$$

Thus, the proof of Lemma 3.6 is complete. \square

Similar to the proof in [5], we can prove that the local solution to problem (1.1)–(1.3) can be extended to a global one.

Theorem 3.7. *Let the assumptions of Theorem 3.2 hold. Then, there exist $\epsilon_0 > 0$ and a nonincreasing positive function $R : (0, \epsilon_0) \rightarrow (0, +\infty)$ with the property*

$$\lim_{\epsilon \searrow 0} R(\epsilon) = +\infty, \quad (3.51)$$

such that, for every $\epsilon \in (0, \epsilon_0)$ and every initial condition $\xi_u(0) \in \mathcal{H}_2$ satisfying

$$\|\xi_u(0)\|_{\mathcal{H}_2} \leq R(\epsilon), \quad (3.52)$$

there exists a (unique) global weak solution $u(t)$ to problem (1.1)–(1.3) such that

$$\|\xi_u(t)\|_{\mathcal{H}_2} \leq Q(\|\xi_u(0)\|_{\mathcal{H}_2}) e^{-\kappa t} + Q(\|g\|_{H_1}), \quad t \geq 0, \quad (3.53)$$

for some $\kappa > 0$ and a positive increasing function Q both independent of ϵ .

Proof. We first note that, due to the embedding $H_2 \subset C(\bar{\Omega})$, the strong solution $\xi_u(t) \in \mathcal{H}_2$ exists locally (for $t \leq T(\xi_u(0))$) and is unique on the existence interval, so, we may multiply Eq. (1.1) by $u_t + \eta u$. We also note that, since Au_t is not to be regular, so we could not multiply directly Eq. (1.1) by Au_t according to Definition 3.1. That is to say, we need to verify the global well-posedness and dissipativity of solutions using the Galerkin approximation. Just for the sake of simplicity, we restrict ourselves by only formal derivation of estimate (3.53).

Setting

$$\phi_4 = \|Au\|^2 + \epsilon \frac{\|A^{\frac{1}{2}}u_t\|^2}{1 + \|A^{\frac{1}{2}}u\|^2} + \frac{\eta \epsilon (\nabla u, \nabla u_t) + \frac{\eta}{2} \|A^{\frac{1}{2}}u\|^2}{1 + \|A^{\frac{1}{2}}u\|^2}. \quad (3.54)$$

Obviously, for $\eta > 0$ small enough, the function ϕ_4 satisfies

$$\frac{1}{2} \left[\|Au\|^2 + \epsilon \frac{\|A^{\frac{1}{2}}u_t\|^2}{1 + \|A^{\frac{1}{2}}u\|^2} \right] \leq \phi_4 \leq C \left[\|Au\|^2 + \epsilon \frac{\|A^{\frac{1}{2}}u_t\|^2}{1 + \|A^{\frac{1}{2}}u\|^2} \right]. \quad (3.55)$$

Taking H -inner product by $2Au_t + \eta Au$ in (2.8), we get

$$\begin{aligned} \frac{d}{dt} \phi_4 + \frac{\|A^{\frac{1}{2}}u_t\|^2}{1 + \|A^{\frac{1}{2}}u\|^2} \left[2 - \epsilon \eta + 2\epsilon \frac{(\nabla u, \nabla u_t)}{1 + \|A^{\frac{1}{2}}u\|^2} \right] + \eta \|Au\|^2 + \frac{\eta(f'(u), |\nabla u|^2)}{1 + \|A^{\frac{1}{2}}u\|^2} \\ = \frac{(\nabla g, 2\nabla u_t + \eta \nabla u)}{1 + \|A^{\frac{1}{2}}u\|^2} - \frac{2(\nabla u, \nabla u_t)[\eta \epsilon (\nabla u, \nabla u_t) + \frac{\eta}{2} \|A^{\frac{1}{2}}u\|^2]}{(1 + \|A^{\frac{1}{2}}u\|^2)^2} - 2 \frac{(\nabla f(u), \nabla u_t)}{1 + \|A^{\frac{1}{2}}u\|^2}. \end{aligned} \quad (3.56)$$

In light of (2.2), (3.31), (3.46), the embedding $H_2 \subset L^\infty(\bar{\Omega})$ and the interpolation inequality, we have

$$\begin{aligned}
 \|\nabla f(u)\|^2 &\leq 2\|f(u) - f(v)\|_{H_1}^2 + 2\|f(v)\|_{H_1}^2 \\
 &\leq C\|u - v\|_{H_1}^2(1 + \|u\|_{H_2}^{2(p+1)} + \|v\|_{H_2}^{2(p+1)}) + 2\|f(v)\|_{H_1}^2 \\
 &\leq C\|u - v\|\|u - v\|_{H_2}(1 + \|u\|_{H_2}^{2(p+1)} + \|v\|_{H_2}^{2(p+1)}) + 2\|f(v)\|_{H_1}^2 \\
 &\leq C\|u - v\|(1 + \|u\|_{H_2}^{2(p+3/2)} + \|v\|_{H_2}^{2(p+3/2)}) + 2\|f(v)\|_{H_1}^2 \\
 &\leq \epsilon^{1/2}[Q(\|\xi_u(0)\|_{\mathcal{H}_2})e^{-\kappa t} + Q(\|g\|_{H_1})](1 + \|u\|_{H_2}^{2(p+3/2)} + \|v\|_{H_2}^{2(p+3/2)}) \\
 &\quad + Q(\|\xi_u(0)\|_{\mathcal{H}_2})e^{-\kappa t} + Q(\|g\|_{H_1}) \\
 &\leq \epsilon^{1/2}[Q(\|\xi_u(0)\|_{\mathcal{H}_2})e^{-\kappa t} + Q(\|g\|_{H_1})]\phi_4^{(p+3/2)} + Q(\|\xi_u(0)\|_{\mathcal{H}_2})e^{-\kappa t} + Q(\|g\|_{H_1}),
 \end{aligned} \tag{3.57}$$

where v is the solution of problem (3.28)–(3.29). Using (3.1) and (3.57), the r.h.s. of (3.56) can be controlled by

$$\begin{aligned}
 &\frac{C\|\nabla g\|^2 + \eta\|A^{\frac{1}{2}}u_t\|^2 + \eta\|A^{\frac{1}{2}}u\|^2}{1 + \|A^{\frac{1}{2}}u\|^2} + 2\eta\epsilon\frac{\|A^{\frac{1}{2}}u\|^2\|A^{\frac{1}{2}}u_t\|^2}{(1 + \|A^{\frac{1}{2}}u\|^2)^2} + \eta\frac{\|A^{\frac{1}{2}}u_t\|^2 + \|A^{\frac{1}{2}}u\|^6}{(1 + \|A^{\frac{1}{2}}u\|^2)^2} + \frac{C\|f(u)\|_{H_1}^2 + \eta\|A^{\frac{1}{2}}u_t\|^2}{1 + \|A^{\frac{1}{2}}u\|^2} \\
 &\leq C\|\nabla g\|^2 + 2\eta\|A^{\frac{1}{2}}u\|^2 + 5\eta\frac{\|A^{\frac{1}{2}}u_t\|^2}{1 + \|A^{\frac{1}{2}}u\|^2} + C\|f(u)\|_{H_1}^2 \\
 &\leq C\|\nabla g\|^2 + 2\eta\|A^{\frac{1}{2}}u\|^2 + 5\eta\frac{\|A^{\frac{1}{2}}u_t\|^2}{1 + \|A^{\frac{1}{2}}u\|^2} + C\|f(v)\|_{H_1}^2 + C\|f(u) - f(v)\|_{H_1}^2 \\
 &\leq Q(\|\xi_u(0)\|_{\mathcal{H}_2})e^{-\kappa t} + Q(\|g\|_{H_1}) + 5\eta\frac{\|A^{\frac{1}{2}}u_t\|^2}{1 + \|A^{\frac{1}{2}}u\|^2} + \epsilon^{1/2}[Q(\|\xi_u(0)\|_{\mathcal{H}_2})e^{-\kappa t} + Q(\|g\|_{H_1})]\phi_4^{(p+3/2)}.
 \end{aligned} \tag{3.58}$$

Substituting (3.58) into (3.56), and using (2.3), we have

$$\begin{aligned}
 &\frac{d}{dt}\phi_4 + \frac{\|A^{\frac{1}{2}}u_t\|^2}{1 + \|A^{\frac{1}{2}}u\|^2}\left[2 - \epsilon\eta - 5\eta + 2\frac{\epsilon(\nabla u, \nabla u_t)}{1 + \|A^{\frac{1}{2}}u\|^2}\right] + \eta\|Au\|^2 \\
 &\leq \epsilon^{1/2}P_1\phi_4^{(p+3/2)} + Q_3(\|\xi_u(0)\|_{\mathcal{H}_2})e^{-\kappa t} + Q_4(\|g\|_{H_1}),
 \end{aligned} \tag{3.59}$$

where

$$P_1 := Q_1(\|\xi_u(0)\|_{\mathcal{H}_2})e^{-\kappa t} + Q_2(\|g\|_{H_1}).$$

By (3.55), one of the second term of the l.h.s. of (3.59) can be estimated as follows

$$\begin{aligned}
 2\frac{\epsilon(\nabla u, \nabla u_t)}{1 + \|A^{\frac{1}{2}}u\|^2} &\leq \frac{1}{2} + 2\epsilon^2\frac{\|A^{\frac{1}{2}}u_t\|^2}{1 + \|A^{\frac{1}{2}}u\|^2} \leq \frac{1}{2} + 4\epsilon\phi_4 \leq \frac{3}{2} + (4\epsilon\phi_4)^{(p+3/2)} \\
 &\leq \frac{3}{2} + \epsilon^{1/2}4^{(p+3/2)}\phi_4^{(p+3/2)} \leq \frac{3}{2} + \epsilon^{1/2}P_2\phi_4^{(p+3/2)},
 \end{aligned} \tag{3.60}$$

where

$$P_2 = P_1 + 4^{(p+3/2)}.$$

Substituting (3.60) into (3.59), we get

$$\begin{aligned}
 &\frac{d}{dt}\phi_4 + \frac{\|A^{\frac{1}{2}}u_t\|^2}{1 + \|A^{\frac{1}{2}}u\|^2}\left[\frac{1}{2} - \epsilon\eta - 5\eta - \epsilon^{1/2}P_2\phi_4^{(p+3/2)}\right] + \eta\|Au\|^2 \\
 &\leq \epsilon^{1/2}P_2\phi_4^{(p+3/2)} + Q_3(\|\xi_u(0)\|_{\mathcal{H}_2})e^{-\kappa t} + Q_4(\|g\|_{H_1}).
 \end{aligned} \tag{3.61}$$

We now set $\rho = \phi_4(0) > 0$ and observe that for a fixed $R > 0$, we have

$$\|\xi_u(0)\|_{\mathcal{H}_2} \leq R \quad \Rightarrow \quad \rho \leq CR. \tag{3.62}$$

Now, for some $z^* > \rho$ whose value will be chosen later, we put

$$\epsilon^{1/2} := \frac{1}{4[Q_1(R) + Q_2(\|g\|_{H_1}) + 4^{(p+3/2)}]z^{*(p+3/2)}}. \quad (3.63)$$

Hence, if $\epsilon > 0$ is above (or smaller), and $\phi_4 \leq z^*$, then (3.61) and (3.63) entail that

$$\frac{d}{dt}\phi_4 + \frac{\|A^{\frac{1}{2}}u_t\|^2}{1 + \|A^{\frac{1}{2}}u\|^2} \left(\frac{1}{4} - \epsilon\eta - 5\eta \right) + \eta\|Au\|^2 \leq \frac{1}{4} + Q_3(\|\xi_u(0)\|_{\mathcal{H}_2})e^{-\kappa t} + Q_4(\|g\|_{H_1}).$$

Using (3.55), the above inequality can be rewritten as

$$\frac{d}{dt}\phi_4 + \kappa\phi_4 \leq \frac{1}{4} + Q_3(\|\xi_u(0)\|_{\mathcal{H}_2})e^{-\kappa t} + Q_4(\|g\|_{H_1}) =: Q_3(\|\xi_u(0)\|_{\mathcal{H}_2})e^{-\kappa t} + Q_5(\|g\|_{H_1}),$$

for $\kappa > 0$ small enough. Applying the Gronwall lemma to the above inequality and using (3.62) yield

$$\begin{aligned} \phi_4(t) &\leq \rho e^{-\kappa t} + e^{-\kappa t} \left[\int_0^t e^{\kappa s} (Q_3(R)e^{-\kappa s} + Q_5(\|g\|_{H_1})) ds \right] \\ &\leq \rho + \kappa^{-1} (Q_3(R) + Q_5(\|g\|_{H_1})) \\ &\leq CR + \kappa^{-1} (Q_3(R) + Q_5(\|g\|_{H_1})). \end{aligned} \quad (3.64)$$

To make $\phi_4 \leq z^*$ true uniformly in time, we may set

$$z^* = CR + \kappa^{-1} (Q_3(R) + Q_5(\|g\|_{H_1})). \quad (3.65)$$

This choice of z^* , together with (3.63) allows us to take ϵ as

$$\begin{aligned} \epsilon &= \left(\frac{1}{4[Q_1(R) + Q_2(\|g\|_{H_1}) + 4^{(p+3/2)}][CR + \kappa^{-1}(Q_3(R) + Q_5(\|g\|_{H_1}))]^{p+3/2}} \right)^2 \\ &=: \Phi(R), \end{aligned} \quad (3.66)$$

where Φ is a monotone function of R . The desired R is then chosen simply as Φ^{-1} , and the value ϵ_0 is then given by $\lim_{R \searrow 0} \Phi(R)$.

Making use of (3.64), we know that ϕ_4 is bounded in a way that does not depend on T . Hence, $\|\xi_u(t)\|_{\mathcal{H}_2}$ cannot explode in finite times, which implies that we could extend $\xi_u(t)$ to a global solution (unique in its class). Again, (3.64) can be estimated as follows

$$\begin{aligned} \phi_4(t) &\leq \rho e^{-\kappa t} + \left[\int_0^T e^{-\kappa(t-s)} (Q_3(R)e^{-\kappa s} + Q_5(\|g\|_{H_1})) ds + \int_T^t e^{-\kappa(t-s)} Q(\|g\|_{H_1}) ds \right] \\ &\leq Q_6(R)e^{-\kappa t} + Q_7(\|g\|_{H_1}), \end{aligned} \quad (3.67)$$

where T is a suitable time depending on ρ , Q_3 and Q_5 , such that, for $t \geq T$, $Q_3(R)e^{-\kappa s}$ is smaller than some computable quantity $Q(\|g\|_{H_1})$. The combination of (3.67) with (3.1), (3.55) yields

$$\begin{aligned} \|Au\|^2 + \epsilon \|A^{1/2}u_t\|^2 &\leq 2[Q_6(R)e^{-\kappa t} + Q_7(\|g\|_{H_1})] + 2(1 + \|A^{1/2}u\|^2)[Q_6(R)e^{-\kappa t} + Q_7(\|g\|_{H_1})] \\ &\leq Q(R)e^{-\kappa t} + Q(\|g\|_{H_1}). \end{aligned}$$

This completes the proof of Theorem 3.7. \square

4. The finite-dimensionality of the global attractor

Thanks to Theorem 3.7, we get that

$$S_\epsilon(t)\xi_u(0) := \xi_u(t) = (u(t), u_t(t)), \quad \forall \xi_u(0) \in \mathcal{B}_\epsilon(R), \quad \forall t \geq 0,$$

is a semigroup, where

$$\mathcal{B}_\epsilon(R) = \{\xi_u \in \mathcal{H}_2 \mid \|\xi_u\|_{\mathcal{H}_2} \leq R\}, \quad \epsilon \in (0, \epsilon_0),$$

and $R = R(\epsilon)$ is given by (3.52). Define

$$\mathbb{B}_\epsilon = \left[\bigcup_{t \geq 0} S_\epsilon(t) \mathcal{B}_\epsilon(R) \right]_{\mathcal{H}_1},$$

where $[\cdot]_W$ denotes the closure in the space W . Obviously, \mathbb{B}_ϵ is bounded and closed in \mathcal{H}_2 , and

$$S_\epsilon(t) \mathbb{B}_\epsilon \subset \mathbb{B}_\epsilon, \quad t \geq 0.$$

Thus, it is sufficient to discuss the attractor for the dynamical system $(S_\epsilon(t), \mathbb{B}_\epsilon)$.

The main result of the paper is the following theorem.

Theorem 4.1. *Let $g(x) \in H_1$, and assumptions (2.1)–(2.3) hold. Then, there exists $0 < \epsilon_0 < 1$, such that for all $\epsilon \in (0, \epsilon_0)$, the solution semigroup $S_\epsilon(t)$ of problem (1.1)–(1.3) on the phase space \mathbb{B}_ϵ possess a global attractor \mathcal{A}_ϵ , i.e.*

- (1) \mathcal{A}_ϵ is bounded in \mathcal{H}_2 and compact in \mathcal{H}_1 ;
- (2) \mathcal{A}_ϵ is strictly invariant: $S_\epsilon(t) \mathcal{A}_\epsilon = \mathcal{A}_\epsilon$, $t \geq 0$;
- (3) for every bounded in \mathcal{H}_2 set $B \subset \mathbb{B}_\epsilon$ and every neighborhood $O(\mathcal{A}_\epsilon)$ of \mathcal{A}_ϵ in \mathcal{H}_1 , there is a time $T = T(B, O)$ such that

$$S_\epsilon(t) B \subset O(\mathcal{A}_\epsilon), \quad t \geq T.$$

Moreover, the global attractor \mathcal{A}_ϵ has finite fractal dimension in \mathcal{H}_1 , that is

$$\dim_F(\mathcal{A}_\epsilon, \mathcal{H}_1) < \infty.$$

Lemma 4.2. *For any $\xi_u(0) = (u_0, u_1)$, $\xi_v(0) = (v_0, v_1) \in \mathbb{B}_\epsilon$,*

$$\|S_\epsilon(t) \xi_u(0) - S_\epsilon(t) \xi_v(0)\|_{\mathcal{H}_1} \leq \gamma_1 e^{-\gamma_2 t} \|\xi_u(0) - \xi_v(0)\|_{\mathcal{H}_1} + \gamma_3 \int_0^t \|u(\tau) - v(\tau)\| \, d\tau, \quad (4.1)$$

where $S_\epsilon(t) \xi_u(0) = (u, u_t)$, $S_\epsilon(t) \xi_v(0) = (v, v_t)$, $\gamma_1, \gamma_2, \gamma_3$ are positive constants independent of ϵ .

Proof. It is easy to know that $w = u - v$ satisfies (3.19). In view of the interpolation theorem, the boundedness of \mathbb{B}_ϵ in \mathcal{H}_2 , $S_\epsilon(t) \mathbb{B}_\epsilon \subset \mathbb{B}_\epsilon$ and (3.21), we have

$$\begin{aligned} & |-(\Delta u, u_t) \|\nabla w\|^2 + (\|\nabla u\|^2 - \|\nabla v\|^2)(\Delta v, w_t)| \\ & \leq \|\Delta u\| \|u_t\| \|\nabla w\|^2 + (\|\nabla u\| + \|\nabla v\|) \|\nabla w\| \|\Delta v\| \|w_t\| \\ & \leq C \|u_t\| Y(t) + C \|\nabla w\| \|w_t\| \\ & \leq C \|u_t\| Y(t) + \eta \|w_t\|^2 + C \|\nabla w\|^2 \\ & \leq C \|u_t\| Y(t) + \eta \|w_t\|^2 + C \|\Delta w\| \|w\| \\ & \leq C \|u_t\| Y(t) + \eta \|w_t\|^2 + C \|w\|, \end{aligned} \quad (4.2)$$

$$\begin{aligned} |(f(u) - f(v), w_t + \eta w)| & \leq \|f(u) - f(v)\| (\|w_t\| + \|w\|) \\ & \leq \left\| \int_0^1 \frac{d}{ds} f(su + (1-s)v) \, ds \right\| (\|w_t\| + \|w\|) \\ & = \left\| \int_0^1 f'(su + (1-s)v) \, ds (u - v) \right\| (\|w_t\| + \|w\|) \\ & \leq C \left\| \int_0^1 [1 + |su + (1-s)v|^{p+1}] \, ds (u - v) \right\| (\|w_t\| + \|w\|) \\ & \leq C [1 + (|u| + |v|)^{p+1}] |u - v| (\|w_t\| + \|w\|) \\ & \leq C [1 + (\|\Delta u\|^{p+1} + \|\Delta v\|^{p+1})] \|w\| (\|w_t\| + \|w\|) \\ & \leq \eta \|w_t\|^2 + C \|w\|^2 \leq \eta \|w_t\|^2 + C \|w\|. \end{aligned} \quad (4.3)$$

Using multiplier $w_t + \eta w$ in (3.19), and substituting (3.22), (4.2) and (4.3) into (3.20) yield

$$\frac{d}{dt} Y + 2\kappa Y \leq C \|u_t\| Y + C \|w\|, \quad (4.4)$$

where κ denotes a small positive constant. Thanks to (3.1),

$$\int_s^t \|u_t\| d\tau \leq \left(\int_s^t \|u_t\|^2 d\tau \right)^{\frac{1}{2}} (t-s)^{\frac{1}{2}} \leq \kappa(t-s) + C.$$

Applying Lemma 3.1 to (4.4) yields

$$Y(t) \leq CY(0)e^{-\kappa t} + C \int_0^t \|w(\tau)\| d\tau. \quad (4.5)$$

The combination of (4.5) with (3.21) gives (4.1). Lemma 4.2 is proved. \square

For any fixed $t^* > 0$, we define the function

$$d_{t^*}((u_0, u_1), (v_0, v_1)) = \left(\int_0^{t^*} \|u(t) - v(t)\| dt \right)^{\frac{1}{2}}, \quad (4.6)$$

where $u(t) = PS_\epsilon(t)(u_0, u_1)$, $v(t) = PS_\epsilon(t)(v_0, v_1)$, P is the projection onto the first component of $H_2 \times H_1$.

Proposition 4.3. d_{t^*} is a pseudo-metric on $\mathbb{B}_\epsilon \subset \mathcal{H}_1$, pre-compact with respect to the norm of \mathcal{H}_1 .

Proof. For any $(u_0, u_1) \in \mathbb{B}_\epsilon$, since $S_\epsilon(t)(u_0, u_1) \in \mathbb{B}_\epsilon(t \geq 0)$, we have $\{u(t), 0 \leq t \leq t^*\}$ is bounded in H_1 and hence in H . That is, d_{t^*} is well-defined and it is obviously a pseudo-metric on \mathcal{H}_1 .

For any bounded sequence $\{u_0^n, u_1^n\} \subset \mathbb{B}_\epsilon$,

$$S_\epsilon(t)(u_0^n, u_1^n) = (u^n, u_t^n) \subset \mathbb{B}_\epsilon,$$

that is

$$\|u^n(t)\|_{H_1} + \|u_t^n(t)\| \leq C, \quad t \in [0, t^*].$$

By the Aubin–Lions theorem

$$W = \{u \in L^2(0, t^*; H_1), u_t \in L^2(0, t^*; H)\} \hookrightarrow L^2(0, t^*; H),$$

so we can extract a subsequence $\{u^n\}$ such that

$$\|u^n - u^m\|_{L^2(0, t^*; H)} \rightarrow 0 \quad \text{as } n, m \rightarrow \infty.$$

Hence,

$$\left(\int_0^{t^*} \|u^n - u^m\| d\tau \right)^{\frac{1}{2}} \leq (t^*)^{\frac{1}{4}} \|u^n - u^m\|_{L^2(0, t^*; H)}^{\frac{1}{2}} \rightarrow 0 \quad \text{as } n, m \rightarrow \infty.$$

That is

$$d_t((u_0^n, u_1^n), (u_0^m, u_1^m)) \rightarrow 0 \quad \text{as } n, m \rightarrow \infty,$$

which gives the conclusion of Proposition 4.3. \square

Lemma 4.2 can be rewritten as

Lemma 4.4. For any $\xi_u(0) = (u_0, u_1)$, $\xi_v(0) = (v_0, v_1) \in \mathbb{B}_\epsilon$,

$$\|S_\epsilon(t)\xi_u(0) - S_\epsilon(t)\xi_v(0)\|_{\mathcal{H}_1} \leq \gamma_1 e^{-\gamma_2 t} \|\xi_u(0) - \xi_v(0)\|_{\mathcal{H}_1} + \gamma_3 d_t(\xi_u(0), \xi_v(0)), \quad (4.7)$$

where d_t is shown by (4.6), $\gamma_1, \gamma_2, \gamma_3$ are positive constants independent of ϵ .

For further use, let us write down explicitly the continuity for $S_\epsilon(t)$ on \mathbb{B}_ϵ .

Lemma 4.5. For any $\xi_u(0), \xi_v(0) \in \mathbb{B}_\epsilon$,

$$\|\xi_u(t) - \xi_v(t)\|_{\mathcal{H}_1} \leq C e^{kt} \|\xi_u(0) - \xi_v(0)\|_{\mathcal{H}_1}, \quad \forall t \in \mathbb{R}^+, \quad (4.8)$$

for some $k = k(R)$, where $\xi_u(t) = S_\epsilon(t)\xi_u(0)$, $\xi_v(t) = S_\epsilon(t)\xi_v(0)$, C is a constant independent of ϵ .

Proof. See the proof of Lemma 3.4, or [10, Proposition 2.1]. \square

Proof of Theorem 4.1. Choosing $t^* > 0$ such that $\gamma_1 e^{-\gamma_2 t^*} \equiv q < 1$, we deduce from Proposition 4.3 and Lemma 4.4 that $S_\epsilon(t^*)$ is an α contraction on \mathbb{B}_ϵ . Obviously, $S_\epsilon(t^*)$ is point dissipative and the orbits of the bounded sets are bounded, so by [6, Theorem 2.8.1], $\omega(\mathbb{B}_\epsilon) = \mathcal{A}_\epsilon$ is a global attractor for the discrete dynamical system $(S_\epsilon^n(t^*), \mathbb{B}_\epsilon)$, which is bounded in \mathcal{H}_2 and

$$\dim_F(\mathcal{A}_\epsilon, \mathcal{H}_1) < \infty. \quad (4.9)$$

By the Lipschitz continuity of the mapping $S_\epsilon(t)$ (see Lemma 4.5), it is not difficult to see that \mathcal{A}_ϵ is a global attractor for the continuous dynamical system $(S_\epsilon(t), \mathbb{B}_\epsilon)$. Theorem 4.1 is proved. \square

Remark 4.6. If conditions (2.1)–(2.3) are replaced by

$$f \in C^2(\mathbb{R}, \mathbb{R}), \quad \text{with } f(0) = 0, \quad (4.10)$$

$$|f''(s)| \leq C(1 + |s|^p), \quad (4.11)$$

$$\liminf_{s \rightarrow \infty} f'(s) > -\lambda_1, \quad (4.12)$$

where $p > 0$, $C > 0$ are given constants. Then, the results of Theorem 4.1 still hold true.

Acknowledgments

The authors thank Professor Zhijian Yang and the reviewer for their valuable comments. This work is supported by the Nature Science Foundation of China (No. 11271336, No. 11171311).

References

- [1] J.J. Bae, M. Nakao, Existence problem of global solutions of the Kirchhoff type wave equations with a localized weakly nonlinear dissipation in exterior domains, *Discrete Contin. Dyn. Syst.* 11 (2004) 731–743.
- [2] M. Ghisi, M. Gobbino, Hyperbolic–parabolic singular perturbation for Kirchhoff equations with weak dissipation, *Rend. Istit. Mat. Univ. Trieste* 42 (Suppl.) (2010) 67–88.
- [3] M. Ghisi, M. Gobbino, Hyperbolic–parabolic singular perturbation for nondegenerate Kirchhoff equations with critical weak dissipation, *Math. Ann.* 354 (3) (2012) 1079–1102.
- [4] M. Ghisi, M. Gobbino, Hyperbolic–parabolic singular perturbation for mildly degenerate Kirchhoff equations: Decay-error estimates, *J. Differential Equations* 252 (2012) 6099–6132.
- [5] M. Grasselli, G. Schimperna, A. Segatti, S. Zelik, On the 3D Cahn–Hilliard equation with inertial term, *J. Evol. Equ.* 9 (2009) 371–404.
- [6] J.K. Hale, *Asymptotic Behavior of Dissipative Systems*, Amer. Math. Soc., Providence, RI, 1988.
- [7] H. Hashimoto, T. Yamazaki, Hyperbolic–parabolic singular perturbation for quasilinear equations of Kirchhoff type, *J. Differential Equations* 237 (2007) 491–525.
- [8] G. Kirchhoff, *Vorlesungen über Mechanik*, Teubner, Leipzig, 1883.
- [9] T. Matsuyama, R. Ikehata, On global solutions and energy decay for the wave equations of Kirchhoff type with nonlinear damping terms, *J. Math. Anal. Appl.* 204 (1996) 729–753.
- [10] M. Nakao, An attractor for a nonlinear dissipative wave equation of Kirchhoff type, *J. Math. Anal. Appl.* 353 (2009) 652–659.
- [11] M. Nakao, Z.J. Yang, Global attractors for some quasilinear wave equations with a strong dissipation, *Adv. Math. Sci. Appl.* 17 (2007) 89–105.
- [12] K. Ono, Global existence, decay and blowup of solutions for some mildly degenerate nonlinear Kirchhoff strings, *J. Differential Equations* 137 (1995) 273–301.
- [13] K. Ono, On global existence, asymptotic stability and blowing up of solutions for some degenerate non-linear wave equations of Kirchhoff type with a strong dissipation, *Math. Methods Appl. Sci.* 20 (1997) 151–177.
- [14] T. Yamazaki, Global solvability for the Kirchhoff equations in exterior domain of dimension three, *J. Differential Equations* 210 (2005) 290–316.
- [15] Z.J. Yang, Y.Q. Wang, Global attractor for the Kirchhoff equation with a strong dissipation, *J. Differential Equations* 249 (2010) 3258–3278.
- [16] Z.J. Yang, X. Li, Finite dimensional attractor for the Kirchhoff equation with a strong dissipation, *J. Math. Anal. Appl.* 375 (2011) 579–593.
- [17] S. Zelik, The attractor for a nonlinear reaction–diffusion system with a supercritical nonlinearity and its dimension, *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl.* 24 (2000) 1–25.