



Matrix representations of truncated Toeplitz operators



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ABSTRACT

In this paper we describe matrix representations of truncated Toeplitz operators on the model space K_B , where B is an infinite Blaschke product satisfying some additional conditions. Our results are extensions of that obtained by Cima, Ross and Wogen in 2008.

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1. Introduction

Let H^2 be the Hardy space of the unit disk $\mathbb{D} = \{z: |z| < 1\}$, that is, the space of functions $f(z) = \sum_{k=0}^{\infty} \hat{f}(k)z^k$ analytic in $\mathbb{D}$ and such that

$$\|f\|^2 = \sum_{k=0}^{\infty} |\hat{f}(k)|^2 < \infty.$$

As usual, H^2 is identified with the subspace of $L^2(\partial\mathbb{D})$ consisting of the functions whose Fourier coefficients with negative indices vanish.

The unilateral shift operator on H^2 is defined by $Sf(z) = zf(z)$. Its adjoint, the backward shift S^* , is given by

$$S^*f(z) = \frac{f(z) - f(0)}{z}.$$

Let H^∞ denote the algebra of bounded analytic functions on $\mathbb{D}$. By the theorem of A. Beurling (see, for example, [3] Thm. 8.1.1) every nontrivial S^* -invariant subspace of H^2 is of the form

$$K_u = H^2 \ominus uH^2,$$

where u is an inner function, that is $u \in H^\infty$ and $|u| = 1$ a.e. on $\partial\mathbb{D}$. The space K_u is called the model space corresponding to the inner function u .

The reproducing kernel for K_u is

$$k_\lambda^u(z) = \frac{1 - \overline{u(\lambda)}u(z)}{1 - \bar{\lambda}z}.$$

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So, for every $f \in K_u$ and $\lambda \in \mathbb{D}$ we have $f(\lambda) = \langle f, k_\lambda^u \rangle$, where $\langle \cdot, \cdot \rangle$ is the usual integral inner product. Since each k_λ^u is bounded, we see that the subspace of all bounded functions in K_u is dense in K_u . Notice that if $u(\lambda) = 0$, then $k_\lambda^u(z) = k_\lambda(z) = (1 - \bar{\lambda}z)^{-1}$ is the reproducing kernel for H^2 at λ .

Let P_u be the orthogonal projection of $L^2(\partial\mathbb{D})$ onto K_u . A truncated Toeplitz operator with a symbol $\varphi \in L^2(\partial\mathbb{D})$ is defined on the model space K_u by

$$A_\varphi f = P_u(\varphi f).$$

The operator A_φ is densely defined on bounded functions in the model space and can be seen as a compression to K_u of the classical Toeplitz operator. The set of all bounded truncated Toeplitz operators on K_u will be denoted by $\mathcal{T}(K_u)$. Truncated Toeplitz operators have been studied in, for example [1,2,7–9,11].

The model space K_u carries a natural conjugation (an isometric, conjugate-linear involution) $C : K_u \rightarrow K_u$, defined in terms of the boundary values by

$$Cf(z) = u(z)\overline{zf(z)}, \quad |z| = 1 \quad (1.1)$$

(see [11] Section 2.3, for details). We will usually write $\tilde{f}$ in place of Cf . A short calculation shows that the conjugate kernel is given by the formula

$$\tilde{k}_\lambda^u(z) = \frac{u(z) - u(\lambda)}{z - \lambda}.$$

It is known that every $A \in \mathcal{T}(K_u)$ is C -symmetric, that is,

$$CA = A^*C. \quad (1.2)$$

In [11] Sarason proved the following characterization of $\mathcal{T}(K_u)$: a bounded linear operator A on K_u is in $\mathcal{T}(K_u)$ if and only if there exist functions $\psi, \chi \in K_u$ such that

$$A - S_u A S_u^* = \psi \otimes k_0^u + k_0^u \otimes \chi, \quad (1.3)$$

where $S_u = A_z$ is the truncated shift operator on K_u and $f \otimes g$ denotes the rank-one operator $f \otimes g(h) = \langle h, g \rangle f$.

We note that if u is a finite Blaschke product of degree n , then K_u has dimension n . If u has distinct zeros $a_1, \dots, a_n$, then the set $\{k_{a_1}, \dots, k_{a_n}\}$ as well as $\{\tilde{k}_{a_1}, \dots, \tilde{k}_{a_n}\}$ is a (non-orthonormal) basis for K_u . J.A. Cima, W.T. Ross and W.R. Wogen [4] proved that then the matrix representation of a truncated Toeplitz operator with respect to each of these bases is completely determined by entries along the main diagonal and the first row. As remarked in [4], there is nothing special about the first row and a similar result can be obtained where the representing matrix is determined by the entries along the main diagonal and any other row or column. A similar result for matrix representations of a truncated Toeplitz operator with respect to the so-called Clark bases was also given in [4].

Here we show that these results can be extended to the case when u is an infinite Blaschke product satisfying some additional conditions. In particular, we describe matrix representations of truncated Toeplitz operators with respect to the conjugate kernel bases for interpolating Blaschke products. The proofs of sufficiency given in [4] rely on the fact that if B is a Blaschke product of degree n , then the dimension of $\mathcal{T}(K_B)$ is $2n - 1$. Our proofs are based on Sarason's characterization (1.3), and its more general version (see (3.1) below).

2. Conjugate kernel function bases

In this section we will consider K_B , where B is an infinite Blaschke product,

$$B(z) = \prod_{m=1}^{\infty} \frac{\bar{a}_m}{|a_m|} \frac{a_m - z}{1 - \bar{a}_m z},$$

with uniformly separated zeros, i.e.,

$$\inf_k \prod_{m \neq k} \left| \frac{a_m - a_k}{1 - \bar{a}_m a_k} \right| \geq \delta$$

for some $\delta > 0$.

It is known that if the sequence $\{a_m\}$ is uniformly separated, then

$$\sum_{m=1}^{\infty} |f(a_m)|^2 (1 - |a_m|^2) < \infty$$

for every $f \in H^2$ (see, e.g., [6] p. 152). Carleson's interpolation theorem for H^2 asserts that the operator

$$Tf = f(a_m)\sqrt{1 - |a_m|^2}$$

maps H^2 onto l^2 if and only if $\{a_m\}$ is uniformly separated. Then a solution of the interpolation problem $g(a_m) = g_m$, where $\sum_{m=1}^{\infty} |g_m|^2(1 - |a_m|^2) < \infty$, is given by the formula

$$g = \sum_{m=1}^{\infty} \frac{g_m}{B'(a_m)} \tilde{k}_{a_m} + Bh,$$

where $h \in H^2$. Moreover, the series

$$f = \sum_{m=1}^{\infty} \frac{g_m}{B'(a_m)} \tilde{k}_{a_m} \quad (2.1)$$

converges in norm and f is the unique solution of the interpolation problem in the space K_B . This also means that the conjugate kernel functions form a Riesz basis for K_B (for proofs see [10] and [7]). In particular, each function $f \in K_B$ can be written as

$$f = \sum_{m=1}^{\infty} \frac{f(a_m)}{B'(a_m)} \tilde{k}_{a_m} = \sum_{m=1}^{\infty} \frac{\langle f, k_{a_m} \rangle}{B'(a_m)} \tilde{k}_{a_m}. \quad (2.2)$$

Applying the conjugation C , given by (1.1), to both sides of the formula (2.2) with f replaced by $\tilde{f} = Cf$, we find that

$$f = \sum_{m=1}^{\infty} \left(\frac{\tilde{f}(a_m)}{B'(a_m)} \right) k_{a_m} = \sum_{m=1}^{\infty} \frac{\langle f, \tilde{k}_{a_m} \rangle}{\overline{B'(a_m)}} k_{a_m}.$$

Since $\{\tilde{k}_{a_m}\}$ is a basis for the space K_B , any bounded linear operator A on K_B has a matrix representation $M_A = (r_{s,p})$ with respect to this basis. Moreover, since

$$\langle \tilde{k}_{a_m}, k_{a_n} \rangle = \begin{cases} 0 & \text{if } n \neq m, \\ B'(a_m) & \text{if } n = m, \end{cases}$$

we have $r_{s,p} = \langle A\tilde{k}_{a_p}, k_{a_s}^* \rangle$, where $k_{a_s}^* = (\overline{B'(a_s)})^{-1} k_{a_s}$.

For $|\alpha| < 1$ let the Möbius transformation φ_α be given by $\varphi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$ and let $B_\alpha = B \circ \varphi_\alpha$. Then B_α is a Blaschke product with zeros $b_m = \varphi_\alpha(a_m)$, and since

$$\left| \frac{\varphi_\alpha(a_m) - \varphi_\alpha(a_k)}{1 - \overline{\varphi_\alpha(a_m)}\varphi_\alpha(a_k)} \right| = \left| \frac{a_m - a_k}{1 - \bar{a}_m a_k} \right|,$$

the sequence $\{b_m\}$ is uniformly separated. Moreover, the operator defined by

$$\mathfrak{U}_\alpha f = \sqrt{\varphi'_\alpha} f \circ \varphi_\alpha \quad (2.3)$$

maps K_B unitarily onto K_{B_α} and, by the proof of Proposition 4.1 in [2], the map

$$A \mapsto \mathfrak{U}_\alpha A \mathfrak{U}_\alpha^*, \quad A \in \mathcal{T}(K_B)$$

carries $\mathcal{T}(K_B)$ onto $\mathcal{T}(K_{B_\alpha})$.

In the proof of our main result we will use the following lemma that may be also of independent interest.

Lemma 2.1. Let A be a bounded linear operator on K_B and let $M_A = (r_{s,p})$ be its matrix representation with respect to the basis $\{\tilde{k}_{a_m}\}$. If $A_\alpha = \mathfrak{U}_\alpha A \mathfrak{U}_\alpha^*$, where $\mathfrak{U}_\alpha$ is given by (2.3), and $M_{A_\alpha} = (t_{s,p})$ is its matrix representation with respect to $\{\tilde{k}_{b_m}\}$, where $b_m = \varphi_\alpha(a_m)$, then for all $s, p \geq 1$,

$$t_{s,p} = \frac{1 - \bar{\alpha}a_p}{1 - \bar{\alpha}a_s} r_{s,p}. \quad (2.4)$$

Proof. Clearly, we have

$$r_{s,p} = (B'(a_s))^{-1} \langle A\tilde{k}_{a_p}, k_{a_s} \rangle,$$

and

$$t_{s,p} = (B'_\alpha(b_s))^{-1} \langle A_\alpha \tilde{k}_{b_p}, k_{b_s} \rangle.$$

Hence,

$$t_{s,p} = \frac{\varphi'_\alpha(a_s)}{B'(a_s)} \langle A \mathfrak{L}_\alpha^* \tilde{k}_{b_p}, \mathfrak{L}_\alpha^* k_{b_s} \rangle.$$

Moreover, using (2.3) one can find that

$$\mathfrak{L}_\alpha^* k_{b_s}(z) = i \frac{1}{1 - \overline{\varphi_\alpha(a_s)} \varphi_\alpha(z)} \frac{\sqrt{1 - |\alpha|^2}}{1 - \bar{\alpha}z} = i \frac{1 - \alpha \bar{a}_s}{\sqrt{1 - |\alpha|^2}} k_{a_s}(z).$$

Analogously,

$$\mathfrak{L}_\alpha^* \tilde{k}_{b_p}(z) = -i \frac{1 - \bar{\alpha}a_p}{\sqrt{1 - |\alpha|^2}} \tilde{k}_{a_p}(z).$$

Therefore,

$$t_{s,p} = -\frac{\varphi'_\alpha(a_s)}{B'(a_s)} \frac{(1 - \bar{\alpha}a_p)(1 - \bar{\alpha}a_s)}{1 - |\alpha|^2} \langle A \tilde{k}_{a_p}, k_{a_s} \rangle = -\frac{|\alpha|^2 - 1}{(1 - \bar{\alpha}a_s)^2} \frac{(1 - \bar{\alpha}a_p)(1 - \bar{\alpha}a_s)}{1 - |\alpha|^2} r_{s,p} = \frac{1 - \bar{\alpha}a_p}{1 - \bar{\alpha}a_s} r_{s,p}. \quad \square$$

We now state our main result which is a generalization of the result obtained in [4] for finite Blaschke products.

Theorem 2.2. Let B be an infinite Blaschke product with uniformly separated zeros $\{a_m\}$ and let A be a bounded linear operator on K_B . If $M_A = (r_{s,p})$ is the matrix representation of A with respect to the basis $\{\tilde{k}_{a_m}\}$, then $A \in \mathcal{T}(K_B)$ if and only if

$$r_{s,p} = \frac{B'(a_1)}{B'(a_s)} \frac{(a_p - a_1)r_{1,p} + (a_1 - a_s)r_{1,s}}{a_p - a_s}, \quad (2.5)$$

for all $s \neq p$.

Proof. The proof of necessity is analogous to that given in [4] for the finite dimensional case.

Let us assume that $A = A_\varphi$ is a truncated Toeplitz operator with symbol $\varphi \in L^2(\partial\mathbb{D})$. Let us compute the matrix representation $M_{A_\varphi} = (r_{s,p})$ of A_φ with respect to the basis $\{\tilde{k}_{a_m}\}$.

By the Corollary to Theorem 3.1 in [11], A_φ can be written as

$$A_\varphi = A_{\psi + \bar{\chi}},$$

where $\psi, \chi \in K_B$. It follows from our previous observations (see (2.1)) that these functions can be written as

$$\psi = \sum_{m=1}^{\infty} c_m \tilde{k}_{a_m}, \quad \chi = \sum_{m=1}^{\infty} d_m \tilde{k}_{a_m}.$$

We first find a formula for $r_{s,p}$ in terms of $\{c_m\}$ and $\{d_m\}$. Since $\tilde{k}_{a_p}$ is a bounded function,

$$A_\varphi \tilde{k}_{a_p} = A_{\psi + \bar{\chi}} \tilde{k}_{a_p} = \sum_{m=1}^{\infty} c_m A_{\tilde{k}_{a_m}} \tilde{k}_{a_p} + \sum_{m=1}^{\infty} \bar{d}_m A_{\tilde{k}_{a_m}} \tilde{k}_{a_p}.$$

By the proof of Theorem 5.1 in [11], we have

$$A_{\tilde{k}_{a_m}} = \tilde{k}_{a_m} \otimes k_{a_m}, \quad \text{and} \quad A_{\tilde{k}_{a_m}} = k_{a_m} \otimes \tilde{k}_{a_m}.$$

Hence,

$$A_\varphi \tilde{k}_{a_p} = \sum_{m=1}^{\infty} c_m \langle \tilde{k}_{a_p}, k_{a_m} \rangle \tilde{k}_{a_m} + \sum_{m=1}^{\infty} \bar{d}_m \langle \tilde{k}_{a_p}, \tilde{k}_{a_m} \rangle k_{a_m} = c_p B'(a_p) \tilde{k}_{a_p} + \sum_{m=1}^{\infty} \frac{\bar{d}_m}{1 - \bar{d}_m a_p} k_{a_m},$$

where the last equality follows from $\langle \tilde{k}_{a_p}, \tilde{k}_{a_s} \rangle = \langle k_{a_s}, k_{a_p} \rangle$.

Since the last series in the preceding string of equalities is norm convergent, we see that

$$r_{s,p} = \langle A_\varphi \tilde{k}_{a_p}, k_{a_s}^* \rangle = c_p B'(a_p) \delta_{s,p} + \sum_{m=1}^{\infty} \frac{\bar{d}_m}{1 - \bar{d}_m a_p} \langle k_{a_m}, k_{a_s}^* \rangle = c_p B'(a_p) \delta_{s,p} + \frac{1}{B'(a_s)} \sum_{m=1}^{\infty} \frac{\bar{d}_m}{(1 - \bar{d}_m a_p)(1 - \bar{d}_m a_s)}.$$

We now show that $M_{A_\varphi} = (r_{s,p})$ satisfies (2.5). We have

$$\frac{a_p - a_s}{(1 - \bar{a}_m a_p)(1 - \bar{a}_m a_s)} = \frac{a_p}{1 - \bar{a}_m a_p} - \frac{a_s}{1 - \bar{a}_m a_s}.$$

Consequently, for $s \neq p$,

$$\begin{aligned} r_{s,p} &= \frac{1}{B'(a_s)} \sum_{m=1}^{\infty} \frac{\bar{d}_m}{(1 - \bar{a}_m a_p)(1 - \bar{a}_m a_s)} \\ &= \frac{1}{B'(a_s)} \sum_{m=1}^{\infty} \frac{\bar{d}_m}{a_p - a_s} \left(\frac{a_p}{1 - \bar{a}_m a_p} - \frac{a_s}{1 - \bar{a}_m a_s} \right) \\ &= \frac{1}{B'(a_s)} \sum_{m=1}^{\infty} \frac{\bar{d}_m}{a_p - a_s} \left(\frac{a_p - a_1}{(1 - \bar{a}_m a_p)(1 - \bar{a}_m a_1)} + \frac{a_1 - a_s}{(1 - \bar{a}_m a_1)(1 - \bar{a}_m a_s)} \right) \\ &= \frac{B'(a_1)}{B'(a_s)} \frac{(a_p - a_1)r_{1,p} + (a_1 - a_s)r_{1,s}}{a_p - a_s}. \end{aligned}$$

Now let us assume that A is a bounded linear operator on K_u and that its matrix representation $M_A = (r_{s,p})$ with respect to the basis $\{\tilde{k}_{a_m}\}$ satisfies (2.5). Let us prove that A is a truncated Toeplitz operator.

We first assume that $B(0) \neq 0$.

By Sarason's result, to show that $A \in \mathcal{T}(K_B)$ it is enough to find functions $\psi, \chi \in K_B$ such that (1.3) is satisfied. But since $\{\tilde{k}_{a_m}\}$ and $\{k_{a_m}\}$ form (non-orthogonal) bases for K_B , condition (1.3) can be replaced with

$$\langle A\tilde{k}_{a_p}, k_{a_s}^* \rangle - \langle S_B A S_B^* \tilde{k}_{a_p}, k_{a_s}^* \rangle = \langle \tilde{k}_{a_p}, k_0^B \rangle \langle \psi, k_{a_s}^* \rangle + \langle \tilde{k}_{a_p}, \chi \rangle \langle k_0^B, k_{a_s}^* \rangle, \quad s, p \geq 1. \quad (2.6)$$

We now construct functions $\psi, \chi \in K_B$ satisfying (2.6). To do this, we note that the above equations can be rewritten as

$$r_{s,p} B'(a_s) - \langle S_B A S_B^* \tilde{k}_{a_p}, k_{a_s} \rangle = \tilde{k}_{a_p}(0) \psi_s + \tilde{\chi}_p, \quad s, p \geq 1, \quad (2.7)$$

with $\psi_s = \langle \psi, k_{a_s} \rangle$ and $\tilde{\chi}_p = \langle \tilde{k}_{a_p}, \chi \rangle = \langle \tilde{\chi}, k_{a_p} \rangle$. Therefore, it suffices to find sequences $\{\psi_s\}$ and $\{\tilde{\chi}_p\}$ satisfying (2.7) and such that

$$\sum_{s=1}^{\infty} |\psi_s|^2 (1 - |a_s|^2) < \infty \quad \text{and} \quad \sum_{p=1}^{\infty} |\tilde{\chi}_p|^2 (1 - |a_p|^2) < \infty. \quad (2.8)$$

Then the desired functions ψ and χ are solutions of the corresponding interpolation problems and they are given by

$$\psi = \sum_{s=1}^{\infty} \frac{\psi_s}{B'(a_s)} \tilde{k}_{a_s}, \quad \chi = \sum_{p=1}^{\infty} \frac{\tilde{\chi}_p}{B'(a_p)} k_{a_p}$$

(see (2.1)). Let us find such $\{\psi_s\}$ and $\{\tilde{\chi}_p\}$.

By Lemma 2.2 in [11], we know that for $a_p \neq 0$,

$$S_B^* \tilde{k}_{a_p} = \frac{1}{a_p} \tilde{k}_{a_p} - \frac{1}{a_p} \tilde{k}_0^B.$$

It then follows from the last equation and from $S_B^* k_{a_s} = \bar{a}_s k_{a_s}$ that

$$\langle A S_B^* \tilde{k}_{a_p}, S_B^* k_{a_s} \rangle = \frac{a_s}{a_p} \langle A \tilde{k}_{a_p} - A \tilde{k}_0^B, k_{a_s} \rangle = \frac{a_s}{a_p} r_{s,p} B'(a_s) - \frac{a_s}{a_p} \langle A \tilde{k}_0^B, k_{a_s} \rangle.$$

Consequently, under assumption that $B(0) \neq 0$, (2.7) can be written in the form

$$\tilde{k}_{a_p}(0) \psi_s + \tilde{\chi}_p = \left(1 - \frac{a_s}{a_p} \right) r_{s,p} B'(a_s) + \frac{a_s}{a_p} \langle A \tilde{k}_0^B, k_{a_s} \rangle, \quad s, p \geq 1.$$

Using (2.5) one can check that this system of equations is equivalent to the following

$$\begin{cases} \tilde{k}_{a_p}(0) \psi_1 + \tilde{\chi}_p = \left(1 - \frac{a_1}{a_p} \right) r_{1,p} B'(a_1) + \frac{a_1}{a_p} \langle A \tilde{k}_0^B, k_{a_1} \rangle, & p \geq 2, \\ \tilde{k}_{a_p}(0) \psi_p + \tilde{\chi}_p = \langle A \tilde{k}_0^B, k_{a_p} \rangle, & p \geq 1. \end{cases} \quad (2.9)$$

Fix an arbitrary ψ_1 . The corresponding solution of (2.9) is given by

$$\begin{cases} \tilde{\chi}_p = \left(1 - \frac{a_1}{a_p}\right) r_{1,p} B'(a_1) + \frac{a_1}{a_p} \langle A \tilde{k}_0^B, k_{a_1} \rangle - \tilde{k}_{a_p}(0) \psi_1, & p \geq 1, \\ \psi_p = (\tilde{k}_{a_p}(0))^{-1} (\langle A \tilde{k}_0^B, k_{a_p} \rangle - \tilde{\chi}_p), & p \geq 2. \end{cases}$$

So, to end our proof it is enough to show that conditions (2.8) are fulfilled. Indeed, we have

$$\begin{aligned} \sum_{p=1}^{\infty} |\tilde{\chi}_p|^2 (1 - |a_p|^2) &\leq C \sum_{p=1}^{\infty} (|r_{1,p} B'(a_1)| + |a_1 \langle A \tilde{k}_0^B, k_{a_1} \rangle + \psi_1 B(0)|)^2 (1 - |a_p|^2) \\ &\leq 2C \sum_{p=1}^{\infty} |\langle A \tilde{k}_{a_p}, k_{a_1} \rangle|^2 (1 - |a_p|^2) + 2C |a_1 \langle A \tilde{k}_0^B, k_{a_1} \rangle + \psi_1 B(0)|^2 \sum_{p=1}^{\infty} (1 - |a_p|^2) \\ &\leq C' + 2C \sum_{p=1}^{\infty} |\tilde{f}_A(a_p)|^2 (1 - |a_p|^2) < \infty \end{aligned}$$

where $f_A = A^* k_{a_1} \in K_B$. Similarly,

$$\sum_{s=1}^{\infty} |\psi_s|^2 (1 - |a_s|^2) \leq C \sum_{s=1}^{\infty} |g_A(a_s)|^2 (1 - |a_s|^2) + C \sum_{s=1}^{\infty} |\tilde{\chi}_s|^2 (1 - |a_s|^2) < \infty,$$

where $g_A = A \tilde{k}_0^B \in K_B$.

This completes the proof in the case when $B(0) \neq 0$.

Assume now that $B(0) = 0$. Let $\alpha \in \mathbb{D}$ be such that $B(\alpha) \neq 0$ and define $B_\alpha = B \circ \varphi_\alpha$, where φ_α is a Möbius transformation such that $\varphi_\alpha(0) = \alpha$. By the result in [2], to show that $A \in \mathcal{T}(K_B)$ it is enough to show that $A_\alpha = \mathfrak{U}_\alpha A \mathfrak{U}_\alpha^*$, where $\mathfrak{U}_\alpha$ is given by (2.3), is in $\mathcal{T}(K_{B_\alpha})$.

Since $B_\alpha(0) \neq 0$ we already know that $A_\alpha \in \mathcal{T}(K_{B_\alpha})$ if and only if its matrix representation $M_{A_\alpha} = (t_{s,p})$ with respect to $\{\tilde{k}_{b_m}\}$, where $b_m = \varphi_\alpha(a_m)$, is such that for all $s \neq p$,

$$t_{s,p} = \frac{B'_\alpha(b_1)}{B'_\alpha(b_s)} \frac{(b_p - b_1)t_{1,p} + (b_1 - b_s)t_{1,s}}{b_p - b_s}. \quad (2.10)$$

Hence, we only need to show that (2.10) holds. To do this, we use Lemma 2.1. If $r_{s,p}$ is as in (2.5) then by (2.4),

$$\begin{aligned} &\frac{B'_\alpha(b_1)}{B'_\alpha(b_s)} \frac{(b_p - b_1)t_{1,p} + (b_1 - b_s)t_{1,s}}{b_p - b_s} \\ &= \frac{\varphi'_\alpha(a_s)}{\varphi'_\alpha(a_1)} \frac{B'(a_1)}{B'(a_s)} \frac{(\varphi_\alpha(a_p) - \varphi_\alpha(a_1)) \frac{1 - \bar{\alpha}a_p}{1 - \bar{\alpha}a_1} r_{1,p} + (\varphi_\alpha(a_1) - \varphi_\alpha(a_s)) \frac{1 - \bar{\alpha}a_s}{1 - \bar{\alpha}a_1} r_{1,s}}{\varphi_\alpha(a_p) - \varphi_\alpha(a_s)} \\ &= \frac{\varphi'_\alpha(a_s)}{\varphi'_\alpha(a_1)} \frac{B'(a_1)}{B'(a_s)} \frac{(1 - \bar{\alpha}a_p)(1 - \bar{\alpha}a_s)}{(1 - \bar{\alpha}a_1)^2} \frac{(a_1 - a_p)r_{1,p} + (a_s - a_1)r_{1,s}}{a_s - a_p} \\ &= \frac{|\alpha|^2 - 1}{(1 - \bar{\alpha}a_s)^2} \frac{(1 - \bar{\alpha}a_1)^2}{|\alpha|^2 - 1} \frac{(1 - \bar{\alpha}a_p)(1 - \bar{\alpha}a_s)}{(1 - \bar{\alpha}a_1)^2} r_{s,p} \\ &= \frac{1 - \bar{\alpha}a_p}{1 - \bar{\alpha}a_s} r_{s,p} = t_{s,p}. \end{aligned}$$

This completes the proof. $\square$

Remark 2.3. Observe that every bounded linear operator A with matrix representation satisfying (2.5) is C -symmetric for C given by (1.1). Indeed, from (2.5) we get

$$r_{s,p} B'(a_s) = r_{p,s} B'(a_p),$$

or, in other words,

$$\langle A k_{a_p}, k_{a_s} \rangle = \langle A k_{a_s}, k_{a_p} \rangle = \langle C k_{a_s}, A^* k_{a_p} \rangle = \langle C A^* k_{a_p}, k_{a_s} \rangle,$$

for all $s, p \geq 1$. Finally, by the density argument we find that (1.2) holds on the whole model space.

Remark 2.4. Theorem 2.2 can be reformulated in terms of the matrix representation $\tilde{M}_A = (t_{s,p})$ with respect to $\{k_{a_m}\}$. In this case we have $t_{s,p} = (\overline{B'(a_s)})^{-1} \langle A k_{a_p}, \tilde{k}_{a_s} \rangle$. Moreover, if $M_{A^*} = (r_{s,p})$ is the matrix representation of A^* with respect to $\{\tilde{k}_{a_m}\}$, then $r_{s,p} = (B'(a_s))^{-1} \langle A^* \tilde{k}_{a_p}, k_{a_s} \rangle$. So

$$t_{s,p} = \left(\frac{\overline{B'(a_p)}}{B'(a_s)} \right) \tilde{r}_{p,s}. \quad (2.11)$$

If $A \in \mathcal{T}(K_B)$, then $A^* \in \mathcal{T}(K_B)$ and $r_{s,p}$ satisfies (2.5). From this,

$$r_{p,s} = \frac{B'(a_s)}{B'(a_p)} r_{s,p},$$

and, consequently,

$$t_{s,p} = \tilde{r}_{s,p}.$$

Hence,

$$t_{s,p} = \left(\frac{\overline{B'(a_1)}}{B'(a_s)} \right) \left(\frac{(\overline{a_p - a_1}) t_{1,p} + (\overline{a_1 - a_s}) t_{1,s}}{a_p - a_s} \right). \quad (2.12)$$

Now we show that (2.12) is also a sufficient condition for A to be a truncated Toeplitz operator. We will be done if we prove that under condition (2.12), $A^* \in \mathcal{T}(K_B)$. Observe that (2.12) implies

$$t_{p,s} = \left(\frac{\overline{B'(a_s)}}{B'(a_p)} \right) t_{s,p}.$$

Hence, by (2.11), $t_{s,p} = \tilde{r}_{s,p}$, and by Theorem 2.2, $A^* \in \mathcal{T}(K_B)$.

3. Clark bases

We first briefly describe the construction of the so-called Clark bases. In [5] D.N. Clark proved that all one-dimensional unitary perturbations of the truncated shift S_u are of the form

$$U_\alpha = S_u + \frac{u(0) + \alpha}{1 - \overline{u(0)}} k_0^u \otimes \tilde{k}_0^u, \quad \alpha \in \partial \mathbb{D}.$$

He also proved that $\zeta \in \partial \mathbb{D}$ is an eigenvalue of U_α if and only if u has a finite angular derivative at ζ and the non-tangential limit of u at ζ is

$$u(\zeta) = \beta_\alpha = \frac{u(0) + \alpha}{1 + \overline{u(0)}\alpha}$$

([5], Thm. 3.2). Moreover,

$$k_\zeta^u(z) = \frac{1 - \bar{\beta}_\alpha u(z)}{1 - \bar{\zeta} z}$$

is an eigenvector corresponding to ζ . Let $v_\zeta = \|k_\zeta^u\|^{-1} k_\zeta^u$. Since U_α is unitary, the set $\{v_\zeta\}$ of normalized eigenvectors corresponding to different eigenvalues of U_α must be orthogonal and countable.

It is known that if U_α has a pure point spectrum, then the set $\{v_{\zeta_m}\}$, where $u(\zeta_m) = \beta_\alpha$ and the angular derivative of u at ζ_m exists, forms an orthonormal basis for K_u (see, e.g. [5] or [3], p. 193). This happens, for example, if u is a Blaschke product whose zeros have at most countably many limit points ([5], p. 185). Basis $\{v_{\zeta_m}\}$ is called the Clark basis.

To prove our next theorem we need to use a more general characterization of $\mathcal{T}(K_u)$ (due also to Sarason [11]) than the one applied in the proof of Theorem 2.2. It says that a bounded linear operator A on K_u is a truncated Toeplitz operator if and only if there are functions ψ, χ in K_u such that

$$A - S_{u,c} A S_{u,c}^* = \psi \otimes k_0^u + k_0^u \otimes \chi, \quad (3.1)$$

where for any complex number c , $S_{u,c}$ is a one-dimensional perturbation of S_u , given by

$$S_{u,c} = S_u + c k_0^u \otimes \tilde{k}_0^u.$$

Clearly,

$$U_\alpha = S_{u,c_\alpha},$$

with

$$c_\alpha = \frac{u(0) + \alpha}{1 - |u(0)|^2}.$$

Hence, a bounded linear operator A on K_u is in $\mathcal{T}(K_u)$ if and only if

$$A - U_\alpha A U_\alpha^* = \psi \otimes k_0^u + k_0^u \otimes \chi \quad (3.2)$$

for some $\psi, \chi \in K_u$.

Theorem 3.1. Let u be an inner function such that K_u has a Clark basis $\{v_{\zeta_m}\}$. If $M_A = (r_{s,p})$ is the matrix representation of a bounded linear operator A on K_u with respect to the basis $\{v_{\zeta_m}\}$, then $A \in \mathcal{T}(K_u)$ if and only if

$$r_{s,p} = \frac{\sqrt{|u'(\zeta_1)|}}{\zeta_p - \zeta_s} \left(\frac{\zeta_p}{\zeta_s} \frac{\zeta_1 - \zeta_s}{\sqrt{|u'(\zeta_p)|}} r_{1,s} + \frac{\zeta_p - \zeta_1}{\sqrt{|u'(\zeta_s)|}} r_{1,p} \right), \quad (3.3)$$

for all $s \neq p$.

The proof, based on the equality (3.2), is analogous to that of Theorem 2.2. The details are left to the reader.

We can also find a matrix representation of a truncated Toeplitz operator with respect to the so-called modified Clark basis $\{e_{\zeta_m}\}$, where $e_{\zeta_m} = \omega_m v_{\zeta_m}$ with

$$\omega_m = e^{-\frac{i}{2}(\arg \zeta_m - \arg \beta_\alpha)}.$$

This basis has the property $Ce_{\zeta_m} = e_{\zeta_m}$. It is also known that the matrix representation of any A_φ with respect to this basis is a complex symmetric (i.e., self-transpose) matrix (see, e.g., [7] and [8]).

Theorem 3.2. Let u be an inner function such that K_u has a Clark basis. If $M_A = (t_{s,p})$ is the matrix representation of a bounded linear operator A on K_u with respect to the modified Clark basis $\{e_{\zeta_m}\}$, then $A \in \mathcal{T}(K_u)$ if and only if

$$t_{s,p} = \frac{\sqrt{|u'(\zeta_1)|}}{\bar{\omega}_1} \frac{1}{\zeta_p - \zeta_s} \left(\frac{\bar{\omega}_p}{\sqrt{|u'(\zeta_p)|}} (\zeta_1 - \zeta_s) t_{1,s} + \frac{\bar{\omega}_s}{\sqrt{|u'(\zeta_s)|}} (\zeta_p - \zeta_1) t_{1,p} \right), \quad (3.4)$$

for all $s \neq p$.

Proof. If $(r_{s,p})$ is a matrix representation of A with respect to the Clark basis $\{v_{\zeta_m}\}$, then

$$\omega_p r_{s,p} = \omega_s t_{s,p}.$$

Next, using the equality

$$\omega_p^2 \zeta_p = \beta_\alpha$$

one can check that (3.4) is equivalent to (3.3). $\square$

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