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Factorization of a class of symbols with outer functions

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ABSTRACT

In this paper the Wiener–Hopf (or Riemann–Hilbert) factorization of a class of symbols important in applications is studied. The symbols in this class involve outer functions that appear in applications such as diffraction by strip gratings and infinite-dimensional integrable systems. The method proposed is based on the reduction of a vector Riemann–Hilbert to a scalar problem on an appropriate Riemann surface. Two examples are given leading to the Riemann sphere and to an elliptic curve.

1. Introduction

Riemann–Hilbert problems appear in many applications, in particular, in diffraction theory and integrable systems – both continuous and discrete [2–5,9,11,12,15]. The solution of these problems involves the Wiener–Hopf (or Riemann–Hilbert) factorization of matrix-valued functions [6,7]. Contrary to the case for the factorization of scalar-valued functions no general method exists in the matrix case, even for 2×2 matrix-valued functions. A large class of 2×2 matrix-valued functions was studied in [6], where it is shown that the matrix factorization problem for this class is equivalent in some sense to a scalar Riemann–Hilbert problem in an appropriate Riemann surface. The factorization problem that arises in the study of finite-dimensional integrable systems can be solved through this theory [4,8]. However, for infinite-dimensional systems, both continuous and discrete, and in diffraction theory for strip gratings [2,3], the corresponding symbols (matrix functions) do not belong to the class treated in [6].

The class of symbols we consider in this paper appears in the study of the KdV equation [10,14] and in the study of discrete integrable systems such as the Toda lattice [9] and also in the solution of some diffraction problems [2,3]. In both cases the explicit solution of the factorization problem leads to obtaining a family of solutions to the corresponding integrable systems [10,14], which is the main objective of many papers on the subject.

This paper is organized as follows. In Section 2 we outline the method showing how the scalar Riemann–Hilbert problem on an appropriate Riemann surface appears. In Section 3 we consider an example where the Riemann surface involved is the Riemann sphere. In Section 4 we study an example in which the Riemann surface is an elliptic curve and where several aspects of complex analysis on Riemann surfaces come into play. Both in Sections 3 and 4 the explicit factors of the symbol's factorization are given.

2. Outline of the method

To begin with we define the class of symbols whose factorization will be studied in this and the following sections.

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Notation 2.1. By C_α $(0<\alpha<1)$ we denote the space of Hölder continuous functions on the circle $S^1=\partial\mathbb{D}$, where $\mathbb{D}\subset\mathbb{C}$ is an open disk centered at the origin. As is known this space possesses the direct sum decomposition $C_\alpha=C_\alpha^+\oplus C_\alpha^-$, where C_α^+ denotes the subspace of functions admitting a holomorphic extension to $\mathbb{D}$ and C_α^- denotes the subspace of functions with holomorphic extension to $\mathbb{C}\setminus\overline{\mathbb{D}}$ vanishing at infinity.

Definition 2.2. Let $\rho=\sqrt{p_n}$ where p_n is a polynomial of even degree with no zeros outside an open disk $\mathbb{D}\subset\mathbb{C}$ centered at the origin and let E denote a function such that E and E^{-1} belong to C_α^+ $(0<\alpha<1$ and $S^1=\partial\mathbb{D})$. We denote by $\mathscr{C}_\alpha$ the class of matrix-valued functions of the form

$$G=\begin{bmatrix}1 & \rho E\\ -\rho^{-1}E^{-1} & 1\end{bmatrix}, \tag{2.1}$$

with ρ, E defined as above.

Remark 2.3. The condition that E and E^{-1} belong to C_α^+ implies that E is an *outer function* [13, Chapter 17].

Remark 2.4. The condition that p_n has even degree is necessary to guarantee that we can choose a continuous branch of $\sqrt{p_n}$ on $S^1=\partial\mathbb{D}$. The function ρ denotes one of the two possible such choices.

Definition 2.5. A matrix $G\in C_\alpha^{2\times 2}$ is said to possess a bounded *Wiener–Hopf* (or *Riemann–Hilbert*) factorization if it has a representation of the form

$$G=G_-DG_+ \tag{2.2}$$

with $[G_\pm]^{\pm1}\in[C_\alpha^\pm]^{2\times2}$ and $D=\operatorname{diag}(z^{k_1},z^{k_2})$, $k_i\in\mathbb{Z}$ with $k_1\geqslant k_2$. The factorization is said to be canonical if $D=I_2$ (identity in $\mathbb{C}^{2\times2}$).

In what follows we concentrate on the calculation of the factors of a canonical factorization although a condition on its existence will appear in the analysis. Keeping in mind the relation $GG_+^{-1}=G_-$ we see that the factors in (2.2) for $D=I_2$ can be calculated by solving the vector Riemann–Hilbert problem

$$G\phi^+=\phi^- \tag{2.3}$$

which yields the columns in the factors G_+^{-1} and G_- for appropriate normalization conditions (*cf.* Remark 2.6 at the end of this section).

The vector Riemann–Hilbert problem (2.3) can be written for G of the class $\mathscr{C}_\alpha$ as

$$\begin{aligned}&\phi_1^++\rho E\phi_2^+=\phi_1^-,\\ &\phi_1^+-\rho E\phi_2^+=-\rho E\phi_2^-.\end{aligned} \tag{2.4}$$

Consider now the *hyperelliptic* Riemann surface Σ obtained from the plane algebraic curve given by the equation

$$w^2=p_n(\xi) \tag{2.5}$$

by adding two "points at infinity" (for general reference on Riemann surfaces see *e.g.* [16]).

This surface comes equipped with its hyperelliptic involution σ given by $(\xi,w)\mapsto(\xi,-w)$ and two meromorphic functions $(\xi,w)\mapsto\xi$ and $(\xi,w)\mapsto w$ – denoted by π_1 and π_2, respectively. Often we abuse notation and denote these functions by ξ and w, respectively.

Also, for a meromorphic function f defined on a region $U\subset\mathbb{C}$ we identify it with the meromorphic function $\pi_1^*f:=f\circ\pi_1$ on $\pi_1^{-1}(U)$. Under this identification we can write any meromorphic function g on $\pi_1^{-1}(U)$ uniquely as

$$g=g_{ev}+wg_{odd}, \tag{2.6}$$

with g_{ev} and g_{odd} meromorphic on U: we have $\pi_1^*g_{ev}=(g+\sigma^*g)/2$ and $\pi_1^*g_{odd}=(g-\sigma^*g)/2w$.

Bearing in mind the condition on the zeros of p_n it follows that the open set obtained by adding the points at infinity to $\pi_1^{-1}(\mathbb{C}\setminus\overline{\mathbb{D}})$ is the union of two disjoint regions Ω_1^-, Ω_2^-. We choose the labeling so that, for $\xi\in S^1$, we have $(\xi,\rho(\xi))\in\partial\Omega_1^-$. Also, we write ∞_i for the point at infinity belonging to Ω_i.

From this and (2.4) it follows that the Riemann–Hilbert problem (2.3) on S^1 is transformed to a Riemann–Hilbert problem on the contour $\Gamma=\pi_1^{-1}(S^1)$, which we write in the form

$$\phi_1^++wE\phi_2^+=h\psi^-, \tag{2.7}$$

with

$$\psi^- = \begin{cases} \phi_1^- & \text{on } \Omega_1^-, \\ \phi_2^- & \text{on } \Omega_2^-, \end{cases} \qquad h = \begin{cases} 1 & \text{on } \Gamma_1, \\ -\rho E & \text{on } \Gamma_2. \end{cases} \tag{2.8}$$

Here $\Gamma_i = \partial\Omega_i^-$, $i = 1, 2$, so that Γ is the composite contour $\Gamma_1 + \Gamma_2$.

Remark 2.6. In Section 4 we will need to consider an orientation on $\Gamma = \Gamma_1 + \Gamma_2$. The orientation on Γ_i we will consider there is not the one induced from Ω_i^-. As an oriented contour, we define $\Gamma = \partial\Omega^+$, where $\Omega^+ = \Sigma \setminus \overline{\Omega_1^- \cup \Omega_2^-}$.

Notation 2.7. We denote by $C_\alpha^\pm(\Gamma)$, $0 < \alpha < 1$, the space of Hölder continuous functions on Γ with holomorphic extensions to Ω^+ and $\Omega_1^- \cup \Omega_2^-$.

We may now state the following result.

Proposition 2.8. *Let $\phi = (\phi_1^+, \phi_2^+)$ be a solution to the Riemann–Hilbert problem (2.4) on S^1. Then the function $\psi^+ := \phi_1^+ + wE\phi_2^+$ is a solution of the scalar Riemann–Hilbert on $\Gamma \subset \Sigma$*

$$\psi^+ = h\psi^- \quad \left(\psi^\pm \in C_\alpha^\pm(\Gamma)\right)$$

where h is given in (2.8).

Proof. It is a consequence of the above argument. $\square$

Next we state a converse to Proposition 2.8.

Proposition 2.9. *Let $\psi = (\psi^+, \psi^-)$ be a solution to the Riemann–Hilbert problem*

$$\psi^+ = h\psi^- \quad \left(\psi^\pm \in C_\alpha^\pm(\Gamma)\right)$$

on the contour $\Gamma = \Gamma_1 + \Gamma_2$ on Σ with

$$h = \begin{cases} 1 & \text{on } \Gamma_1, \\ -\rho(\xi)E(\xi) & \text{on } \Gamma_2. \end{cases}$$

Then the functions

$$\begin{aligned} \phi_1^+ &= \psi_{ev}^+, \qquad \phi_1^- = \psi^-|_{\Omega_1^-}, \\ \phi_2^+ &= E^{-1}\psi_{odd}^+, \qquad \phi_2^- = \psi^-|_{\Omega_2^-} \end{aligned}$$

(cf. (2.6)) *satisfy the Riemann–Hilbert problem (2.4).*

Proof. We have

$$\begin{aligned} 2\phi_1^+ &= 2\psi_{ev}^+ = h\psi^- + \sigma^*\left(h\psi^-\right) \\ &= \phi_1^- - \rho E\phi_2^-, \\ 2\phi_2^+ &= 2E^{-1}\psi_{odd}^+ = \rho^{-1}E^{-1}\left(h\psi^- - \sigma^*\left(h\psi^-\right)\right) \\ &= \rho^{-1}E^{-1}\left(\phi_1^- + \rho E\phi_2^-\right). \end{aligned}$$

From the above relations it follows that

$$\begin{aligned} \phi_1^+ + \rho E\phi_2^+ &= \phi_1^-, \\ \phi_1^+ - \rho E\phi_2^+ &= -\rho E\phi_2^-, \end{aligned}$$

as stated. $\square$

Remark 2.10. Although in the following analysis we concentrate on calculating the factors of a canonical factorization it should be noted that the method to be developed actually applies to the study of the solvability and calculation of explicit solutions to Eq. (2.3). In particular, since $\operatorname{ind}(G) = 0$ the existence of nontrivial solutions to (2.3) subject to the condition $\phi_i^-(\infty) = 0$, $i = 1, 2$, implies that the factorization is noncanonical. The dimension of the space of solutions gives, then, the partial indices of the factorization.

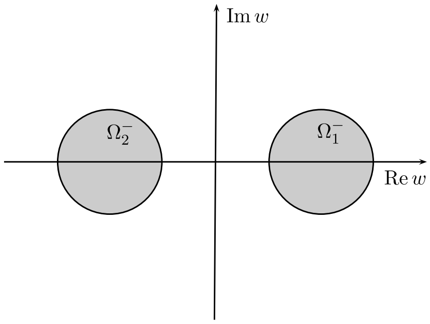

Fig. 1. The w plane.

3. Example 1 (Riemann sphere)

In this section we study the factorization problem for the symbol (2.1), where we take

$$E(\xi)=\exp(\xi), \qquad \rho(\xi)=\left(\frac{\xi-\alpha}{\xi+\alpha}\right)^{\frac{1}{2}}, \quad |\alpha|<1 \tag{3.1}$$

on the unit circle S^1. Here ρ is the square root of a rational function instead of a polynomial for the sake of simplicity of the computations that follow.[2]

In this case the Riemann surface considered in the previous section is isomorphic to the Riemann sphere. Thus taking

$$w=\sqrt{\frac{\xi-\alpha}{\xi+\alpha}}, \quad \operatorname{Re} w>0 \tag{3.2}$$

the Riemann sphere is realized as the closure of the two half-planes $\operatorname{Re} w>0$ and $\operatorname{Re} w<0$. We will denote by Ω_1 and Ω_2 the components of $\overline{\pi_1^{-1}\mathbb{C}\setminus\mathbb{D}}$ in the half-planes $\operatorname{Re} w>0$ and $\operatorname{Re} w<0$, respectively. The general picture is represented in Fig. 1 (we keep the notation of Section 2).

Thus we have to solve the scalar Riemann–Hilbert problem on $\Gamma=\Gamma_1+\Gamma_2$ (see (2.7))

$$\phi_1^+ + wE\phi_2^+ = h\psi^- \tag{3.3}$$

where

$$\psi^-(w)=\begin{cases}\phi_1^-(\xi), & w\in\Omega_1^-,\\ \phi_2^-(\xi), & w\in\Omega_2^-,\end{cases} \tag{3.4}$$

$$h(w)=\begin{cases}1, & w\in\Gamma_1,\\ -wE(w), & w\in\Gamma_2,\end{cases} \tag{3.5}$$

and, as in Section 2, we identify the functions ϕ_i^+ and E in the complex plane with the "even" functions $\pi_1^*\phi_1^+$, π^*E they determine on Σ (see (2.6)). For example, noting that $\xi=-\alpha\frac{w^2+1}{w^2-1}$ by (3.2), we set

$$E(w)=\exp\left(-\alpha\frac{w^2+1}{w^2-1}\right).$$

To solve (3.3) our first task is to find a factorization for h relative to the contour Γ.

This is easily obtained by noting that

$$E(w)=e^{-\alpha}\exp\left(-\frac{\alpha}{w-1}\right)\exp\left(-\frac{\alpha}{w+1}\right).$$

Then for $w\in\Gamma_2$ we have

$$h(w)=h^-(w)h^+(w),$$

where

[2] The choice of a branch of the square root in the definition of ρ will be given in the computations below.

$$h^-(w) = -we^{-\alpha}\exp\left(-\frac{\alpha}{w-1}\right),$$
$$h^+(w) = \exp\left(\frac{\alpha}{w+1}\right).$$

For $w \in \Gamma_1$ we write

$$h(w) = 1 = \tilde{h}^-(w)h^+(w),$$

which yields

$$\tilde{h}^-(w) = \exp\left(-\frac{\alpha}{w+1}\right).$$

We summarize these results in the proposition that follows.

Proposition 3.1. *The function h defined in (3.5) has the following factorization relative to the contour Γ*

$$h(w) = h^-(w)h^+(w)$$

where

$$h^+(w) = \exp\left(\frac{\alpha}{w+1}\right), \quad w \in \Omega^+,$$
$$h^-(w) = \begin{cases} \exp(\frac{\alpha}{w+1}), & w \in \Omega_1^-, \\ -we^{-\alpha}\exp(-\frac{\alpha}{w-1}), & w \in \Omega_2^-. \end{cases}$$

We are now in a position to obtain a solution to the Riemann–Hilbert problem (3.3).
Using the above factorization of h we write Eq. (3.3) in the form

$$\left(w^{-1}\phi_1^+ + E\phi_2^+\right)/h^+ = h^- w^{-1}\psi^-.$$

Taking into account that the left-hand side has a pole at $w = 0$ we have

$$\left(w^{-1}\phi_1^+ + E\phi_2^+\right)/h^+ = h^- w^{-1}\psi^- = \gamma_0 + \gamma_1 w^{-1} \tag{3.6}$$

where γ_0, γ_1 are constants.
From (3.6) we get

$$\begin{aligned} \phi_1^+ &= \left(\gamma_0 h^+ w + \gamma_1 h^+\right)_{ev} = \gamma_0 h^+_{odd}\rho^2 + \gamma_1 h^+_{ev}, \\ \phi_2^+ &= E^{-1}\left(\gamma_0 h^+ w + \gamma_1 h^+\right)_{odd} = \gamma_0 E^{-1} h^+_{ev} + \gamma_1 E^{-1} h^+_{odd}, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \phi_1^- &= w\left(\gamma_0 + \gamma_1 w^{-1}\right)/h^-\big|_{\Omega_1^-} = (\gamma_0 w + \gamma_1)\exp\left(-\frac{\alpha}{w+1}\right), \\ \phi_2^- &= w\left(\gamma_0 + \gamma_1 w^{-1}\right)/h^-\big|_{\Omega_2^-} = -\left(\gamma_0 + \gamma_1 w^{-1}\right)e^{\alpha}\exp\left(\frac{\alpha}{w-1}\right). \end{aligned} \tag{3.8}$$

To obtain from the above relations two linearly independent solutions to our original Riemann–Hilbert problem, $G\phi^+ = \phi^-$, we introduce the normalizations: (i) $\phi_1^-(\infty) = 0$; (ii) $\phi_2^-(\infty) = 0$.
Noting that $\xi = \infty$ corresponds to $w = 1$ if $w \in \Omega_1^-$ and to $w = -1$ if $w \in \Omega_2^-$, we get

$$\phi_1^-(\infty) = 0 \quad \Rightarrow \quad \gamma_1 = -\gamma_0,$$
$$\phi_2^-(\infty) = 0 \quad \Rightarrow \quad \gamma_1 = \gamma_0.$$

We may now state the explicit factorization for the symbol given in (2.2). Note that, apart from the constant γ_0, which for convenience we take to be equal to 2, the factorization is unique (*i.e.*, canonical $D = I_2$ in (2.2)).

Proposition 3.2. *The symbol G given in (2.1) has a canonical factorization in $\mathscr{C}_\alpha$ $(0 < \alpha < 1)$ with factors given by $G_+^{-1} = [\phi_{ij}^+]$, $G_- = [\phi_{ij}^-]$, where*

$$\phi_{1j}^+ = \rho\left[h^+(\rho) - h^+(-\rho)\right] + (-1)^{j-1}\left[h^+(\rho) + h^+(-\rho)\right],$$
$$\phi_{2j}^+ = E^{-1}\left[h^+(\rho) + h^+(-\rho) + (-1)^{j-1}\rho^{-1}\left(h^+(\rho) - h^+(-\rho)\right)\right].$$

For the minus factor we have

$$\phi_{1j}^- = \left(\rho + (-1)^{j-1}\right)/h^-\big|_{\Omega_1^-}(\rho) = \left(\rho + (-1)^{j-1}\right)\exp\left(\frac{\alpha}{\rho+1}\right),$$
$$\phi_{2j}^- = \left(1 - \rho^{-1}(-1)^{j-1}\right)/h^-\big|_{\Omega_2^-}(-\rho) = \left(1 - \rho^{-1}(-1)^{j-1}\right)e^{\alpha}\rho\exp\left(\frac{\alpha}{\rho-1}\right).$$

4. Example 2 (elliptic curve)

In this section we study the factorization of a symbol of the form (2.1) where ρ is given by

$$\rho(z) = \sqrt{\left(1 - z^2\right)\left(1 - z^2\kappa^2\right)}, \tag{4.1}$$

where $0 < \kappa < 1$ and we choose the branch of the square root that is continuous on $\mathbb{C} \setminus \{z \in \mathbb{R} \mid 1 \leqslant |z| \leqslant 1/\kappa\}$ and satisfies $\rho(0) = 1$.

In this case, the Riemann surface to consider is the elliptic curve defined by

$$w^2 = p(z) = \left(1 - z^2\right)\left(1 - z^2\kappa^2\right), \tag{4.2}$$

which we denote by Σ. To satisfy the conditions of Definition 2.2 we let $\mathbb{D} \subset \mathbb{C}$ be a disk of radius greater than $1/\kappa$ and, as before, we denote $S^1 = \partial\mathbb{D}$.

According to the outline of the method given in Section 2 we have to study a scalar Riemann–Hilbert problem on $\Gamma \subset \Sigma$ of the form (2.5), *i.e.*,

$$\phi_1^+ + wE\phi_2^+ = h\psi^-, \tag{4.3}$$

in which ϕ_1^+, ϕ_2^+ and E are functions on $\overline{\mathbb{D}} \subset \mathbb{C}$ and $\Gamma = \pi_1^{-1}(S^1) = \Gamma_1 + \Gamma_2$. As explained in Section 2, the left-hand side of (4.3) is the general form of a meromorphic function on $\pi_1^{-1}(\mathbb{D})$ (see (2.6)).

The function h is given by

$$h(z, w) = \begin{cases} 1, & (z, w) \in \Gamma_1, \\ wE(z), & (z, w) \in \Gamma_2. \end{cases} \tag{4.4}$$

As in Section 3 we consider the case $E(z) := \exp(z)$.

To solve (4.3) we have to study the factorization of h relative to the contour Γ.

A general method to calculate the factorization of h is given in [6]. It involves computing $P_\Gamma^\pm \log h$ where $P_\Gamma^\pm$ are complementary projections given by (see [6, Theorem 3.4 and Appendix A])

$$\left(P_\Gamma^\pm\phi\right)(z, w) = \pm\frac{1}{4\pi i}\int\limits_{\partial\Omega^+} \frac{w+\eta}{\eta}\frac{\phi(\xi,\eta)}{\xi - z}\,d\xi, \quad (z, w) \in \Omega^\pm,$$

where $\Omega^- = \Omega_1^- \cup \Omega_2^-$, $\Omega^+ = \Sigma \setminus \overline{\Omega^-}$ so that $\Gamma = \partial\Omega^+$ (with the boundary orientation).

From (4.4) we see that $\log h = 0$ on Γ_1, so $P_\Gamma^\pm \log h$ is given by

$$\left(P_\Gamma^\pm \log h\right)(z, w) = \mp\frac{1}{4\pi i}\int\limits_{\partial\Omega_2^-} \frac{w+\eta}{\eta}\frac{\log h(\xi,\eta)}{\xi - z}\,d\xi, \tag{4.5}$$

where $\partial\Omega_2^-$ has the boundary orientation.

Instead of calculating the value of the above expressions we compute the simpler expressions $P_\Gamma^\pm\tilde{h}$, where

$$\tilde{h}(z, w) = \begin{cases} 1, & (z, w) \in \Gamma_1, \\ \exp(z), & (z, w) \in \Gamma_2 \end{cases} \tag{4.6}$$

from which we shall obtain the factorization of h. For $P_\Gamma^\pm \log\tilde{h}$ we get

$$\begin{aligned} \left(P_\Gamma^+ \log\tilde{h}\right)(z, w) &= \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}} \frac{\xi}{\xi - z}\,d\xi - \frac{w}{4\pi i}\int\limits_{\partial\Omega_2^-} \frac{\xi}{\xi - z}\frac{d\xi}{\eta} \\ &= \frac{z}{2}, \quad (z, w) \in \Omega^+, \end{aligned} \tag{4.7}$$

where we used the fact that $z \in \mathbb{D}$.

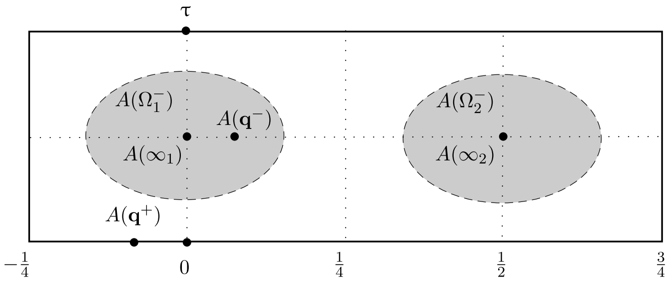

Fig. 2. Image of Ω^+ and Ω^- under the Abel–Jacobi map.

We now calculate $P_\Gamma^- \log\tilde{h}$. We have from (4.5)

$$\begin{aligned}\left(P_\Gamma^- \log\tilde{h}\right)(z,w) &= \frac{1}{4\pi i}\int\limits_{\partial\Omega_2^-}\frac{\eta+w}{\eta}\frac{\log\tilde{h}(\xi,\eta)}{\xi-z}\,d\xi, \quad (z,w)\in\Omega^- \\ &= \frac{1}{4\pi i}\int\limits_{\partial\Omega_2^-}\frac{\eta+w}{\eta}\frac{\xi}{\xi-z}\frac{d\xi}{\eta} \\ &= -\frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}}\frac{\xi}{\xi-z}\,d\xi + \frac{w}{4\pi i}\int\limits_{\partial\Omega_2^-}\frac{\xi}{\xi-z}\frac{d\xi}{\eta}\end{aligned} \tag{4.8}$$

where the first term is zero since $z\in\mathbb{C}\setminus\overline{\mathbb{D}}$. For the second term we note that the form $\frac{\xi}{(\xi-z)}\frac{d\xi}{\eta}$ has a pole at $\mathbf{p}_z := (z,-\chi w)\in\Omega_2^-$, where $\chi=1$ for $(z,w)\in\Omega_1^-$ and $\chi=-1$ for $(z,w)\in\Omega_2^-$. Using the local parameter ξ at $\mathbf{p}_z$, we obtain

$$\begin{aligned}\frac{w}{4\pi i}\int\limits_{\partial\Omega_2^-}\frac{\xi}{\xi-z}\frac{d\xi}{\eta} &= \frac{w}{2}\operatorname{Res}\left(\frac{\xi}{\xi-z}\frac{d\xi}{\eta};\mathbf{p}_z\right) \\ &= -\frac{z\chi}{2} \\ &= -\frac{w}{2z}+\frac{w-z^2\chi}{2z}.\end{aligned} \tag{4.9}$$

Note that the term $[w-z^2\chi]/2z$ in (4.9) is a minus function, *i.e.*, it extends holomorphically to Ω^-. Hence we will focus our attention on the first term.

In Appendix A we give the factorization of $\exp(-\frac{w}{2z})$. It has the form

$$g=\exp\left(-\frac{w}{z}\right)=g^- r g^+, \tag{4.10}$$

where $g^\pm$ are invertible in $C_\alpha^\pm(\Gamma)$ and r is a rational function on Σ. We will not need explicit expressions for the factors g^-, g^+ right now (but these are given in Appendix A below). The description of the rational factor r requires the complete elliptic integrals

$$\mathbf{K} := \int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-\kappa^2\xi^2)}}, \qquad \mathbf{K}' := \int_1^{1/\kappa}\frac{d\xi}{\sqrt{(\xi^2-1)(1-\kappa^2\xi^2)}},$$

and the Abel–Jacobi map $A\colon\Sigma\to\mathbb{C}/(\mathbb{Z}+\tau\cdot\mathbb{Z})$; $\mathbf{p}\mapsto\frac{1}{4\mathbf{K}}\int_{\mathbf{0}_1}^{\mathbf{p}}\frac{dz}{w}$ (where $\tau=\mathbf{K}'i/2\mathbf{K}$).

The rational factor r is determined by its divisor of zeros and poles as given in Appendix A: (i) r has simple zeros at ∞_1, $\mathbf{0}_1$ and simple poles at $\mathbf{q}^+\in\Omega^+$ and $\mathbf{q}^-\in\Omega^-$; (ii) $A(\mathbf{q}^+)=-1/4\mathbf{K}$ and $A(\mathbf{q}^-)=1/4\mathbf{K}+\tau/2$ (see Fig. 2).

Remark 4.1. It is easy to check that $A(\mathbf{p})\in\mathbb{R}$ [3] *iff* $\mathbf{p}=(z,w)$ with $z\in[-1,1]$, hence we always have $A^{-1}(-1/4\mathbf{K})\subset\Omega^+$. But in order to obtain $A^{-1}(1/4\mathbf{K}+\tau/2)\in\Omega^-$ we need to assume that the radius of $\mathbb{D}$ is close enough to $1/\kappa$, which is always possible since G has no singularities outside the disk of radius $1/\kappa$ (see Fig. 2).

[3] We will frequently abuse notation by writing expressions such as $A(\mathbf{p})=z$ instead of $A(\mathbf{p})\equiv z \bmod \mathbb{Z}+\tau\cdot\mathbb{Z}$.

We may now summarize the above results.

Proposition 4.2. *The function $\tilde{h}$ given by (4.6) has the following factorization relative to $\Gamma \subset \Sigma$*

$$\tilde{h} = \tilde{h}^+ r \tilde{h}^-,$$

where r is a rational function on Σ with simple zeros at ∞_1, $\mathbf{0}_1$ and simple poles at points $\mathbf{q}^+ \in \Omega^+$ and $\mathbf{q}^- \in \Omega^-$ such that $A(\mathbf{q}^+) = -1/4\mathbf{K}$, $A(\mathbf{q}^-) = 1/4\mathbf{K} + \tau/2$. The factors $\tilde{h}^\pm$ are given by

$$\tilde{h}^+(z,w) = \exp\left(\frac{z}{2}\right) g^+,$$

$$\tilde{h}^-(z,w) = \exp\left[\frac{w - z^2\chi}{z}\right] g^-,$$

where g^+, g^- are the invertible factors in (4.10) (see Definition A.1 for explicit formulas).

We are now in a position to study the solutions to Eq. (4.3). Note that from (4.4) and Proposition 4.2 we have

$$w^{-1}h = h_0 \tilde{h} = h_0 \tilde{h}^+ r \tilde{h}^-,$$

where h_0 is a minus function given by

$$h_0(\xi, w) = \begin{cases} w^{-1}, & (\xi, w) \in \Gamma_1, \\ 1, & (\xi, w) \in \Gamma_2. \end{cases}$$

Then we have, from (4.3)

$$\left(\tilde{h}^+\right)^{-1}\left(w^{-1}\phi_1^+ + E\phi_2^+\right) = r h_0 \tilde{h}^- \psi^- = R, \tag{4.11}$$

where R is a rational function on Σ.

To determine R we have to identify its divisor of zeros and poles. From Eq. (4.11) we see that R has:

- 4 simple poles: at $(\pm 1, 0), (\pm 1/\kappa, 0) \in \Omega^+$;
- 1 triple zero at ∞_1 coming from the factors h_0 and r in (4.11);
- 1 simple pole at $\mathbf{q}^-$ coming from the factor r in (4.11);
- 1 zero $\mathbf{p}_1$ that, by normalization, we can impose in Ω^+;
- 1 remaining zero $\mathbf{p}_2$ determined by Abel's relation:

$$A(\mathbf{p}_2) \equiv A(-1/\kappa, 0) + A(-1, 0) + A(1, 0) + A(1/\kappa, 0) + A\left(\mathbf{q}^-\right) - 3A(\infty_1) - A(\mathbf{p}_1).$$

From this we conclude that $R(\mathbf{p}) = \tilde{R}(A(\mathbf{p}))$ where $\tilde{R}(z)$ is the following elliptic function:

$$\tilde{R}(z) = \frac{\vartheta_1^3(z - \frac{\tau}{2})\vartheta_1(z - z_1)\vartheta_1(z - z_2)}{\vartheta_1(z - \frac{1}{4})\vartheta_1(z + \frac{1}{4})\vartheta_1(z - \frac{1}{4} - \frac{\tau}{2})\vartheta_1(z + \frac{1}{4} - \frac{\tau}{2})\vartheta_1(z - \frac{1}{4\mathbf{K}} - \frac{\tau}{2})};$$

here $\vartheta_1(z) = \vartheta_1(z;\tau)$ is the theta function of characteristic $(1, 1)$:

$$\vartheta_1(z;\tau) = \sum_{n \in \mathbb{Z}} \exp\left(\pi i\left(n + \frac{1}{2}\right)^2 \tau + 2\pi i\left(n + \frac{1}{2}\right)\left(z + \frac{1}{2}\right)\right),$$

$z_1 = A(\mathbf{p}_1)$ is a parameter that we fix, and z_2 is obtained imposing equality in Abel's relation: $z_2 = \frac{1}{4\mathbf{K}} - z_1$ (see *e.g.* [1]).

From (4.11) and Proposition 4.2 we now obtain

$$w^{-1}\phi_1^+ + E\phi_2^+ = \tilde{h}^+ R = \exp\left(\frac{z}{2}\right) g^+ R \quad \Rightarrow \quad \begin{cases} \phi_1^+ = p(z)\exp\left(\frac{z}{2}\right)\left(g^+ R\right)_{odd}, \\ \phi_2^+ = \exp\left(-\frac{z}{2}\right)\left(g^+ R\right)_{ev}. \end{cases}$$

Remark 4.3. As noted in Remark 2.10 we can use the above analysis to investigate the existence of noncanonical factorizations. This is done by examining the space of solutions to (4.11) imposing on R two additional zeros (at ∞_1 and ∞_2) and dropping the fourth condition above ($R(\mathbf{p}_1) = 0$). It follows that such a rational function must be zero (otherwise its divisor would not satisfy Abel's relation), hence the factorization is canonical.

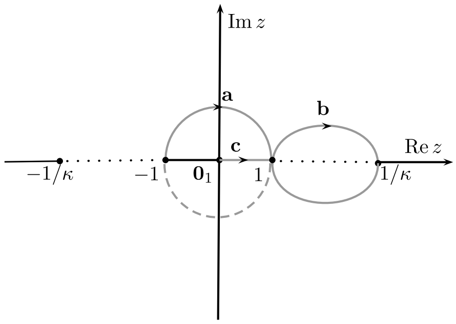

Fig. 3. Basis of $H_1(\Sigma;\mathbb{Z})$ and the path $\mathbf{c}$ from $\mathbf{0}_1$ to $\mathbf{1}$. The dashed line denotes an arc lying on the second sheet.

Remark 4.4. If in the example we replace $\rho(\xi)$ by $\varrho(\xi)=(\xi-\alpha)\rho(\xi)$, $\alpha\in\mathbb{D}$, it can be seen that the factorization is noncanonical. This is a consequence of the fact that ϱ introduces two poles in the left-hand side of (4.11) resulting in a rational function R that is allowed to have up to seven poles with only five prescribed zeros. From the Riemann–Roch theorem it follows that the space of such rational functions has dimension 2, hence the factorization is noncanonical.

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Appendix A. Factorization of $\exp(-\frac{w}{2z})$

In this appendix we describe the main steps needed to compute the factorization of the function $\exp(-\frac{w}{2z})$. The computation follows closely that of [6, Appendix B], where more details are given.

As in [6, Appendix B], we consider the basis $\{\mathbf{a},\mathbf{b}\}$ for the homology group $H_1(\Sigma;\mathbb{Z})$ depicted in Fig. 3. In this figure the solid lines denote paths lying on the first sheet and dashed line denotes an arc on the second sheet.[4]

Consider also the meromorphic forms

$$\gamma_+=\left(\kappa^2z^2-\frac{\mathbf{K}-\mathbf{E}}{\mathbf{K}}\right)\frac{dz}{w},\qquad \gamma_-=\left(z^{-2}-\frac{\mathbf{K}-\mathbf{E}}{\mathbf{K}}\right)\frac{dz}{w}$$

where $\mathbf{E}$ is the complete elliptic integral $\int_0^1\sqrt{(1-\kappa^2x^2)/(1-z^2)}\,dz$.

It can easily be checked that γ_+ is holomorphic on Ω^+ (with poles at ∞_1, ∞_2) and γ_- is holomorphic on Ω^- (with poles at $\mathbf{0}_1$, $\mathbf{0}_2$). A direct computation gives $d(\frac{w}{z})=\gamma_+-\gamma_-$ and

$$\int_{\mathbf{a}}\gamma_+=\int_{\mathbf{a}}\gamma_-=0,\qquad \int_{\mathbf{b}}\gamma_+=\int_{\mathbf{b}}\gamma_-=\frac{\pi i}{\mathbf{K}} \tag{A.1}$$

(see [6, Appendix B]).

We can now introduce the following definition.

Definition A.1. Define

$$g^+(\mathbf{p})=\exp\left(-\frac{1}{2}\int_{\mathbf{1}}^{\mathbf{p}}\gamma_+\right)\frac{\vartheta_1(A(\mathbf{p})-\frac{\tau}{2}-\frac{1}{4\mathbf{K}})}{\vartheta_1(A(\mathbf{p})-\frac{\tau}{2})},$$

$$g^-(\mathbf{p})=\exp\left(\frac{1}{2}\int_{\mathbf{1}}^{\mathbf{p}}\gamma_-\right)\frac{\vartheta_1(A(\mathbf{p})+\frac{1}{4\mathbf{K}})}{\vartheta_1(A(\mathbf{p}))}$$

where $\mathbf{1}=(1,0)\in\Sigma$ and the path used to compute $A(\mathbf{p})$ is obtained by composing the path used to compute the integrals on right-hand sides (the same for g^+ and g^-) with the path $\mathbf{c}: t\mapsto(t,+\sqrt{p(t)})\in\Sigma$, $t\in[0,1]$, from $\mathbf{0}_1$ to $\mathbf{1}$ (see Fig. 3).

Remark A.2. Using (A.1) and $\vartheta_1(u+n+m\tau)=\exp(-2\pi i(\frac{1}{2}m^2+mu))\vartheta_1(u)$ it is easily seen that the expressions above yield well-defined functions g^+, g^-, holomorphic on Ω^+ and Ω^-, respectively.

[4] The i-th sheet is the component of $\Sigma\setminus\{(z,\pm\sqrt{p(z)})\mid 1\leqslant|z|\leqslant 1/\kappa\}\cup\{\infty_1,\infty_2\}$ containing $\Omega_i^-\setminus\{\infty_i\}$.

From $d(\frac{w}{z}) = \gamma_+ - \gamma_-$ it follows that

$$\exp\left(-\frac{w}{2z}\right) = g^+ r g^-$$

where r is the rational function on Σ given by the following quotient of theta functions

$$r(\mathbf{p}) = \frac{\vartheta_1(A(\mathbf{p}) - \frac{\tau}{2})\vartheta_1(A(\mathbf{p}))}{\vartheta_1(A(\mathbf{p}) - \frac{\tau}{2} - \frac{1}{4\mathbf{K}})\vartheta_1(A(\mathbf{p}) + \frac{1}{4\mathbf{K}})}.$$

Denoting by $\mathbf{q}^+ = A^{-1}(-1/4\mathbf{K}) \in \Omega^+$, $\mathbf{q}^- = A^{-1}(1/4\mathbf{K} + \tau/2) \in \Omega^-$ and noting that $A(\mathbf{0}_1) = 0$, $A(\infty_1) = \tau/2$ it follows that r has simple zeros at ∞_1, $\mathbf{0}_1$ and simple poles at $\mathbf{q}^\pm$, as stated in Section 3 (see Fig. 2).

References

[1] N.I. Akhiezer, Elements of the Theory of Elliptic Functions, Transl. Math. Monogr., vol. 79, American Mathematical Society, Providence, RI, 1990, translated from the second Russian edition by H.H. McFaden.

[2] M.A. Bastos, A.F. dos Santos, Convolution operators on a finite interval with periodic kernel—Fredholm property and invertibility, Integral Equations Operator Theory 16 (2) (1993) 186–223.

[3] M.A. Bastos, A. Moura Santos, A.F. dos Santos, Wave diffraction by a strip grating: the two-straight line approach, Math. Nachr. 269/270 (2004) 39–58.

[4] C. Câmara, A.F dos Santos, P.F. dos Santos, Lax equations, factorization and Riemann–Hilbert problems, Port. Math. 64 (2006) 509–533.

[5] M.C. Câmara, A.F. dos Santos, M.A. Bastos, Generalized factorization for Daniele–Khrapkov matrix functions—explicit formulas, J. Math. Anal. Appl. 190 (2) (1995) 295–328.

[6] M.C. Câmara, A.F. dos Santos, Pedro F. dos Santos, Matrix Riemann–Hilbert problems and factorization on Riemann surfaces, J. Funct. Anal. 255 (1) (2008) 228–254.

[7] Kevin F. Clancey, Israel Gohberg, Factorization of Matrix Functions and Singular Integral Operators, Oper. Theory Adv. Appl., vol. 3, Birkhäuser Verlag, Basel, 1981.

[8] António F. dos Santos, Pedro F. dos Santos, Lax equations, singularities and Riemann–Hilbert problems, Math. Phys. Anal. Geom. 15 (3) (2012) 203–229.

[9] H. Flaschka, On the Toda lattice. II. Inverse-scattering solution, Progr. Theoret. Phys. 51 (1974) 703–716.

[10] N.J. Hitchin, G.B. Segal, R.S. Ward, Integrable systems, in: Twistors, Loop Groups, and Riemann Surfaces, Lectures from the Instructional Conference held at the University of Oxford, Oxford, September 1997, in: Oxf. Grad. Texts Math., vol. 4, The Clarendon Press Oxford University Press, New York, 1999.

[11] Alexander R. Its, The Riemann–Hilbert problem and integrable systems, Notices Amer. Math. Soc. 50 (11) (2003) 1389–1400.

[12] E. Meister, F.-O. Speck, Modern Wiener–Hopf methods in diffraction theory, in: Ordinary and Partial Differential Equations, vol. II, Dundee, 1988, in: Pitman Res. Notes Math. Ser., vol. 216, Longman Sci. Tech., Harlow, 1988, pp. 130–171.

[13] Walter Rudin, Real and Complex Analysis, second ed., McGraw–Hill Ser. Higher Math., McGraw–Hill Book Co., New York, 1974.

[14] Graeme Segal, George Wilson, Loop groups and equations of KdV type, Publ. Math. Inst. Hautes Études Sci. (61) (1985) 5–65.

[15] M.A. Semenov-Tian-Shansky, Integrable systems and factorization problems, in: Factorization and Integrable Systems, Faro, 2000, in: Oper. Theory Adv. Appl., vol. 141, Birkhäuser, Basel, 2003, pp. 155–218.

[16] George Springer, Introduction to Riemann Surfaces, Addison–Wesley Publishing Company, Inc., Reading, MA, 1957.