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Order of Selberg's and Ruelle's zeta functions for compact even-dimensional locally symmetric spaces

Muharem Avdispahić, Dženan Gušić *

University of Sarajevo, Department of Mathematics, Zmaja od Bosne 35, 71000 Sarajevo, Bosnia and Herzegovina

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ABSTRACT

We prove that Selberg's and Ruelle's zeta functions considered by U. Bunke and M. Olbrich can be represented as quotients of two entire functions of order not larger than the dimension of the underlaying compact, even-dimensional, locally symmetric space.

1. Introduction

Let Y be a compact, n-dimensional (n even), locally symmetric Riemannian manifold with negative sectional curvature given by $Y = \Gamma\backslash G/K = \Gamma\backslash X$, where G is a connected semisimple Lie group of real rank one, K is a maximal compact subgroup of G and Γ is a discrete co-compact torsion free subgroup of G.

We require G to be linear in order to have complexification available.

In [3], Bunke and Olbrich studied the zeta functions of Selberg and Ruelle associated with a locally homogeneous vector bundle of the unit sphere bundle of Y. It is well known that the classical Selberg zeta function is an entire function of order two (see, [6]). The main purpose of this paper is to give an analogous result for the zeta functions considered in [3].

2. Preliminaries

In the sequel we follow the notation of [3].

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$ and M the centralizer of $\mathfrak{a}$ in K with Lie algebra $\mathfrak{m}$. We normalize the $\mathrm{Ad}(G)$-invariant inner product $(.,.)$ on $\mathfrak{g}$ to restrict to the metric on $\mathfrak{p}$. Let $SX = G/M$ be the unit sphere bundle of X. Hence $SY = \Gamma\backslash G/M$.

Let $\Phi(\mathfrak{g}, \mathfrak{a})$ be the root system and $W = W(\mathfrak{g}, \mathfrak{a}) \cong \mathbb{Z}_2$ its Weyl group. Fix a system of positive roots $\Phi^+(\mathfrak{g}, \mathfrak{a}) \subset \Phi(\mathfrak{g}, \mathfrak{a})$. Let

$$\mathfrak{n} = \sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{a})} \mathfrak{n}_\alpha$$

* Corresponding author.
E-mail addresses: mavdispa@pmf.unsa.ba (M. Avdispahić), dzenang@pmf.unsa.ba (Dž. Gušić).

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be the sum of the root spaces corresponding to elements of $\Phi^+(\mathfrak{g},\mathfrak{a})$. The decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}\oplus\mathfrak{n}$ corresponds to the Iwasawa decomposition of the group $G=KAN$. Define $\rho\in\mathfrak{a}^*_{\mathbb{C}}$ by

$$\rho=\frac{1}{2}\sum_{\alpha\in\Phi^+(\mathfrak{g},\mathfrak{a})}\dim(\mathfrak{n}_\alpha)\alpha.$$

The positive Weyl chamber $\mathfrak{a}^+$ is the half line in $\mathfrak{a}$ on which the positive roots take positive values. Let $A^+=\exp(a^+)\subset A$.

The symmetric space X has a compact dual space $X_d=G_d/K$, where G_d is the analytic subgroup of $GL(n,\mathbb{C})$ corresponding to $\mathfrak{g}_d=\mathfrak{k}\oplus\mathfrak{p}_d$, $\mathfrak{p}_d=\mathrm{i}\mathfrak{p}$. We normalize the metric on X_d in such a way that the multiplication by i induces an isometry between $\mathfrak{p}$ and $\mathfrak{p}_d$.

Let $i^*:R(K)\to R(M)$ be the restriction map induced by the embedding $i:M\hookrightarrow K$, where $R(K)$ and $R(M)$ are the representation rings over $\mathbb{Z}$ of K and M, respectively.

Since n is even, every $\sigma\in\hat{M}$ is invariant under the action of the Weyl group W (see, [3, p. 27]). Let $\sigma\in\hat{M}$. We choose $\gamma\in R(K)$ such that $i^*(\gamma)=\sigma$ and represent it by $\Sigma a_i\gamma_i$, $a_i\in\mathbb{Z}$, $\gamma_i\in\hat{K}$. Set

$$V_\gamma^\pm=\sum_{\operatorname{sign}(a_i)=\pm1}\sum_{m=1}^{|a_i|}V_{\gamma_i},$$

where V_{γ_i} is the representation space of γ_i. Define $V(\gamma)^\pm=G\times_K V_\gamma^\pm$ and $V_d(\gamma)^\pm=G_d\times_K V_\gamma^\pm$. To γ we associate $\mathbb{Z}_2$-graded homogeneous vector bundles $V(\gamma)=V(\gamma)^+\oplus V(\gamma)^-$ and $V_d(\gamma)=V_d(\gamma)^+\oplus V_d(\gamma)^-$ on X and X_d, respectively. Let

$$V_{Y,\chi}(\gamma)=\Gamma\backslash\big(V_\chi\otimes V(\gamma)\big)$$

be a $\mathbb{Z}_2$-graded vector bundle on Y, where (χ,V_χ) is a finite-dimensional unitary representation of Γ.

Reasoning as in the beginning of Subsection 1.1.2 in [3], we choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{m}$ and a system of positive roots $\Phi^+(\mathfrak{m}_{\mathbb{C}},\mathfrak{t})$. Then, $\rho_{\mathfrak{m}}\in\mathrm{i}\mathfrak{t}^*$, where

$$\rho_{\mathfrak{m}}=\frac{1}{2}\sum_{\alpha\in\Phi^+(\mathfrak{m}_{\mathbb{C}},\mathfrak{t})}\alpha.$$

Let $\mu_\sigma\in\mathrm{i}\mathfrak{t}^*$ be the highest weight of σ. Set

$$c(\sigma)=|\rho|^2+|\rho_{\mathfrak{m}}|^2-|\mu_\sigma+\rho_{\mathfrak{m}}|^2,$$

where the norms are induced by the complex bilinear extension to $\mathfrak{g}_{\mathbb{C}}$ of the inner product $(.,.)$. Finally, we introduce the operators (see, [3, p. 28])

$$A_d(\gamma,\sigma)^2=\Omega+c(\sigma):C^\infty\big(X_d,V_d(\gamma)\big)\to C^\infty\big(X_d,V_d(\gamma)\big),$$
$$A_{Y,\chi}(\gamma,\sigma)^2=-\Omega-c(\sigma):C^\infty\big(Y,V_{Y,\chi}(\gamma)\big)\to C^\infty\big(Y,V_{Y,\chi}(\gamma)\big),$$

Ω being the Casimir element of the complex universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$.

Let $m_\chi(s,\gamma,\sigma)=\operatorname{Tr}E_{A_{Y,\chi}(\gamma,\sigma)}(\{s\})$, $m_d(s,\gamma,\sigma)=\operatorname{Tr}E_{A_d(\gamma,\sigma)}(\{s\})$, where $E_A(.)$ denotes the family of spectral projections of a normal operator A.

Now, we choose a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{m}$. Then, $\mathfrak{h}=\mathfrak{t}_{\mathbb{C}}\oplus\mathfrak{a}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let $\Phi^+(\mathfrak{g}_{\mathbb{C}},\mathfrak{h})$ be a positive root system having the property that, for $\alpha\in\Phi(\mathfrak{g}_{\mathbb{C}},\mathfrak{h})$, $\alpha_{|\mathfrak{a}}\in\Phi^+(\mathfrak{g},\mathfrak{a})$ implies $\alpha\in\Phi^+(\mathfrak{g}_{\mathbb{C}},\mathfrak{h})$. Let

$$\delta=\frac{1}{2}\sum_{\Phi^+(\mathfrak{g}_{\mathbb{C}},\mathfrak{h})}\alpha.$$

We set $\rho_{\mathfrak{m}}=\delta-\rho$. Define the root vector $H_\alpha\in\mathfrak{a}$ for $\alpha\in\Phi^+(\mathfrak{g},\mathfrak{a})$ by

$$\lambda(H_\alpha)=\frac{(\lambda,\alpha)}{(\alpha,\alpha)},$$

where $\lambda\in\mathfrak{a}^*$.

For $\alpha\in\Phi^+(\mathfrak{g},\mathfrak{a})$, introduce $\varepsilon_\alpha(\sigma)\in\{0,\frac{1}{2}\}$ by

$$e^{2\pi i\varepsilon_\alpha(\sigma)}=\sigma\big(e^{2\pi iH_\alpha}\big)\in\{\pm1\}.$$

According to [3, p. 47], the root system $\Phi^+(\mathfrak{g},\mathfrak{a})$ is of the form $\Phi^+(\mathfrak{g},\mathfrak{a})=\{\alpha\}$ or $\Phi^+(\mathfrak{g},\mathfrak{a})=\{\frac{\alpha}{2},\alpha\}$ for the long root α. Let α be the long root in $\Phi^+(\mathfrak{g},\mathfrak{a})$. We set $T=|\alpha|$. For $\sigma\in\hat{M}$, $\epsilon_\sigma\in\{0,\frac{1}{2}\}$ is given by

$$\epsilon_\sigma\equiv\frac{|\rho|}{T}+\varepsilon_\alpha(\sigma)\bmod\mathbb{Z}.$$

We define the lattice $L(\sigma)\subset\mathbb{R}\cong\mathfrak{a}^*$ by $L(\sigma)=T(\epsilon_\sigma+\mathbb{Z})$. Finally, for $\lambda\in\mathfrak{a}^*_{\mathbb{C}}\cong\mathbb{C}$ we set

$$P_\sigma(\lambda) = \prod_{\beta \in \Phi^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h})} \frac{(\lambda + \mu_\sigma + \rho_\mathfrak{m}, \beta)}{(\delta, \beta)}.$$

Since n is even, there exists a σ-admissible $\gamma \in R(K)$ for every $\sigma \in \hat{M}$ (see, [3, p. 49, Lemma 1.18]). Here, $\gamma \in R(K)$ is called σ-admissible if $i^*(\gamma) = \sigma$ and $m_d(s, \gamma, \sigma) = P_\sigma(s)$ for all $0 \leqslant s \in L(\sigma)$.

3. Zeta functions and the geodesic flow

Since $\Gamma \subset G$ is co-compact and torsion free, there are only two types of conjugacy classes – the class of the identity $1 \in \Gamma$ and classes of hyperbolic elements.

Let $g \in G$ be hyperbolic. Then there is an Iwasawa decomposition $G = NAK$ such that $g = am \in A^+M$. Following [3, p. 59], we define

$$l(g) = \left|\log(a)\right|.$$

Let Γ_h, resp. $\mathrm{P}\Gamma_\mathrm{h}$ denote the set of the Γ-conjugacy classes of hyperbolic resp. primitive hyperbolic elements in Γ.

Let φ be the geodesic flow on SY determined by the metric of Y. In the representation $SY = \Gamma \backslash G/M$ it is given by

$$\varphi : \mathbb{R} \times SY \ni (t, \Gamma gM) \to \Gamma \exp(-tH)M \in SY,$$

where H is the unit vector in $\mathfrak{a}^+$. If $V_\chi(\sigma) = \Gamma \backslash (G \times_M V_\sigma \otimes V_\chi)$ is the vector bundle corresponding to finite-dimensional unitary representations (σ, V_σ) of M and (χ, V_χ) of Γ, then we define a lift $\varphi_{\chi,\sigma}$ of φ to $V_\chi(\sigma)$ by (see, [3, p. 95])

$$\varphi_{\chi,\sigma} : \mathbb{R} \times V_\chi(\sigma) \ni \bigl(t, [g, v \otimes w]\bigr) \to \bigl[g \exp(-tH), v \otimes w\bigr] \in V_\chi(\sigma).$$

For $\mathrm{Re}\,(s) > 2\rho$, the Ruelle zeta function for the flow $\varphi_{\chi,\sigma}$ is the infinite product

$$Z_{R,\chi}(s, \sigma) = \prod_{\gamma_0 \in \mathrm{P}\Gamma_\mathrm{h}} \det\bigl(1 - \bigl(\sigma(m) \otimes \chi(\gamma_0)\bigr)e^{-sl(\gamma_0)}\bigr)^{(-1)^{n-1}}.$$

The Selberg zeta function for the flow $\varphi_{\chi,\sigma}$ is given by

$$Z_{S,\chi}(s, \sigma) = \prod_{\gamma_0 \in \mathrm{P}\Gamma_\mathrm{h}} \prod_{k=0}^{+\infty} \det\bigl(1 - \bigl(\sigma(m) \otimes \chi(\gamma_0) \otimes S^k\bigl(\mathrm{Ad}\,(ma)_{\bar{\mathfrak{n}}}\bigr)\bigr)e^{-(s+\rho)l(\gamma_0)}\bigr),$$

for $\mathrm{Re}\,(s) > \rho$, where S^k denotes the k-th symmetric power of an endomorphism, $\bar{\mathfrak{n}} = \theta\mathfrak{n}$ is the sum of negative root spaces of $\mathfrak{a}$ as usual, and θ is the Cartan involution of $\mathfrak{g}$.

Let $\mathfrak{n}_\mathbb{C}$ be the complexification of $\mathfrak{n}$. For $\lambda \in \mathbb{C} \cong \mathfrak{a}^*_\mathbb{C}$ let $\mathbb{C}_\lambda$ denote the one-dimensional representation of A given by $A \ni a \to a^\lambda$. Let $p \geqslant 0$. There exist sets

$$I_p = \bigl\{(\tau, \lambda) \bigm| \tau \in \hat{M},\ \lambda \in \mathbb{R}\bigr\}$$

such that $\Lambda^p \mathfrak{n}_\mathbb{C}$ (as a representation of MA) decomposes with respect to MA as

$$\Lambda^p \mathfrak{n}_\mathbb{C} = \sum_{(\tau,\lambda) \in I_p} V_\tau \otimes \mathbb{C}_\lambda,$$

where V_τ is the space of the representation τ. Bunke and Olbrich proved that the Ruelle zeta function $Z_{R,\chi}(s, \sigma)$ has the following representation (see, [3, p. 99, Prop. 3.4])

$$Z_{R,\chi}(s, \sigma) = \prod_{p=0}^{n-1} \prod_{(\tau,\lambda) \in I_p} Z_{S,\chi}(s + \rho - \lambda, \tau \otimes \sigma)^{(-1)^p}. \tag{3.1}$$

Let $d_Y = -(-1)^{\frac{n}{2}}$. The following theorem holds true (see, [3, p. 113, Th. 3.15]).

Theorem A. *The Selberg zeta function $Z_{S,\chi}(s, \sigma)$ has a meromorphic continuation to all of $\mathbb{C}$. If γ is σ-admissible, then the singularities (zeros and poles) of $Z_{S,\chi}(s, \sigma)$ are the following ones:*

(1) *at $\pm is$ of order $m_\chi(s, \gamma, \sigma)$ if $s \neq 0$ is an eigenvalue of $A_{Y,\chi}(\gamma, \sigma)$,*
(2) *at $s = 0$ of order $2m_\chi(0, \gamma, \sigma)$ if 0 is an eigenvalue of $A_{Y,\chi}(\gamma, \sigma)$,*
(3) *at $-s$, $s \in T(\mathbb{N} - \epsilon_\sigma)$ of order $2\frac{d_Y \dim(\chi)\,\mathrm{vol}(Y)}{\mathrm{vol}(X_d)} m_d(s, \gamma, \sigma)$.*
Then $s > 0$ is an eigenvalue of $A_d(\gamma, \sigma)$.

If two such points coincide, then the orders add up.

4. Main result

The main result of this paper is the following theorem.

Theorem 4.1. *If γ is σ-admissible, then*

(a) $|Z_{S,\chi}(s,\sigma)| = O(e^{|s|^{n+\alpha}})$, $\alpha > 0$, $s \to \infty$,
(b) *there exist entire functions $Z_S^1(s)$, $Z_S^2(s)$ of order at most n such that*

$$Z_{S,\chi}(s,\sigma) = \frac{Z_S^1(s)}{Z_S^2(s)},$$

where the zeros of $Z_S^1(s)$ correspond to the zeros of $Z_{S,\chi}(s,\sigma)$ and the zeros of $Z_S^2(s)$ correspond to the poles of $Z_{S,\chi}(s,\sigma)$. The orders of the zeros of $Z_S^1(s)$ resp. $Z_S^2(s)$ equal the orders of the corresponding zeros resp. poles of $Z_{S,\chi}(s,\sigma)$.

Proof. (a) By [3, p. 118, Th. 3.19], $Z_{S,\chi}(s,\sigma)$ has the representation

$$Z_{S,\chi}(s,\sigma) = \det\bigl(A_{Y,\chi}(\gamma,\sigma)^2 + s^2\bigr)\det\bigl(A_d(\gamma,\sigma)+s\bigr)^{-\frac{2\dim(\chi)\chi(Y)}{\chi(X_d)}} \cdot \exp\left(\frac{\dim(\chi)\chi(Y)}{\chi(X_d)}\sum_{m=1}^{\frac{n}{2}} c_{-m}\frac{s^{2m}}{m!}\left(\sum_{r=1}^{m-1}\frac{1}{r} - 2\sum_{r=1}^{2m-1}\frac{1}{r}\right)\right), \tag{4.1}$$

where the coefficients c_k are defined by the asymptotic expansion

$$\operatorname{Tr} e^{-tA_d(\gamma,\sigma)^2} \overset{t\to 0}{\sim} \sum_{k=-\frac{n}{2}}^{\infty} c_k t^k.$$

Furthermore (see, [3, pp. 120–123]),

$$\log\det\bigl(A_{Y,\chi}(\gamma,\sigma)^2 + s^2\bigr) \overset{s\to\infty}{\sim} \sum_{k=\frac{n}{2}}^{1} e_{-k}\left(\log(s^2) - \left(\sum_{r=1}^{k}\frac{1}{r}\right)\right)(-1)^k\frac{s^{2k}}{k!} + e_0\log(s^2) + o(1), \tag{4.2}$$

$$\log\det\bigl(A_d(\gamma,\sigma)+s\bigr) \overset{s\to\infty}{\sim} \sum_{k=n,\ even}^{1} d_{-k}\left(\log(s) - \left(\sum_{r=1}^{k}\frac{1}{r}\right)\right)(-1)^k\frac{s^{k}}{k!} + d_0\log(s) + o(1), \tag{4.3}$$

where e_k resp. d_k are defined by

$$\operatorname{Tr} e^{-tA_{Y,\chi}(\gamma,\sigma)^2} \overset{t\to 0}{\sim} \sum_{k=-\frac{n}{2}}^{\infty} e_k t^k$$

resp.

$$\operatorname{Tr} e^{-tA_d(\gamma,\sigma)} \overset{t\to 0}{\sim} \sum_{k=-n}^{\infty} d_k t^k.$$

Substituting (4.2) and (4.3) into (4.1) we obtain

$$\begin{aligned}\bigl|Z_{S,\chi}(s,\sigma)\bigr| &= \left|\exp\left(\sum_{k=\frac{n}{2}}^{1} E^1(k)\log(s^2)s^{2k}\right)\right|\left|\exp\left(\sum_{k=\frac{n}{2}}^{1} E^2(k)s^{2k}\right)\right| \\ &\quad\cdot\bigl|\exp\bigl(e_0\log(s^2)\bigr)\bigr|\bigl|\exp\bigl(o(1)\bigr)\bigr|\left|\exp\left(\sum_{k=n,\ even}^{1} D^1(k)\log(s)s^k\right)\right| \\ &\quad\cdot\left|\exp\left(\sum_{k=n,\ even}^{1} D^2(k)s^k\right)\right|\bigl|\exp\bigl(D^3\log(s)\bigr)\bigr|\bigl|\exp\bigl(o(1)\bigr)\bigr| \\ &\quad\cdot\left|\exp\left(\sum_{m=1}^{\frac{n}{2}} C^1(m)s^{2m}\right)\right| = |E_1||E_2||E_3||E_4||D_1||D_2||D_3||D_4||C_1|,\end{aligned} \tag{4.4}$$

as $s \to \infty$, where

$$E^1(k) = e_{-k}\frac{(-1)^k}{k!}, \qquad E^2(k) = -e_{-k}\frac{(-1)^k}{k!}\left(\sum_{r=1}^{k}\frac{1}{r}\right),$$

$$D^1(k) = \frac{-2\dim(\chi)\chi(Y)}{\chi(X_d)}d_{-k}\frac{(-1)^k}{k!},$$

$$D^2(k) = \frac{2\dim(\chi)\chi(Y)}{\chi(X_d)}d_{-k}\frac{(-1)^k}{k!}\left(\sum_{r=1}^{k}\frac{1}{r}\right),$$

$$D^3 = \frac{-2\dim(\chi)\chi(Y)}{\chi(X_d)}d_0$$

and

$$C^1(m) = \frac{\dim(\chi)\chi(Y)}{\chi(X_d)}\frac{c_{-m}}{m!}\left(\sum_{r=1}^{m-1}\frac{1}{r} - 2\sum_{r=1}^{2m-1}\frac{1}{r}\right).$$

Let $l \in \mathbb{N}$, $a, z \in \mathbb{C}$. Noting that

$$\left|\exp\left(az^l\log\left(z^2\right)\right)\right| = \exp\left(O\left(|z|^{l+\alpha}\right)\right), \quad \alpha > 0,$$
$$\left|\exp\left(az^l\right)\right| = \exp\left(O\left(|z|^l\right)\right),$$
$$\left|\exp\left(a\log\left(z^2\right)\right)\right| = \exp\left(O\left(|z|^\alpha\right)\right), \quad \alpha > 0,$$
$$\left|\exp\left(o(1)\right)\right| = \exp\left(O(1)\right),$$
$$\left|\exp\left(az^l\log(z)\right)\right| = \exp\left(O\left(|z|^{l+\alpha}\right)\right), \quad \alpha > 0$$

and

$$\left|\exp\left(a\log(z)\right)\right| = \exp\left(O\left(|z|^\alpha\right)\right), \quad \alpha > 0,$$

one easily concludes that $|E_1|$ is $\exp(O(|s|^{n+\alpha}))$, $\alpha > 0$, $|E_2|$, $|D_2|$ and $|C_1|$ are $\exp(O(|s|^n))$, $|E_3|$ is $\exp(O(|s|^\alpha))$, $\alpha > 0$, $|E_4|$ and $|D_4|$ are $\exp(O(1))$ and $|D_3|$ is $\exp(O(|s|^\alpha))$, $\alpha > 0$. Substituting these estimates into (4.4) we obtain the assertion.

(b) Let $N_1(r) = \#\{s \in \operatorname{spec} A_{Y,\chi}(\gamma,\sigma) \mid |s| \leqslant r\}$. By the Weyl asymptotic law (see, [3, p. 66])

$$N_1(r) \sim C_1 r^n, \tag{4.5}$$

as $r \to +\infty$.

The points $s \in T(\mathbb{N} - \epsilon_\sigma)$ are eigenvalues of $A_d(\gamma,\sigma)$ with multiplicity $m_d(s,\gamma,\sigma)$. By [3, p. 109], $A_d(\gamma,\sigma)$ may have more eigenvalues, but the weighted multiplicities of these additional eigenvalues are zero. Since $s > 0$ for $s \in T(\mathbb{N} - \epsilon_\sigma)$, we let $N_2(r) = \#\{s \in \operatorname{spec} A_d(\gamma,\sigma) \mid s \leqslant r\}$. Recall that $A_d(\gamma,\sigma)$ is elliptic and of second order. Hence, reasoning as in [4, p. 21], we obtain the estimate

$$N_2(r) \sim C_2 r^n, \tag{4.6}$$

as $r \to +\infty$.

Let S_1, S_2 denote respectively the sets consisting of the singularities of $Z_{S,\chi}(s,\sigma)$ appearing in (1) and (3) of Theorem A. By (4.5), we have

$$\sum_{s\in S_1}|s|^{-(n+\varepsilon)} = \sum_{\substack{s\in S_1\\0<|s|<1}}|s|^{-(n+\varepsilon)} + \sum_{\substack{s\in S_1\\|s|\geqslant 1}}|s|^{-(n+\varepsilon)} = O(1) + \int_1^{+\infty} t^{-(n+\varepsilon)}\,dN_1(t) = O\left(\int_1^{+\infty} t^{-(1+\varepsilon)}\,dt\right) = O(1), \tag{4.7}$$

for any $\varepsilon > 0$. Similarly, by (4.6), we have

$$\begin{aligned}\sum_{s\in S_2}|s|^{-(n+\varepsilon)} &= \sum_{s\in T(\mathbb{N}-\epsilon_\sigma)}\left|2\frac{d_Y\dim(\chi)\operatorname{vol}(Y)}{\operatorname{vol}(X_d)}m_d(s,\gamma,\sigma)\right||s|^{-(n+\varepsilon)}\\
&= \left|2\frac{d_Y\dim(\chi)\operatorname{vol}(Y)}{\operatorname{vol}(X_d)}\right|\sum_{s\in T(\mathbb{N}-\epsilon_\sigma)}\left|m_d(s,\gamma,\sigma)\right||s|^{-(n+\varepsilon)}\\
&= O\left(\int_{T(1-\epsilon_\sigma)}^{+\infty} t^{-(n+\varepsilon)}\,dN_2(t)\right) = O\left(\int_{T(1-\epsilon_\sigma)}^{+\infty} t^{-(1+\varepsilon)}\,dt\right) = O(1),\end{aligned} \tag{4.8}$$

for any $\varepsilon > 0$. By (4.7) and (4.8) we get

$$\sum_{s\in S\setminus\{0\}} |s|^{-(n+\varepsilon)} = \sum_{s\in S_1} |s|^{-(n+\varepsilon)} + \sum_{s\in S_2} |s|^{-(n+\varepsilon)} < \infty, \tag{4.9}$$

where S denotes the set of singularities of $Z_{S,\chi}(s,\sigma)$.

Let R_1 resp. R_2 denote the sets of zeros resp. poles of $Z_{S,\chi}(s,\sigma)$. For simplicity, the point $s=0$ will be considered separately. Assume that $0 \notin R_i$, $i=1,2$. Put $m_0 = 2m_\chi(0,\gamma,\sigma)$.

It follows from (4.9) that

$$\sum_{s\in R_i} |s|^{-(n+\varepsilon)} < \infty, \tag{4.10}$$

for $i=1,2$ and for any $\varepsilon > 0$.

Let ρ_1^i resp. p_i denote the convergence exponent resp. the genus of the set R_i for $i=1,2$ (see, [2, p. 14]). By (4.10), ρ_1^i, $p_i \leqslant n$ for $i=1,2$. Note that (4.10) does not necessarily imply that $\rho_1^1 = \rho_1^2 = n$. However, taking into account (4.9) as well as (4.7) and (4.8), we conclude that (4.10) implies that there is $i \in \{1,2\}$ such that $\rho_1^i = n$ and hence $p_i = n$. We may assume, without a loss of generality, that $\rho_1^1 = p_1 = n$. Hence, by [2, p. 19, Th. 2.6.5.], (see also, [5, p. 93]), the following canonical product over R_1 is an entire function of order n over $\mathbb{C}$,

$$W_1(s) = \prod_{z\in R_1} E\left(\frac{s}{z}, n\right),$$

where,

$$E(u,k) = (1-u)\exp\left(u + \frac{u^2}{2} + \cdots + \frac{u^k}{k}\right).$$

Similarly,

$$W_2(s) = \prod_{z\in R_2} E\left(\frac{s}{z}, p_2\right)$$

is an entire function of order ρ_1^2 over $\mathbb{C}$.

Now, we see that $Z_{S,\chi}(s,\sigma)W_1(s)^{-1}W_2(s)s^{-m_0}$ is an entire function and has no zeros over $\mathbb{C}$. Hence, by Hadamard's factorization theorem there exists a polynomial $g(s)$ such that

$$Z_{S,\chi}(s,\sigma)W_1(s)^{-1}W_2(s)s^{-m_0} = e^{g(s)}$$

for $s \in \mathbb{C}$. By taking logarithms of both sides we obtain

$$g(s) = \log Z_{S,\chi}(s,\sigma) + \log W_2(s) - \log W_1(s) - m_0 \log s.$$

Differentiating $n+1$ times and having in mind that the logarithmic derivative of $Z_{S,\chi}(s,\sigma)$ is given by a Dirichlet series absolutely convergent for $\mathrm{Re}(s) \gg 0$ (see, [3]), we conclude that

$$\lim_{|s|\to+\infty} g^{(n+1)}(s) = 0.$$

Therefore, the degree of $g(s)$ is at most n. Now, the assertion follows from the representation

$$Z_{S,\chi}(s,\sigma) = s^{m_0} e^{g(s)} \frac{W_1(s)}{W_2(s)}.$$

This completes the proof. $\square$

Corollary 4.2. *A meromorphic extension over $\mathbb{C}$ of the Ruelle zeta function $Z_{R,\chi}(s,\sigma)$ can be expressed as*

$$Z_{R,\chi}(s,\sigma) = \frac{Z_R^1(s)}{Z_R^2(s)},$$

where $Z_R^1(s)$, $Z_R^2(s)$ are entire functions of order at most n over $\mathbb{C}$.

Proof. An immediate consequence of the formula (3.1) and Theorem 4.1(b). $\square$

Remark 4.3. It is well known that in case of dimensions larger than 3, a meromorphic extension of the Ruelle zeta function can be used to obtain error terms in the prime geodesic theorem that are not achievable by the Selberg zeta function approach (see, [1,5]).

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