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Representation of the inverse of a frame multiplier ☆

P. Balazs, D.T. Stoeva *

Acoustics Research Institute, Wohllebengasse 12-14, Vienna A-1040, Austria

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ABSTRACT

Certain mathematical objects appear in a lot of scientific disciplines, like physics, signal processing and, naturally, mathematics. In a general setting they can be described as frame multipliers, consisting of analysis, multiplication by a fixed sequence (called the symbol), and synthesis. In this paper we show a surprising result about the inverse of such operators, if any, as well as new results about a core concept of frame theory, dual frames. We show that for semi-normalized symbols, the inverse of any invertible frame multiplier can always be represented as a frame multiplier with the reciprocal symbol and dual frames of the given ones. Furthermore, one of those dual frames is uniquely determined and the other one can be arbitrarily chosen. We investigate sufficient conditions for the special case, when both dual frames can be chosen to be the canonical duals. In connection to the above, we show that the set of dual frames determines a frame uniquely. Furthermore, for a given frame, the union of all coefficients of its dual frames is dense in ℓ^2. We also introduce a class of frames (called pseudo-coherent frames), which includes Gabor frames and coherent frames, and investigate invertible pseudo-coherent frame multipliers, allowing a classification for frame-type operators for these frames. Finally, we give a numerical example for the invertibility of multipliers in the Gabor case.

1. Introduction, notation, and motivation

In many scientific disciplines, certain objects play an important role. Those systems are described by an analysis procedure followed by a multiplication, followed by a synthesis. Those operators are of utmost importance in

- mathematics, where they are used for the diagonalization of operators [28];

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* Corresponding author.
E-mail addresses: peter.balazs@oeaw.ac.at (P. Balazs), dstoeva@kfs.oeaw.ac.at (D.T. Stoeva).

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- physics, where they are a link between classical and quantum mechanics, so called quantization operators [1];
- signal processing, where they are a particular way to implement time-variant filters [25];
- acoustics, where those time-frequency filters are used in several fields, for example in computational auditory scene analysis [34].

In this paper we show a surprising result about the shape of the inverse of such operators, if any. This also leads us to new results concerning dual frames, a concept at the core of frame theory.

To be able to describe those operators in a general setting, as an extension of Gabor multipliers [17], multipliers for general Bessel sequences were introduced by one of the authors [4]. Further properties of multipliers for general sequences and in particular, for Bessel sequences, frames, Riesz bases, Bessel fusion sequences, were investigated in [2,30–33]. Multipliers are operators defined by

$$M_{m,\Phi,\Psi} h = \sum_{n=1}^{\infty} m_n \langle h, \psi_n \rangle \phi_n, \tag{1}$$

for given sequences $\Phi = (\phi_n)$ and $\Psi = (\psi_n)$ with elements from a Hilbert space $\mathcal{H}$, and a given complex scalar sequence $m = (m_n)$ called the *symbol*. Such operators are also investigated for continuous transforms – in a general [5] (continuous frame multipliers), wavelet [27] (Calderón–Toeplitz operators) and short-time Fourier setting [12] (localization operators). Here we stick to the discrete version. Multipliers are interesting not only from a theoretical point of view, but also for applications. They are applied for example in psychoacoustical modeling [7] and denoising [24]. Multipliers are a particular way to implement time-variant filters [25]. Therefore, for some applications it is important to find the inverse of a multiplier if it exists. The paper [31] is devoted to invertibility of multipliers, necessary conditions for invertibility, sufficient conditions, and representation for the inverse via Neumann series.

In the present paper our attention is on how to express the inverse of an invertible frame multiplier as a multiplier with the reciprocal symbol and dual frames of the given ones. We show a result for all frames, namely, the inverse of any invertible frame multiplier with a semi-normalized symbol can always be represented as a multiplier with the reciprocal symbol and dual frames of the given ones, where one of these dual frames is uniquely determined and the other one can be arbitrarily chosen:

Theorem 1.1. *Let Φ and Ψ be frames for $\mathcal{H}$, and let the symbol m satisfy $0 < \inf_n |m_n| \le \sup_n |m_n| < \infty$. Assume that $M_{m,\Phi,\Psi}$ is invertible. Then*

- *there exists a unique dual frame $\Phi^\dagger$ of Φ, so that for any dual frame Ψ^d of Ψ we have*

$$M_{m,\Phi,\Psi}^{-1} = M_{1/m,\Psi^d,\Phi^\dagger}; \tag{2}$$

- *there exists a unique dual frame $\Psi^\dagger$ of Ψ, so that for any dual frame Φ^d of Φ we have*

$$M_{m,\Phi,\Psi}^{-1} = M_{1/m,\Psi^\dagger,\Phi^d}. \tag{3}$$

The investigation of this topic led us to surprising new results about dual frames. We show that a frame is uniquely determined by the set of its dual frames. Furthermore, for a given frame, the union of all coefficients of its dual frames is dense in ℓ^2:

Theorem 1.2. *Let Φ be a frame for $\mathcal{H}$. Then the following statements hold.*

(i) *The closure of the union of all sets $\mathcal{R}(U_{\Phi^d})$, where Φ^d runs through all dual frames of Φ, is ℓ^2.*
(ii) *Let Ψ be a frame for $\mathcal{H}$. If every dual frame Φ^d of Φ is a dual frame of Ψ, then $\Psi = \Phi$.*

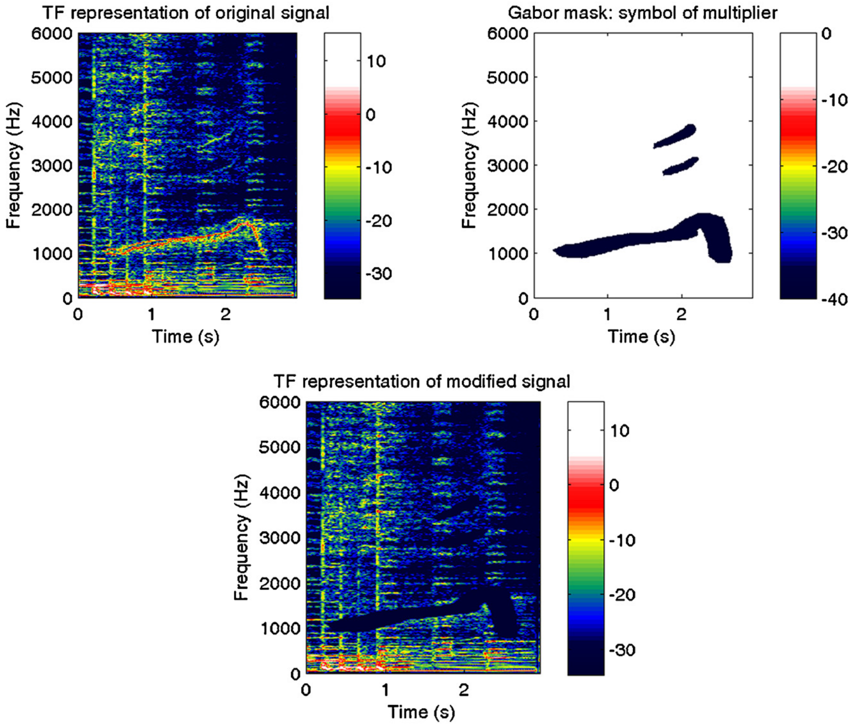

Fig. 1. *An illustrative example to visualize a multiplier.* (Top left) The time-frequency representation of the music signal f. (Top right) The symbol m, found by a (manual) estimation of the time-frequency region of the singer's voice. (Bottom) Time-frequency representation of $M_{m,\widetilde{\Psi},\Psi} f$.

In Fig. 1 we show a visualization of a multiplier $M_{m,\Phi,\Psi}$ in the time-frequency plane, which will again become interesting in the last section of the paper. The visualization is done using algorithms in the LTFAT toolbox [29]. We consider a music signal f and the action of a multiplier $M_{m,\Phi,\Psi}$ on f. For f we use a 2 seconds long excerpt of the "Jump" from Van Halen. For a time-frequency representation of the musical signal f (top left) we use a 'painless' Gabor frame Ψ (an 80 ms Hanning window with 12.5% overlap). By a manual estimation, we determine the symbol m that should describe the time-frequency region of the singer's voice. This region is then multiplied by 0.01, the rest by 1 (top right). Finally, we show a time-frequency representation of the modified signal (bottom).

For implementations and scripts producing Figs. 1 and 2, see http://www.kfs.oeaw.ac.at/RepresentationInverseMultiplier.

1.1. Motivation

In [4] it is proved that, if m is semi-normalized, then a Riesz multiplier $M_{m,\Phi,\Psi}$ is automatically invertible and

$$M^{-1}_{m,\Phi,\Psi} = M_{1/m,\widetilde{\Psi},\widetilde{\Phi}}, \tag{4}$$

where $\widetilde{\Phi}$ and $\widetilde{\Psi}$ denote the canonical duals of Φ and Ψ, respectively.

The result on Riesz multipliers has opened the following questions:

[Q1] *Are there other invertible frame multipliers $M_{m,\Phi,\Psi}$ whose inverses can be represented using the inverted symbol $1/m$ and appropriate dual frames of Φ and Ψ?*

[$Q2$] *Are there other invertible frame multipliers $M_{m,\Phi,\Psi}$ whose inverses can be written as $M_{1/m,\widetilde{\Psi},\widetilde{\Phi}}$ using the canonical duals?*

The paper is devoted to these two questions. First note that every bounded (resp. bounded surjective) operator can be written as a Bessel (resp. frame) multiplier. Thus, the inverse of every invertible multiplier can be written as a frame multiplier. The aim of the present paper is to represent the inverse of an invertible frame multiplier as described in [$Q1$].

We give an affirmative answer to Question [$Q1$]. We show in Theorem 1.1 that the inverse of every invertible frame multiplier with semi-normalized symbol can be represented as a multiplier with the reciprocal symbol and dual frames of the given ones. One of the dual frames is uniquely determined, while the other one can be arbitrarily chosen.

We also give an affirmative answer to Question [$Q2$]. We determine frame multipliers $M_{m,\Phi,\Psi}$ (not necessarily being Riesz multipliers) which are invertible and their inverses can be written as $M_{1/m,\widetilde{\Psi},\widetilde{\Phi}}$.

In Section 5, we introduce a class of frames (called pseudo-coherent frames), which includes Gabor frames and coherent frames. Classification of invertible pseudo-coherent frame-type operators is given and representations of the inverses using pseudo-coherent frames are determined.

The last section contains a numerical example with an invertible Gabor frame multiplier.

1.2. Notation and definitions

Throughout the paper, $\mathcal{H}$ denotes a separable Hilbert space, $\Phi = (\phi_n)_{n=1}^{\infty}$ and $\Psi = (\psi_n)_{n=1}^{\infty}$ are sequences with elements from $\mathcal{H}$. The sequence $(e_n)_{n=1}^{\infty}$ denotes an orthonormal basis of $\mathcal{H}$ and $(\delta_n)_{n=1}^{\infty}$ denotes the canonical basis of ℓ^2. When the index set is omitted, $\mathbb{N}$ should be understood as the index set. The letter m is used to denote a complex valued scalar sequence (m_n). Furthermore, $\overline{m} = (\overline{m}_n)$ and $1/m = (1/m_n)$. The sequence m is called *semi-normalized* if $0 < \inf_n|m_n| \leq \sup_n|m_n| < \infty$. For $m \in \ell^\infty$, we will use the operator $\mathcal{M}_m : \ell^2 \to \ell^2$ given by $\mathcal{M}_m(c_n) = (m_n c_n)$, which is bounded with $\|\mathcal{M}_m\| = \|m\|_{\ell^\infty}$. An operator $M : \mathcal{H} \to \mathcal{H}$ is called *invertible* if it is a bounded bijection from $\mathcal{H}$ onto $\mathcal{H}$. The identity operator on $\mathcal{H}$ is denoted by $\mathrm{Id}_{\mathcal{H}}$.

Recall that Φ is called a *frame for $\mathcal{H}$ with bounds A_Φ, B_Φ* if $0 < A_\Phi \leq B_\Phi < \infty$ and $A_\Phi\|h\|^2 \leq \sum_{n=1}^{\infty}|\langle h, \phi_n\rangle|^2 \leq B_\Phi\|h\|^2$ for every $h \in \mathcal{H}$. For a given frame Φ for $\mathcal{H}$, the analysis operator is denoted by U_Φ, the synthesis operator by T_Φ, the frame operator by S_Φ, a dual frame of Φ by $\Phi^d = (\phi_n^d)$, and the canonical dual by $\widetilde{\Phi} = (\widetilde{\phi}_n)$. For the definition of all these frame-related concepts, as well as for the definition of a Bessel sequence and a Riesz basis, we refer to [10]. Recall that two frames Φ and Ψ for $\mathcal{H}$ are called *equivalent* if there exists an invertible operator $G : \mathcal{H} \to \mathcal{H}$ so that $\psi_n = G\phi_n$ for all $n \in \mathbb{N}$. When Φ is a frame for $\mathcal{H}$, then a dual frame Φ^d of Φ is equivalent to Φ if and only if $\Phi^d = \widetilde{\Phi}$ [23, Section 1.2].

For given m, Φ, and Ψ, the operator $M_{m,\Phi,\Psi}$ given by Eq. (1) is called a *multiplier*. The operator $M_{m,\Phi,\Psi}$ is called *unconditionally convergent* if the series in Eq. (1) converges unconditionally for every $h \in \mathcal{H}$. When Φ and Ψ are Bessel sequences, frames, Riesz bases for $\mathcal{H}$, then $M_{m,\Phi,\Psi}$ will be called a *Bessel multiplier, frame multiplier, Riesz multiplier,* respectively. When $m \in \ell^\infty$, then a Bessel multiplier is a well defined operator from $\mathcal{H}$ into $\mathcal{H}$ [4].

Note that frame multipliers with a constant symbol are the so-called frame-type operators (see, e.g., [16,14,35]) or mixed frame operators (see, e.g., [11]), as the frame multiplier $M_{(1),\Phi,\Psi}$ corresponds to the frame-type operator denoted as $S_{\Psi,\Phi}$ and for $c \neq 0$, $M_{(c),\Phi,\Psi} = cS_{\Psi,\Phi} = S_{\Psi,c\Phi}$.

2. The set of dual frames

In order to prove Theorem 1.1 and Proposition 3.1, we need Theorem 1.2, stated on page 982. This is a result which is of independent interest for frame theory, showing new properties of the set of dual frames.

Proof of Theorem 1.2. (i) Let the sequence $c = (c_n) \in \ell^2$ fulfill $c \perp \mathcal{R}(U_{\Phi^d})$ for every dual frame Φ^d of Φ. Then

$$T_{\Phi^d} c = 0, \quad \forall \text{dual frame } \Phi^d \text{ of } \Phi. \tag{5}$$

The dual frames of Φ are precisely the sequences

$$\left(\widetilde{\phi}_n + h_n - \sum_{j=1}^{\infty} \langle \widetilde{\phi}_n, \phi_j \rangle h_j \right)_{n=1}^{\infty},$$

where $(h_n)_{n=1}^{\infty}$ is a Bessel sequence in $\mathcal{H}$ (see, e.g., [10, Theorem 5.6.5]). Therefore,

$$\sum_{n=1}^{\infty} c_n \left(\widetilde{\phi}_n + h_n - \sum_{j=1}^{\infty} \langle \widetilde{\phi}_n, \phi_j \rangle h_j \right) = 0$$

for every Bessel sequence $\{h_n\}_{n=1}^{\infty}$ in $\mathcal{H}$. By Eq. (5) we have $T_{\widetilde{\Phi}} c = 0$, which implies that

$$\sum_{n=1}^{\infty} c_n \left(h_n - \sum_{j=1}^{\infty} \langle \widetilde{\phi}_n, \phi_j \rangle h_j \right) = 0 \tag{6}$$

for every Bessel sequence $\{h_n\}_{n=1}^{\infty}$ in $\mathcal{H}$. Using Eq. (6) with the Bessel sequence $(h_n)_{n=1}^{\infty} = (e_1, 0, 0, 0, \ldots)$, we obtain

$$c_1 e_1 - \sum_{n=1}^{\infty} c_n \langle \widetilde{\phi}_n, \phi_1 \rangle e_1 = 0.$$

Since $\sum_{n=1}^{\infty} c_n \widetilde{\phi}_n = 0$, it now follows that $c_1 = 0$. In a similar way, using Eq. (6) with the Bessel sequence $(h_n)_{n=1}^{\infty} = (0, \ldots, 0, e_j, 0, 0, 0, \ldots)$, where e_j stands at the j-th position, we obtain $c_j = 0$ for every $j \geq 2$. Therefore, $c = (0)$, which completes the proof.

(ii) Assume that all dual frames Φ^d of Φ are dual frames of Ψ. Then $T_\Phi U_{\Phi^d} = \mathrm{Id}_{\mathcal{H}} = T_\Psi U_{\Phi^d}$, which by (i) implies that $T_\Phi = T_\Psi$ and hence, $\Phi = \Psi$. □

By the above result, different frames have different sets of dual frames; if two frames Φ and Ψ for $\mathcal{H}$ have the same sets of dual frames, then $\Phi = \Psi$. In particular, two different frames cannot have sets of dual frames which are included into one another.

3. Inversion of multipliers by inverted symbol [$Q1$] and dual frames

Here we give an affirmative answer to Question [$Q1$]. The result about the inverses of invertible frame multipliers with semi-normalized symbols is stated in Theorem 1.1. In addition, we show the following:

Proposition 3.1. *For the assumptions in Theorem 1.1, we have the additional properties:*

- *If $F = (f_n)$ is a Bessel sequence in $\mathcal{H}$ which fulfills $M_{m,\Phi,\Psi}^{-1} = M_{1/m,\Psi^\dagger,F}$ (resp. $M_{m,\Phi,\Psi}^{-1} = M_{1/m,F,\Phi^\dagger}$), then F must be a dual frame of Φ (resp. Ψ).*
- *$\Psi^\dagger$ is the only Bessel sequence in $\mathcal{H}$ which satisfies $M_{m,\Phi,\Psi}^{-1} = M_{1/m,\Psi^\dagger,\Phi^d}$ for all dual frames Φ^d of Φ.*
- *$\Phi^\dagger$ is the only Bessel sequence in $\mathcal{H}$ which satisfies $M_{m,\Phi,\Psi}^{-1} = M_{1/m,\Psi^d,\Phi^\dagger}$ for all dual frames Ψ^d of Ψ.*

Proof of Theorem 1.1 and Proposition 3.1. Denote $M := M_{m,\Phi,\Psi}$. First observe that the sequence $(M^{-1}(m_n\phi_n))$ is a dual frame of Ψ. Denote it by $\Psi^\dagger$. Therefore, $M^{-1}T_\Phi\delta_n = T_{\Psi^\dagger}\mathcal{M}_{1/m}\delta_n$, $n \in \mathbb{N}$. Now the boundedness of the operators implies that $M^{-1}T_\Phi = T_{\Psi^\dagger}\mathcal{M}_{1/m}$ on ℓ^2. Using any dual frame Φ^d of Φ we get $M^{-1} = T_{\Psi^\dagger}\mathcal{M}_{1/m}U_{\Phi^d} = M_{1/m,\Psi^\dagger,\Phi^d}$ on $\mathcal{H}$.

In a similar way as above, it follows that the sequence $((M^{-1})^*(\overline{m}_n\psi_n))$ is a dual frame of Φ (denoted by $\Phi^\dagger$) and hence,

$$\left(M^{-1}\right)^* T_\Psi = T_{\Phi^\dagger}\mathcal{M}_{1/\overline{m}} \quad \text{on } \ell^2. \tag{7}$$

Therefore, $M^{-1} = T_{\Psi^d}\mathcal{M}_{1/m}U_{\Phi^\dagger} = M_{1/m,\Psi^d,\Phi^\dagger}$.

Now assume that $F = (f_n)$ is a Bessel sequence in $\mathcal{H}$ which satisfies $M^{-1}_{m,\Phi,\Psi} = M_{1/m,F,\Phi^\dagger}$. By Eq. (7), it follows that $T_\Psi U_F = M^* T_{\Phi^\dagger}\mathcal{M}_{1/\overline{m}}U_F = M^*(M^{-1})^* = \mathrm{Id}_\mathcal{H}$, which implies that F is a dual frame of Ψ. In a similar way, every Bessel sequence F in $\mathcal{H}$ which satisfies $M^{-1}_{m,\Phi,\Psi} = M_{1/m,\Psi^\dagger,F}$ must be a dual frame of Φ.

On the other hand, assume that F is a Bessel sequence in $\mathcal{H}$ which satisfies $M_{1/m,F,\Phi^d} = M_{1/m,\Psi^\dagger,\Phi^d}$ for all dual frames Φ^d of Φ. Then $T_F\mathcal{M}_{1/m}U_{\Phi^d} = T_{\Psi^\dagger}\mathcal{M}_{1/m}U_{\Phi^d}$ for all dual frames Φ^d of Φ, which by Theorem 1.2(i) implies that $T_F\mathcal{M}_{1/m} = T_{\Psi^\dagger}\mathcal{M}_{1/m}$. Since m is semi-normalized (so, $\mathcal{M}_{1/m}$ is invertible on ℓ^2), it follows that $T_F = T_{\Psi^\dagger}$ and hence, $F = \Psi^\dagger$.

The statement for $\Phi^\dagger$ follows in a similar way. □

Remark 3.2. Concerning Theorem 1.1, it is natural to ask whether the frame $\Psi^\dagger$ (resp $\Phi^\dagger$) is the canonical dual of Ψ (resp. Φ). Observe that in this context we have $\Psi^\dagger = \widetilde{\Psi}$ (resp. $\Phi^\dagger = \widetilde{\Phi}$) if and only if Ψ is equivalent to $(m_n\phi_n)$ (resp. Φ is equivalent to $(\overline{m}_n\psi_n)$).

Note that Eqs. (2) and (3) are *not* constructive approaches leading to an implementation for the inversion of M. For the dual frame $\Psi^\dagger$ (resp. $\Phi^\dagger$) we already had to apply M^{-1}. For more constructive approaches to the inversion of multipliers see the next sections and [31].

A sub-result of Theorem 1.1, the representation of the inverse for the particular case of finite-dimensional spaces and $\Psi = \Phi^d$, has been independently found in the context of frame diagonalization of matrices [19].

Remark 3.3. In [32] the following conjecture is formulated: For an unconditionally convergent multiplier $M_{m,\Phi,\Psi}$, there always exist sequences (c_n) and (d_n) so that $M_{m,\Phi,\Psi} = M_{(1),(c_n\phi_n),(d_n\psi_n)}$ and the sequences $(c_n\phi_n)$, $(d_n\psi_n)$ are Bessel sequences.

If this conjecture is true, then any invertible, unconditionally convergent multiplier $M_{m,\Phi,\Psi}$ can be rewritten as $M_{m,\Phi,\Psi} = M_{(1),(c_n\phi_n),(d_n\psi_n)}$ where the sequences $(c_n\phi_n)$, $(d_n\psi_n)$ are frames for $\mathcal{H}$, and thus, by Theorem 1.1, $M^{-1}_{m,\Phi,\Psi}$ can be written as $M_{(1),(d_n\psi_n)^\dagger,(c_n\phi_n)^d}$ and $M_{(1),(d_n\psi_n)^d,(c_n\phi_n)^\dagger}$.

4. Inversion of multipliers using the canonical duals $[Q2]$

The following example shows cases where Question $[Q2]$ is answered affirmatively.

Example 4.1. Every frame Φ for $\mathcal{H}$ fulfills $M^{-1}_{(1),\Phi,\Phi} = M_{(1),\widetilde{\Phi},\widetilde{\Phi}}$.

Example 4.2 shows a case when $M_{m,\Phi,\Psi}$ is invertible but the inverse is not equal to $M_{1/m,\widetilde{\Psi},\widetilde{\Phi}}$.

Example 4.2. Let $\Phi = (e_1, e_1, e_1, e_2, e_2, e_2, e_3, e_3, e_3, \dots)$ and $\Psi = (e_1, e_1, -e_1, e_2, e_2, -e_2, e_3, e_3, -e_3, \dots)$. Then $M_{(1),\widetilde{\Psi},\widetilde{\Phi}} = \frac{1}{9}\mathrm{Id}_\mathcal{H} \neq M^{-1}_{(1),\Phi,\Psi} = \mathrm{Id}_\mathcal{H}$.

The next proposition determines a class of multipliers which are invertible and whose inverses can be written as in Eq. (4). While in Theorem 1.1 it is assumed that the frame multiplier is invertible, in Proposition 4.3 we investigate the invertibility of frame multipliers – we give sufficient conditions for invertibility and sufficient conditions for non-invertibility. For the rest of the section the letter c means a non-zero constant.

Proposition 4.3. *Let Φ and Ψ be frames for $\mathcal{H}$ and $(m_n) = (c)$. Then the following assertions hold.*

(i) *If $\mathcal{R}(U_\Phi) \subseteq \mathcal{R}(U_\Psi)$, then $M_{(1/c),\widetilde{\Psi},\widetilde{\Phi}}$ is a bounded right inverse of $M_{(c),\Phi,\Psi}$.*
(ii) *If $\mathcal{R}(U_\Psi) \subseteq \mathcal{R}(U_\Phi)$, then $M_{(1/c),\widetilde{\Psi},\widetilde{\Phi}}$ is a bounded left inverse of $M_{(c),\Phi,\Psi}$.*
(iii) *If $\mathcal{R}(U_\Phi) = \mathcal{R}(U_\Psi)$, then $M_{(c),\Phi,\Psi}$ is invertible and $M^{-1}_{(c),\Phi,\Psi} = M_{(\frac{1}{c}),\widetilde{\Psi},\widetilde{\Phi}}$.*
(iv) *If $\mathcal{R}(U_\Phi) \subsetneq \mathcal{R}(U_\Psi)$, then $M_{(c),\Phi,\Psi}$ is not invertible.*
(v) *If $\mathcal{R}(U_\Psi) \subsetneq \mathcal{R}(U_\Phi)$, then $M_{(c),\Phi,\Psi}$ is not invertible.*

Proof. (i) Assume that $\mathcal{R}(U_\Phi) \subseteq \mathcal{R}(U_\Psi)$. For every $h \in \mathcal{H}$, the element $U_\Phi S_\Phi^{-1} h$ can be written as $U_\Psi g^h$ for some $g^h \in \mathcal{H}$ and

$$M_{(c),\Phi,\Psi} M_{(1/c),\widetilde{\Psi},\widetilde{\Phi}} h = T_\Phi U_\Psi S_\Psi^{-1} T_\Psi U_\Phi S_\Phi^{-1} h = T_\Phi U_\Psi g^h = h.$$

(ii) can be proved in a similar way as (i).

(iii) follows from (i) and (ii).

(iv) Assume that $\mathcal{R}(U_\Phi) \subset \mathcal{R}(U_\Psi)$ with $\mathcal{R}(U_\Phi) \neq \mathcal{R}(U_\Psi)$. By (i), the operator $M_{(1/c),\widetilde{\Psi},\widetilde{\Phi}}$ is a bounded right inverse of $M_{(c),\Phi,\Psi}$. We will prove that $M_{(1/c),\widetilde{\Psi},\widetilde{\Phi}}$ is not a left inverse of $M_{(c),\Phi,\Psi}$, which will imply that $M_{(c),\Phi,\Psi}$ cannot be invertible. Consider an arbitrary element $g \in \mathcal{R}(U_\Psi)$, $g \notin \mathcal{R}(U_\Phi)$. Write $g = U_\Psi h$ for some $h \in \mathcal{H}$. Since $\ell^2 = \mathcal{R}(U_\Phi) \oplus \ker(T_\Phi)$, we can also write $g = U_\Phi f + d$ for some $f \in \mathcal{H}$ and some $d \in \ker(T_\Phi)$, $d \neq 0$. Since $d = g - U_\Phi f \in \mathcal{R}(U_\Psi)$, it follows that $d \notin \ker T_\Psi$, which implies that $S_\Psi^{-1} T_\Psi d \neq 0$. Then

$$\begin{aligned} M_{(1/c),\widetilde{\Psi},\widetilde{\Phi}} M_{(c),\Phi,\Psi} h &= S_\Psi^{-1} T_\Psi U_\Phi S_\Phi^{-1} T_\Phi U_\Psi h \\ &= S_\Psi^{-1} T_\Psi U_\Phi S_\Phi^{-1} T_\Phi (U_\Phi f + d) \\ &= S_\Psi^{-1} T_\Psi (U_\Psi h - d) = h - S_\Psi^{-1} T_\Psi d \neq h, \end{aligned}$$

which implies that $M_{(1/c),\widetilde{\Psi},\widetilde{\Phi}}$ is not a left inverse of $M_{(c),\Phi,\Psi}$.

(v) Assume that $\mathcal{R}(U_\Psi) \subset \mathcal{R}(U_\Phi)$ with $\mathcal{R}(U_\Psi) \neq \mathcal{R}(U_\Phi)$. By (i), $M_{(1/c),\widetilde{\Psi},\widetilde{\Phi}}$ is a bounded left inverse of $M_{(c),\Phi,\Psi}$. In a similar way as in (iv), one can prove that $M_{(1/c),\widetilde{\Psi},\widetilde{\Phi}}$ is not a right inverse of $M_{(c),\Phi,\Psi}$, which implies that $M_{(c),\Phi,\Psi}$ cannot be invertible. □

Concerning the statements in Proposition 4.3, note that if none of $\mathcal{R}(U_\Phi)$ and $\mathcal{R}(U_\Psi)$ is a subset of the other one, then both invertibility and non-invertibility of $M_{(c),\Phi,\Psi}$ are possible. For a case of invertibility, consider the frame multiplier $M_{(1),\Phi,\Psi}$, where $\Phi = (e_1, e_1, e_2, e_2, e_3, e_3, e_4, e_4, \dots)$ and $\Psi = (\frac{1}{2}e_1, \frac{1}{2}e_1, \frac{1}{2}e_2, \frac{1}{2}e_2, \frac{1}{3}e_3, \frac{2}{3}e_3, \frac{1}{4}e_4, \frac{3}{4}e_4, \dots)$, and thus, $M_{(1),\Phi,\Psi}$ is the identity operator on $\mathcal{H}$. For a case of non-invertibility, consider the frame multiplier $M_{(1),\Phi,\Psi}$, where $\Phi = (e_1, e_1, e_2, e_2, e_3, e_3, e_4, e_4, \dots)$ and $\Psi = (e_1, e_1, e_2, e_3, e_4, \dots)$.

Remark 4.4. Let Φ and Ψ be frames for $\mathcal{H}$. The condition $\mathcal{R}(U_\Phi) = \mathcal{R}(U_\Psi)$ corresponds to Φ and Ψ being equivalent frames [9, Corollary 4.5]. The condition $\mathcal{R}(U_\Phi) \subseteq \mathcal{R}(U_\Psi)$ is identical to Ψ being partial equivalent to Φ, i.e. to the existence of a bounded operator $Q : \mathcal{H} \to \mathcal{H}$, such that $\phi_k = Q\psi_k$, $\forall k \in \mathbb{N}$, see [3].

Corollary 4.5. *If Φ and Ψ are equivalent frames, then $M_{(c),\Phi,\Psi}$ is invertible and $M^{-1}_{(c),\Phi,\Psi} = M_{(\frac{1}{c}),\widetilde{\Psi},\widetilde{\Phi}}$.*

Now it is natural to pose the inverse question: *If $M^{-1}_{(c),\Phi,\Psi} = M_{(\frac{1}{c}),\widetilde{\Psi},\widetilde{\Phi}}$, does it follow that Φ and Ψ are equivalent*? We give an affirmative answer in the next theorem.

Theorem 4.6. *Let Φ and Ψ be frames for $\mathcal{H}$. The following statements are equivalent.*

- (a) $M_{(c),\Phi,\Psi}$ *is invertible and* $M^{-1}_{(c),\Phi,\Psi} = M_{(\frac{1}{c}),\widetilde{\Psi},\widetilde{\Phi}}$.
- (b) Φ *and* Ψ *are equivalent frames.*
- (c1) $M_{(c),\Phi,\Psi}$ *is invertible and the unique frame* $\Psi^\dagger$ *in Theorem 1.1 is* $\widetilde{\Psi}$.
- (c2) $M_{(c),\Phi,\Psi}$ *is invertible and the unique frame* $\Phi^\dagger$ *in Theorem 1.1 is* $\widetilde{\Phi}$.
- (d1) $M_{(c),\Phi,\Psi}$ *is invertible and* $M^{-1}_{(c),\Phi,\Psi} = M_{(\frac{1}{c}),\widetilde{\Psi},\Phi^d}$ *for all dual frames* Φ^d *of* Φ.
- (d2) $M_{(c),\Phi,\Psi}$ *is invertible and* $M^{-1}_{(c),\Phi,\Psi} = M_{(\frac{1}{c}),\Psi^d,\widetilde{\Phi}}$ *for all dual frames* Ψ^d *of* Ψ.

Proof. Without loss of generality, we may consider $c = 1$. For a closed subspace U of ℓ^2, the orthogonal projection on U will be denoted by P_U.

(a) $\Rightarrow$ (b) By (a), we have $T_{\widetilde{\Psi}} U_{\widetilde{\Phi}} T_\Phi U_\Psi = \mathrm{Id}_{\mathcal{H}}$ and hence, $U_\Psi T_{\widetilde{\Psi}} U_{\widetilde{\Phi}} T_\Phi U_\Psi T_{\widetilde{\Psi}} = U_\Psi T_{\widetilde{\Psi}}$. Then $P_{\mathcal{R}(U_\Psi)} P_{\mathcal{R}(U_{\widetilde{\Phi}})} P_{\mathcal{R}(U_\Psi)} = P_{\mathcal{R}(U_\Psi)}$, which implies that $\mathcal{R}(U_\Psi) \subseteq \mathcal{R}(U_{\widetilde{\Phi}})$.

In an analog way, it follows that $\mathcal{R}(U_{\widetilde{\Phi}}) \subseteq \mathcal{R}(U_\Psi)$.

Therefore, $\mathcal{R}(U_{\widetilde{\Phi}}) = \mathcal{R}(U_\Psi)$. This implies that Φ and Ψ are equivalent.

(b) $\Rightarrow$ (c1) and (c2) Since $\psi_n^\dagger = M^{-1}(\phi_n)$, $n \in \mathbb{N}$, it follows that $\Psi^\dagger$ is equivalent to Ψ. Therefore, $\Psi^\dagger = \widetilde{\Psi}$. The validity of (c2) follows in a similar way.

(c1) $\Rightarrow$ (d1) and (c2) $\Rightarrow$ (d2) Use Theorem 1.1.

(d1) $\Rightarrow$ (a) and (d2) $\Rightarrow$ (a) Clear. □

For the more general case of semi-normalized symbols, it is not difficult to prove the following sufficient condition for validity of Eq. (4).

Proposition 4.7. *Let Φ and Ψ be frames for $\mathcal{H}$, and let the symbol m be semi-normalized. Assume that $M_{m,\Phi,\Psi}$ is invertible. If Ψ is equivalent to $(m_n\phi_n)$ or Φ is equivalent to $(\overline{m}_n\psi_n)$, then $M^{-1}_{m,\Phi,\Psi} = M_{1/m,\widetilde{\Psi},\widetilde{\Phi}}$.*

5. Multipliers for pseudo-coherent frames

5.1. General results

To show a result valid for a lot of classes of frames, let us define the following general concept.

Definition 5.1. Let Λ be a discrete set. Consider $\theta : \Lambda \to \mathcal{B}(\mathcal{H})$ (i.e., $\theta(\lambda)$ being a bounded operator from $\mathcal{H}$ into $\mathcal{H}$ for $\lambda \in \Lambda$). Assume that there exist $\phi : \Lambda \times \Lambda \to \mathbb{C}$ and $\mu : \Lambda \times \Lambda \to \Lambda$, which satisfy

$$\forall \lambda \in \Lambda, \quad \text{the mapping } \lambda' \to \mu(\lambda, \lambda') \text{ is a bijection from } \Lambda \text{ onto } \Lambda, \tag{8}$$

$$\theta(\lambda)^*\theta(\lambda') = \phi(\lambda, \lambda')\theta(\mu(\lambda, \lambda')), \tag{9}$$

$$\theta(\lambda)\theta(\mu(\lambda, \lambda')) = \overline{\phi(\lambda, \lambda')}\theta(\lambda'). \tag{10}$$

Then a sequence of the form $(g_\lambda)_{\lambda\in\Lambda} = (\theta(\lambda)g)_{\lambda\in\Lambda}$, $g \in \mathcal{H}$, $g \neq 0$, is called a θ-pseudo-coherent system.

A θ-pseudo-coherent system which is a frame for $\mathcal{H}$ is called a θ-pseudo-coherent frame for $\mathcal{H}$. A multiplier for θ-pseudo-coherent frames is called a θ-pseudo-coherent frame multiplier. When $m = (1)$, a θ-pseudo-coherent frame multiplier is also called θ-pseudo-coherent frame-type operator or θ-pseudo-coherent mixed frame operator.

Note that Gabor frames and coherent frames are pseudo-coherent frames (see Subsections 5.2 and 5.3) and thus, the results in this subsection hold in the particular case of multipliers for Gabor frames and coherent frames.

In this section we are interested in invertible θ-pseudo-coherent frame multipliers, whose inverses can be written not only as frame multipliers, but also as θ-pseudo-coherent frame multipliers. When $M_{m,\Phi,\Psi}$ is a θ-pseudo-coherent frame multiplier, Theorem 1.1 naturally leads to the question: when the frames used in the representations in (2) and (3) are θ-pseudo-coherent frames? Here we give answers to the above questions in the case $m = (1)$. First we consider invertible operators and give equivalent conditions to the operator being a θ-pseudo-coherent frame-type operator.

Proposition 5.2. *Let $g \in \mathcal{H}$ and $(g_\lambda)_{\lambda\in\Lambda} = (\theta(\lambda)g)_{\lambda\in\Lambda}$ be a θ-pseudo-coherent frame for $\mathcal{H}$. Then the frame operator S_g for $(g_\lambda)_{\lambda\in\Lambda}$ commutes with all the operators $\theta(\lambda)$, $\lambda \in \Lambda$, and thus the canonical dual of $(g_\lambda)_{\lambda\in\Lambda}$ is $(\theta(\lambda)\widetilde{g})_{\lambda\in\Lambda}$ with $\widetilde{g} = S_g^{-1}g$. Furthermore, if $V : \mathcal{H} \to \mathcal{H}$ is a bounded bijective operator, then the following statements are equivalent.*

($\mathcal{A}_1$) *For every $\lambda \in \Lambda$, $V\theta(\lambda)g = \theta(\lambda)Vg$.*
($\mathcal{A}_2$) *For every $\lambda \in \Lambda$ and every $f \in \mathcal{H}$, $V\theta(\lambda)f = \theta(\lambda)Vf$ (i.e., V commutes with $\theta(\lambda)$ for every $\lambda \in \Lambda$).*
($\mathcal{A}_3$) *V is a θ-pseudo-coherent frame-type operator for the same set Λ.*
($\mathcal{A}_4$) *V^{-1} is a θ-pseudo-coherent frame-type operator for the same set Λ.*
($\mathcal{A}_5$) *For every $\lambda \in \Lambda$, V^* commutes with $\theta(\lambda)$.*

Proof. $(\mathcal{A}_3) \Rightarrow (\mathcal{A}_2)$ Let V be the θ-coherent frame-type operator $M_{(1),(\theta(\lambda)v)_{\lambda\in\Lambda},(\theta(\lambda)u)_{\lambda\in\Lambda}}$ for some $u, v \in \mathcal{H}$ and let ϕ and μ satisfy (8)–(10). For every $f \in \mathcal{H}$ and every $\lambda \in \Lambda$,

$$\begin{aligned}
V\theta(\lambda)f &= \sum_{\lambda'\in\Lambda} \langle \theta(\lambda)f, \theta(\lambda')u\rangle \theta(\lambda')v \\
&= \sum_{\lambda'\in\Lambda} \langle f, \phi(\lambda,\lambda')\theta(\mu(\lambda,\lambda'))u\rangle \theta(\lambda')v \\
&= \sum_{\lambda'\in\Lambda} \langle f, \theta(\mu(\lambda,\lambda'))u\rangle \overline{\phi(\lambda,\lambda')}\theta(\lambda')v \\
&= \sum_{\lambda'\in\Lambda} \langle f, \theta(\mu(\lambda,\lambda'))u\rangle \theta(\lambda)\theta(\mu(\lambda,\lambda'))v = \theta(\lambda)Vf.
\end{aligned}$$

This also implies the commutative property of the frame operator S_g and its inverse.

$(\mathcal{A}_2) \Rightarrow (\mathcal{A}_1)$ is obvious.

$(\mathcal{A}_1) \Rightarrow (\mathcal{A}_3)$ For every $f \in \mathcal{H}$,

$$Vf = V\left(\sum_{\lambda\in\Lambda} \langle f, \widetilde{g}_\lambda\rangle g_\lambda\right) = \sum_{\lambda\in\Lambda} \langle f, \theta(\lambda)\widetilde{g}\rangle \theta(\lambda)Vg,$$

which means that V is a θ-pseudo-coherent frame-type operator.

$(\mathcal{A}_1) \Rightarrow (\mathcal{A}_4)$ For $\lambda \in \Lambda$, denote $h_\lambda = \theta(\lambda)Vg$. By what is already proved, V can be written as the multiplier $M_{(1),(h_\lambda),(\widetilde{g}_\lambda)}$. Since (h_λ) and $(\widetilde{g}_\lambda)$ are equivalent frames, because $h_\lambda = \theta(\lambda)Vg = V\theta(\lambda)g = VS_g\widetilde{g}_\lambda$, Corollary 4.5 implies that $M^{-1}_{(1),(h_\lambda),(\widetilde{g}_\lambda)} = M_{(1),(g_\lambda),(\widetilde{h}_\lambda)}$. By what is already proved, $(\widetilde{h}_\lambda)$ is a θ-pseudo-coherent frame.

$(\mathcal{A}_4) \Rightarrow (\mathcal{A}_2)$ Having in mind the implication $(\mathcal{A}_3) \Rightarrow (\mathcal{A}_2)$ applied to V^{-1}, it follows that V^{-1} commutes with $\theta(\lambda)$, $\forall\lambda \in \Lambda$. Therefore, V also commutes with $\theta(\lambda)$, $\forall\lambda \in \Lambda$.

$(\mathcal{A}_2) \Leftrightarrow (\mathcal{A}_5)$ Assume that $(\mathcal{A}_2)$ holds. Let ϕ and μ satisfy (8)–(10). Fix $\lambda \in \Lambda$. For every $\lambda' \in \Lambda$,

$$\theta(\lambda)^*V\theta(\lambda')g = \theta(\lambda)^*\theta(\lambda')Vg = \phi(\lambda,\lambda')\theta(\mu(\lambda,\lambda'))Vg = V\phi(\lambda,\lambda')\theta(\mu(\lambda,\lambda'))g = V\theta(\lambda)^*\theta(\lambda')g.$$

Since $(\theta(\lambda')g)_{\lambda'\in\Lambda}$ is complete in $\mathcal{H}$, it follows that $\theta(\lambda)^*V = V\theta(\lambda)^*$ and thus, $V^*\theta(\lambda) = \theta(\lambda)V^*$.

The converse implication is now also clear. $\square$

For an invertible θ-pseudo-coherent frame-type operator $V = M_{(1),(\theta(\lambda)v)_{\lambda\in\Lambda},(\theta(\lambda)u)_{\lambda\in\Lambda}}$, one can now conclude that the frames $(\theta(\lambda)v)^{\dagger}_{\lambda\in\Lambda}$ and $(\theta(\lambda)u)^{\dagger}_{\lambda\in\Lambda}$ from Theorem 1.1 have a θ-pseudo-coherent structure and thus V^{-1} can be written as a θ-pseudo-coherent frame-type operator as follows:

$$V^{-1} = M_{(1),(\theta(\lambda)V^{-1}v)_{\lambda\in\Lambda},(\theta(\lambda)v)^d_{\lambda\in\Lambda}} = M_{(1),(\theta(\lambda)u)^d_{\lambda\in\Lambda},(\theta(\lambda)(V^{-1})^*u)_{\lambda\in\Lambda}}$$

using dual frames $(\theta(\lambda)u)^d_{\lambda\in\Lambda}$ and $(\theta(\lambda)v)^d_{\lambda\in\Lambda}$ with the θ-pseudo-coherent structure (for example, the canonical duals). However, the above formulas involve the inverse V^{-1} in the representation. Other representations can be derived as a consequence of the proof of Proposition 5.2 using any given θ-pseudo-coherent frame:

Corollary 5.3. *Let $V : \mathcal{H} \to \mathcal{H}$ be an invertible θ-pseudo-coherent frame-type operator $M_{(1),(\theta(\lambda)v)_{\lambda\in\Lambda},(\theta(\lambda)u)_{\lambda\in\Lambda}}$ and let $(g_\lambda)_{\lambda\in\Lambda} = (\theta(\lambda)g)_{\lambda\in\Lambda}$ be any θ-pseudo-coherent frame for $\mathcal{H}$. Then V^{-1} can be written as the θ-pseudo-coherent frame-type operator $M_{(1),(g_\lambda)_{\lambda\in\Lambda},(\widetilde{h}_\lambda)_{\lambda\in\Lambda}}$, where $h_\lambda = \theta(\lambda)Vg$, $\lambda \in \Lambda$.*

Concerning Proposition 5.2, note that if weaker assumptions on V are made, then a similar proof can be used to show the following statements.

Lemma 5.4. *As in Proposition 5.2, let $g \in \mathcal{H}$ and let $(\theta(\lambda)g)_{\lambda\in\Lambda}$ be a θ-pseudo-coherent frame for $\mathcal{H}$. Let $V : \mathcal{H} \to \mathcal{H}$ be an operator. Consider the condition*

($\mathcal{A}_3'$) *V can be written as a θ-pseudo-coherent Bessel multiplier with a constant symbol* (1) *for the same set Λ.*

Then the following statements hold.

(i) *If V is bounded, then $(\mathcal{A}_3) \Rightarrow (\mathcal{A}_3') \Leftrightarrow (\mathcal{A}_2) \Leftrightarrow (\mathcal{A}_1)$;*
(ii) *If V is bounded and surjective, then $(\mathcal{A}_3) \Leftrightarrow (\mathcal{A}_3') \Leftrightarrow (\mathcal{A}_2) \Leftrightarrow (\mathcal{A}_1)$.*

So, when a θ-pseudo-coherent Bessel multiplier $V = M_{(1),(\theta(\lambda)v)_{\lambda\in\Lambda},(\theta(\lambda)u)_{\lambda\in\Lambda}}$ is surjective, it can always be written as the θ-pseudo-coherent frame multiplier $M_{(1),(\theta(\lambda)Vg),(\widetilde{g}_\lambda)}$, using any θ-pseudo-coherent frame $(g_\lambda)_{\lambda\in\Lambda} = (\theta(\lambda)g)_{\lambda\in\Lambda}$ for $\mathcal{H}$. Note that when a θ-pseudo-coherent Bessel multiplier $M_{(1),(\theta(\lambda)v)_{\lambda\in\Lambda},(\theta(\lambda)u)_{\lambda\in\Lambda}}$ is surjective (resp. invertible), then $(\theta(\lambda)v)_{\lambda\in\Lambda}$ is already a frame (resp. $(\theta(\lambda)u)_{\lambda\in\Lambda}$ and $(\theta(\lambda)v)_{\lambda\in\Lambda}$ are frames) for $L^2(\mathbb{R}^d)$ (see e.g. [8, Proposition 4.2] (resp. see [31, Proposition 3.1])).

5.2. Gabor multipliers

Consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$. Let $\Lambda = \{(\omega,\tau)\}$ be a lattice in $\mathbb{R}^{2d}$, i.e., a discrete subgroup of $\mathbb{R}^{2d}$ of the form $A\mathbb{Z}^{2d}$ for some invertible matrix A. For $\omega \in \mathbb{R}^d$ and $\tau \in \mathbb{R}^d$, recall the modulation operator $E_\omega : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ and the translation operator $T_\tau : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ given by $(E_\omega f)(x) = e^{2\pi i\omega x}f(x)$ and $(T_\tau f)(x) = f(x-\tau)$. For $\lambda = (\omega,\tau) \in \Lambda$ and $\lambda' = (\omega',\tau') \in \Lambda$, take $\theta(\lambda) = \pi(\lambda) = E_\omega T_\tau$, $\mu(\lambda,\lambda') = \lambda' - \lambda$, and $\phi(\lambda,\lambda') = e^{2\pi i\tau(\omega'-\omega)}$. For every $\lambda \in \Lambda$, the mapping $\lambda' \to \mu(\lambda,\lambda')$ is a bijection of Λ onto Λ. Furthermore, the conditions (9) and (10) hold. Indeed, for every $\lambda, \lambda' \in \Lambda$,

$$\theta(\lambda)^*\theta(\lambda') = e^{-2\pi i\tau\omega}\theta(-\lambda)\theta(\lambda') = e^{-2\pi i\tau\omega}e^{2\pi i\tau\omega'}\theta(\lambda'-\lambda) = \phi(\lambda,\lambda')\theta(\lambda'-\lambda) = \phi(\lambda,\lambda')\theta(\mu(\lambda,\lambda')),$$
$$\theta(\lambda)\theta(\mu(\lambda,\lambda')) = \theta(\lambda)\theta(\lambda'-\lambda) = e^{-2\pi i\tau(\omega'-\omega)}\theta(\lambda') = \overline{\phi(\lambda,\lambda')}\theta(\lambda').$$

Thus, the θ-pseudo-coherent frames $(\theta(\lambda)g)_{\lambda\in\Lambda}$ for $\mathcal{H}$ in this case are the Gabor frames $(E_\omega T_\tau g)_{(\omega,\tau)\in\Lambda}$ for $L^2(\mathbb{R}^d)$. Therefore, all the results in Subsection 5.1 hold for the case of Gabor frames. We restate Proposition 5.2 for Gabor frames, having in mind their applicability.

Proposition 5.5. *Let $g \in L^2(\mathbb{R}^d)$ and let $(\pi(\lambda)g)_{\lambda\in\Lambda}$ be a Gabor frame for $L^2(\mathbb{R}^d)$. Let $V : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be a bounded bijective operator. Then the following statements are equivalent.*

($\mathcal{A}_1$) *For every $\lambda \in \Lambda$, $V\pi(\lambda)g = \pi(\lambda)Vg$.*
($\mathcal{A}_2$) *For every $\lambda \in \Lambda$ and every $f \in L^2(\mathbb{R}^d)$, $V\pi(\lambda)f = \pi(\lambda)Vf$ (i.e., V commutes with $\pi(\lambda)$ for every $\lambda \in \Lambda$).*
($\mathcal{A}_3$) *V can be written as a Gabor frame multiplier with the constant symbol* (1) *and lattice Λ, i.e., V is a Gabor frame-type operator with respect to the lattice Λ.*
($\mathcal{A}_4$) *V^{-1} can be written as a Gabor frame multiplier with the constant symbol* (1) *and lattice Λ, i.e., V^{-1} is a Gabor frame-type operator with respect to the lattice Λ.*
($\mathcal{A}_5$) *For every $\lambda \in \Lambda$, V^* commutes with $\pi(\lambda)$.*

Remark 5.6. This result gives a nice representation and criterion for TF-lattice invariant operators [15], which correspond to condition ($\mathcal{A}_2$). Motivated by [13], the condition ($\mathcal{A}_1$) can be considered to define ‘locally TF-lattice invariant’ operators. We have shown that this local property already implies the global one.

Note that Proposition 5.5 (as well as Lemma 5.4(ii)) also answers the question for a characterization of operators, which can be represented as Gabor frame-type operators. Based on another approach, the parallel work [26] concerns characterization of operators, which can be written as Gabor frame operators.

For representations of Gabor frame-type operators, using the adjoint lattice, see e.g. [16,18]. Necessary and sufficient conditions for the invertibility of Gabor frame-type operators are given in [35].

In the finite-dimensional case, for matrices commuting with certain time-frequency shifts and fast algorithms for approximate or exact inversion of Gabor frame operators, see e.g. [6,22].

5.3. Multipliers for coherent frames

Let $(G, \cdot)$ be a locally compact group. Let $(\pi, \mathcal{H})$ be a unitary representation of G, i.e., a strongly continues homomorphism from G into the group of unitary operators on $\mathcal{H}$, which means that π satisfies the properties $\pi(\lambda \cdot \lambda') = \pi(\lambda)\pi(\lambda')$, $\pi(\lambda)^* = \pi(\lambda^{-1}) = \pi(\lambda)^{-1}$, and $\lim_{n\to\infty} \lambda_n = \lambda$ and $h \in \mathcal{H}$ imply $\lim_{n\to\infty} \pi(\lambda_n)h = \pi(\lambda)h$ (see, e.g., [20, Section 9.2]). Consider a discrete subgroup Λ of G. For $\lambda \in \Lambda$ and $\lambda' \in \Lambda$, take $\theta(\lambda) = \pi(\lambda)$, $\mu(\lambda, \lambda') = \lambda^{-1} \cdot \lambda'$, and $\phi(\lambda, \lambda') = 1$, which implies that

$$\theta(\lambda)^*\theta(\lambda') = \pi(\lambda^{-1})\pi(\lambda') = \pi(\lambda^{-1} \cdot \lambda') = \phi(\lambda, \lambda')\theta(\mu(\lambda, \lambda')),$$
$$\theta(\lambda)\theta(\mu(\lambda, \lambda')) = \pi(\lambda)\pi(\lambda^{-1} \cdot \lambda') = \pi(\lambda') = \overline{\phi(\lambda, \lambda')}\theta(\lambda').$$

Thus, for $g \in \mathcal{H}$, $g \neq 0$, a sequence $(g_\lambda)_{\lambda\in\Lambda} = (\pi(\lambda)g)_{\lambda\in\Lambda}$ is a π-pseudo-coherent system and a frame $(g_\lambda)_{\lambda\in\Lambda}$ is a π-pseudo-coherent frame (a coherent frame, see e.g. [21]). Thus, all the results in Subsection 5.1 hold for the case of multipliers for coherent frames.

As a particular example of coherent frames, consider frames of translates:

Example 5.7. Consider the locally compact group $(\mathbb{R}^d, +)$. Let $\pi(\lambda) = T_\lambda$, $\lambda \in \mathbb{R}^d$, and let $\mathcal{H}$ be a closed subspace of $L^2(\mathbb{R}^d)$. Then $(\pi, \mathcal{H})$ is a unitary representation of $\mathbb{R}^d$. Let Λ be a discrete subgroup of $\mathbb{R}^d$. Then a frame of translates $(\pi(\lambda)g)_{\lambda\in\Lambda}$, $g \in \mathcal{H}$, $g \neq 0$, is a π-pseudo-coherent frame and all the results in Subsection 5.1 hold for the case of multipliers for frames of translates.

By Proposition 5.2, one obtains a characterization of the bounded bijective operators on $\mathcal{H}$ which commute with the operators of translates on a discrete subgroup. For another structure theorem for operators from

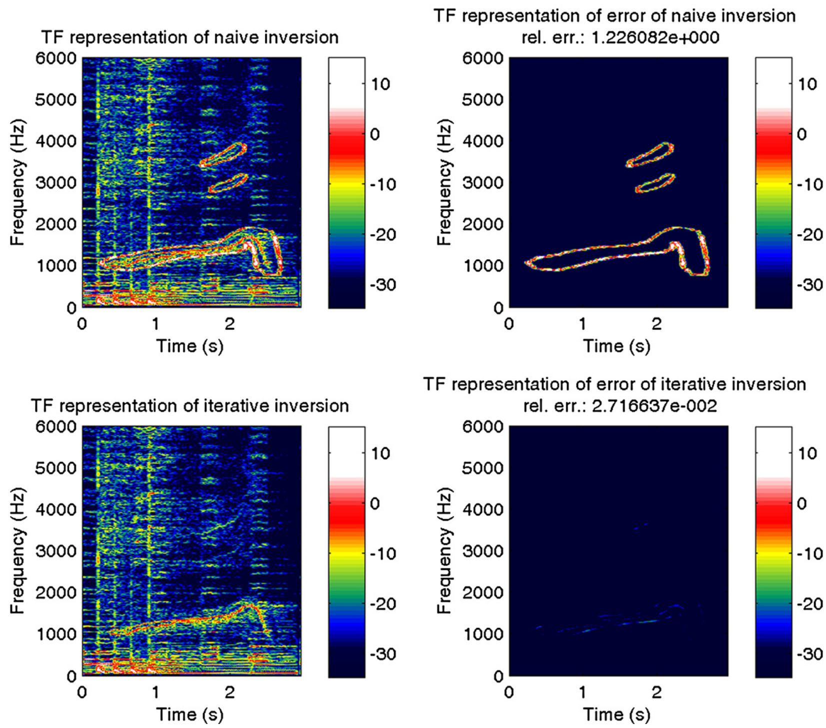

Fig. 2. *Inversion of multipliers.* Time-frequency representation of (top left) the result of the 'naive' inversion $\dot{f}$, (top right) the error of the 'naive' inversion, i.e. $\dot{f} - f$, (bottom left) the iterative inversion $\ddot{f}$, (bottom right) the error of the iterative inversion $\ddot{f} - f$.

the Schwartz space $S(\mathbb{R}^d)$ into the tempered distributions $S'(\mathbb{R}^d)$, which commute with a discrete subgroup of translations, we refer to [14].

6. Numerical visualization of results

Let us come back to the example in Fig. 1. Here we use the same signal f, Gabor frame Ψ, and symbol m, as in Fig. 1. Note that all the elements of the symbol m fulfill $m_{n,k} \in \{1, 10^{-2}\}$ and denote $M = M_{m,\widetilde{\Psi},\Psi}$. Since m is semi-normalized, the multiplier M is analytically invertible [31, Proposition 4.3]. However, the operator is badly conditioned, the condition number is around 99. As mentioned before, the signal f is approximately 2 seconds long, using a sampling rate of 44 100. So the signal is a 128 148-dimensional vector.

Starting from $g = Mf$, we compare two approaches numerically:

1. a 'naive' inversion $\dot{f} = M_{1/m,\widetilde{\Psi},\Psi} g$ (corresponding to the approach raised in $[Q2]$),
2. and the 'iterative' inversion $\ddot{f} = M^{-1}g$. For numerical efficiency and in particular, for memory constraints, we use the iterative inversion in LTFAT, using a conjugate gradient method (for M^*M). Note that by Theorem 1.1, $M^{-1}g$ corresponds to $M_{1/m,\Psi^\dagger,\Psi}g$, where $\psi^\dagger_{n,k} = (M^{-1}(m_{n,k}\widetilde{\psi}_{n,k}))$.

For results see Fig. 2. Clearly the naive approach has strong artifacts. The error is especially big at the boundaries of the constant region of the symbols. The chosen atoms are well localized in time-frequency, so that within the interior of the constant regions, this inversion works well. This could be expected as we have shown in Corollary 4.5 that constant symbols allow this kind of inversion for equivalent frames (so, in particular for Ψ and $\widetilde{\Psi}$).

The iterative inversion worked well with an error of 3%. This could, naturally, be decreased by investing more calculation time. But also in the chosen setting for the iterative inversion (100 iterations in iframemul [29]) no difference can be seen in the time-frequency representation, as well as no audible difference can be detected.

Similar results can also be shown for other redundancies and other sound files.

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