

Stability and convergence of two-level iterative methods for the stationary incompressible magnetohydrodynamics with different Reynolds numbers [☆]



ZhenZhen Tao ^a, Tong Zhang ^{a,b,*}

^a School of Mathematics & Information Science, Henan Polytechnic University, Jiaozuo 454003, PR China

^b Departamento de Matemática, Universidade Federal do Paraná, Curitiba 81531-990, Brazil

ARTICLE INFO

Article history:

Received 13 January 2015
Available online 20 March 2015
Submitted by W. Layton

Keywords:

Two-level methods
Stationary incompressible magnetohydrodynamics
Reynolds numbers
Stability
Convergence

ABSTRACT

In the paper we develop some two-level finite element iterative methods and use these methods to solve the stationary incompressible magnetohydrodynamics (MHD) with different Reynolds numbers under some different uniqueness conditions. Firstly, we use the Stokes type iterative method, Newton type iterative method and Oseen type iterative method of m times on a coarse mesh with mesh size H and then solve a linear problem with the Stokes type, Newton type and Oseen type correction of one time on a fine grid with mesh sizes $h \ll H$. Furthermore, we analyze the uniform stability and convergence of these methods with respect to Reynolds numbers, mesh sizes h and H and iterative times m . Finally, the stationary incompressible MHD equations with large Reynolds number are researched by the one-level finite element method based on the Oseen type iteration on a fine mesh under a weak uniqueness condition.

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1. Introduction

In this paper we consider the following stationary incompressible MHD equations [4]:

$$\begin{cases} -R_e^{-1} \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - S_c \operatorname{curl} \mathbf{B} \times \mathbf{B} = \mathbf{f} & \text{in } \Omega, \\ S_c R_m^{-1} \operatorname{curl}(\operatorname{curl} \mathbf{B}) - S_c \operatorname{curl}(\mathbf{u} \times \mathbf{B}) = \mathbf{g} & \text{in } \Omega, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ \nabla \cdot \mathbf{B} = 0 & \text{in } \Omega, \end{cases} \quad (1.1)$$

[☆] This work was supported by the NSF of China (No. 11301157), CAPES and CNPq of Brazil, the Natural Science Foundation of Education Department of Henan Province (No. 14A110008, 15A110001), the Doctor Fund of Henan Polytechnic University (B2012-098) and the Foundation of Distinguished Young Scientists of Henan Polytechnic University (J2015-05).

* Corresponding author at: School of Mathematics & Information Science, Henan Polytechnic University, Jiaozuo 454003, PR China.

E-mail address: tzhang@hpu.edu.cn (T. Zhang).

subject to boundary conditions

$$\begin{cases} \mathbf{u} = 0 & \text{on } \partial\Omega, \\ \mathbf{B} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \\ \mathbf{n} \times \operatorname{curl} \mathbf{B} = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

where $\mathbf{n}$ is outward normal unit vector of $\partial\Omega$, p is hydrodynamic pressure, R_e, R_m and S_c are the hydrodynamic Reynolds number, magnetic Reynolds number and coupling number, respectively. We set $\Omega \subset R^d (d = 2 \text{ or } 3)$ is a convex polygonal/polyhedral domain (see [11,12,34]). $\mathbf{u}$ is the velocity field, $\mathbf{B}$ denotes the magnetic field, $\mathbf{f}$ and $\mathbf{g}$ are the source terms.

The stationary incompressible MHD is the system that can be used to describe the interaction between a viscous, incompressible, electrically conducting field and an external magnetic field. Obviously, the stationary incompressible MHD equations are a coupled equation system, which is composed of Navier–Stokes equations of fluid dynamics and Maxwell’s equations that couple Lorentz’s force with Ohm’s law. For the researches of problem (1.1), a lot of literature were reported in the last decades. For example, the detailed existence and uniqueness of the solutions for MHD problem were established [11,12]. Numerical approximation of the steady MHD problem was researched by Discacciati [3]. For the Galerkin finite element method for MHD problem, we can refer to [10,13,35] and the references therein. Layton et al. used the two-level Newton finite element method and the two-level discretization method to solve the incompressible MHD problem [31,32]. Aydin et al. combined the stabilized subgrid method with the two-level finite element method for solving the incompressible MHD problem [2]. Combining the iterative methods with the Galerkin finite element method, stability and convergence of the 2D/3D stationary incompressible MHD were presented [5]. Later, Dong and He considered the two-level Newton iterative method for the MHD problem under some regularity conditions [4].

Recently, the two-level method was studied in many problems. For example, we can refer to [7,18,19,24,25,28–30,37] for the research of the steady Navier–Stokes equations. For the time-dependent Navier–Stokes equations, we refer to [8,14,15,23,33] in both semi-discretization and full discretization schemes in two level methods. For the multi-level strategy, we can read [20–22,38] for the time-dependent Navier–Stokes equations. From above mentioned literature, we know that the combination of two-level method and iterative method is quite efficient for solving the nonlinear system. Therefore, in this paper we try to extend the two-level iterative methods for the stationary MHD equations and establish the corresponding stability and convergence. Our numerical schemes are very comprehensive because they are inclusive of all situations of applying the three classical iterations (the Stokes type iteration, Newton type iteration and Oseen type iteration) for two-level methods. Furthermore, we provide the most suitable methods under different uniqueness conditions.

We choose a fixed $\alpha > 0$ such that $\alpha(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0})^{\frac{1}{2}} < \frac{2}{5}$ and denote $\sigma = \frac{\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} \|\mathbf{F}\|_{-1}}{(\min\{R_e^{-1}, S_c C_1 R_m^{-1}\})^2}$. $\sigma \in (0, 1)$ is a fixed constant. If σ satisfies $0 < \sigma \leq 1 - \alpha(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0})^{\frac{1}{2}}$, we call it as the strong uniqueness condition. If σ satisfies $1 - \alpha(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0})^{\frac{1}{2}} < \sigma < 1$ we call it as the weak uniqueness condition. In this paper, under different stability conditions, we firstly analyze the stability and convergence of the one-level Stokes type, Newton type and Oseen type iterative methods for the steady incompressible MHD equations on the mixed finite element spaces $\mathbf{W}_{0n}^\mu \times M_\mu$ (μ takes h or H and $H \gg h$). Secondly, for the strong uniqueness condition, we structure three different two-level iterative methods in which on a coarse mesh solving the iterative solution $((\mathbf{u}_H^m, \mathbf{B}_H^m), p_H^m)$ with mesh size H and finding a correction solution $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ by solving the Stokes type, Newton type and Oseen type correction problems on a fine grid with mesh size h . In more detail, for $0 < \sigma \leq \frac{2}{5}$, $((\mathbf{u}_H^m, \mathbf{B}_H^m), p_H^m)$ is gained by applying the Stokes type, Newton type and Oseen type iterative methods of m times; For $\frac{2}{5} < \sigma \leq \frac{5}{11}$, $((\mathbf{u}_H^m, \mathbf{B}_H^m), p_H^m)$ is gained by applying the Newton type and Oseen type iterative methods of m times; For $\frac{5}{11} < \sigma \leq 1 - \alpha(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0})^{\frac{1}{2}}$, $((\mathbf{u}_H^m, \mathbf{B}_H^m), p_H^m)$ is gained by applying the

Oseen type iterative method of m times. We establish the stability and convergence of two-level iterative methods. Finally, under the weak uniqueness condition, we consider the one-level Oseen iterative method with mesh size h and present the corresponding stability and convergence for the numerical solutions.

The rest of this paper is organized as follows. Section 2 introduces some notations and basic results for the steady MHD equations. The stability and convergence of Galerkin finite element solutions are presented in Section 3. In Section 4 we analyze the stability and convergence of the Stokes type, Newton type and Oseen type iterative methods of m times. The uniform stability and convergence of two-level finite element iterative methods are considered with $0 < \sigma \leq 1 - \alpha(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0})^{\frac{1}{2}}$ in Sections 5, 6, and 7, respectively. The uniform stability and convergence of one-level Oseen type iterative method is provided with $1 - \alpha(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0})^{\frac{1}{2}} < \sigma < 1$ in Section 8. Some numerical results are presented in Section 9 to verify the established theoretical results. Finally, conclusions are given in Section 10.

2. The well-posedness and stability for problem (1.1)

To present the weak variational formulation for the stationary incompressible MHD equations, the standard Sobolev space $H^k(\Omega) = W^{k,2}(\Omega)$ is recalled for any nonnegative integer k with norm $\|v\|_k = (\sum_{\gamma=0}^k \|D^\gamma v\|_0^2)^{\frac{1}{2}}$. The standard Sobolev space $\mathbf{H}^k(\Omega) = (H^k(\Omega))^d$ equipped with the corresponding norm $\|\mathbf{v}\|_k = (\sum_{i=1}^d \|v_i\|_k^2)^{\frac{1}{2}}$ are used [1,9]. Furthermore, we introduce the following spaces

$$\begin{aligned}\mathbf{X} &= \mathbf{H}_0^1(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v}|_{\partial\Omega} = 0\}, \\ M &= L_0^2(\Omega) = \{q \in L^2(\Omega) : (q, 1)_\Omega = 0\}, \\ \mathbf{W} &= \{\mathbf{w} \in \mathbf{H}^1(\Omega) : \mathbf{w} \cdot \mathbf{n}|_{\partial\Omega} = 0\}, \\ \mathbf{V} &= \{\mathbf{v} \in \mathbf{X} : \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega\}, \\ \mathbf{V}_n &= \{\mathbf{w} \in \mathbf{W} : \nabla \cdot \mathbf{w} = 0 \text{ in } \Omega\}.\end{aligned}$$

With the equivalent norm $\|\nabla \mathbf{w}\|_0 = \|\mathbf{w}\|_{\mathbf{H}_0^1(\Omega)}$ of $\mathbf{X}$, we define the product space $\mathbf{W}_{0n} = \mathbf{X} \times \mathbf{W}$ equipped with the usual graph norm $\|(\mathbf{w}, \Phi)\|_1, \forall (\mathbf{w}, \Phi) \in \mathbf{W}_{0n}$, where $\|(\mathbf{w}, \Phi)\|_i = (\|\mathbf{w}\|_i^2 + \|\Phi\|_i^2)^{1/2} (i = 0, 1, 2)$. The dual space of $\mathbf{H}_0^1(\Omega)$ is denoted as $\mathbf{H}^{-1}(\Omega)$ equipped with the norm $\|\cdot\|_{-1}$, see [5,9] for details.

Moreover, the following two formulas

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \cdot \mathbf{d} = (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = -(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{d} \times \mathbf{c}), \quad (2.1)$$

and

$$\int_{\Omega} (\nabla \times \Phi) \cdot \Psi dx = - \int_{\partial\Omega} (\Phi \times \mathbf{n}) \cdot \Psi ds + \int_{\Omega} \Phi \cdot (\nabla \times \Psi) dx, \quad (2.2)$$

imply that

$$\begin{aligned}(\operatorname{curl}(\mathbf{w} \times \Phi), \Psi) &= -\langle (\mathbf{w} \times \Phi) \times \mathbf{n}, \Psi \rangle|_{\partial\Omega} + (\mathbf{w} \times \Phi, \operatorname{curl} \Psi) \\ &= (\mathbf{w} \times \Phi, \operatorname{curl} \Psi) = -(\operatorname{curl} \Psi \times \Phi, \mathbf{w}), \quad \forall \mathbf{w} \in \mathbf{X}, \Phi, \Psi \in \mathbf{W}.\end{aligned} \quad (2.3)$$

We define the trilinear form as follows:

$$a_1(\mathbf{u}, \mathbf{w}, \mathbf{v}) = (\mathbf{u} \cdot \nabla \mathbf{w} + \frac{1}{2}(\nabla \cdot \mathbf{u})\mathbf{w}, \mathbf{v}) = \frac{1}{2}(\mathbf{u} \cdot \nabla \mathbf{w}, \mathbf{v}) - \frac{1}{2}(\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}), \quad \forall \mathbf{u}, \mathbf{w}, \mathbf{v} \in \mathbf{X}. \quad (2.4)$$

With above notations, the weak form of the steady incompressible MHD problem (1.1) can be rewritten as: Find $((\mathbf{u}, \mathbf{B}), p) \in \mathbf{W}_{0\mathbf{n}} \times M$ such that

$$\begin{aligned} A_0((\mathbf{u}, \mathbf{B}), (\mathbf{v}, \Psi)) + A_1((\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B}), (\mathbf{v}, \Psi)) - d_0((\mathbf{v}, \Psi), p) + d_0((\mathbf{u}, \mathbf{B}), q) \\ = \langle \mathbf{F}, (\mathbf{v}, \Psi) \rangle, \quad \forall ((\mathbf{v}, \Psi), q) \in \mathbf{W}_{0\mathbf{n}} \times M, \end{aligned} \quad (2.5)$$

where

$$\begin{aligned} A_0((\mathbf{u}, \mathbf{B}), (\mathbf{v}, \Psi)) &= a_0(\mathbf{u}, \mathbf{v}) + b_0(\mathbf{B}, \Psi), \\ A_1((\mathbf{w}, \Phi), (\mathbf{u}, \mathbf{B}), (\mathbf{v}, \Psi)) &= a_1(\mathbf{w}, \mathbf{u}, \mathbf{v}) - c(\mathbf{B}, \Phi, \mathbf{v}) + c(\Psi, \Phi, \mathbf{u}), \\ d_0((\mathbf{v}, \Psi), q) &= (\nabla \cdot \mathbf{v}, q), \quad a_0(\mathbf{u}, \mathbf{v}) = R_e^{-1}(\nabla \mathbf{u}, \nabla \mathbf{v}), \\ b_0(\mathbf{B}, \Psi) &= S_c R_m^{-1}(\nabla \times \mathbf{B}, \nabla \times \Psi) + S_c R_m^{-1}(\nabla \cdot \mathbf{B}, \nabla \cdot \Psi), \\ c(\mathbf{B}, \Phi, \mathbf{v}) &= S_c(\text{curl} \mathbf{B} \times \Phi, \mathbf{v}), \quad \langle \mathbf{F}, (\mathbf{v}, \Psi) \rangle = \langle \mathbf{f}, \mathbf{v} \rangle + \langle \mathbf{g}, \Psi \rangle. \end{aligned}$$

In addition, we define

$$\|\mathbf{F}\|_{-1} = \sup_{(\mathbf{0}, \mathbf{0}) \neq (\mathbf{v}, \Psi) \in \mathbf{W}_{0\mathbf{n}}} \frac{\langle \mathbf{F}, (\mathbf{v}, \Psi) \rangle}{\|(\mathbf{v}, \Psi)\|_1}.$$

The following properties of trilinear form $a_1(\cdot, \cdot, \cdot)$ will be often used in following sections:

$$a_1(\mathbf{w}, \mathbf{u}, \mathbf{v}) = -a_1(\mathbf{w}, \mathbf{v}, \mathbf{u}), \quad \forall \mathbf{w}, \mathbf{u}, \mathbf{v} \in X, \quad (2.6)$$

$$|a_1(\mathbf{w}, \mathbf{u}, \mathbf{v})| \leq C_0^2 \|\nabla \mathbf{w}\|_0 \|\nabla \mathbf{u}\|_0 \|\nabla \mathbf{v}\|_0, \quad \forall \mathbf{w}, \mathbf{u}, \mathbf{v} \in \mathbf{X}. \quad (2.7)$$

Here C_0 (only dependent on Ω) is an embedding constant from the inequality:

$$\|\mathbf{w}\|_{\mathbf{L}^4} \leq C_0 \|\nabla \mathbf{w}\|_0, \quad \forall \mathbf{w} \in \mathbf{X}.$$

It is easy to see that the trilinear form $A_1(\cdot, \cdot, \cdot)$ is skew symmetric with respect to the later two variables, and it satisfies

$$A_1((\mathbf{w}, \Phi), (\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B})) = 0, \quad \forall (\mathbf{w}, \Phi), (\mathbf{u}, \mathbf{B}) \in \mathbf{W}_{0\mathbf{n}}. \quad (2.8)$$

In order to gain the well-posedness of the problem (2.5), we recall the coercivity and continuity of $A_0(\cdot, \cdot)$ and continuous property of $A_1(\cdot, \cdot, \cdot)$ (see [12]): For all $(\mathbf{w}, \Phi), (\mathbf{u}, \mathbf{B}), (\mathbf{v}, \Psi) \in \mathbf{W}_{0\mathbf{n}}$, such that

$$A_0((\mathbf{u}, \mathbf{B}), (\mathbf{v}, \Psi)) \leq \max\{R_e^{-1}, (2+d)S_c R_m^{-1}\} \|(\mathbf{u}, \mathbf{B})\|_1 \|(\mathbf{v}, \Psi)\|_1, \quad (2.9)$$

$$A_0((\mathbf{u}, \mathbf{B}), (\mathbf{u}, \mathbf{B})) \geq \min\{R_e^{-1}, S_c C_1 R_m^{-1}\} \|(\mathbf{u}, \mathbf{B})\|_1^2, \quad (2.10)$$

$$A_1((\mathbf{w}, \Phi), (\mathbf{u}, \mathbf{B}), (\mathbf{v}, \Psi)) \leq \sqrt{2} C_0^2 \max\{1, \sqrt{2} S_c\} \|(\mathbf{w}, \Phi)\|_1 \|(\mathbf{u}, \mathbf{B})\|_1 \|(\mathbf{v}, \Psi)\|_1, \quad (2.11)$$

where C_1 (depending only on Ω) is the constant that comes from the following inequality:

$$\|\nabla \times \Psi\|_0^2 + \|\nabla \cdot \Psi\|_0^2 \geq C_1 \|\Psi\|_1^2, \quad \forall \Psi \in \mathbf{W},$$

$\sqrt{2}$ and d come from two inequalities as follows (d is the dimension of the considered domain):

$$\|\text{curl} \mathbf{v}\|_0 \leq \sqrt{2} \|\nabla \mathbf{v}\|_0, \quad \|\nabla \cdot \mathbf{v}\|_0 \leq \sqrt{d} \|\nabla \mathbf{v}\|_0.$$

Furthermore, the following properties of $A_1(\cdot, \cdot, \cdot)$ hold [5,9].

$$|A_1((\mathbf{w}, \Phi), (\mathbf{u}, \mathbf{B}), (\mathbf{v}, \Psi))| \leq C\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} \|(\mathbf{w}, \Phi)\|_0 \|(\mathbf{u}, \mathbf{B})\|_2 \|(\mathbf{v}, \Psi)\|_1, \\ \forall (\mathbf{w}, \Phi) \in \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega), (\mathbf{u}, \mathbf{B}) \in \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega), (\mathbf{v}, \Psi) \in \mathbf{W}_{0n}, \quad (2.12)$$

$$|A_1((\mathbf{w}, \Phi), (\mathbf{u}, \mathbf{B}), (\mathbf{v}, \Psi))| \leq C\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} \|(\mathbf{w}, \Phi)\|_2 \|(\mathbf{u}, \mathbf{B})\|_1 \|(\mathbf{v}, \Psi)\|_0, \\ \forall (\mathbf{w}, \Phi) \in \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega), (\mathbf{u}, \mathbf{B}) \in \mathbf{W}_{0n}, (\mathbf{v}, \Psi) \in \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega). \quad (2.13)$$

The letter $C > 0$ denotes different constants at different places, and C is independent of the mesh size μ and the iterative times m .

The bilinear form $d_0(\cdot, \cdot)$ is continuous on $\mathbf{W}_{0n} \times M$, and it satisfies the inf-sup condition (see [9,12]):

$$\sup_{(\mathbf{v}, \Psi) \in \mathbf{W}_{0n}} \frac{|d_0((\mathbf{v}, \Psi), q)|}{\|(\mathbf{v}, \Psi)\|_1} \geq \beta_0 \|q\|_0, \quad \forall q \in M. \quad (2.14)$$

Furthermore, for all $\mathbf{w} \in \mathbf{H}^i(\Omega) \cap \mathbf{X}$, $\Phi \in \mathbf{H}^i(\Omega) \cap \mathbf{W}$, $i = 0, 1, 2$, we set

$$|||(\mathbf{w}, \Phi)|||_i = \min\{R_e^{-1}, S_c C_1 R_m^{-1}\} (\|\mathbf{w}\|_i^2 + \|\Phi\|_i^2)^{\frac{1}{2}}.$$

Assumption A0. The steady Stokes problem

$$-\Delta \mathbf{v} + \nabla q = \mathbf{f}, \quad \nabla \cdot \mathbf{v} = 0, \quad \text{in } \Omega, \quad \mathbf{v}|_{\partial\Omega} = 0,$$

with function $\mathbf{f} \in \mathbf{L}^2(\Omega)$ has a unique solution $(\mathbf{v}, q)$, which satisfies

$$\|\mathbf{v}\|_2 + \|q\|_1 \leq C\|\mathbf{f}\|_0,$$

and Maxwell's equations

$$\nabla \times (\nabla \times \Psi) = \mathbf{g}, \quad \nabla \cdot \Psi = 0 \quad \text{in } \Omega, \quad \nabla \times \Psi \times \mathbf{n} = 0, \quad \Psi \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega,$$

with function $\mathbf{g} \in \mathbf{L}^2(\Omega)$ exists a unique solution $\Psi \in \mathbf{V}_n$ equipped with

$$\|\Psi\|_2 \leq C\|\mathbf{g}\|_0.$$

If Ω is a convex polygon/polyhedron or $\partial\Omega$ is of C^2 , [Assumption A0](#) holds.

We end this section by recalling two important theorems, which can be found in [4,39].

Theorem 2.1. Suppose that R_e, R_m, S_c and C_1 satisfy the uniqueness condition

$$0 < \sigma = \frac{\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} \|\mathbf{F}\|_{-1}}{(\min\{R_e^{-1}, S_c C_1 R_m^{-1}\})^2} < 1, \quad (2.15)$$

then problem (2.5) admits a unique solution $((\mathbf{u}, \mathbf{B}), p) \in \mathbf{W}_{0n} \times M$. Moreover,

$$|||(\mathbf{u}, \mathbf{B})|||_1 \leq \|\mathbf{F}\|_{-1}. \quad (2.16)$$

Theorem 2.2. If $\mathbf{f}, \mathbf{g} \in \mathbf{L}^2(\Omega)$, under the assumptions of $0 < \sigma < 1$ and [Assumption A0](#), the solution $((\mathbf{u}, \mathbf{B}), p)$ of (2.5) satisfies

$$|||(\mathbf{u}, \mathbf{B})|||_2 + \|p\|_1 \leq C\|\mathbf{F}\|_0. \quad (2.17)$$

3. The stability and convergence of Galerkin finite element method

We set $\{\tau_\mu\}$ is a family of triangulations or tetrahedrons of Ω , and τ_μ is a shape-regular partition of Ω with mesh size μ . $\mu > 0$ is real parameters h or H ($h \ll H$) tending to 0. We take the fine grid partition τ_h as a mesh refinement generated from the coarse grid τ_H . Based on the regular partitions τ_h and τ_H , we can construct conforming finite element spaces $(\mathbf{X}_h, M_h, \mathbf{W}_h)$ and $(\mathbf{X}_H, M_H, \mathbf{W}_H) \subset (\mathbf{X}_h, M_h, \mathbf{W}_h)$.

We set that $\mathbf{W}_{0n}^\mu = \mathbf{X}_\mu \times \mathbf{W}_\mu$ and $\mathbf{W}_{0n}^\mu \times M_\mu$ satisfy the following assumptions which are needed in the following analysis (see [5,9,25]).

Assumption A1. There exist a mapping $r_\mu \in \mathcal{L}(\mathbf{H}^2(\Omega) \cap \mathbf{V}, \mathbf{X}_\mu)$ which satisfies

$$(\nabla \cdot (\mathbf{v} - r_\mu \mathbf{v}), q) = 0, \quad \|\nabla(\mathbf{v} - r_\mu \mathbf{v})\|_0 \leq C\mu \|\mathbf{v}\|_2, \quad \forall \mathbf{v} \in \mathbf{H}^2(\Omega) \cap \mathbf{V}, \forall q \in M_\mu,$$

and an L^2 -orthogonal projection operator $\rho_\mu : M \rightarrow M_\mu$ which satisfies

$$\|q - \rho_\mu q\|_0 \leq C\mu \|q\|_1, \quad \forall q \in H^1(\Omega) \cap M,$$

and a mapping $R_\mu \in \mathcal{L}(\mathbf{H}^2(\Omega) \cap \mathbf{V}_n, \mathbf{W}_\mu)$ which satisfies

$$\begin{aligned} (\nabla \times R_\mu \Phi, \nabla \times \Psi) + (\nabla \cdot R_\mu \Phi, \nabla \cdot \Psi) &= (\nabla \times \Phi, \nabla \times \Psi) + (\nabla \cdot \Phi, \nabla \cdot \Psi) \\ &= (\nabla \times \Phi, \nabla \times \Psi), \quad \forall \Psi \in \mathbf{W}_\mu, \\ \|\Phi - R_\mu \Phi\|_0 + \mu \|\Phi - R_\mu \Phi\|_1 &\leq C\mu^2 \|\Phi\|_2, \quad \forall \Phi \in \mathbf{H}^2(\Omega) \cap \mathbf{V}_n. \end{aligned}$$

Assumption A2. Assume that the finite element space $\mathbf{W}_{0n}^\mu$ satisfies the discrete inf-sup condition, namely, there exists a positive constant β_0 such that:

$$\sup_{(\mathbf{v}, \Psi) \in \mathbf{W}_{0n}^\mu} \frac{|d_0((\mathbf{v}, \Psi), q)|}{\|(\mathbf{v}, \Psi)\|_1} \geq \beta_0 \|q\|_0, \quad \forall q \in M_\mu.$$

We should underscore that there are a variety of finite element spaces satisfying [Assumption A2](#) when Ω is a convex polygonal or polyhedral domain. Here, we choose the stable finite element spaces that have been used traditionally for the Navier–Stokes equations to approximate velocity and pressure. For example, we choose the velocity and pressure finite element spaces as follows:

$$\mathbf{X}_\mu = (P_{1,\mu}^b)^d \cap \mathbf{X}, \quad M_\mu = \{q_\mu \in C^0(\Omega) : q_\mu|_K \in P_1(K), \quad \forall K \in \tau_\mu\},$$

where

$$P_{1,\mu}^b = \{v_\mu \in C^0(\Omega) : v_\mu|_K \in P_1(K) \oplus \text{span}\{\hat{b}\}, \quad \forall K \in \tau_\mu\},$$

$P_1(K)$ is denoted as the space of polynomials of degree (the degree ≤ 1 on K), and $\hat{b}$ is a bubble function. From [12] we know there is unrestricted on the magnetic field approximation space $\mathbf{W}_\mu$, hence we can choose any approximate subspace of $\mathbf{W}$. For the sake of convenience, we choose the same finite element space for the magnetic field space as the one for velocity field, such as $\mathbf{W}_\mu = (P_{1,\mu}^b)^d \cap \mathbf{W}$.

The Galerkin finite element method for problem (2.5) reads as: Find $((\mathbf{u}_\mu, \mathbf{B}_\mu), p_\mu) \in \mathbf{W}_{0n}^\mu \times M_\mu$ such that

$$\begin{aligned} A_0((\mathbf{u}_\mu, \mathbf{B}_\mu), (\mathbf{v}, \Psi)) + A_1((\mathbf{u}_\mu, \mathbf{B}_\mu), (\mathbf{u}_\mu, \mathbf{B}_\mu), (\mathbf{v}, \Psi)) - d_0((\mathbf{v}, \Psi), p_\mu) \\ + d_0((\mathbf{u}_\mu, \mathbf{B}_\mu), q) = \langle \mathbf{F}, (\mathbf{v}, \Psi) \rangle, \quad \forall ((\mathbf{v}, \Psi), q) \in \mathbf{W}_{0n}^\mu \times M_\mu. \end{aligned} \quad (3.1)$$

Using the similar argument to [Theorem 2.1](#) we can obtain the following conclusion.

Theorem 3.1. *Under the conditions of (2.15) and Assumption A2, the discrete problem (3.1) admits a unique solution $((\mathbf{u}_\mu, \mathbf{B}_\mu), p_\mu) \in \mathbf{W}_{0n}^\mu \times M_\mu$, which satisfies*

$$|||(\mathbf{u}_\mu, \mathbf{B}_\mu)|||_1 \leq \|\mathbf{F}\|_{-1}. \quad (3.2)$$

Now we define the discrete form of the divergence-free space $\mathbf{V}$ as:

$$\mathbf{V}_\mu = \{\mathbf{v} \in \mathbf{X}_\mu : d_0((\mathbf{v}, \Psi), q) = 0, \quad \forall q \in M_\mu, \quad \forall \Psi \in \mathbf{W}_\mu\}.$$

Introduce two L^2 -orthogonal projectors $P_\mu : \mathbf{L}^2(\Omega) \rightarrow \mathbf{V}_\mu$ and $R_{0\mu} : \mathbf{L}^2(\Omega) \rightarrow \mathbf{W}_\mu$. Define the discrete Stokes operator $A_{1\mu} = -P_\mu \Delta_\mu$, where Δ_μ is defined by (see [27,36])

$$-(\Delta_\mu \mathbf{u}_\mu, \mathbf{v}_\mu) = (\nabla \mathbf{u}_\mu, \nabla \mathbf{v}_\mu), \quad \forall \mathbf{u}_\mu, \mathbf{v}_\mu \in \mathbf{X}_\mu,$$

and its corresponding discrete norm is $\|\mathbf{v}_\mu\|_{k,\mu} = \|A_{1\mu}^{\frac{k}{2}} \mathbf{v}_\mu\|_0$ with the order $k \in R$, in which

$$\|\mathbf{v}_\mu\|_{1,\mu} = \|\nabla \mathbf{v}_\mu\|_0, \quad \|\mathbf{v}_\mu\|_{2,\mu} = \|A_{1\mu} \mathbf{v}_\mu\|_0, \quad \forall \mathbf{v}_\mu \in \mathbf{V}_\mu.$$

Similarly, define the discrete operator $A_{2\mu} \mathbf{B}_\mu = R_{0\mu}(\nabla_\mu \times \nabla \times \mathbf{B}_\mu + \nabla_\mu \nabla \cdot \mathbf{B}_\mu) \in \mathbf{W}_\mu$ as follows (see [16,36])

$$(A_{2\mu} \mathbf{B}_\mu, \Psi) = (A_{2\mu}^{\frac{1}{2}} \mathbf{B}_\mu, A_{2\mu}^{\frac{1}{2}} \Psi) = (\nabla \times \mathbf{B}_\mu, \nabla \times \Psi) + (\nabla \cdot \mathbf{B}_\mu, \nabla \cdot \Psi), \quad \forall \mathbf{B}_\mu, \Psi \in \mathbf{W}_\mu,$$

and its corresponding discrete norm is $\|\mathbf{B}_\mu\|_{k,\mu} = \|A_{2\mu}^{\frac{k}{2}} \mathbf{B}_\mu\|_0$ with the order $k \in R$, in which

$$\begin{aligned} \|\mathbf{B}_\mu\|_{1,\mu}^2 &= \|A_{2\mu}^{\frac{1}{2}} \mathbf{B}_\mu\|_0^2 = \|\nabla \times \mathbf{B}_\mu\|_0^2 + \|\nabla \cdot \mathbf{B}_\mu\|_0^2, \\ \|\mathbf{B}_\mu\|_{2,\mu} &= \|\nabla_\mu \times \nabla \times \mathbf{B}_\mu + \nabla_\mu \nabla \cdot \mathbf{B}_\mu\|_0. \end{aligned}$$

Moreover, we also introduce some discrete estimates as follows (see [27])

$$\|\nabla \mathbf{v}_\mu\|_{\mathbf{L}^3} + \|\mathbf{v}_\mu\|_{\mathbf{L}^\infty} \leq C \|\nabla \mathbf{v}_\mu\|_0^{\frac{1}{2}} \|A_{1\mu} \mathbf{v}_\mu\|_0^{\frac{1}{2}}, \quad \|\nabla \mathbf{v}_\mu\|_{\mathbf{L}^6} \leq C \|A_{1\mu} \mathbf{v}_\mu\|_0, \quad \forall \mathbf{v}_\mu \in \mathbf{V}_\mu.$$

Thanks to the above preparations, we can obtain the following results (see [4,5]).

Lemma 3.2. *For all $(\mathbf{w}_\mu, \Phi_\mu), (\mathbf{u}_\mu, \mathbf{B}_\mu), (\mathbf{v}_\mu, \Psi_\mu) \in \mathbf{W}_{0n}^\mu$, the trilinear $A_1(\cdot, \cdot, \cdot)$ satisfies*

$$\begin{aligned} &|A_1((\mathbf{w}_\mu, \Phi_\mu), (\mathbf{u}_\mu, \mathbf{B}_\mu), (\mathbf{v}_\mu, \Psi_\mu))| + |A_1((\mathbf{u}_\mu, \mathbf{B}_\mu), (\mathbf{w}_\mu, \Phi_\mu), (\mathbf{v}_\mu, \Psi_\mu))| \\ &\leq C \sqrt{2} C_0^2 \max\{1, \sqrt{2} S_c\} \|(\mathbf{w}_\mu, \Phi_\mu)\|_1^{\frac{1}{2}} \|(A_{1\mu} \mathbf{w}_\mu, A_{2\mu} \Phi_\mu)\|_0^{\frac{1}{2}} \|(\mathbf{u}_\mu, \mathbf{B}_\mu)\|_1 \|(\mathbf{v}_\mu, \Psi_\mu)\|_0. \end{aligned} \quad (3.3)$$

Theorem 3.3. *Under Assumptions A0–A2 and the condition (2.15), the solutions of problem (3.1) satisfy*

$$|||(A_{1\mu} \mathbf{u}_\mu, A_{2\mu} \mathbf{B}_\mu)|||_0 \leq C \|\mathbf{F}\|_0. \quad (3.4)$$

Furthermore, the following convergence result holds

$$\delta |||(\mathbf{u} - \mathbf{u}_\mu, \mathbf{B} - \mathbf{B}_\mu)|||_0 + \mu(C_r |||(\mathbf{u} - \mathbf{u}_\mu, \mathbf{B} - \mathbf{B}_\mu)|||_1 + \|p - p_\mu\|_0) \leq C C_r^2 \delta^{-1} \mu^2 \|\mathbf{F}\|_0, \quad (3.5)$$

where $C_r = \max\{R_e^{-1}, S_c C_1 R_m^{-1}\} / \min\{R_e^{-1}, S_c C_1 R_m^{-1}\}$ and $\delta = 1 - \sigma$.

Following the way that used by Dong and He in [5], we can obtain the following conclusions.

Lemma 3.4. For all $(\mathbf{w}_\mu, \Phi_\mu), (\mathbf{u}_\mu, \mathbf{B}_\mu), (\mathbf{v}_\mu, \Psi_\mu) \in \mathbf{W}_{0n}^\mu$, the trilinear $A_1(\cdot, \cdot, \cdot)$ satisfies

$$|A_1((\mathbf{w}_\mu, \Phi_\mu), (\mathbf{u}_\mu, \mathbf{B}_\mu), (\mathbf{v}_\mu, \Psi_\mu))| + |A_1((\mathbf{u}_\mu, \mathbf{B}_\mu), (\mathbf{w}_\mu, \Phi_\mu), (\mathbf{v}_\mu, \Psi_\mu))| \\ \leq C\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} \|(\mathbf{w}_\mu, \Phi_\mu)\|_0 \|(\mathbf{u}_\mu, \mathbf{B}_\mu)\|_1^{\frac{1}{2}} \|(A_{1\mu}\mathbf{u}_\mu, A_{2\mu}\mathbf{B}_\mu)\|_0^{\frac{1}{2}} \|(\mathbf{v}_\mu, \Psi_\mu)\|_1, \quad (3.6)$$

$$|A_1((\mathbf{w}_\mu, \Phi_\mu), (\mathbf{u}_\mu, \mathbf{B}_\mu), (\mathbf{v}_\mu, \Psi_\mu))| + |A_1((\mathbf{u}_\mu, \mathbf{B}_\mu), (\mathbf{w}_\mu, \Phi_\mu), (\mathbf{v}_\mu, \Psi_\mu))| \\ \leq C\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} \|(\mathbf{w}_\mu, \Phi_\mu)\|_0^{\frac{1}{2}} \|(\mathbf{w}_\mu, \Phi_\mu)\|_1^{\frac{1}{2}} \|(\mathbf{u}_\mu, \mathbf{B}_\mu)\|_1 \|(\mathbf{v}_\mu, \Psi_\mu)\|_1. \quad (3.7)$$

4. The stability and convergence of three iterative methods

Recently, the classical iterative methods (Stokes, Newton and Oseen type iterative methods) have been analyzed for the steady Navier–Stokes equations in [17]. Furthermore, these iterative schemes have been designed and considered for the steady MHD problem in [5]. Here we continue to consider the Stokes and Newton iterative methods for the steady MHD equations based on the strong uniqueness conditions $0 < \sigma \leq \frac{2}{5}$ and $0 < \sigma \leq \frac{5}{11}$, respectively. Furthermore, the Oseen iterative method based on the weak uniqueness condition $0 < \sigma < 1$ for problem (1.1) is analyzed. Moreover, in this section we provide the optimal error estimates between the exact solutions and the iterative solutions. The Stokes type, Newton type and Oseen type iterative methods are described as follows.

Iterative method 1 (*The Stokes type iterative method*). For all $n = 1, 2, \dots$, define the iterative solution $((\mathbf{u}_\mu^n, \mathbf{B}_\mu^n), p_\mu^n) \in \mathbf{W}_{0n}^\mu \times M_\mu$ such that

$$A_0((\mathbf{u}_\mu^n, \mathbf{B}_\mu^n), (\mathbf{v}, \Psi)) + A_1((\mathbf{u}_\mu^{n-1}, \mathbf{B}_\mu^{n-1}), (\mathbf{u}_\mu^{n-1}, \mathbf{B}_\mu^{n-1}), (\mathbf{v}, \Psi)) - d_0((\mathbf{v}, \Psi), p_\mu^n) \\ + d_0((\mathbf{u}_\mu^n, \mathbf{B}_\mu^n), q) = \langle \mathbf{F}, (\mathbf{v}, \Psi) \rangle, \quad \forall ((\mathbf{v}, \Psi), q) \in \mathbf{W}_{0n}^\mu \times M_\mu. \quad (4.1)$$

Iterative method 2 (*The Newton type iterative method*). For all $n = 1, 2, \dots$, define the iterative solution $((\mathbf{u}_\mu^n, \mathbf{B}_\mu^n), p_\mu^n) \in \mathbf{W}_{0n}^\mu \times M_\mu$ such that

$$A_0((\mathbf{u}_\mu^n, \mathbf{B}_\mu^n), (\mathbf{v}, \Psi)) + A_1((\mathbf{u}_\mu^{n-1}, \mathbf{B}_\mu^{n-1}), (\mathbf{u}_\mu^n, \mathbf{B}_\mu^n), (\mathbf{v}, \Psi)) + A_1((\mathbf{u}_\mu^n, \mathbf{B}_\mu^n), (\mathbf{u}_\mu^{n-1}, \mathbf{B}_\mu^{n-1}), (\mathbf{v}, \Psi)) \\ - A_1((\mathbf{u}_\mu^{n-1}, \mathbf{B}_\mu^{n-1}), (\mathbf{u}_\mu^{n-1}, \mathbf{B}_\mu^{n-1}), (\mathbf{v}, \Psi)) - d_0((\mathbf{v}, \Psi), p_\mu^n) + d_0((\mathbf{u}_\mu^n, \mathbf{B}_\mu^n), q) \\ = \langle \mathbf{F}, (\mathbf{v}, \Psi) \rangle, \quad \forall ((\mathbf{v}, \Psi), q) \in \mathbf{W}_{0n}^\mu \times M_\mu. \quad (4.2)$$

Iterative method 3 (*The Oseen type iterative method*). For all $n = 1, 2, \dots$, define the iterative solution $((\mathbf{u}_\mu^n, \mathbf{B}_\mu^n), p_\mu^n) \in \mathbf{W}_{0n}^\mu \times M_\mu$ such that

$$A_0((\mathbf{u}_\mu^n, \mathbf{B}_\mu^n), (\mathbf{v}, \Psi)) + A_1((\mathbf{u}_\mu^{n-1}, \mathbf{B}_\mu^{n-1}), (\mathbf{u}_\mu^n, \mathbf{B}_\mu^n), (\mathbf{v}, \Psi)) - d_0((\mathbf{v}, \Psi), p_\mu^n) \\ + d_0((\mathbf{u}_\mu^n, \mathbf{B}_\mu^n), q) = \langle \mathbf{F}, (\mathbf{v}, \Psi) \rangle, \quad \forall ((\mathbf{v}, \Psi), q) \in \mathbf{W}_{0n}^\mu \times M_\mu. \quad (4.3)$$

Here $((\mathbf{u}_\mu^0, \mathbf{B}_\mu^0), p_\mu^0)$ is defined by

$$A_0((\mathbf{u}_\mu^0, \mathbf{B}_\mu^0), (\mathbf{v}, \Psi)) - d_0((\mathbf{v}, \Psi), p_\mu^0) + d_0((\mathbf{u}_\mu^0, \mathbf{B}_\mu^0), q) = \langle \mathbf{F}, (\mathbf{v}, \Psi) \rangle, \quad \forall ((\mathbf{v}, \Psi), q) \in \mathbf{W}_{0n}^\mu \times M_\mu. \quad (4.4)$$

For convenience, we denote $(e^n, b^n) = (\mathbf{u}_\mu - \mathbf{u}_\mu^n, \mathbf{B}_\mu - \mathbf{B}_\mu^n)$ and $\eta^n = p_\mu - p_\mu^n$. Subtracting (4.1), (4.2), (4.3) and (4.4) from (3.1) respectively, we obtain the corresponding error equations for **Iterative methods 1, 2 and 3**: For all $((\mathbf{v}, \Psi), q) \in \mathbf{W}_{0n}^\mu \times M_\mu$, the error $((e^n, b^n), \eta^n)$ satisfies:

Error equation of iterative method 1.

$$A_0((e^n, b^n), (\mathbf{v}, \Psi)) + A_1((e^{n-1}, b^{n-1}), (\mathbf{u}_\mu, \mathbf{B}_\mu), (\mathbf{v}, \Psi)) \\ + A_1((\mathbf{u}_\mu^{n-1}, \mathbf{B}_\mu^{n-1}), (e^{n-1}, b^{n-1}), (\mathbf{v}, \Psi)) - d_0((\mathbf{v}, \Psi), \eta^n) + d_0((e^n, b^n), q) = 0, \quad (4.5)$$

Error equation of iterative method 2.

$$A_0((e^n, b^n), (\mathbf{v}, \Psi)) + A_1((\mathbf{u}_\mu^{n-1}, \mathbf{B}_\mu^{n-1}), (e^n, b^n), (\mathbf{v}, \Psi)) + A_1((e^n, b^n), (\mathbf{u}_\mu^{n-1}, \mathbf{B}_\mu^{n-1}), (\mathbf{v}, \Psi)) \\ + A_1((e^{n-1}, b^{n-1}), (e^{n-1}, b^{n-1}), (\mathbf{v}, \Psi)) - d_0((\mathbf{v}, \Psi), \eta^n) + d_0((e^n, b^n), q) = 0, \quad (4.6)$$

Error equation of iterative method 3.

$$A_0((e^n, b^n), (\mathbf{v}, \Psi)) + A_1((e^{n-1}, b^{n-1}), (\mathbf{u}_\mu, \mathbf{B}_\mu), (\mathbf{v}, \Psi)) + A_1((\mathbf{u}_\mu^{n-1}, \mathbf{B}_\mu^{n-1}), (e^n, b^n), (\mathbf{v}, \Psi)) \\ - d_0((\mathbf{v}, \Psi), \eta^n) + d_0((e^n, b^n), q) = 0, \quad (4.7)$$

and

$$A_0((e^0, b^0), (\mathbf{v}, \Psi)) + A_1((\mathbf{u}_\mu, \mathbf{B}_\mu), (\mathbf{u}_\mu, \mathbf{B}_\mu), (\mathbf{v}, \Psi)) - d_0((\mathbf{v}, \Psi), \eta^0) + d_0((e^0, b^0), q) = 0. \quad (4.8)$$

Theorem 4.1. For all $m \geq 0$, under the conditions of [Theorem 3.3](#), if $0 < \sigma \leq \frac{2}{5}$, the iterative solution $((\mathbf{u}_\mu^m, \mathbf{B}_\mu^m), p_\mu^m)$ defined by [Iterative method 1](#) satisfies

$$\|(\mathbf{u}_\mu^m, \mathbf{B}_\mu^m)\|_1 \leq \frac{6}{5} \|\mathbf{F}\|_{-1}, \quad \|(A_{1\mu} \mathbf{u}_\mu^m, A_{2\mu} \mathbf{B}_\mu^m)\|_0 \leq C \|\mathbf{F}\|_0, \quad (4.9)$$

$$\|(e^m, b^m)\|_1 \leq \left(\frac{11}{5}\sigma\right)^m \frac{2}{5} \|\mathbf{F}\|_{-1}, \quad \|\eta^m\|_0 \leq CC_r \left(\frac{11}{5}\sigma\right)^m \frac{2}{5} \|\mathbf{F}\|_{-1}. \quad (4.10)$$

If $0 < \sigma \leq \frac{5}{11}$, the iterative solution $((\mathbf{u}_\mu^m, \mathbf{B}_\mu^m), p_\mu^m)$ defined by [Iterative method 2](#) satisfies

$$\|(\mathbf{u}_\mu^m, \mathbf{B}_\mu^m)\|_1 \leq \frac{4}{3} \|\mathbf{F}\|_{-1}, \quad \|(A_{1\mu} \mathbf{u}_\mu^m, A_{2\mu} \mathbf{B}_\mu^m)\|_0 \leq C \|\mathbf{F}\|_0, \quad (4.11)$$

$$\|(e^m, b^m)\|_1 \leq \left(\frac{15}{13}\sigma\right)^{2^m-1} \frac{5}{11} \|\mathbf{F}\|_{-1}, \quad \|\eta^m\|_0 \leq CC_r \left(\frac{15}{13}\sigma\right)^{2^m-1} \frac{5}{11} \|\mathbf{F}\|_{-1}. \quad (4.12)$$

If $0 < \sigma < 1$, the iterative solution $((\mathbf{u}_\mu^m, \mathbf{B}_\mu^m), p_\mu^m)$ defined by [Iterative method 3](#) satisfies

$$\|(\mathbf{u}_\mu^m, \mathbf{B}_\mu^m)\|_1 \leq \|\mathbf{F}\|_{-1}, \quad \|(A_{1\mu} \mathbf{u}_\mu^m, A_{2\mu} \mathbf{B}_\mu^m)\|_0 \leq C \|\mathbf{F}\|_0, \quad (4.13)$$

$$\|(e^m, b^m)\|_1 \leq \sigma^m \|\mathbf{F}\|_{-1}, \quad \|\eta^m\|_0 \leq CC_r \sigma^m \|\mathbf{F}\|_{-1}. \quad (4.14)$$

Proof. Eqs. (4.9), (4.11), (4.13) and the first inequality of (4.10), (4.12) and (4.14) can be shown by the similar methods as in the proofs of Lemmas 3, 4 and 5 in [5]. The second inequality of (4.12) has been proven by a similar method as in the proofs of Theorem 4.1 in [4].

Now we prove the second inequality of (4.10). Choosing $(\mathbf{v}, \Psi) = (e^m, b^m)$ and $q = 0$ in (4.5), using the first inequality of (4.10) to derive that

$$\|\eta^m\|_0 \leq C \frac{\max\{R_e^{-1}, S_c C_1 R_m^{-1}\}}{\min\{R_e^{-1}, S_c C_1 R_m^{-1}\}} \|(\mathbf{u}_\mu^m, \mathbf{B}_\mu^m)\|_1 + \frac{\sqrt{2} C_0^2 \max\{1, \sqrt{2} S_c\}}{\min\{R_e^{-1}, S_c C_1 R_m^{-1}\}} \|(\mathbf{u}_\mu, \mathbf{B}_\mu)\|_1 \\ + \|(\mathbf{u}_\mu^{m-1}, \mathbf{B}_\mu^{m-1})\|_1 \|(\mathbf{u}_\mu^{m-1}, \mathbf{B}_\mu^{m-1})\|_1 \\ \leq CC_r \|(\mathbf{u}_\mu^m, \mathbf{B}_\mu^m)\|_1 + \frac{11}{5} \sigma \|(\mathbf{u}_\mu^{m-1}, \mathbf{B}_\mu^{m-1})\|_1 \leq CC_r \left(\frac{11}{5}\sigma\right)^m \frac{2}{5} \|\mathbf{F}\|_{-1}.$$

Next, taking $(\mathbf{v}, \Psi) = (e^m, b^m)$ and $q = 0$ in (4.7), one finds

$$\begin{aligned} \|\eta^m\|_0 &\leq C \frac{\max\{R_e^{-1}, S_c C_1 R_m^{-1}\}}{\min\{R_e^{-1}, S_c C_1 R_m^{-1}\}} \| (e^m, b^m) \|_1 \\ &\quad + \frac{\sqrt{2} C_0^2 \max\{1, \sqrt{2} S_c\}}{\min\{R_e^{-1}, S_c C_1 R_m^{-1}\}} \|(\mathbf{u}_\mu, \mathbf{B}_\mu)\|_1 \| (e^{m-1}, b^{m-1}) \|_1 \\ &\leq C C_r \| (e^m, b^m) \|_1 + \sigma \| (e^{m-1}, b^{m-1}) \|_1. \end{aligned}$$

Thanks to the first inequality of (4.14), we complete the proof. $\square$

Based on Theorems 3.3 and 4.1, we derive the following optimal error estimates.

Theorem 4.2. For all $m \geq 0$, under the assumptions of Theorem 4.1, if $0 < \sigma \leq \frac{2}{5}$, the errors between the exact solution $((\mathbf{u}, \mathbf{B}), p)$ and the iterative solution $((\mathbf{u}_\mu^m, \mathbf{B}_\mu^m), p_\mu^m)$ defined by Iterative method 1 satisfy

$$\|(\mathbf{u} - \mathbf{u}_\mu^m, \mathbf{B} - \mathbf{B}_\mu^m)\|_1 + \|p - p_\mu^m\|_0 \leq C C_r \left\{ C_r \delta^{-1} \mu \|\mathbf{F}\|_0 + \left(\frac{11}{5}\sigma\right)^m \frac{2}{5} \|\mathbf{F}\|_{-1} \right\}. \quad (4.15)$$

If $0 < \sigma \leq \frac{5}{11}$, the errors between the exact solution $((\mathbf{u}, \mathbf{B}), p)$ and the iterative solution $((\mathbf{u}_\mu^m, \mathbf{B}_\mu^m), p_\mu^m)$ defined by Iterative method 2 satisfy

$$\|(\mathbf{u} - \mathbf{u}_\mu^m, \mathbf{B} - \mathbf{B}_\mu^m)\|_1 + \|p - p_\mu^m\|_0 \leq C C_r \left\{ C_r \delta^{-1} \mu \|\mathbf{F}\|_0 + \left(\frac{15}{13}\sigma\right)^{2^m-1} \frac{5}{11} \|\mathbf{F}\|_{-1} \right\}. \quad (4.16)$$

If $0 < \sigma < 1$, the errors between the exact solution $((\mathbf{u}, \mathbf{B}), p)$ and the iterative solution $((\mathbf{u}_\mu^m, \mathbf{B}_\mu^m), p_\mu^m)$ defined by Iterative method 3 satisfy

$$\|(\mathbf{u} - \mathbf{u}_\mu^m, \mathbf{B} - \mathbf{B}_\mu^m)\|_1 + \|p - p_\mu^m\|_0 \leq C C_r \left\{ C_r \delta^{-1} \mu \|\mathbf{F}\|_0 + \sigma^m \|\mathbf{F}\|_{-1} \right\}. \quad (4.17)$$

Remark 4.1. In the case of $0 < \sigma \leq \frac{2}{5}$, Iterative method 1 is the simplest one and Iterative method 2 has the best convergence rate for large m . In the case of $\frac{2}{5} < \sigma \leq \frac{5}{11}$, Iterative method 3 is simpler than Iterative method 2, while Iterative method 2 has better convergence rate than Iterative method 3 for large m . In the case of $\frac{5}{11} < \sigma < 1$, Iterative method 3 is a unique suitable iterative method.

5. Two-level iterative methods with $0 < \sigma \leq \frac{2}{5}$

In the case of $0 < \sigma \leq \frac{2}{5}$, from Theorem 4.1 we know that Iterative methods 1, 2 and 3 are stable and convergent. In this section, we consider the two-level iterative methods in which the iterative solutions $((\mathbf{u}_H^m, \mathbf{B}_H^m), p_H^m)$ of Iterative methods 1, 2 and 3 are obtained on a coarse grid τ_H with mesh size H and the correction solutions $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ of the Stokes type, Newton type and Oseen type correction are presented on a fine grid τ_h with mesh size h . Our two-level iterative methods consist of the following two steps.

Step 1. Find a global coarse grid iterative solution $((\mathbf{u}_H^m, \mathbf{B}_H^m), p_H^m) \in \mathbf{W}_{0n}^H \times M_H$ defined by Iterative methods 1, 2 and 3 on τ_H , respectively.

Step 2. Find a fine grid solution $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh}) \in \mathbf{W}_{0n}^h \times M_h$ defined by the Stokes type, Newton type and Oseen type correction problems on τ_h .

Correction 1 (*Stokes type correction*). Find a fine grid solution $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh}) \in \mathbf{W}_{0n}^h \times M_h$ provided by the following Stokes type correction problem:

$$\begin{aligned} & A_0((\mathbf{u}_{mh}, \mathbf{B}_{mh}), (\mathbf{v}, \Psi)) - d_0((\mathbf{v}, \Psi), p_{mh}) + d_0((\mathbf{u}_{mh}, \mathbf{B}_{mh}), q) \\ & + A_1((\mathbf{u}_H^m, \mathbf{B}_H^m), (\mathbf{u}_{mh}, \mathbf{B}_{mh}), (\mathbf{v}, \Psi)) = \langle \mathbf{F}, (\mathbf{v}, \Psi) \rangle, \quad \forall ((\mathbf{v}, \Psi), q) \in \mathbf{W}_{0n}^h \times M_h. \end{aligned} \quad (5.1)$$

Correction 2 (*Newton type correction*). Find a fine grid solution $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh}) \in \mathbf{W}_{0n}^h \times M_h$ provided by the following Newton type correction problem:

$$\begin{aligned} & A_0((\mathbf{u}_{mh}, \mathbf{B}_{mh}), (\mathbf{v}, \Psi)) - d_0((\mathbf{v}, \Psi), p_{mh}) + d_0((\mathbf{u}_{mh}, \mathbf{B}_{mh}), q) \\ & + A_1((\mathbf{u}_H^m, \mathbf{B}_H^m), (\mathbf{u}_{mh}, \mathbf{B}_{mh}), (\mathbf{v}, \Psi)) + A_1((\mathbf{u}_{mh}, \mathbf{B}_{mh}), (\mathbf{u}_H^m, \mathbf{B}_H^m), (\mathbf{v}, \Psi)) \\ & - A_1((\mathbf{u}_H^m, \mathbf{B}_H^m), (\mathbf{u}_H^m, \mathbf{B}_H^m), (\mathbf{v}, \Psi)) = \langle \mathbf{F}, (\mathbf{v}, \Psi) \rangle, \quad \forall ((\mathbf{v}, \Psi), q) \in \mathbf{W}_{0n}^h \times M_h. \end{aligned} \quad (5.2)$$

Correction 3 (*Oseen type correction*). Find a fine grid solution $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh}) \in \mathbf{W}_{0n}^h \times M_h$ provided by the following Oseen type correction problem:

$$\begin{aligned} & A_0((\mathbf{u}_{mh}, \mathbf{B}_{mh}), (\mathbf{v}, \Psi)) - d_0((\mathbf{v}, \Psi), p_{mh}) + A_1((\mathbf{u}_H^m, \mathbf{B}_H^m), (\mathbf{u}_{mh}, \mathbf{B}_{mh}), (\mathbf{v}, \Psi)) \\ & + d_0((\mathbf{u}_{mh}, \mathbf{B}_{mh}), q) = \langle \mathbf{F}, (\mathbf{v}, \Psi) \rangle, \quad \forall ((\mathbf{v}, \Psi), q) \in \mathbf{W}_{0n}^h \times M_h. \end{aligned} \quad (5.3)$$

Denote $(e_{mh}, b_{mh}) = (\mathbf{u}_h - \mathbf{u}_{mh}, \mathbf{B}_h - \mathbf{B}_{mh})$ and $\eta_{mh} = p_h - p_{mh}$. Subtracting (5.1), (5.2) and (5.3) from (3.1) respectively, we obtain the corresponding error equations for Corrections 1, 2 and 3: Find $((e_{mh}, b_{mh}), \eta_{mh}) \in \mathbf{W}_{0n}^h \times M_h$ such that for all $((\mathbf{v}, \Psi), q) \in \mathbf{W}_{0n}^h \times M_h$,

Error equation based on Stokes type correction.

$$\begin{aligned} & A_0((e_{mh}, b_{mh}), (\mathbf{v}, \Psi)) - d_0((\mathbf{v}, \Psi), \eta_{mh}) + d_0((e_{mh}, b_{mh}), q) \\ & + A_1((\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H), (\mathbf{u}_h, \mathbf{B}_h), (\mathbf{v}, \Psi)) + A_1((\mathbf{u}_H^m, \mathbf{B}_H^m), (\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H), (\mathbf{v}, \Psi)) \\ & + A_1((\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m), (\mathbf{u}_h, \mathbf{B}_h), (\mathbf{v}, \Psi)) \\ & + A_1((\mathbf{u}_H^m, \mathbf{B}_H^m), (\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m), (\mathbf{v}, \Psi)) = 0. \end{aligned} \quad (5.4)$$

Error equation based on Newton type correction.

$$\begin{aligned} & A_0((e_{mh}, b_{mh}), (\mathbf{v}, \Psi)) - d_0((\mathbf{v}, \Psi), \eta_{mh}) + d_0((e_{mh}, b_{mh}), q) \\ & + A_1((e_{mh}, b_{mh}), (\mathbf{u}_H^m, \mathbf{B}_H^m), (\mathbf{v}, \Psi)) + A_1((\mathbf{u}_H^m, \mathbf{B}_H^m), (e_{mh}, b_{mh}), (\mathbf{v}, \Psi)) \\ & + A_1((\mathbf{u}_h - \mathbf{u}_H^m, \mathbf{B}_h - \mathbf{B}_H^m), (\mathbf{u}_h - \mathbf{u}_H^m, \mathbf{B}_h - \mathbf{B}_H^m), (\mathbf{v}, \Psi)) = 0. \end{aligned} \quad (5.5)$$

Error equation based on Oseen type correction.

$$\begin{aligned} & A_0((e_{mh}, b_{mh}), (\mathbf{v}, \Psi)) - d_0((\mathbf{v}, \Psi), \eta_{mh}) + d_0((e_{mh}, b_{mh}), q) \\ & + A_1((\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H), (\mathbf{u}_h, \mathbf{B}_h), (\mathbf{v}, \Psi)) + A_1((\mathbf{u}_H^m, \mathbf{B}_H^m), (e_{mh}, b_{mh}), (\mathbf{v}, \Psi)) \\ & + A_1((\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m), (\mathbf{u}_h, \mathbf{B}_h), (\mathbf{v}, \Psi)) = 0. \end{aligned} \quad (5.6)$$

Theorem 5.1. Under the conditions of Theorem 3.3 and $0 < \sigma \leq \frac{2}{5}$, $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ defined by Iterative methods 1, 2 and 3 with Correction 1 satisfies the following stability and error estimates:

$$\begin{aligned} |||(\mathbf{u}_{mh}, \mathbf{B}_{mh})|||_1 &\leq 2\|\mathbf{F}\|_{-1}, \quad |||(A_{1h}\mathbf{u}_{mh}, A_{2h}\mathbf{B}_{mh})|||_0 \leq C\|\mathbf{F}\|_0, \\ |||(e_{mh}, b_{mh})|||_1 + \|\eta_{mh}\|_0 &\leq CC_r(\tilde{C}H^2\|\mathbf{F}\|_0 + |||(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)|||_1), \end{aligned}$$

where $\tilde{C} = C_r^2\delta^{-2}(\frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}})^{\frac{1}{2}}$.

Proof. Choosing $(\mathbf{v}, \Psi) = (\mathbf{u}_{mh}, \mathbf{B}_{mh})$, $q = p_{mh}$ and $(\mathbf{v}, \Psi) = (A_{1h}\mathbf{u}_{mh}, A_{2h}\mathbf{B}_{mh})$, $q = p_{mh}$ in (5.1), respectively, one finds

$$\begin{aligned} |||(\mathbf{u}_{mh}, \mathbf{B}_{mh})|||_1 &\leq \|\mathbf{F}\|_{-1} + \sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} \|(\mathbf{u}_H^m, \mathbf{B}_H^m)\|_1^2 \\ &\leq \|\mathbf{F}\|_{-1} + \frac{\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\}}{(\min\{R_e^{-1}, S_c C_1 R_m^{-1}\})^2} (\frac{4}{3})^2 \|\mathbf{F}\|_{-1}^2 \leq (1 + \frac{16}{9}\sigma)\|\mathbf{F}\|_{-1}, \end{aligned} \quad (5.7)$$

and

$$\begin{aligned} |||(A_{1h}\mathbf{u}_{mh}, A_{2h}\mathbf{B}_{mh})|||_0 &\leq \|\mathbf{F}\|_0 + C\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} \|(\mathbf{u}_H^m, \mathbf{B}_H^m)\|_1^{\frac{3}{2}} \|(A_{1h}\mathbf{u}_H^m, A_{2h}\mathbf{B}_H^m)\|_0^{\frac{1}{2}} \\ &\leq \|\mathbf{F}\|_0 + C\sigma\|\mathbf{F}\|_1^{\frac{1}{2}}\|\mathbf{F}\|_0^{\frac{1}{2}} \leq C\|\mathbf{F}\|_0. \end{aligned} \quad (5.8)$$

Setting $(\mathbf{v}, \Psi) = (e_{mh}, b_{mh})$, $q = \eta_{mh}$ in (5.4), we have

$$\begin{aligned} |||(e_{mh}, b_{mh})|||_1 &\leq C\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} \|(\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H)\|_0 \|(\mathbf{u}_h, \mathbf{B}_h)\|_1^{\frac{1}{2}} \|(A_{1h}\mathbf{u}_h, A_{2h}\mathbf{B}_h)\|_0^{\frac{1}{2}} \\ &\quad + \|(\mathbf{u}_H^m, \mathbf{B}_H^m)\|_1^{\frac{1}{2}} \|(A_{1h}\mathbf{u}_H^m, A_{2h}\mathbf{B}_H^m)\|_0^{\frac{1}{2}} \\ &\quad + \sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1 (\|(\mathbf{u}_h, \mathbf{B}_h)\|_1 + \|(\mathbf{u}_H^m, \mathbf{B}_H^m)\|_1) \\ &\leq C\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} \frac{C_r^2\delta^{-2}H^2\|\mathbf{F}\|_0}{(\min\{R_e^{-1}, S_c C_1 R_m^{-1}\})^2} \|\mathbf{F}\|_0^{\frac{1}{2}} \|\mathbf{F}\|_{-1}^{\frac{1}{2}} \\ &\quad + \frac{\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\}}{(\min\{R_e^{-1}, S_c C_1 R_m^{-1}\})^2} \frac{7}{3} \|\mathbf{F}\|_{-1} |||(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)|||_1 \\ &\leq C\sigma C_r^2\delta^{-2}H^2(\frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}})^{\frac{1}{2}}\|\mathbf{F}\|_0 + \frac{7}{3}\sigma |||(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)|||_1 \\ &\leq C\tilde{C}H^2\|\mathbf{F}\|_0 + \frac{7}{3}\sigma |||(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)|||_1. \end{aligned} \quad (5.9)$$

Moreover, taking $q = 0$ in (5.4) and using Assumption A2 to obtain

$$\begin{aligned} \|\eta_{mh}\|_0 &\leq C \max\{R_e^{-1}, S_c C_1 R_m^{-1}\} \|(e_{mh}, b_{mh})\|_1 \\ &\quad + C\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} \|(\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H)\|_0 \|(\mathbf{u}_h, \mathbf{B}_h)\|_1^{\frac{1}{2}} \|(A_{1h}\mathbf{u}_h, A_{2h}\mathbf{B}_h)\|_0^{\frac{1}{2}} \\ &\quad + \|(\mathbf{u}_H^m, \mathbf{B}_H^m)\|_1^{\frac{1}{2}} \|(A_{1h}\mathbf{u}_H^m, A_{2h}\mathbf{B}_H^m)\|_0^{\frac{1}{2}} \\ &\quad + \sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1 (\|(\mathbf{u}_h, \mathbf{B}_h)\|_1 + \|(\mathbf{u}_H^m, \mathbf{B}_H^m)\|_1) \\ &\leq CC_r |||(e_{mh}, b_{mh})|||_1 + C\tilde{C}H^2\|\mathbf{F}\|_0 + C\frac{7}{3}\sigma |||(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)|||_1 \\ &\leq CC_r(\tilde{C}H^2\|\mathbf{F}\|_0 + |||(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)|||_1). \end{aligned} \quad (5.10)$$

Combining (5.7), (5.8), (5.9) with (5.10), we complete the proof. $\square$

Theorem 5.2. Under the conditions of [Theorem 5.1](#), $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ defined by [Iterative methods 1, 2 and 3](#) with [Correction 2](#) satisfies the following stability and error estimates:

$$\begin{aligned} \|(\mathbf{u}_{mh}, \mathbf{B}_{mh})\|_1 &\leq \|\mathbf{F}\|_{-1} + C_R \|(\mathbf{u}_{mh} - \mathbf{u}_H^m, \mathbf{B}_{mh} - \mathbf{B}_H^m)\|_1^2, \\ \| (A_{1h}\mathbf{u}_{mh}, A_{2h}\mathbf{B}_{mh}) \|_0 &\leq C\|\mathbf{F}\|_0 + CC_R \|(\mathbf{u}_{mh} - \mathbf{u}_H^m, \mathbf{B}_{mh} - \mathbf{B}_H^m)\|_1^2, \\ \| (e_{mh}, b_{mh}) \|_1 + \|\eta_{mh}\|_0 &\leq CC_r(\tilde{C}H^{\frac{5}{2}}\|\mathbf{F}\|_0 + C_R \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1^2), \end{aligned}$$

where $\tilde{C} = C_r^3 \delta^{-3} \frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}$, $C_R = \frac{\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\}}{(\min\{R_e^{-1}, S_e C_1 R_m^{-1}\})^2}$.

Proof. Setting $(\mathbf{v}, \Psi) = (\mathbf{u}_{mh}, \mathbf{B}_{mh})$, $q = p_{mh}$ and $(\mathbf{v}, \Psi) = (A_{1h}\mathbf{u}_{mh}, A_{2h}\mathbf{B}_{mh})$, $q = p_{mh}$ in [\(5.2\)](#), respectively, one obtains

$$\begin{aligned} \|(\mathbf{u}_{mh}, \mathbf{B}_{mh})\|_1 &\leq \|\mathbf{F}\|_{-1} + \sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} \|(\mathbf{u}_{mh} - \mathbf{u}_H^m, \mathbf{B}_{mh} - \mathbf{B}_H^m)\|_1^2 \\ &\leq \|\mathbf{F}\|_{-1} + C_R \|(\mathbf{u}_{mh} - \mathbf{u}_H^m, \mathbf{B}_{mh} - \mathbf{B}_H^m)\|_1^2, \end{aligned} \quad (5.11)$$

and

$$\begin{aligned} \| (A_{1h}\mathbf{u}_{mh}, A_{2h}\mathbf{B}_{mh}) \|_0 &\leq \|\mathbf{F}\|_0 + C\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} \|(\mathbf{u}_H^m, \mathbf{B}_H^m)\|_1^{\frac{3}{2}} \| (A_{1h}\mathbf{u}_H^m, A_{2h}\mathbf{B}_H^m) \|_0^{\frac{1}{2}} \\ &\quad + C\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} \|(\mathbf{u}_{mh}, \mathbf{B}_{mh})\|_1^{\frac{1}{2}} \| (A_{1h}\mathbf{u}_{mh}, A_{2h}\mathbf{B}_{mh}) \|_0^{\frac{1}{2}} \|(\mathbf{u}_H^m, \mathbf{B}_H^m)\|_1 \\ &\leq \|\mathbf{F}\|_0 + \frac{1}{2} \| (A_{1h}\mathbf{u}_{mh}, A_{2h}\mathbf{B}_{mh}) \|_0 + \frac{1}{2} \| (A_{1h}\mathbf{u}_H^m, A_{2h}\mathbf{B}_H^m) \|_0 \\ &\quad + C \frac{(\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\})^2}{\min\{R_e^{-1}, S_e C_1 R_m^{-1}\}} (\|(\mathbf{u}_{mh}, \mathbf{B}_{mh})\|_1 \|(\mathbf{u}_H^m, \mathbf{B}_H^m)\|_1^2 + \|(\mathbf{u}_H^m, \mathbf{B}_H^m)\|_1^3) \\ &\leq C\|\mathbf{F}\|_0 + C\sigma^2\|\mathbf{F}\|_{-1} + C\sigma^2 \|(\mathbf{u}_{mh}, \mathbf{B}_{mh})\|_1 + \frac{1}{2} \| (A_{1h}\mathbf{u}_{mh}, A_{2h}\mathbf{B}_{mh}) \|_0. \end{aligned} \quad (5.12)$$

Next, choosing $(\mathbf{v}, \Psi) = (e_{mh}, b_{mh})$, $q = \eta_{mh}$ in [\(5.5\)](#), thanks to the following identity

$$\begin{aligned} A_1((\mathbf{u}_h - \mathbf{u}_H^m, \mathbf{B}_h - \mathbf{B}_H^m), (\mathbf{u}_h - \mathbf{u}_H^m, \mathbf{B}_h - \mathbf{B}_H^m), (\mathbf{v}, \Psi)) \\ = A_1((\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H), (\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H), (\mathbf{v}, \Psi)) \\ + A_1((\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H), (\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m), (\mathbf{v}, \Psi)) \\ + A_1((\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m), (\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H), (\mathbf{v}, \Psi)) \\ + A_1((\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m), (\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m), (\mathbf{v}, \Psi)), \end{aligned}$$

we have

$$\begin{aligned} \| (e_{mh}, b_{mh}) \|_1 &\leq \sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} \| (e_{mh}, b_{mh}) \|_1 \|(\mathbf{u}_H^m, \mathbf{B}_H^m)\|_1 \\ &\quad + C\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} (\|(\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H)\|_1^{\frac{3}{2}} \|(\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H)\|_0^{\frac{1}{2}} \\ &\quad + \|(\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H)\|_1^{\frac{1}{2}} \|(\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H)\|_0^{\frac{1}{2}} \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1) \\ &\quad + \sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1^2 \\ &\leq \frac{4}{3}\sigma \| (e_{mh}, b_{mh}) \|_1 + C\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} (\|(\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H)\|_1^{\frac{3}{2}} \\ &\quad + \|(\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H)\|_0^{\frac{1}{2}} + \|(\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H)\|_1 \|(\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H)\|_0) \\ &\quad + 2\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1^2 \\ &\leq \frac{4}{3}\sigma \| (e_{mh}, b_{mh}) \|_1 + C\sigma C_r^3 \delta^{-3} H^{\frac{5}{2}} \frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}} \|\mathbf{F}\|_0 + 2C_R \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1^2. \end{aligned} \quad (5.13)$$

Noting the fact that $1 - \frac{4}{3}\sigma \geq \frac{7}{15}$, choosing $q = 0$ in (5.5) and using Assumption A2, one finds

$$\begin{aligned}
\|\eta_{mh}\|_0 &\leq C \max\{R_e^{-1}, S_c C_1 R_m^{-1}\} \|(e_{mh}, b_{mh})\|_1 + \sqrt{2} C_0^2 \max\{1, \sqrt{2} S_c\} \|(e_{mh}, b_{mh})\|_1 \|(\mathbf{u}_H^m, \mathbf{B}_H^m)\|_1 \\
&\quad + C \sqrt{2} C_0^2 \max\{1, \sqrt{2} S_c\} \|(\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H)\|_1^{\frac{3}{2}} \|(\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H)\|_0^{\frac{1}{2}} \\
&\quad + \|(\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H)\|_1^{\frac{1}{2}} \|(\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H)\|_0^{\frac{1}{2}} \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_0 \\
&\quad + \sqrt{2} C_0^2 \max\{1, \sqrt{2} S_c\} \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1^2 \\
&\leq C C_r \| (e_{mh}, b_{mh}) \|_1 + C \sigma C_r^3 \delta^{-3} H^{\frac{5}{2}} \frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}} \|\mathbf{F}\|_0 + C C_R \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1^2 \\
&\leq C C_r (\tilde{C} H^{\frac{5}{2}} \|\mathbf{F}\|_0 + C_R \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1^2).
\end{aligned} \tag{5.14}$$

With the help of (5.11), (5.12), (5.13) and (5.14), we finish the proof. $\square$

Theorem 5.3. Under the conditions of Theorem 5.1, $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ defined by Iterative methods 1, 2 and 3 with Correction 3 satisfies

$$\begin{aligned}
\|(\mathbf{u}_{mh}, \mathbf{B}_{mh})\|_1 &\leq \|\mathbf{F}\|_{-1}, \quad \|(A_{1h} \mathbf{u}_{mh}, A_{2h} \mathbf{B}_{mh})\|_0 \leq C \|\mathbf{F}\|_0, \\
\|(e_{mh}, b_{mh})\|_1 + \|\eta_{mh}\|_0 &\leq C C_r (\tilde{C} H^2 \|\mathbf{F}\|_0 + \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1),
\end{aligned}$$

where $\tilde{C} = C_r^2 \delta^{-2} (\frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}})^{\frac{1}{2}}$.

Proof. Taking $(\mathbf{v}, \Psi) = (\mathbf{u}_{mh}, \mathbf{B}_{mh}), q = p_{mh}$ and $(\mathbf{v}, \Psi) = (A_{1h} \mathbf{u}_{mh}, A_{2h} \mathbf{B}_{mh}), q = p_{mh}$ in (5.3), respectively, we get

$$\|(\mathbf{u}_{mh}, \mathbf{B}_{mh})\|_1 \leq \|\mathbf{F}\|_{-1}, \tag{5.15}$$

and

$$\begin{aligned}
\|(A_{1h} \mathbf{u}_{mh}, A_{2h} \mathbf{B}_{mh})\|_0 &\leq \|\mathbf{F}\|_0 + C \sqrt{2} C_0^2 \max\{1, \sqrt{2} S_c\} \|(\mathbf{u}_{mh}, \mathbf{B}_{mh})\|_1 \\
&\quad \|(\mathbf{u}_H^m, \mathbf{B}_H^m)\|_1^{\frac{1}{2}} \|(A_{1H} \mathbf{u}_H^m, A_{2H} \mathbf{B}_H^m)\|_0^{\frac{1}{2}} \\
&\leq \|\mathbf{F}\|_0 + C \sigma \|\mathbf{F}\|_0 \leq C \|\mathbf{F}\|_0.
\end{aligned} \tag{5.16}$$

On the other hand, choosing $(\mathbf{v}, \Psi) = (e_{mh}, b_{mh}), q = \eta_{mh}$ in (5.6), we have

$$\begin{aligned}
\|(e_{mh}, b_{mh})\|_1 &\leq C \sqrt{2} C_0^2 \max\{1, \sqrt{2} S_c\} \|(\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H)\|_0 \|(\mathbf{u}_h, \mathbf{B}_h)\|_1^{\frac{1}{2}} \|(A_{1h} \mathbf{u}_h, A_{2h} \mathbf{B}_h)\|_0^{\frac{1}{2}} \\
&\quad + \sqrt{2} C_0^2 \max\{1, \sqrt{2} S_c\} \|(\mathbf{u}_h, \mathbf{B}_h)\|_1 \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1 \\
&\leq C \sigma C_r^2 \delta^{-2} H^2 (\frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}})^{\frac{1}{2}} \|\mathbf{F}\|_0 + \sigma \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1 \\
&\leq C \tilde{C} H^2 \|\mathbf{F}\|_0 + \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1.
\end{aligned} \tag{5.17}$$

Moreover, setting $q = 0$ in (5.6) and using Assumption A2 to arrive at

$$\begin{aligned}
\|\eta_{mh}\|_0 &\leq C \max\{R_e^{-1}, S_c C_1 R_m^{-1}\} \|(e_{mh}, b_{mh})\|_1 \\
&\quad + C \sqrt{2} C_0^2 \max\{1, \sqrt{2} S_c\} \|(\mathbf{u}_h - \mathbf{u}_H, \mathbf{B}_h - \mathbf{B}_H)\|_0 \|(\mathbf{u}_h, \mathbf{B}_h)\|_1^{\frac{1}{2}} \|(A_{1h} \mathbf{u}_h, A_{2h} \mathbf{B}_h)\|_0^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
& + \sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} \|(\mathbf{u}_h, \mathbf{B}_h)\|_1 \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1 \\
& \leq CC_r \|(\mathbf{e}_{mh}, \mathbf{b}_{mh})\|_1 + C\tilde{C}H^2 \|\mathbf{F}\|_0 + \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1 \\
& \leq CC_r (\tilde{C}H^2 \|\mathbf{F}\|_0 + \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1).
\end{aligned} \tag{5.18}$$

Thanks to (5.15), (5.16), (5.17) and (5.18), we complete the proof. $\square$

Theorem 5.4. Under the conditions of Theorem 5.1, $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ provided by Iterative methods 1, 2 and 3 with Corrections 1 and 3 satisfies the following error estimates:

$$\|(\mathbf{u} - \mathbf{u}_{mh}, \mathbf{B} - \mathbf{B}_{mh})\|_1 + \|p - p_{mh}\|_0 \leq CC_r^2 \left\{ (h + \tilde{C}H^2) \|\mathbf{F}\|_0 + \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1 \right\},$$

where $\tilde{C} = C_r^2 \delta^{-2} (\frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}})^{\frac{1}{2}}$. For the numerical solution $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ provided by Iterative methods 1, 2 and 3 with Correction 2 satisfies the following estimates:

$$\|(\mathbf{u} - \mathbf{u}_{mh}, \mathbf{B} - \mathbf{B}_{mh})\|_1 + \|p - p_{mh}\|_0 \leq CC_r^2 \left\{ (h + \tilde{C}H^{\frac{5}{2}}) \|\mathbf{F}\|_0 + C_R \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1^2 \right\},$$

where $\tilde{C} = C_r^3 \delta^{-3} \frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}$, $C_R = \frac{\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\}}{(\min\{R_e^{-1}, S_c C_1 R_m^{-1}\})^2}$. Here, for Iterative method 1, it holds

$$\|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1 \leq (\frac{11}{5}\sigma)^m \frac{2}{5} \|\mathbf{F}\|_{-1}.$$

For Iterative method 3, it holds

$$\|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1 \leq \sigma^m \|\mathbf{F}\|_{-1}.$$

For Iterative method 2, it holds

$$\|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1 \leq (\frac{15}{13}\sigma)^{2^m-1} \frac{5}{11} \|\mathbf{F}\|_{-1}.$$

Remark 5.1. For the small σ with $0 < \sigma \leq \frac{2}{5}$, from Theorems 5.1, 5.2, 5.3 and 5.4, we know that the fine grid solution $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ based on the Iterative methods i ($i = 1, 2, 3$) with Corrections j ($j = 1, 2, 3$) is stable and convergent. Furthermore, the combination of Iterative method 2 and Correction 2 is the better choice than others because the combination has the second order convergence on m in the case of $0 < \sigma \leq \frac{2}{5}$.

Remark 5.2. Based on Theorem 5.4, we choose m such that

$$\|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1 \leq (\frac{11}{5}\sigma)^m \frac{2}{5} \|\mathbf{F}\|_{-1} \leq h \|\mathbf{F}\|_0,$$

for Iterative method 1, and

$$\|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1 \leq \sigma^m \|\mathbf{F}\|_{-1} \leq h \|\mathbf{F}\|_0,$$

for Iterative method 3, and

$$\|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1 \leq (\frac{15}{13}\sigma)^{2^m-1} \frac{5}{11} \|\mathbf{F}\|_{-1} \leq \sqrt{h \|\mathbf{F}\|_0},$$

for Iterative method 2. Namely,

$$m \geq \frac{\ln(\frac{5h\|\mathbf{F}\|_0}{2\|\mathbf{F}\|_{-1}})}{\ln(\frac{11}{5}\sigma)}, \quad m \geq \frac{\ln(\frac{h\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}})}{\ln(\sigma)}, \quad m \geq \frac{1}{\ln 2} \ln \frac{\ln \frac{33\sigma\sqrt{h\|\mathbf{F}\|_0}}{13\|\mathbf{F}\|_{-1}}}{\ln \frac{15\sigma}{13}},$$

on [Iterative method 1](#), [Iterative method 3](#) and [Iterative method 2](#), respectively.

6. Two-level iterative methods with $\frac{2}{5} < \sigma \leq \frac{5}{11}$

For the slightly large $\frac{2}{5} < \sigma \leq \frac{5}{11}$, we know from [Theorem 4.1](#) that [Iterative methods 2 and 3](#) are stable and convergent. Thus we consider the two-level iterative methods in which the iterative solutions $((\mathbf{u}_H^m, \mathbf{B}_H^m), p_H^m)$ of [Iterative methods 2 and 3](#) are obtained on a coarse grid τ_H and the correction solutions $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ of the Stokes type, Newton type and Oseen type corrections are given on the fine grid τ_h . Our two-level iterative methods can be designed as follows.

Step 1. Find a global coarse grid iterative solution $((\mathbf{u}_H^m, \mathbf{B}_H^m), p_H^m) \in \mathbf{W}_{0n}^H \times M_H$ defined by [Iterative methods 2 and 3](#), respectively. The value of m will be discussed in [Remark 6.2](#).

Step 2. Find a fine grid solution $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh}) \in \mathbf{W}_{0n}^h \times M_h$ defined by (5.1), (5.2) and (5.3) on τ_h , respectively.

Theorem 6.1. Under the conditions of [Theorem 3.3](#) and $\frac{2}{5} < \sigma \leq \frac{5}{11}$, $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ defined by [Iterative methods 2 and 3](#) with [Correction 1](#) satisfies:

$$\|(\mathbf{u}_{mh}, \mathbf{B}_{mh})\|_1 \leq 2\|\mathbf{F}\|_{-1}, \quad \|(A_{1h}\mathbf{u}_{mh}, A_{2h}\mathbf{B}_{mh})\|_0 \leq C\|\mathbf{F}\|_0, \quad (6.1)$$

$$\|(e_{mh}, b_{mh})\|_1 + \|\eta_{mh}\|_0 \leq CC_r(\tilde{C}H^2\|\mathbf{F}\|_0 + \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1), \quad (6.2)$$

where $\tilde{C} = C_r^2\delta^{-2}(\frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}})^{\frac{1}{2}}$.

Proof. Noting the fact that $1 + \frac{16}{9}\sigma \leq 2$, using (5.7) and (5.8) to obtain (6.1). Applying (5.9), (5.10) and $1 - \sigma \geq \frac{6}{11}$, we obtain the desired result (6.2).

Theorem 6.2. Under the conditions of [Theorem 6.1](#), $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ defined by [Iterative methods 2 and 3](#) with [Correction 2](#) satisfies the following stability and error estimates:

$$\|(\mathbf{u}_{mh}, \mathbf{B}_{mh})\|_1 \leq \|\mathbf{F}\|_{-1} + C_R \|(\mathbf{u}_{mh} - \mathbf{u}_H^m, \mathbf{B}_{mh} - \mathbf{B}_H^m)\|_1^2, \quad (6.3)$$

$$\|(A_{1h}\mathbf{u}_{mh}, A_{2h}\mathbf{B}_{mh})\|_0 \leq C\|\mathbf{F}\|_0 + CC_R \|(\mathbf{u}_{mh} - \mathbf{u}_H^m, \mathbf{B}_{mh} - \mathbf{B}_H^m)\|_1^2, \quad (6.3)$$

$$\|(e_{mh}, b_{mh})\|_1 + \|\eta_{mh}\|_0 \leq CC_r(\tilde{C}H^{\frac{5}{2}}\|\mathbf{F}\|_0 + C_R \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1^2), \quad (6.4)$$

where $\tilde{C} = C_r^3\delta^{-3}\frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}$, $C_R = \frac{\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\}}{(\min\{R_e^{-1}, S_c C_1 R_m^{-1}\})^2}$.

Proof. Using (5.11) and (5.12) to derive (6.3). Applying (5.13), (5.14) and noting the fact that $1 - \sigma \geq \frac{6}{11}$ and $1 - \frac{4}{3}\sigma \geq \frac{13}{33}$, we complete the proof of (6.4). $\square$

Theorem 6.3. Under the conditions of [Theorem 6.1](#), $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ defined by [Iterative methods 2 and 3](#) with [Correction 3](#) satisfies the following stability and error estimates:

$$\|(\mathbf{u}_{mh}, \mathbf{B}_{mh})\|_1 \leq \|\mathbf{F}\|_{-1}, \quad \|(A_{1h}\mathbf{u}_{mh}, A_{2h}\mathbf{B}_{mh})\|_0 \leq C\|\mathbf{F}\|_0, \quad (6.5)$$

$$\|(e_{mh}, b_{mh})\|_1 + \|\eta_{mh}\|_0 \leq CC_r(\tilde{C}H^2\|\mathbf{F}\|_0 + \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1), \quad (6.6)$$

where $\tilde{C} = C_r^2\delta^{-2}(\frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}})^{\frac{1}{2}}$.

Proof. Thanks to (5.15) and (5.16) we obtain (6.5). Furthermore, with the help of (5.17) and (5.18), we complete the proof of (6.6). $\square$

Theorem 6.4. Under the conditions of Theorem 6.1, $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ provided by Iterative methods 2 and 3 with Corrections 1 and 3 satisfies the following error estimates:

$$\|(\mathbf{u} - \mathbf{u}_{mh}, \mathbf{B} - \mathbf{B}_{mh})\|_1 + \|p - p_{mh}\|_0 \leq CC_r^2 \left\{ (h + \tilde{C}H^2) \|\mathbf{F}\|_0 + \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1 \right\}, \quad (6.7)$$

where $\tilde{C} = C_r^2 \delta^{-2} (\frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}})^{\frac{1}{2}}$. For the numerical solution $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ provided by Iterative methods 2 and 3 with Correction 2 satisfies the following error estimates:

$$\|(\mathbf{u} - \mathbf{u}_{mh}, \mathbf{B} - \mathbf{B}_{mh})\|_1 + \|p - p_{mh}\|_0 \leq CC_r^2 \left\{ (h + \tilde{C}H^{\frac{5}{2}}) \|\mathbf{F}\|_0 + C_R \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1^2 \right\}, \quad (6.8)$$

where $\tilde{C} = C_r^3 \delta^{-3} \frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}$, $C_R = \frac{\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\}}{(\min\{Re^{-1}, S_e C_1 R_m^{-1}\})^2}$. For Iterative method 3, it holds

$$\|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1 \leq \sigma^m \|\mathbf{F}\|_{-1}. \quad (6.9)$$

For Iterative method 2, it holds

$$\|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1 \leq \left(\frac{15}{13}\sigma\right)^{2^m-1} \frac{5}{11} \|\mathbf{F}\|_{-1}. \quad (6.10)$$

Remark 6.1. For the slightly large σ with $\frac{2}{5} < \sigma \leq \frac{5}{11}$, from Theorems 6.1, 6.2, 6.3 and 6.4, we know that the fine grid solution $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ based on Iterative methods i ($i = 2, 3$) with Corrections j ($j = 1, 2, 3$) is stable and convergent. Furthermore, we find that the combination of Iterative method 2 and Correction 2 is better than others because this combination has the second order convergence on m in the case of $\frac{2}{5} < \sigma \leq \frac{5}{11}$.

Remark 6.2. Based on Theorem 6.4 we choose m such that

$$\|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1 \leq \sigma^m \|\mathbf{F}\|_{-1} \leq h \|\mathbf{F}\|_0, \quad (6.11)$$

for Iterative method 3, and

$$\|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1 \leq \left(\frac{15}{13}\sigma\right)^{2^m-1} \frac{5}{11} \|\mathbf{F}\|_{-1} \leq \sqrt{h \|\mathbf{F}\|_0}, \quad (6.12)$$

for Iterative method 2. Namely,

$$m \geq \frac{\ln(\frac{h \|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}})}{\ln(\sigma)}, \quad m \geq \frac{1}{\ln 2} \ln \frac{\ln \frac{33\sigma \sqrt{h \|\mathbf{F}\|_0}}{13 \|\mathbf{F}\|_{-1}}}{\ln \frac{15\sigma}{13}}, \quad (6.13)$$

respectively, on Iterative method 3 and Iterative method 2.

7. Two-level iterative methods with $\frac{5}{11} < \sigma \leq 1 - \alpha(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0})^{\frac{1}{2}}$

For the large σ with $\frac{5}{11} < \sigma \leq 1 - \alpha(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0})^{\frac{1}{2}}$, from Theorem 4.1 we know that Iterative method 3 is stable and convergent. Thus we consider the two-level iterative methods in which the iterative solution $((\mathbf{u}_H^m, \mathbf{B}_H^m), p_H^m)$ of Iterative method 3 is obtained on a coarse grid τ_H and the correction solution

$((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ of the Stokes type, Newton type and Oseen type corrections is given on a fine grid τ_h . The considered two-level iterative methods can be divided into the following two steps.

Step 1. Find a global coarse grid iterative solution $((\mathbf{u}_H^m, \mathbf{B}_H^m), p_H^m) \in \mathbf{W}_{0n}^H \times M_H$ defined by [Iterative method 3](#) on τ_H . Here m is chosen such that

$$m \geq \ln\left(\frac{h\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}\right)/\ln(\sigma).$$

Step 2. Find a fine grid solution $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh}) \in \mathbf{W}_{0n}^h \times M_h$ defined by the corrections [\(5.1\)](#), [\(5.2\)](#) and [\(5.3\)](#) on τ_h , respectively.

Theorem 7.1. Under the conditions of [Theorem 3.3](#) and $\frac{5}{11} < \sigma \leq 1 - \alpha\left(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0}\right)^{\frac{1}{2}}$, for the numerical solution $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ defined by [Iterative method 3](#) with [Correction 1](#) satisfies:

$$|||(\mathbf{u}_{mh}, \mathbf{B}_{mh})|||_1 \leq 2\|\mathbf{F}\|_{-1}, \quad |||(A_{1h}\mathbf{u}_{mh}, A_{2h}\mathbf{B}_{mh})|||_0 \leq C\|\mathbf{F}\|_0, \quad (7.1)$$

$$|||(e_{mh}, b_{mh})|||_1 + \|\eta_{mh}\|_0 \leq CC_r(\tilde{C}H^2\|\mathbf{F}\|_0 + |||(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)|||_1), \quad (7.2)$$

where $\tilde{C} = C_r^2\delta^{-2}\left(\frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}\right)^{\frac{1}{2}}$.

Proof. Taking $(\mathbf{v}, \Psi) = (\mathbf{u}_{mh}, \mathbf{B}_{mh}), q = p_{mh}$ in [\(5.1\)](#), we have

$$\begin{aligned} |||(\mathbf{u}_{mh}, \mathbf{B}_{mh})|||_1 &\leq \|\mathbf{F}\|_{-1} + \sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\} |||(\mathbf{u}_H^m, \mathbf{B}_H^m)|||_1^2 \\ &\leq \|\mathbf{F}\|_{-1} + \frac{\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\}}{(\min\{R_e^{-1}, S_c C_1 R_m^{-1}\})^2} \|\mathbf{F}\|_{-1}^2 \leq (1 + \sigma)\|\mathbf{F}\|_{-1}, \end{aligned}$$

which and [\(5.8\)](#) yield [\(7.1\)](#). Thanks to [\(5.9\)](#), [\(5.10\)](#) and noting the fact that $1 - \sigma \geq \alpha\left(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0}\right)^{\frac{1}{2}}$, we obtain the desired result [\(7.2\)](#). $\square$

Theorem 7.2. Under the conditions of [Theorem 7.1](#), $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ defined by [Iterative method 3](#) with [Correction 2](#) satisfies the following stability and error estimates:

$$\begin{aligned} |||(\mathbf{u}_{mh}, \mathbf{B}_{mh})|||_1 &\leq \|\mathbf{F}\|_{-1} + C_R |||(\mathbf{u}_{mh} - \mathbf{u}_H^m, \mathbf{B}_{mh} - \mathbf{B}_H^m)|||_1^2, \\ |||(A_{1h}\mathbf{u}_{mh}, A_{2h}\mathbf{B}_{mh})|||_0 &\leq C\|\mathbf{F}\|_0 + CC_R |||(\mathbf{u}_{mh} - \mathbf{u}_H^m, \mathbf{B}_{mh} - \mathbf{B}_H^m)|||_1^2, \end{aligned} \quad (7.3)$$

$$|||(e_{mh}, b_{mh})|||_1 + \|\eta_{mh}\|_0 \leq CC_r(\tilde{C}H^{\frac{5}{2}}\|\mathbf{F}\|_0 + C_R |||(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)|||_1^2), \quad (7.4)$$

where $\tilde{C} = C_r^3\delta^{-3}\frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}$, $C_R = \frac{\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\}}{(\min\{R_e^{-1}, S_c C_1 R_m^{-1}\})^2}$.

Proof. Using [\(5.11\)](#) and [\(5.12\)](#) to derive [\(7.3\)](#). Applying [\(5.13\)](#) and [\(5.14\)](#), and noting the fact that $1 - \sigma \geq \alpha\left(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0}\right)^{\frac{1}{2}}$, we finish the proof of [\(7.4\)](#). $\square$

Theorem 7.3. Under the conditions of [Theorem 7.1](#), the numerical solution $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ defined by [Iterative method 3](#) with [Correction 3](#) satisfies:

$$|||(\mathbf{u}_{mh}, \mathbf{B}_{mh})|||_1 \leq \|\mathbf{F}\|_{-1}, \quad |||(A_{1h}\mathbf{u}_{mh}, A_{2h}\mathbf{B}_{mh})|||_0 \leq C\|\mathbf{F}\|_0, \quad (7.5)$$

$$|||(e_{mh}, b_{mh})|||_1 + \|\eta_{mh}\|_0 \leq CC_r(\tilde{C}H^2\|\mathbf{F}\|_0 + |||(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)|||_1), \quad (7.6)$$

where $\tilde{C} = C_r^2\delta^{-2}\left(\frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}\right)^{\frac{1}{2}}$.

Proof. Using (5.15) and (5.16) to obtain (7.5). Thanks to (5.17), (5.18) and $1 - \sigma \geq \alpha(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0})^{\frac{1}{2}}$, we complete the proof of (7.6). $\square$

Theorem 7.4. Under the conditions of Theorem 7.1, $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ provided by Iterative method 3 with Corrections 1 and 3 satisfies the following error estimates:

$$\|(\mathbf{u} - \mathbf{u}_{mh}, \mathbf{B} - \mathbf{B}_{mh})\|_1 + \|p - p_{mh}\|_0 \leq CC_r^2 \{(h + \tilde{C}H^2)\|\mathbf{F}\|_0 + \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1\}, \quad (7.7)$$

where $\tilde{C} = C_r^2 \delta^{-2} (\frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}})^{\frac{1}{2}}$. For the numerical solution $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ provided by Iterative method 3 with Correction 2 satisfies the following error estimates:

$$\|(\mathbf{u} - \mathbf{u}_{mh}, \mathbf{B} - \mathbf{B}_{mh})\|_1 + \|p - p_{mh}\|_0 \leq CC_r^2 \{(h + \tilde{C}H^{\frac{5}{2}})\|\mathbf{F}\|_0 + C_R \|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1^2\}, \quad (7.8)$$

where $\tilde{C} = C_r^2 \delta^{-3} \frac{\|\mathbf{F}\|_0}{\|\mathbf{F}\|_{-1}}$, $C_R = \frac{\sqrt{2}C_0^2 \max\{1, \sqrt{2}S_c\}}{(\min\{R_e^{-1}, S_c C_1 R_m^{-1}\})^2}$. For Iterative method 3, it holds

$$\|(\mathbf{u}_H - \mathbf{u}_H^m, \mathbf{B}_H - \mathbf{B}_H^m)\|_1 \leq \sigma^m \|\mathbf{F}\|_{-1}. \quad (7.9)$$

Remark 7.1. For the slightly large σ with $\frac{5}{11} < \sigma \leq 1 - \alpha(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0})^{\frac{1}{2}}$, from Theorems 7.1, 7.2, 7.3 and 7.4, we know that the fine grid solution $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ based on Iterative method 3 with Corrections j ($j = 1, 2, 3$) is stable and convergent. Furthermore, we can see that the combination of Iterative method 3 and Correction 2 is a better choice for the steady incompressible MHD equations since Iterative method 3 is a unique choice among three iterative methods in the case of $\frac{5}{11} < \sigma \leq 1 - \alpha(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0})^{\frac{1}{2}}$ and this combination is of the second order convergence on m .

8. One-level iterative method with $1 - \alpha(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0})^{\frac{1}{2}} < \sigma < 1$

From Theorem 4.1, we know that Iterative method 3 is a unique stable method under the weak unique condition $1 - \alpha(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0})^{\frac{1}{2}} < \sigma < 1$. In this section, we consider one-level finite element iterative solution $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh}) = ((\mathbf{u}_h^m, \mathbf{B}_h^m), p_h^m)$ on a fine grid τ_h , and establish the stability and convergence for the numerical solutions.

Step 1. Find a fine grid iterative solution $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh}) \in \mathbf{W}_{0n}^h \times M_h$ defined by Iterative method 3 on τ_h .

It follows from Theorem 4.1 with $1 - \alpha(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0})^{\frac{1}{2}} < \sigma < 1$, we know that the following stability and error estimates hold.

Theorem 8.1. Under the conditions of Theorem 3.3 and $1 - \alpha(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0})^{\frac{1}{2}} < \sigma < 1$, the iterative solution $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ defined by Iterative method 3 satisfies:

$$\begin{aligned} \|(\mathbf{u}_{mh}, \mathbf{B}_{mh})\|_1 &\leq \|\mathbf{F}\|_{-1}, \quad \|(A_{1h}\mathbf{u}_{mh}, A_{2h}\mathbf{B}_{mh})\|_0 \leq C\|\mathbf{F}\|_0, \\ \|(\mathbf{u}_{mh} - \mathbf{u}_h, \mathbf{B}_{mh} - \mathbf{B}_h)\|_1 &\leq \sigma^m \|\mathbf{F}\|_{-1}, \quad \|p_h - p_{mh}\|_0 \leq CC_r \sigma^m \|\mathbf{F}\|_{-1}. \end{aligned}$$

Combining Theorem 8.1 with Theorem 3.3, we obtain the following conclusion.

Theorem 8.2. Under the conditions of Theorem 8.1, the iterative solution $((\mathbf{u}_{mh}, \mathbf{B}_{mh}), p_{mh})$ defined by Iterative method 3 on the fine mesh τ_h satisfies

$$\|(\mathbf{u} - \mathbf{u}_{mh}, \mathbf{B} - \mathbf{B}_{mh})\|_1 + \|p - p_{mh}\|_0 \leq CC_r \left\{ C_r \delta^{-1} h \|\mathbf{F}\|_0 + \sigma^m \|\mathbf{F}\|_{-1} \right\}.$$

Table 1The standard Galerkin finite element method for incompressible MHD problem with $R_e = R_m = S_c = 1$.

$1/h$	$\frac{\ \nabla(\mathbf{u}-\mathbf{u}_h)\ _0}{\ \nabla\mathbf{u}\ _0}$	Rate	$\frac{\ p-p_h\ _0}{\ p\ _0}$	Rate	$\frac{\ \nabla(\mathbf{B}-\mathbf{B}_h)\ _0}{\ \nabla\mathbf{B}\ _0}$	Rate	CPU(S)
9	0.319636		0.0491044		0.163963		1.41
16	0.170385	1.09343	0.0161331	1.93456	0.0928431	0.988471	3.21
25	0.111537	1.04499	0.00736572	1.93367	0.0620036	0.995695	6.88
36	0.0736314	1.02422	0.00338634	1.91653	0.0413681	0.99807	15.49
49	0.0538613	1.01413	0.0018876	1.89571	0.0304017	0.999062	29.53
64	0.0411348	1.00934	0.00114322	1.87768	0.0232795	0.999473	50.41
81	0.032451	1.00661	0.000737113	1.86302	0.0183951	0.999682	83.69
100	0.0265245	1.00494	0.000508291	1.85223	0.0150512	0.999795	126.08

9. Numerical experiments

In this section, we present two numerical examples to verify the performances of the two level iterative methods for MHD equations. These investigations are focused on validating the optimal convergence rates of Theorems 4.2, 5.4, 6.4 and 7.4 and computational cost. The software FreeFem++ developed by Hecht et al. [26] is used in our experiments. We use the UMFPACK routine to solve the linear systems arising from the discrete algebraic equations. The mesh consists of triangular elements that obtained by dividing Ω into subsquares of equal size and then drawing the diagonal in each sub-square. The (P_1b, P_1, P_1b) finite element pair is used for discretization and iterative tolerance 10^{-8} is used in all numerical implementations.

9.1. MHD problem with smooth solution

In this example, the steady incompressible MHD equations are defined on a convex domain $\Omega = [0, 1]^2$. The boundary and initial conditions and right-hand side functions $\mathbf{f}$ and $\mathbf{g}$ are selected such that the exact solutions are given by

$$\begin{aligned} u_1 &= x^2(x-1)^2y(y-1)(2y-1); & u_2 &= -y^2(y-1)^2x(x-1)(2x-1); \\ p &= (2x-1)(2y-1); & B_1 &= \sin(\pi x)\cos(\pi y); & B_2 &= -\sin(\pi y)\cos(\pi x); \end{aligned}$$

where the components of $\mathbf{u}$ and $\mathbf{B}$ are denoted by (u_1, u_2) and (B_1, B_2) for convenience. In order to show the prominent features of the two level iterative methods (5.1), (5.2) and (5.3), we compare our numerical results with that obtained by standard Galerkin FEM (3.1).

Firstly, we choose the parameters $R_e = R_m = S_c = 1$, the relative errors between the exact solution and the numerical approximations of standard Galerkin finite element method (3.1) are shown in Table 1. As observed from Table 1, the errors of $\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_0}{\|\nabla\mathbf{u}\|_0}$ and $\frac{\|\nabla(\mathbf{B}-\mathbf{B}_h)\|_0}{\|\nabla\mathbf{B}\|_0}$ become smaller and smaller as the mesh is refined, and the corresponding errors are of the order of $\mathcal{O}(h)$. The orders of $\frac{\|p-p_h\|_0}{\|p\|_0}$ are nearly $\mathcal{O}(h^{1.8})$, which show some superconvergences.

In contrast, we show the relative errors between the exact solution and the numerical solutions obtained from two level iterative methods with different corrections in Tables 2–4. Here, TLSSCM, TLSOCM and TLNSCM denote two level Stokes iteration with Stokes correction method, two level Stokes iteration with Oseen correction method and two level Stokes iteration with Newton correction method, respectively. Similarly, we use TLNSCM, TLNNCM, TLNOCM, TLOSCM, TLONCM and TLOOCM to denote two level Newton iteration with Stokes correction method, two level Newton iteration with Newton correction method, two level Newton iteration with Oseen correction method, two level Oseen iteration with Stokes correction method, two level Oseen iteration with Newton correction method and two level Oseen iteration with Oseen correction method, respectively. Apparently, the accuracy of two level iterative methods with different corrections is comparable with that of standard Galerkin FEM (3.1) with the same mesh sizes $\frac{1}{h} = 9, 16, 25, 36, 49, 64, 81, 100$ and $H = h^{\frac{1}{2}}$.

Table 2MHD problem with smooth solution with $R_e = R_m = S_c = 1$.

$1/H$	$1/h$	Type	$\frac{\ \nabla(\mathbf{u}-\mathbf{u}_h)\ _0}{\ \nabla\mathbf{u}\ _0}$	Rate	$\frac{\ p-p_h\ _0}{\ p\ _0}$	Rate	$\frac{\ \nabla(\mathbf{B}-\mathbf{B}_h)\ _0}{\ \nabla\mathbf{B}\ _0}$	Rate	CPU(S)
3	9	TLSSCM	0.583151		0.229787		0.1639640		0.57
4	16	TLSSCM	0.386064	0.716837	0.137281	0.895313	0.0928436	0.988472	1.44
5	25	TLSSCM	0.271492	0.788889	0.090987	0.921603	0.0595302	0.995845	3.16
6	36	TLSSCM	0.199694	0.842314	0.064506	0.943302	0.0413682	0.998159	6.07
7	49	TLSSCM	0.152283	0.879165	0.048010	0.957983	0.0304017	0.999063	9.83
8	64	TLSSCM	0.119599	0.904638	0.037075	0.967823	0.0232796	0.999475	16.53
9	81	TLSSCM	0.096234	0.922728	0.029469	0.974647	0.0183951	0.999683	26.06
10	100	TLSSCM	0.079009	0.935915	0.023973	0.979571	0.0149007	0.999798	40.41
3	9	TLNCSM	0.339653		0.0793987		0.163963		0.66
4	16	TLNCSM	0.177003	1.13277	0.0278921	1.81823	0.092843	0.988472	1.64
5	25	TLNCSM	0.109382	1.07850	0.0120814	1.87473	0.059530	0.995842	3.18
6	36	TLNCSM	0.074629	1.04851	0.0060430	1.89987	0.041368	0.998156	6.97
7	49	TLNCSM	0.054308	1.03095	0.0033532	1.91038	0.030402	0.999062	12.51
8	64	TLNCSM	0.041352	1.02054	0.0020120	1.91268	0.023280	0.999473	22.70
9	81	TLNCSM	0.032565	1.01417	0.0012831	1.90995	0.018395	0.999682	35.61
10	100	TLNCSM	0.026321	1.01014	0.0008590	1.90395	0.014901	0.999797	54.02
3	9	TLSOCM	0.366741		0.22635		0.163963		0.55
4	16	TLSOCM	0.206411	0.99899	0.133871	0.912828	0.092843	0.988471	1.36
5	25	TLSOCM	0.132057	1.00077	0.088194	0.935141	0.059531	0.995841	3.07
6	36	TLSOCM	0.091664	1.00126	0.062301	0.953194	0.041368	0.998156	5.94
7	49	TLSOCM	0.067323	1.00107	0.046265	0.965178	0.030402	0.999062	9.45
8	64	TLSOCM	0.051533	1.00083	0.035676	0.973236	0.023280	0.999473	16.16
9	81	TLSOCM	0.040711	1.00067	0.028329	0.978854	0.018395	0.999682	26.22
10	100	TLSOCM	0.032972	1.00054	0.023030	0.9829	0.014901	0.999797	45.75

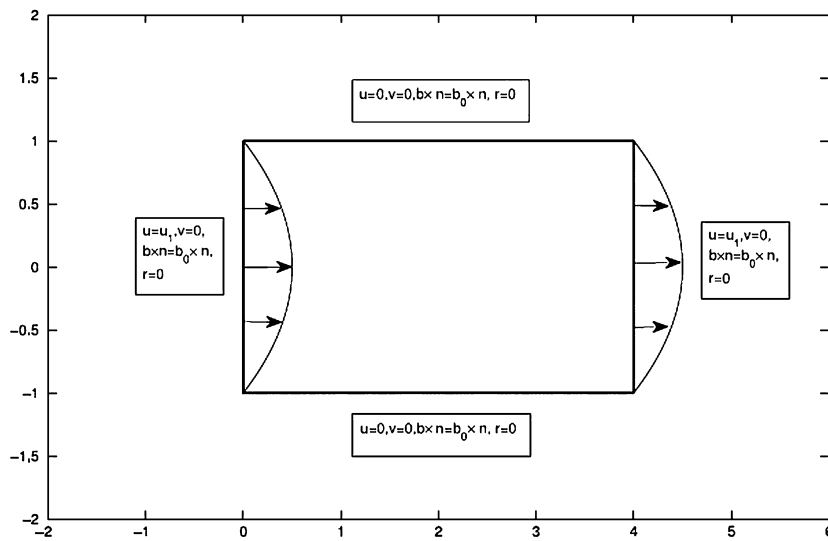
Table 3MHD problem with smooth solution with $R_e = R_m = S_c = 1$.

$1/H$	$1/h$	Type	$\frac{\ \nabla(\mathbf{u}-\mathbf{u}_h)\ _0}{\ \nabla\mathbf{u}\ _0}$	Rate	$\frac{\ p-p_h\ _0}{\ p\ _0}$	Rate	$\frac{\ \nabla(\mathbf{B}-\mathbf{B}_h)\ _0}{\ \nabla\mathbf{B}\ _0}$	Rate	CPU(S)
3	9	TLNSCM	0.568303		0.229914		0.163963		0.51
4	16	TLNSCM	0.347491	0.854971	0.137277	0.89631	0.092843	0.988471	1.15
5	25	TLNSCM	0.242802	0.803268	0.090984	1.01443	0.059531	0.995841	2.98
6	36	TLNSCM	0.179785	0.824053	0.064503	0.84833	0.041368	0.998156	5.54
7	49	TLNSCM	0.138146	0.854532	0.048009	0.95790	0.030402	0.999061	8.49
8	64	TLNSCM	0.109189	0.880794	0.037075	0.96775	0.023280	0.999473	15.52
9	81	TLNSCM	0.088300	0.901401	0.029470	0.97459	0.018395	0.999681	26.11
10	100	TLNSCM	0.072781	0.917262	0.023974	1.02859	0.014901	0.999795	39.66
3	9	TLNNCM	0.339605		0.0793835		0.163963		0.62
4	16	TLNNCM	0.176993	1.13262	0.0278904	1.8180	0.092843	0.988471	1.84
5	25	TLNNCM	0.109379	1.07844	0.0120821	1.8745	0.059531	0.995842	4.03
6	36	TLNNCM	0.074627	1.04846	0.0060434	1.8998	0.041368	0.998156	7.66
7	49	TLNNCM	0.054308	1.03093	0.0033534	1.9104	0.030402	0.999062	13.21
8	64	TLNNCM	0.041352	1.02053	0.0020121	1.9127	0.023280	0.999473	20.61
9	81	TLNNCM	0.032565	1.01416	0.0012831	1.9100	0.018395	0.999682	35.80
10	100	TLNNCM	0.026321	1.01014	0.0008590	1.9040	0.014901	0.999797	54.74
3	9	TLNOCM	0.366773		0.226168		0.163963		0.53
4	16	TLNOCM	0.206411	0.99915	0.133793	0.91245	0.092843	0.988471	1.42
5	25	TLNOCM	0.132057	1.00077	0.0881531	0.93486	0.059532	0.995841	2.90
6	36	TLNOCM	0.091665	1.00123	0.0622803	0.95279	0.041368	0.998156	5.91
7	49	TLNOCM	0.067325	1.00103	0.0462554	0.96486	0.030402	0.999062	10.48
8	64	TLNOCM	0.051534	1.00082	0.0356704	0.97302	0.023280	0.999473	17.71
9	81	TLNOCM	0.040712	1.00067	0.0283259	0.97869	0.018395	0.999682	28.28
10	100	TLNOCM	0.032973	1.00055	0.0230273	0.98278	0.014901	0.999797	45.82

Finally, we compare the CPU time among four kinds of numerical schemes in [Tables 1–4](#). From these data, we can see that the two level methods with different iterations not only keep good accuracy but also speed less time than standard Galerkin finite element method. Furthermore, we compare the results obtained from these two level methods with different corrections. From [Tables 2, 3 and 4](#), we can see that two level Newton iterative scheme has the best accuracy, while two level Stokes iterative scheme spends the least computational time.

Table 4MHD problem with smooth solution with $R_e = R_m = S_c = 1$.

$1/H$	$1/h$	Type	$\frac{\ \nabla(\mathbf{u}-\mathbf{u}_h)\ _0}{\ \nabla\mathbf{u}\ _0}$	Rate	$\frac{\ p-p_h\ _0}{\ p\ _0}$	Rate	$\frac{\ \nabla(\mathbf{B}-\mathbf{B}_h)\ _0}{\ \nabla\mathbf{B}\ _0}$	Rate	CPU(S)
3	9	TLOSCM	0.58339		0.229914		0.163963		0.41
4	16	TLOSCM	0.386017	0.717764	0.137277	0.89631	0.092843	0.98847	1.08
5	25	TLOSCM	0.271463	0.868278	0.090984	1.01443	0.059532	1.09610	2.51
6	36	TLOSCM	0.199682	0.757396	0.064503	0.84833	0.041368	0.89766	4.95
7	49	TLOSCM	0.152281	0.879044	0.048009	0.95790	0.030402	0.99906	9.25
8	64	TLOSCM	0.119599	0.904567	0.037075	0.96775	0.023280	0.99947	15.59
9	81	TLOSCM	0.096235	0.922684	0.029470	0.97459	0.018395	0.99968	24.73
10	100	TLOSCM	0.079010	0.982761	0.023974	1.02859	0.014901	1.04987	37.35
3	9	TLONCM	0.339605		0.0793835		0.1639630		0.64
4	16	TLONCM	0.176993	1.13262	0.0278904	1.81799	0.0928431	0.98847	1.34
5	25	TLONCM	0.113634	1.09291	0.0125919	1.96129	0.0620036	0.99570	3.79
6	36	TLONCM	0.074627	1.03701	0.0060434	1.81048	0.0413681	0.99807	7.49
7	49	TLONCM	0.054308	1.03093	0.0033535	1.91038	0.0304017	0.99906	12.26
8	64	TLONCM	0.041352	1.02053	0.0020121	1.91271	0.0232795	0.99947	20.45
9	81	TLONCM	0.032565	1.01416	0.0012831	1.90996	0.0183951	0.99968	33.81
10	100	TLONCM	0.026585	1.01099	0.0008681	1.94738	0.0150512	0.99980	55.38
3	9	TLOOCM	0.366773		0.226168		0.163963		0.48
4	16	TLOOCM	0.206411	0.99915	0.133793	0.91245	0.0928432	0.98847	1.34
5	25	TLOOCM	0.135892	1.03094	0.089158	1.00103	0.0620036	0.99569	2.87
6	36	TLOOCM	0.091665	0.97102	0.062280	0.88482	0.0413681	0.99807	6.12
7	49	TLOOCM	0.067325	1.00103	0.046255	0.96486	0.0304017	0.99906	9.81
8	64	TLOOCM	0.051534	1.00082	0.035670	0.97301	0.0232796	0.99947	17.44
9	81	TLOOCM	0.040712	1.00067	0.028326	0.97869	0.0183951	0.99968	27.66
10	100	TLOOCM	0.033251	1.00884	0.023086	1.01942	0.0150512	0.99980	43.62

**Fig. 1.** Hartmann flow problem: model setting.

9.2. Hartmann flow

Hartmann flow is a classical MHD problem, which describes the flow of a liquid metal through a channel (see [6]). The liquid flows in the x -direction under the influence of a prescribed pressure gradient G , meanwhile, a uniform magnetic field $b_0 = (0, 1)$ is applied on the boundaries. The problem model is described in Fig. 1. The Hartmann number $Ha = \sqrt{R_e R_m S_c}$, the exact solutions are given by

$$u_1 = \frac{R_e G}{Ha \tanh(Ha)} \left(1 - \frac{\cosh(Hay)}{\cosh(Ha)} \right), \quad u_2 = 0,$$

$$b_1 = \frac{G}{S} \left(\frac{\sinh(Hay)}{\sinh(Ha)} - y \right), \quad b_2 = 1.$$

Table 5

Hartmann flow with $R_e = 1$, $R_m = 0.01$, $S_c = 100$ and $Ha = 1$.

$1/H$	$1/h$	Type	$\frac{\ \nabla(\mathbf{u}-\mathbf{u}_h)\ _0}{\ \nabla\mathbf{u}\ _0}$	Rate	$\frac{\ p-p_h\ _0}{\ p\ _0}$	Rate	$\frac{\ \nabla(\mathbf{B}-\mathbf{B}_h)\ _0}{\ \nabla\mathbf{B}\ _0}$	Rate	CPU(S)
	9	SGFEM	0.0815023		0.0280712		0.000462966		7.75
3	9	TLSSCM	0.0829040		0.0283307		0.000462965		2.86
3	9	TLNSCM	0.0815023		0.0280714		0.000462966		4.62
3	9	TLSOCM	0.0815023		0.0280701		0.000462965		4.01
3	9	TLNSCM	0.0815023		0.0274375		0.000462965		2.82
3	9	TLNNCM	0.0815023		0.0280715		0.000462966		3.96
3	9	TLNOCM	0.0815023		0.0280698		0.000462965		3.48
3	9	TLOSCM	0.0815023		0.0274375		0.000466531		2.75
3	9	TLONCM	0.0815023		0.0280715		0.000462966		4.15
3	9	TLOOCM	0.0815023		0.0280698		0.000462965		3.37
	16	SGFEM	0.0464583	0.9769	0.0143563	1.1654	0.000264697	0.9717	24.53
4	16	TLSSCM	0.0472182	0.9783	0.0144195	1.1738	0.000273921	0.9121	8.75
4	16	TLNSCM	0.0464583	0.9769	0.0143567	1.1654	0.000264697	0.9717	13.11
4	16	TLSOCM	0.0464582	0.9769	0.0143563	1.1654	0.000264697	0.9717	10.58
4	16	TLNSCM	0.0472182	0.9487	0.0144195	1.1738	0.000273917	0.9122	6.77
4	16	TLNNCM	0.0464583	0.9769	0.0143565	1.1654	0.000264697	0.9717	11.36
4	16	TLNOCM	0.0464582	0.9769	0.0143562	1.1654	0.000264697	0.9717	8.88
4	16	TLOSCM	0.0472182	0.9487	0.0144195	1.1738	0.000273917	0.9255	6.66
4	16	TLONCM	0.0464583	0.9769	0.0143565	1.1654	0.000264697	0.9717	10.89
4	16	TLOOCM	0.0464582	0.9769	0.0143562	1.1654	0.000264697	0.9717	8.58
	25	SGFEM	0.0295701	1.0123	0.0087093	1.1199	0.000168843	1.0075	63.37
5	25	TLSSCM	0.0300492	1.0127	0.0087804	1.1115	0.000174838	1.0060	19.22
5	25	TLNSCM	0.0295701	1.0123	0.0087095	1.1199	0.000168843	1.0075	29.78
5	25	TLSOCM	0.0295701	1.0123	0.0087104	1.1196	0.000168843	1.0075	24.01
5	25	TLNSCM	0.0300492	1.0127	0.0087591	1.1170	0.000174838	1.0061	15.02
5	25	TLNNCM	0.0295701	1.0123	0.0087104	1.1196	0.000168843	1.0075	20.97
5	25	TLNOCM	0.0295701	1.0123	0.0087094	1.1199	0.000168843	1.0075	27.38
5	25	TLOSCM	0.0300492	1.0127	0.0087591	1.1170	0.000174838	1.0060	15.02
5	25	TLONCM	0.0295701	1.0123	0.0087094	1.1199	0.000168843	1.0075	25.82
5	25	TLOOCM	0.0295701	1.0123	0.0087104	1.1196	0.000168843	1.0075	20.34

In this test, we set pressure gradient $G = 1$, and choose the following three cases to compute this problem

Case 1: $R_e = 1$, $R_m = 0.01$, $S_c = 100$, $Ha = 1$.

Case 2: $R_e = 5$, $R_m = 0.1$, $S_c = 50$, $Ha = 5$.

Case 3: $R_e = 10$, $R_m = 10$, $S_c = 1$, $Ha = 10$.

The numerical result of Case 1 is shown in Table 5, SGFEM denotes standard Galerkin finite element method. We observe that two level iterative schemes with different corrections can compute in this case, and the rates of $\frac{\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_0}{\|\nabla\mathbf{u}\|_0}$, $\frac{\|p-p_h\|_0}{\|p\|_0}$ and $\frac{\|\nabla(\mathbf{B}-\mathbf{B}_h)\|_0}{\|\nabla\mathbf{B}\|_0}$ all have optimal orders. The two level iterative schemes with different corrections have almost the same numerical accuracy with standard Galerkin FEM, while our algorithms take less CPU time than Galerkin FEM. Next, we consider the numerical results of Case 2 in Table 6. From the data we observe that two level Stokes iteration cannot run in this case. The convergence rates of two level Oseen iteration and two level Newton iteration with different corrections have optimal order as mesh size refines, and two level Oseen iteration spends less CPU time than two level Newton iteration. Finally, we present the numerical results of Case 3 in Table 7. In this case, two level Stokes and Newton iterations are all divergent, while two level Oseen iteration with different corrections still works with high Hartmann number, furthermore, the convergence rates of two level Oseen iteration arrive optimal order.

From this numerical example, we can see that two level Stokes iteration with different corrections can solve the MHD problem in the case of low Hartmann number, two level Newton iteration with different corrections can work in the case of moderate Hartmann number, while two level Oseen iteration with different corrections can run with the high Hartmann number case. These numerical results verify our theoretical analysis about two level iterative schemes which developed in Sections 5, 6 and 7 very well.

Table 6Hartmann flow with $R_e = 5$, $R_m = 0.1$, $S_c = 50$ and $Ha = 5$.

$1/H$	$1/h$	Type	$\frac{\ \nabla(\mathbf{u}-\mathbf{u}_h)\ _0}{\ \nabla\mathbf{u}\ _0}$	Rate	$\frac{\ p-p_h\ _0}{\ p\ _0}$	Rate	$\frac{\ \nabla(\mathbf{B}-\mathbf{B}_h)\ _0}{\ \nabla\mathbf{B}\ _0}$	Rate	CPU(S)
	16	SGFEM	0.351756		0.0244657		0.0075224		35.97
4	16	TLSSCM	N/A		N/A		N/A		
4	16	TLSCNM	N/A		N/A		N/A		
4	16	TLSOCM	N/A		N/A		N/A		
4	16	TLNSCM	0.569908		0.0309162		0.011361		7.85
4	16	TLNNCM	0.351787		0.0245177		0.007523		11.51
4	16	TLNOCM	0.351688		0.0247921		0.007521		9.36
4	16	TLOSCM	0.569908		0.0309162		0.011361		8.99
4	16	TLONCM	0.361663		0.0333228		0.103259		11.44
4	16	TLOOCM	0.351688		0.0247921		0.007521		9.38
	25	SGFEM	0.225151	0.9997	0.014881	1.1142	0.0048246	0.9952	90.98
5	25	TLSSCM	N/A		N/A		N/A		
5	25	TLSCNM	N/A		N/A		N/A		
5	25	TLSOCM	N/A		N/A		N/A		
5	25	TLNSCM	0.360528	1.0260	0.0198842	0.9889	0.0072441	1.0083	19.25
5	25	TLNNCM	0.225160	0.9998	0.0149021	1.1156	0.0048247	0.9953	26.11
5	25	TLNOCM	0.225168	0.9991	0.0152587	1.0876	0.0048245	0.9949	21.11
5	25	TLOSCM	0.360528	1.0260	0.0198842	0.9885	0.0072441	1.0083	20.87
5	25	TLONCM	0.223477	1.0787	0.0186782	1.2971	0.0669657	0.9704	25.87
5	25	TLOOCM	0.225168	0.9991	0.0152587	1.0878	0.0048245	0.9949	21.56
	36	SGFEM	0.154925	1.0252	0.0093174	1.2838	0.0033323	1.0149	193.13
6	36	TLSSCM	N/A		N/A		N/A		
6	36	TLSCNM	N/A		N/A		N/A		
6	36	TLSOCM	N/A		N/A		N/A		
6	36	TLNSCM	0.240252	1.1131	0.0121999	1.3397	0.0048010	1.1281	38.50
6	36	TLNNCM	0.154927	1.0253	0.0093197	1.2872	0.0033323	1.0149	49.73
6	36	TLNOCM	0.154950	1.0250	0.0094099	1.3257	0.0033323	1.0148	41.18
6	36	TLOSCM	0.240252	1.1131	0.0121999	1.3402	0.0048010	1.1281	38.38
6	36	TLONCM	0.154927	1.0047	0.0093197	1.9066	0.0033323	1.9140	49.73
6	36	TLOOCM	0.154950	1.0250	0.0094099	1.3256	0.0033323	1.0148	42.37

Table 7Hartmann flow with $R_e = 10$, $R_m = 10$, $S_c = 1$ and $Ha = 10$.

$1/H$	$1/h$	Type	$\frac{\ \nabla(\mathbf{u}-\mathbf{u}_h)\ _0}{\ \nabla\mathbf{u}\ _0}$	Rate	$\frac{\ p-p_h\ _0}{\ p\ _0}$	Rate	$\frac{\ \nabla(\mathbf{B}-\mathbf{B}_h)\ _0}{\ \nabla\mathbf{B}\ _0}$	Rate	CPU(S)
	25	SGFEM	0.623501		0.020915		0.669664		122.38
5	25	TLSSCM	N/A		N/A		N/A		
5	25	TLSCNM	N/A		N/A		N/A		
5	25	TLSOCM	N/A		N/A		N/A		
5	25	TLNSCM	N/A		N/A		N/A		
5	25	TLNNCM	N/A		N/A		N/A		
5	25	TLNOCM	N/A		N/A		N/A		
5	25	TLOSCM	1.575951		0.143465		1.52212		21.25
5	25	TLONCM	0.623503		0.023115		0.669622		28.89
5	25	TLOOCM	0.627012		0.042810		0.674965		23.63
	36	SGFEM	0.430595	1.0152	0.013184	1.2656	0.46391	1.0067	261.29
6	36	TLSSCM	N/A		N/A		N/A		
6	36	TLSCNM	N/A		N/A		N/A		
6	36	TLSOCM	N/A		N/A		N/A		
6	36	TLNSCM	N/A		N/A		N/A		
6	36	TLNNCM	N/A		N/A		N/A		
6	36	TLNOCM	N/A		N/A		N/A		
6	36	TLOSCM	1.049920	1.1138	0.097669	1.0545	1.01328	1.1159	39.67
6	36	TLONCM	0.430558	1.0154	0.014098	1.3559	0.463923	0.9753	55.65
6	36	TLOOCM	0.433213	1.0140	0.026330	1.3330	0.467568	1.0068	44.23
	49	SGFEM	0.313443	1.0300	0.009102	1.2018	0.338966	1.0178	580.97
7	49	TLSSCM	N/A		N/A		N/A		
7	49	TLSCNM	N/A		N/A		N/A		
7	49	TLSOCM	N/A		N/A		N/A		
7	49	TLNSCM	N/A		N/A		N/A		
7	49	TLNNCM	N/A		N/A		N/A		
7	49	TLNOCM	N/A		N/A		N/A		
7	49	TLOSCM	0.802837	0.8703	0.075205	0.8478	0.77128	0.8852	75.114
7	49	TLONCM	0.313494	1.0042	0.009949	1.1305	0.33903	1.0541	103.96
7	49	TLOOCM	0.315920	1.0241	0.017399	1.3438	0.34222	1.0123	84.02

10. Conclusion

In this paper, we have considered the two-level iterative methods for the steady incompressible MHD equations with different viscosities under the strong uniqueness condition $0 < \sigma \leq 1 - \alpha(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0})^{\frac{1}{2}}$. In the case of $0 < \sigma \leq \frac{5}{11}$, the two-level iterative method based on the Newton type iteration of m times on a coarse grid and Newton type correction of one time on a fine grid is a suitable choice. In the case of $\frac{5}{11} < \sigma \leq 1 - \alpha(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0})^{\frac{1}{2}}$ the two-level iterative method based on Oseen type iteration of m times on a coarse grid and Newton type correction of one time on a fine grid is the best scheme. Under the weak uniqueness condition $1 - \alpha(\frac{\|\mathbf{F}\|_{-1}}{\|\mathbf{F}\|_0})^{\frac{1}{2}} < \sigma < 1$, the one-level Oseen type iterative method on a fine mesh τ_h is the suitable method to solve the steady incompressible MHD equations.

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