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# $L^p$ solutions of multidimensional BSDEs with weak monotonicity and general growth generators ☆

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ABSTRACT

In this paper, we first establish the existence and uniqueness of $L^p$ ($p > 1$) solutions for multidimensional backward stochastic differential equations (BSDEs) under a weak monotonicity condition together with a general growth condition in $y$ for the generator $g$. Then, we overview several conditions related closely to the weak monotonicity condition and compare them in an effective way. Finally, we put forward and prove a stability theorem and a comparison theorem of $L^p$ ($p > 1$) solutions for this kind of BSDEs.



## 1. Introduction

Throughout this paper, let us fix a real number $T > 0$, and two positive integers $k$ and $d$. Let $\mathbb{R}^+ := [0, +\infty)$ and let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space carrying a standard $d$-dimensional Brownian motion $(B_t)_{t\geq 0}$, $(\mathbb{F}_t)_{t\geq 0}$ be the natural $\sigma$-algebra generated by $(B_t)_{t\geq 0}$ and $\mathbb{F} = \mathbb{F}_T$. For each subset $A \subset \Omega \times [0,T]$, let $\mathbb{1}_A = 1$ in case of $(t,\omega) \in A$, otherwise, let $\mathbb{1}_A = 0$. The Euclidean norm of a vector $y \in \mathbb{R}^k$ will be defined by $|y|$, and for a $k \times d$ matrix $z$, we define $|z| = \sqrt{\mathrm{Tr} zz^*}$, where $z^*$ is the transpose of $z$. Let $\langle x, y\rangle$ represent the inner product of $x, y \in \mathbb{R}^k$. For each $p > 1$, we denote by $L^p(\mathbb{R}^k)$ the set of all $\mathbb{R}^k$-valued and $\mathbb{F}_T$-measurable random vectors $\xi$ such that $\mathbb{E}[|\xi|^p] < +\infty$, and by $\mathcal{S}^p(0,T;\mathbb{R}^k)$ the set of $\mathbb{R}^k$-valued, adapted and continuous processes $(Y_t)_{t\in[0,T]}$ such that

$$\|Y\|_{\mathcal{S}^p} := \left(\mathbb{E}[\sup_{t\in[0,T]} |Y_t|^p]\right)^{1/p} < +\infty.$$

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Moreover, let $\mathrm{M}^p(0,T;\mathbb{R}^{k\times d})$ denote the set of $(\mathbb{F}_t)$-progressively measurable $\mathbb{R}^{k\times d}$-valued processes $(Z_t)_{t\in[0,T]}$ such that

$$\|Z\|_{\mathrm{M}^p} := \left\{ \mathbb{E}\left[ \left( \int_0^T |Z_t|^2 \, \mathrm{d}t \right)^{p/2} \right] \right\}^{1/p} < +\infty.$$

Obviously, both $\mathcal{S}^p$ and $\mathrm{M}^p$ are Banach spaces for each $p > 1$.

In this paper, we are concerned with the following multidimensional backward stochastic differential equation (BSDE for short in the remaining):

$$y_t = \xi + \int_t^T g(s, y_s, z_s)\mathrm{d}s - \int_t^T z_s \mathrm{d}B_s, \quad t \in [0,T], \tag{1}$$

where $\xi \in L^p(\mathbb{R}^k)$ is called the terminal condition, $T$ is called the time horizon, the random function

$$g(\omega, t, y, z) : \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \longmapsto \mathbb{R}^k$$

is $(\mathbb{F}_t)$-progressively measurable for each $(y, z)$, called the generator of BSDE (1). This BSDE is usually denoted by the BSDE $(\xi, T, g)$.

For convenience of the following discussion, we introduce the following definitions concerning solutions of BSDE (1).

**Definition 1.** A solution to BSDE (1) is a pair of $(\mathbb{F}_t)$-progressively measurable processes $(y_t, z_t)_{t\in[0,T]}$ with values in $\mathbb{R}^k \times \mathbb{R}^{k\times d}$ such that $\mathrm{d}\mathbb{P} - a.s.$, $t \mapsto y_t$ is continuous, $t \mapsto z_t$ belongs to $\mathrm{L}^2(0,T)$, $t \mapsto g(t, y_t, z_t)$ belongs to $\mathrm{L}^1(0,T)$, and $\mathrm{d}\mathbb{P} - a.s.$, (1) holds true for each $t \in [0,T]$.

**Definition 2.** Assume that $(y_t, z_t)$ is a solution to BSDE (1). If $(y_t, z_t) \in \mathcal{S}^p(0,T;\mathbb{R}^k) \times \mathrm{M}^p(0,T;\mathbb{R}^{k\times d})$ for some $p > 1$, then it will be called an $L^p$ solution of BSDE (1).

Nonlinear BSDEs were firstly introduced in 1990 by Pardoux and Peng [36], who established the existence and uniqueness for $L^2$ solutions of BSDEs under the Lipschitz assumption of the generator $g$. Since then, BSDEs have been studied with great interest, and they have become a powerful tool in many fields above all financial mathematics, stochastic games and optimal control, non-linear PDEs and homogenization. See [4,9,10,14–16,29,34,35,37,38,42,41] and the references therein for applications of BSDEs to PDEs, optimal control, homogenization as well as in mathematical finances.

From the beginning, many authors attempted to improve the result of [36] by weakening the Lipschitz hypothesis on $g$, see [1,2,4,6–8,13–20,22–24,26–29,31–35,37,40,43,45,46], or the $L^2$ integrability assumptions on $\xi$, see [5–7,11,21,23,38,44], or relaxing the finite terminal time $T$ to a stopping time or infinity, see [12, 25,29,34,35,45]. From these results we can see that the case of one-dimensional BSDEs is easier to handle due to the presence of the comparison theorem of solutions (see [6–8,11,12,17–21,25,27–29,31,32,38]).

One of the main purposes of the present paper is to establish an existence and uniqueness of $L^p$ $(p > 1)$ solutions for multidimensional BSDEs under weaker conditions on the generators. Here, we would like to mention the following several results on multidimensional BSDEs, which is related closely to our result. First of all, Mao [33] obtained an existence and uniqueness result of an $L^2$ solution for (1) where $g$ satisfies a particular non-Lipschitz condition in $y$ called usually the Mao condition in the literature, and Fan and Jiang [23] investigated the existence and uniqueness of an $L^p$ $(p > 1)$ solution for (1) where $g$ satisfies a new kind of non-Lipschitz condition in $y$. Second, Peng [37] first introduced a kind of monotonicity condition in $y$ for $g$, and under this monotonicity condition as well as a general growth condition in $y$ for $g$, Pardoux [35] established an existence and uniqueness result of an $L^2$ solution for (1). Using the same monotonicity condition and a more general growth condition in $y$ for $g$, Briand et al. [5] investigated the existence and uniqueness of an $L^p$ $(p \geq 1)$ solution for (1). Furthermore, Situ [40] put forward a kind of weak monotonicity condition in $y$ for $g$ and considered the existence and uniqueness of $L^p$ $(p \geq 1)$ solutions for BSDEs with jumps, but the generator $g$ is forced to also satisfy a linear growth condition in $y$. Recently, Fan and Jiang [22] and Xu and Fan [43] established the existence and uniqueness of an $L^2$ solution for (1) under the weak monotonicity condition and the more general growth condition in $y$ for the generator $g$, which really and truly unifies the Mao condition in $y$ and the monotonicity condition with the general growth condition in $y$.

In this paper, we first establish the existence and uniqueness of $L^p$ $(p > 1)$ solutions for multidimensional BSDEs under the weak monotonicity condition together with the more general growth condition in $y$ for the generator $g$ (see Theorem 1 in Section 2 and its proof in Section 4), which extends some existing results including Theorem 4.2 in Briand et al. [5] and Theorem 1 in Fan and Jiang [23]. Then, we overview several conditions related closely to the weak monotonicity condition and compare them in an effective way (see Proposition 1 in Section 2 and its proof in Appendix A). Finally, we put forward and prove a stability theorem and a comparison theorem of $L^p$ $(p > 1)$ solutions for this kind of BSDEs (see Theorem 2 in Section 4 and Theorem 3 in Section 5).

This paper is organized as follows. In Section 2 we state the assumptions and the existence and uniqueness result for $L^p$ $(p > 1)$ solutions of multidimensional BSDEs and introduce several propositions, corollaries, remarks and examples to show that it generalizes some existing results. In Section 3, we establish two nonstandard a priori estimates for $L^p$ $(p > 1)$ solutions of multidimensional BSDEs, based on which we prove a stability theorem and the existence and uniqueness result in Section 4. Then, we put forward and prove a new comparison theorem for $L^p$ $(p > 1)$ solutions of one dimensional BSDEs in Section 5. Finally, the proof of the relations between the assumptions related closely to the weak monotonicity condition is provided in Appendix A.

## 2. An existence and uniqueness result

In this section, we will state the existence and uniqueness result for $L^p$ $(p > 1)$ solutions of multidimensional BSDEs and introduce several propositions, corollaries, remarks and examples to show that it generalizes some existing results including Theorem 4.2 in Briand et al. [5] and Theorem 1 in Fan and Jiang [23]. Let us start with introducing the following assumptions:

(H1)$_p$ $g$ satisfies the $p$-order weak monotonicity condition in $y$, i.e., there exists a nondecreasing and concave function $\rho(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\rho(0) = 0$, $\rho(u) > 0$ for $u > 0$ and $\int_{0^+} \frac{\mathrm{d}u}{\rho(u)} = +\infty$ such that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall\, y_1, y_2 \in \mathbb{R}^k$, $z \in \mathbb{R}^{k\times d}$,

$$|y_1 - y_2|^{p-1} \langle \tfrac{y_1-y_2}{|y_1-y_2|} \mathbb{1}_{|y_1-y_2|\neq 0}, g(\omega, t, y_1, z) - g(\omega, t, y_2, z)\rangle \leq \rho(|y_1 - y_2|^p);$$

(H2) $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall\, z \in \mathbb{R}^{k\times d}$, $y \longmapsto g(\omega, t, y, z)$ is continuous;

(H3) $g$ has a general growth with respect to $y$, i.e.,

$$\forall \;\; \alpha > 0,\; \phi_\alpha(t) := \sup_{|y|\leq\alpha} |g(\omega, t, y, 0) - g(\omega, t, 0, 0)| \in L^1([0, T] \times \Omega);$$

(H4) $g$ is Lipschitz continuous in $z$, uniformly with respect to $(\omega, t, y)$, i.e., there exists a constant $\bar{\lambda} \geq 0$ such that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall\, y \in \mathbb{R}^k$, $z_1, z_2 \in \mathbb{R}^{k\times d}$,

$$|g(\omega, t, y, z_1) - g(\omega, t, y, z_2)| \leq \bar{\lambda}|z_1 - z_2|;$$

$$(\mathrm{H5})_p \ \mathbb{E}\left[|\xi|^p + \left(\int_0^T |g(\omega,t,0,0)|\,\mathrm{d}t\right)^p\right] < +\infty.$$

The following Theorem 1 is one of the main results of this paper. It's proof will be given in Section 4.

**Theorem 1.** *Assume that $p > 1$, and assumptions $(\mathrm{H1})_{p\wedge 2}$, (H2)–(H4) and $(\mathrm{H5})_p$ hold. Then, the BSDE $(\xi, T, g)$ has a unique $L^p$ solution.*

It should be mentioned that Theorem 1 has been proved in Xu and Fan [43] for the case of $p = 2$. In addition, by Theorem 1 the following corollary is immediate.

**Corollary 1.** *Assume that the generator $g$ satisfies assumptions $(\mathrm{H1})_2$ and (H2)–(H4). Then, if $(\mathrm{H5})_p$ holds for some $p > 2$, then the BSDE $(\xi, T, g)$ has a unique $L^p$ solution.*

In the sequel, let us further introduce the following assumptions on $g$:

$(\mathrm{H1a})_p$ $g$ satisfies the $p$-order one-sided Mao condition in $y$, i.e., there exists a nondecreasing and concave function $\rho(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\rho(0) = 0$, $\rho(u) > 0$ for $u > 0$ and $\int_{0^+} \frac{\mathrm{d}u}{\rho(u)} = +\infty$ such that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall\, y_1, y_2 \in \mathbb{R}^k$, $z \in \mathbb{R}^{k\times d}$,

$$\langle \tfrac{y_1-y_2}{|y_1-y_2|}\mathbb{1}_{|y_1-y_2|\neq 0}, g(\omega,t,y_1,z) - g(\omega,t,y_2,z)\rangle \leq \rho^{\frac{1}{p}}(|y_1-y_2|^p);$$

$(\mathrm{H1b})_p$ $g$ satisfies the $p$-order one-sided Constantin condition in $y$, i.e., there exists a nondecreasing and concave function $\rho(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\rho(0) = 0$, $\rho(u) > 0$ for $u > 0$ and $\int_{0^+} \frac{u^{p-1}}{\rho^p(u)}\mathrm{d}u = +\infty$ such that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall\, y_1, y_2 \in \mathbb{R}^k$, $z \in \mathbb{R}^{k\times d}$,

$$\langle \tfrac{y_1-y_2}{|y_1-y_2|}\mathbb{1}_{|y_1-y_2|\neq 0}, g(\omega,t,y_1,z) - g(\omega,t,y_2,z)\rangle \leq \rho(|y_1-y_2|);$$

(H1*) $g$ satisfies the one-sided Osgood condition in $y$, i.e., there exists a nondecreasing and concave function $\rho(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\rho(0) = 0$, $\rho(u) > 0$ for $u > 0$ and $\int_{0^+} \frac{\mathrm{d}u}{\rho(u)} = +\infty$ such that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall\, y_1, y_2 \in \mathbb{R}^k$, $z \in \mathbb{R}^{k\times d}$,

$$\langle \tfrac{y_1-y_2}{|y_1-y_2|}\mathbb{1}_{|y_1-y_2|\neq 0}, g(\omega,t,y_1,z) - g(\omega,t,y_2,z)\rangle \leq \rho(|y_1-y_2|).$$

**Remark 1.** It is easy to see that the following statements are true.

- When $\rho(x) = \mu x$ for some constant $\mu > 0$, $(\mathrm{H1})_p$, $(\mathrm{H1a})_p$, $(\mathrm{H1b})_p$ and (H1*) are all the known monotonicity condition for each $p \geq 1$;
- In case of $p = 1$, $(\mathrm{H1})_p$, $(\mathrm{H1a})_p$ and $(\mathrm{H1b})_p$ are all the same as (H1*);
- In case of $p = 2$, $(\mathrm{H1})_p$, $(\mathrm{H1a})_p$ and $(\mathrm{H1b})_p$ are respectively the so-called weak monotonicity condition, one-sided Mao condition and one-sided Constantin condition put forward in Fan and Jiang [22].

With respect to the previous assumptions, we have the following important observation. It's proof will be provided in Appendix A.

**Proposition 1.** *For each $1 \leq p \leq q < +\infty$, we have*

$$\begin{array}{llllll} (i) & (\mathrm{H1}^*) & \Longrightarrow & (\mathrm{H1})_p & \Longrightarrow & (\mathrm{H1})_q; \\ (ii) & (\mathrm{H1b})_q & \Longrightarrow & (\mathrm{H1b})_p & \Longrightarrow & (\mathrm{H1}^*); \\ (iii) & (\mathrm{H1a})_p & \Longleftrightarrow & (\mathrm{H1b})_p. & & \end{array}$$

*In addition, we can show that for each $p \geq 1$, the concavity condition of $\rho(\cdot)$ in assumptions* (H1a)$_p$ *and* (H1b)$_p$ *can be replaced with the continuity condition.*

According to Theorem 1 and Proposition 1, the following corollaries follow immediately.

**Corollary 2.** *Assume that the generator $g$ satisfies assumptions* (H1*) *and* (H2)–(H4). *Then, if* (H5)$_p$ *holds for some $p > 1$, then the BSDE $(\xi, T, g)$ has a unique $L^p$ solution.*

**Corollary 3.** *Assume that $p > 1$, and assumptions* (H1a)$_p$ *(or* (H1b)$_p$*),* (H2)–(H4) *and* (H5)$_p$ *hold. Then, the BSDE $(\xi, T, g)$ has a unique $L^p$ solution.*

The following four assumptions (H1')$_p$, (H1a')$_p$, (H1b')$_p$ and (H1'*) are respectively the stronger and two-sided versions of assumptions (H1)$_p$, (H1a)$_p$, (H1b)$_p$ and (H1*):

(H1')$_p$ There exists a nondecreasing and concave function $\rho(\cdot): \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\rho(0) = 0$, $\rho(u) > 0$ for $u > 0$ and $\int_{0^+} \frac{\mathrm{d}u}{\rho(u)} = +\infty$ such that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall\, y_1, y_2 \in \mathbb{R}^k$, $z \in \mathbb{R}^{k \times d}$,

$$|y_1 - y_2|^{p-1} |g(\omega, t, y_1, z) - g(\omega, t, y_2, z)| \leq \rho(|y_1 - y_2|^p);$$

(H1a')$_p$ $g$ satisfies the $p$-order Mao condition in $y$, i.e., there exists a nondecreasing and concave function $\rho(\cdot): \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\rho(0) = 0$, $\rho(u) > 0$ for $u > 0$ and $\int_{0^+} \frac{\mathrm{d}u}{\rho(u)} = +\infty$ such that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall\, y_1, y_2 \in \mathbb{R}^k$, $z \in \mathbb{R}^{k \times d}$,

$$|g(\omega, t, y_1, z) - g(\omega, t, y_2, z)| \leq \rho^{\frac{1}{p}}(|y_1 - y_2|^p);$$

(H1b')$_p$ $g$ satisfies the $p$-order Constantin condition in $y$, i.e., there exists a nondecreasing and concave function $\rho(\cdot): \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\rho(0) = 0$, $\rho(u) > 0$ for $u > 0$ and $\int_{0^+} \frac{u^{p-1}}{\rho^p(u)} \mathrm{d}u = +\infty$ such that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall\, y_1, y_2 \in \mathbb{R}^k$, $z \in \mathbb{R}^{k \times d}$,

$$|g(\omega, t, y_1, z) - g(\omega, t, y_2, z)| \leq \rho(|y_1 - y_2|);$$

(H1'*) $g$ satisfies the Osgood condition in $y$, i.e., there exists a nondecreasing and concave function $\rho(\cdot): \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $\rho(0) = 0$, $\rho(u) > 0$ for $u > 0$ and $\int_{0^+} \frac{\mathrm{d}u}{\rho(u)} = +\infty$ such that $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall\, y_1, y_2 \in \mathbb{R}^k$, $z \in \mathbb{R}^{k \times d}$,

$$|g(\omega, t, y_1, z) - g(\omega, t, y_2, z)| \leq \rho(|y_1 - y_2|).$$

**Remark 2.** It is easy to see that the following statements are true.

- When $\rho(x) = \mu x$ for some constant $\mu > 0$, (H1')$_p$, (H1a')$_p$, (H1b')$_p$ and (H1'*) are all the known Lipschitz condition for each $p \geq 1$;
- In case of $p = 1$, (H1')$_p$, (H1a')$_p$ and (H1b')$_p$ are the same as (H1'*);
- In case of $p = 2$, (H1a')$_p$ and (H1b')$_p$ are respectively the known Mao condition and Constantin condition;
- For each $p \geq 1$, we have (H1')$_p$ $\Longrightarrow$ (H1)$_p$, (H1a')$_p$ $\Longrightarrow$ (H1a)$_p$ + (H2) + (H3), and (H1b')$_p$ $\Longrightarrow$ (H1b)$_p$ + (H2) + (H3);
- Proposition 1 holds also true for assumptions (H1')$_p$, (H1a')$_p$, (H1b')$_p$ and (H1'*).

**Remark 3.** It follows from Remarks 1–2 and Proposition 1 that Theorem 1 and some of its corollaries all improve some existing results for $L^p$ solutions of multidimensional BSDEs including Theorem 4.2 in Briand et al. [5] and Theorem 1 in Fan and Jiang [23].

Now, we give two examples of BSDEs which satisfy the assumptions in Corollary 2. In our knowledge, they are not covered by the previous works.

**Example 1.** Let $k = 1$, $\bar{p} \geq 1$ and

$$g(\omega, t, y, z) = h(|y|) - e^{|B_t(\omega)|\cdot y} + (e^{-y} \wedge 1) \cdot |z| + \frac{1}{\sqrt[3]{t}} \mathbb{1}_{t>0},$$

where

$$h(x) = \begin{cases} -x|\ln x|^{1/\bar{p}} & , \quad 0 < x \leq \delta; \\ h'(\delta-)(x-\delta) + h(\delta) & , \quad x > \delta; \\ 0 & , \quad \text{other cases}, \end{cases}$$

with $\delta > 0$ small enough.

It is easy to see that $g$ satisfies (H2)–(H4) with $\bar{\lambda} = 1$. Furthermore, we can prove that $g$ satisfies $(\mathrm{H1b})_{\bar{p}}$ by verifying that $e^{-\beta y}$ with $\beta \geq 0$ is decreasing in $y$, $h(\cdot)$ is concave and sub-additive on $\mathbb{R}^+$ and then the following inequality holds: $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$,

$$\forall\ y_1, y_2, z, \quad \langle \frac{y_1 - y_2}{|y_1 - y_2|} \mathbb{1}_{|y_1 - y_2| \neq 0}, g(\omega, t, y_1, z) - g(\omega, t, y_2, z) \rangle \leq h(|y_1 - y_2|)$$

with

$$\int_{0^+} \frac{u^{\bar{p}-1}}{h^{\bar{p}}(u)} \mathrm{d}u = +\infty.$$

Thus, (H1*) holds for $g$ by Proposition 1, and then from Corollary 2 we know that if $\xi \in L^p(\mathbb{R}^k)$ for some $p > 1$, then BSDE $(\xi, T, g)$ has a unique $L^p$ solution.

**Example 2.** Let $y = (y_1, \cdots, y_k)$ and $g(t, y, z) = (g_1(t, y, z), \cdots, g_k(t, y, z))$, where for each $i = 1, \cdots, k$,

$$g_i(\omega, t, y, z) := e^{-y_i} + h(|y|) + \sin|z| + |B_t(\omega)|,$$

and $h(x)$ is defined in Example 1.

It is not hard to verify that this generator $g$ satisfies $(\mathrm{H1b})_{\bar{p}}$, (H1*) and (H2)–(H4) with $\bar{\lambda} = 1$. It then follows from Corollary 2 that if $\xi \in L^p(\mathbb{R}^k)$ for some $p > 1$, then BSDE $(\xi, T, g)$ has a unique $L^p$ solution.

Finally, we would like to mention that the function $h(x)$ defined in Example 1 satisfies that

$$\forall\ q > \bar{p}, \quad \int_{0^+} \frac{u^{q-1}}{h^q(u)} \mathrm{d}u < +\infty.$$

And, we can also prove that neither of the two generators $g$ defined in Examples 1–2 satisfies $(\mathrm{H1a})_q$ or $(\mathrm{H1b})_q$ for each $q > \bar{p}$, which means that the inverse version of $(ii)$ of Proposition 1 does not hold.

## 3. Two nonstandard a priori estimates

In this section, we will establish two nonstandard a priori estimates concerning $L^p$ solutions of multi-dimensional BSDE (1), which will play an important role in the proof of our main results. The following assumption on the generator $g$ will be used:

(A1) $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall\ (y, z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$,

$$\langle y, g(\omega, t, y, z)\rangle \le \mu|y|^2 + \lambda|y||z| + |y|f_t + \varphi_t,$$

where $\mu$ and $\lambda$ are two non-negative constants, $f_t$ and $\varphi_t$ are two non-negative and $(\mathbb{F}_t)$-progressively measurable processes with

$$\mathbb{E}\left[\left(\int_0^T f_t\ \mathrm{d}t\right)^p\right] < +\infty \quad \text{and} \quad \mathbb{E}\left[\left(\int_0^T \varphi_t\ \mathrm{d}t\right)^{p/2}\right] < +\infty.$$

**Proposition 2.** *Assume that $p > 0$ and (A1) holds. Let $(y_t, z_t)_{t\in[0,T]}$ be a solution of BSDE (1) such that $y_t$ belongs to $\mathcal{S}^p(0, T; \mathbb{R}^k)$. Then $z_t$ belongs to $\mathrm{M}^p(0, T; \mathbb{R}^{k\times d})$, and for each $0 \le u \le t \le T$, we have*

$$\mathbb{E}\left[\left(\int_t^T |z_s|^2\ \mathrm{d}s\right)^{p/2}\middle|\mathbb{F}_u\right] \le C_{\mu,\lambda,p,T}\mathbb{E}\left[\sup_{s\in[t,T]}|y_s|^p\middle|\mathbb{F}_u\right] + C_p\mathbb{E}\left[\left(\int_t^T f_s\ \mathrm{d}s\right)^p\middle|\mathbb{F}_u\right]$$
$$+ C_p\mathbb{E}\left[\left(\int_t^T \varphi_s\ \mathrm{d}s\right)^{p/2}\middle|\mathbb{F}_u\right],$$

*where $C_{\mu,\lambda,p,T}$ is a nonnegative constant depending on $(\mu, \lambda, p, T)$, and $C_p$ is a nonnegative constant depending only on $p$.*

**Remark 4.** Note that the constant $C_p$ does not depend on $\mu$ and $\lambda$. This fact will play an important role later.

**Proof of Proposition 2.** For each integer $n \ge 1$, let us introduce the stopping time

$$\tau_n = \inf\left\{t \in [0, T] : \int_0^t |z_s|^2\ \mathrm{d}s \ge n\right\} \wedge T.$$

Applying Itô's formula to $|y_t|^2$ leads the equation

$$|y_{t\wedge\tau_n}|^2 + \int_{t\wedge\tau_n}^{\tau_n} |z_s|^2\ \mathrm{d}s = |y_{\tau_n}|^2 + 2\int_{t\wedge\tau_n}^{\tau_n} \langle y_s, g(s, y_s, z_s)\rangle\ \mathrm{d}s - 2\int_{t\wedge\tau_n}^{\tau_n} \langle y_s, z_s\mathrm{d}B_s\rangle,\ \ t \in [0, T].$$

It follows from (A1) that for each $s \in [t \wedge \tau_n, \tau_n]$,

$$2\langle y_s, g(s, y_s, z_s)\rangle \le 2(\mu + \lambda^2)|y_s|^2 + \frac{|z_s|^2}{2} + 2|y_s|f_s + 2\varphi_s.$$

Thus, we have

$$\frac{1}{2}\int_{t\wedge\tau_n}^{\tau_n}|z_s|^2\,\mathrm{d}s\le 2[(\mu+\lambda^2)T+1]\sup_{s\in[t\wedge\tau_n,T]}|y_s|^2+\left(\int_{t\wedge\tau_n}^{T}f_s\,\mathrm{d}s\right)^2+2\int_{t\wedge\tau_n}^{T}\varphi_s\,\mathrm{d}s+2\left|\int_{t\wedge\tau_n}^{\tau_n}\langle y_s,z_s\mathrm{d}B_s\rangle\right|,$$

and the inequality $(a+b)^{p/2}\le 2^p(a^{p/2}+b^{p/2})$ yields the existence of a constant $c_p>0$ depending only on $p$ such that

$$\begin{aligned}\left(\int_{t\wedge\tau_n}^{\tau_n}|z_s|^2\,\mathrm{d}s\right)^{p/2}&\le c_p[(\mu+\lambda^2)T+1]^{p/2}\sup_{s\in[t\wedge\tau_n,T]}|y_s|^p+c_p\left(\int_{t\wedge\tau_n}^{T}f_s\,\mathrm{d}s\right)^p\\&\quad+c_p\left(\int_{t\wedge\tau_n}^{T}\varphi_s\,\mathrm{d}s\right)^{p/2}+c_p\left|\int_{t\wedge\tau_n}^{\tau_n}\langle y_s,z_s\mathrm{d}B_s\rangle\right|^{p/2}.\end{aligned}\tag{2}$$

Furthermore, the Burkholder–Davis–Gundy (BDG) inequality yields that there exists a constant $d_p>0$ depending only on $p$ such that for each $0\le u\le t\le T$,

$$\begin{aligned}c_p\mathbb{E}\left[\left|\int_{t\wedge\tau_n}^{\tau_n}\langle y_s,z_s\mathrm{d}B_s\rangle\right|^{p/2}\middle|\mathbb{F}_u\right]&\le d_p\mathbb{E}\left[\left(\int_{t\wedge\tau_n}^{\tau_n}|y_s|^2|z_s|^2\,\mathrm{d}s\right)^{p/4}\middle|\mathbb{F}_u\right]\\&\le\frac{d_p^2}{2}\mathbb{E}\left[\sup_{s\in[t\wedge\tau_n,T]}|y_s|^p\middle|\mathbb{F}_u\right]+\frac{1}{2}\mathbb{E}\left[\left(\int_{t\wedge\tau_n}^{\tau_n}|z_s|^2\,\mathrm{d}s\right)^{p/2}\middle|\mathbb{F}_u\right].\end{aligned}$$

Finally, taking the conditional mathematical expectation with respect to $\mathbb{F}_u$ in both sides of (2) and using the above inequality together with Fatou's lemma and Lebesgue's dominated convergence theorem yields the desired result. The proof is completed. □

Let us further introduce the following assumption on the generator $g$:

(A2) $\mathrm{d}\mathbb{P}\times\mathrm{d}t-a.e.$, $\forall\,(y,z)\in\mathbb{R}^k\times\mathbb{R}^{k\times d}$,

$$|y|^{p-1}\langle\tfrac{y}{|y|}\mathbb{1}_{|y|\neq 0},g(\omega,t,y,z)\rangle\le\psi(|y|^p)+\lambda|y|^{p-1}|z|+|y|^{p-1}f_t,$$

where $\lambda$ is a non-negative constant, $f_t$ is a non-negative and $(\mathbb{F}_t)$-progressively measurable process with

$$\mathbb{E}\left[\left(\int_0^T f_t\,\mathrm{d}t\right)^p\right]<+\infty,$$

and $\psi(\cdot):\mathbb{R}^+\mapsto\mathbb{R}^+$ is a nondecreasing and concave function with $\psi(0)=0$.

**Proposition 3.** *Assume that $p>1$ and (A2) holds. Let $(y_t,z_t)_{t\in[0,T]}$ be an $L^p$ solution of BSDE (1). Then, there exists a nonnegative constant $C_{\lambda,p,T}$ depending only on $\lambda$, $p$ and $T$ such that for each $0\le u\le t\le T$,*

$$\mathbb{E}\left[\sup_{s\in[t,T]}|y_s|^p\middle|\mathbb{F}_u\right]\le C_{\lambda,p,T}\left\{\mathbb{E}[|\xi|^p|\mathbb{F}_u]+\int_t^T\psi(\mathbb{E}[|y_s|^p|\mathbb{F}_u])\,\mathrm{d}s+\mathbb{E}\left[\left(\int_t^T f_s\,\mathrm{d}s\right)^p\middle|\mathbb{F}_u\right]\right\}.$$

**Proof.** It follows from Corollary 2.3 in Briand et al. [5] that, with $c(p) = p[(p-1)\wedge 1]/2$,

$$|y_t|^p + c(p)\int_t^T |y_s|^{p-2}\mathbb{1}_{|y_s|\neq 0}|z_s|^2\ \mathrm{d}s$$
$$\leq |\xi|^p + p\int_t^T |y_s|^{p-2}\mathbb{1}_{|y_s|\neq 0}\langle y_s, g(s,y_s,z_s)\rangle\ \mathrm{d}s - p\int_t^T |y_s|^{p-2}\mathbb{1}_{|y_s|\neq 0}\langle y_s, z_s\mathrm{d}B_s\rangle.$$

Assumption (A2) yields that, with probability one, for each $t\in[0,T]$,

$$|y_t|^p + c(p)\int_t^T |y_s|^{p-2}\mathbb{1}_{|y_s|\neq 0}|z_s|^2\ \mathrm{d}s$$
$$\leq |\xi|^p + p\int_t^T [\psi(|y_s|^p) + \lambda|y_s|^{p-1}|z_s| + |y_s|^{p-1}f_s]\ \mathrm{d}s - p\int_t^T |y_s|^{p-2}\mathbb{1}_{|y_s|\neq 0}\langle y_s, z_s\mathrm{d}B_s\rangle.$$

First of all, in view of the fact that $\psi(\cdot)$ increases at most linearly since it is a nondecreasing concave function and $\psi(0)=0$, we deduce from the previous inequality that

$$\int_0^T |y_s|^{p-2}\mathbb{1}_{|y_s|\neq 0}|z_s|^2\ \mathrm{d}s < +\infty,\quad \mathrm{d}\mathbb{P}-a.s.$$

Moreover, it follows from the inequality $ab\leq (a^2+b^2)/2$ that

$$p\lambda|y_s|^{p-1}|z_s| = p\left(\frac{\sqrt{2}\lambda}{\sqrt{(p-1)\wedge 1}}|y_s|^{\frac{p}{2}}\right)\left(\sqrt{\frac{(p-1)\wedge 1}{2}}|y_s|^{\frac{p-2}{2}}\mathbb{1}_{|y_s|\neq 0}|z_s|\right)$$
$$\leq \frac{p\lambda^2}{(p-1)\wedge 1}|y_s|^p + \frac{c(p)}{2}|y_s|^{p-2}\mathbb{1}_{|y_s|\neq 0}|z_s|^2.$$

Thus, for each $t\in[0,T]$, we have

$$|y_t|^p + \frac{c(p)}{2}\int_t^T |y_s|^{p-2}\mathbb{1}_{|y_s|\neq 0}|z_s|^2\ \mathrm{d}s \leq X_t - p\int_t^T |y_s|^{p-2}\mathbb{1}_{|y_s|\neq 0}\langle y_s, z_s\mathrm{d}B_s\rangle, \tag{3}$$

where

$$X_t = |\xi|^p + d_{\lambda,p}\int_t^T |y_s|^p\ \mathrm{d}s + p\int_t^T \psi(|y_s|^p)\ \mathrm{d}s + p\int_t^T |y_s|^{p-1}f_s\ \mathrm{d}s$$

with $d_{\lambda,p} = p\lambda^2/[(p-1)\wedge 1] > 0$.

It follows from the BDG inequality that $\{M_t := \int_0^t |y_s|^{p-2}\mathbb{1}_{|y_s|\neq 0}\langle y_s, z_s\mathrm{d}B_s\rangle\}_{t\in[0,T]}$ is a uniformly integrable martingale. In fact, Young's inequality yields

$$\mathbb{E}\left[\langle M,M\rangle_T^{1/2}\right] \leq \mathbb{E}\left[\sup_{s\in[0,T]}|y_s|^{p-1}\cdot\left(\int_0^T |z_s|^2\ \mathrm{d}s\right)^{1/2}\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left\{ \left( \sup_{s\in[0,T]} |y_s|^p \right)^{\frac{p-1}{p}} \cdot \left[ \left( \int_0^T |z_s|^2 \, \mathrm{d}s \right)^{p/2} \right]^{\frac{1}{p}} \right\} \\
&\le \frac{(p-1)}{p} \mathbb{E}\left[ \sup_{s\in[0,T]} |y_s|^p \right] + \frac{1}{p} \mathbb{E}\left[ \left( \int_0^T |z_s|^2 \, \mathrm{d}s \right)^{p/2} \right] \\
&< +\infty.
\end{aligned}
$$

Thus, for each $0 \le u \le t \le T$, taking the conditional mathematical expectation with respect to $\mathbb{F}_u$ in both sides of the inequality (3) yields both

$$
\frac{c(p)}{2} \mathbb{E}\left[ \int_t^T |y_s|^{p-2} \mathbb{1}_{|y_s|\neq 0} |z_s|^2 \, \mathrm{d}s \middle| \mathbb{F}_u \right] \le \mathbb{E}[X_t | \mathbb{F}_u] \tag{4}
$$

and

$$
\mathbb{E}\left[ \sup_{s\in[t,T]} |y_s|^p \middle| \mathbb{F}_u \right] \le \mathbb{E}[X_t | \mathbb{F}_u] + k_p \mathbb{E}\left[ (\langle M, M\rangle_T - \langle M, M\rangle_t)^{1/2} \middle| \mathbb{F}_u \right], \tag{5}
$$

where we have used the BDG inequality in (5), and $k_p$ is a constant depending only on $p$.

On the other hand, it follows from Young's inequality that for each $0 \le u \le t \le T$,

$$
\begin{aligned}
k_p \mathbb{E}\left[ (\langle M, M\rangle_T - \langle M, M\rangle_t)^{1/2} \middle| \mathbb{F}_u \right] &\le k_p \mathbb{E}\left[ \sup_{s\in[t,T]} |y_s|^{p/2} \cdot \left( \int_t^T |y_s|^{p-2} \mathbb{1}_{|y_s|\neq 0} |z_s|^2 \, \mathrm{d}s \right)^{1/2} \middle| \mathbb{F}_u \right] \\
&\le \frac{1}{2} \mathbb{E}\left[ \sup_{s\in[t,T]} |y_s|^p \middle| \mathbb{F}_u \right] + \frac{k_p^2}{2} \mathbb{E}\left[ \int_t^T |y_s|^{p-2} \mathbb{1}_{|y_s|\neq 0} |z_s|^2 \, \mathrm{d}s \middle| \mathbb{F}_u \right].
\end{aligned}
$$

It then follows from inequalities (4) and (5) that there exists a constant $k_p' > 0$ depending only on $p$ such that

$$
\mathbb{E}\left[ \sup_{s\in[t,T]} |y_s|^p \middle| \mathbb{F}_u \right] \le k_p' \mathbb{E}[X_t | \mathbb{F}_u].
$$

For each $0 \le u \le t \le T$, applying once again Young's inequality we get, with $k_p''$ is another constant depending only on $p$,

$$
\begin{aligned}
p k_p' \mathbb{E}\left[ \int_t^T |y_s|^{p-1} f_s \, \mathrm{d}s \middle| \mathbb{F}_u \right] &\le p k_p' \mathbb{E}\left[ \sup_{s\in[t,T]} |y_s|^{p-1} \int_t^T f_s \, \mathrm{d}s \middle| \mathbb{F}_u \right] \\
&= \mathbb{E}\left[ \left( \frac{p}{2(p-1)} \sup_{s\in[t,T]} |y_s|^p \right)^{\frac{p-1}{p}} \cdot \left[ \frac{p k_p''}{2} \left( \int_t^T f_s \, \mathrm{d}s \right)^p \right]^{\frac{1}{p}} \middle| \mathbb{F}_u \right] \\
&\le \frac{1}{2} \mathbb{E}\left[ \sup_{s\in[t,T]} |y_s|^p \middle| \mathbb{F}_u \right] + \frac{k_p''}{2} \mathbb{E}\left[ \left( \int_t^T f_s \, \mathrm{d}s \right)^p \middle| \mathbb{F}_u \right],
\end{aligned}
$$

from which we deduce, combing back to the definition of $X_t$, that

$$\mathbb{E}\left[\sup_{s\in[t,T]}|y_s|^p\middle|\mathbb{F}_u\right] \leq 2k'_p\mathbb{E}\left[|\xi|^p + d_{\lambda,p}\int_t^T |y_s|^p\ \mathrm{d}s + p\int_t^T \psi(|y_s|^p)\ \mathrm{d}s\middle|\mathbb{F}_u\right] + k''_p\mathbb{E}\left[\left(\int_t^T f_s\ \mathrm{d}s\right)^p\middle|\mathbb{F}_u\right].$$

Letting

$$h_t = \mathbb{E}\left[\sup_{s\in[t,T]}|y_s|^p\middle|\mathbb{F}_u\right]$$

in the previous inequality and using Fubini's Theorem and Jensen's inequality yields, in view of the concavity of $\psi(\cdot)$, that for each $0 \leq u \leq t \leq T$,

$$h_t \leq 2k'_p\mathbb{E}[\,|\xi|^p|\,\mathbb{F}_u] + 2pk'_p\int_t^T \psi(\mathbb{E}[\,|y_s|^p|\,\mathbb{F}_u])\ \mathrm{d}s + k''_p\mathbb{E}\left[\left(\int_t^T f_s\ \mathrm{d}s\right)^p\middle|\mathbb{F}_u\right] + 2k'_pd_{\lambda,p}\int_t^T h_s\ \mathrm{d}s.$$

Finally, Gronwall's inequality yields that for each $t \in [0, T]$,

$$h_t \leq e^{2k'_pd_{\lambda,p}(T-t)}\left\{2k'_p\mathbb{E}[\,|\xi|^p|\,\mathbb{F}_u] + 2pk'_p\int_t^T \psi(\mathbb{E}[\,|y_s|^p|\,\mathbb{F}_u])\ \mathrm{d}s + k''_p\mathbb{E}\left[\left(\int_t^T f_s\ \mathrm{d}s\right)^p\middle|\mathbb{F}_u\right]\right\},$$

which completes the proof of Proposition 3. □

## 4. A stability theorem and the proof of Theorem 1

In this section, we shall put forward and prove a stability theorem for $L^p$ ($p > 1$) solutions to multidimensional BSDEs with generators satisfying $(\mathrm{H1})_{p\wedge 2}$ and (H4). Based on this result, we shall further give the proof of Theorem 1 in Section 2.

The following lemma will be used, which comes from Fan and Jiang [17].

**Lemma 1.** *Assume that $\kappa(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a nondecreasing and concave function with $\kappa(0) = 0$. Then, it increases at most linearly, i.e., there exists a constant $A > 0$ such that*

$$\kappa(x) \leq A(x + 1), \quad \forall\ x \geq 0.$$

*Furthermore, for each $m \geq 1$, we have*

$$\kappa(x) \leq (m + 2A)x + \kappa(\frac{2A}{m + 2A}), \quad \forall\ x \in \mathbb{R}^+.$$

In the sequel, let $p > 1$ and for each $n \geq 1$, let $(y_t, z_t)_{t\in[0,T]}$ and $(y^n_t, z^n_t)_{t\in[0,T]}$ be respectively an $L^p$ solution of the BSDE $(\xi, T, g)$ and the following BSDE depending on parameter $n$:

$$y^n_t = \xi^n + \int_t^T g^n(s, y^n_s, z^n_s)\mathrm{d}s - \int_t^T z^n_s\mathrm{d}B_s, \quad t \in [0, T].$$

Furthermore, we introduce the following assumptions:

(B1) $\xi^n \in L^p(\mathbb{R}^k)$ for each $n \geq 1$ and all of $g^n$ satisfy assumptions $(\mathrm{H1})_{p \wedge 2}$ and (H4) with the same $\rho(\cdot)$ and $\bar{\lambda}$.

(B2) $\lim\limits_{n\to\infty} \mathbb{E}\left[|\xi^n - \xi|^p + \left(\int_0^T |g^n(s, y_s, z_s) - g(s, y_s, z_s)|\, \mathrm{d}s\right)^p\right] = 0.$

The following Theorem 2 is one of the main results of this section.

**Theorem 2.** *Under assumptions* (B1) *and* (B2)*, we have*

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{s\in[0,T]} |y_s^n - y_s|^p + \left(\int_0^T |z_s^n - z_s|^2\, \mathrm{d}s\right)^{p/2}\right] = 0. \tag{6}$$

**Proof.** First, in view of (B1), by $(i)$ of Proposition 1 we note that for each $n \geq 1$,

(a) $(\mathrm{H1})_p$ holds true for each $g^n$, together with a new and same function $\hat{\rho}(x)$;
(in case of $1 < p \leq 2$, $\hat{\rho}(x) \equiv \rho(x)$)

(b) $(\mathrm{H1})_2$ holds also true for each $g^n$, together with a new and same function $\bar{\rho}(x)$.
(in case of $p \geq 2$, $\bar{\rho}(x) \equiv \rho(x)$)

In the sequel, for each $n \geq 1$, let $\hat{y}^n_\cdot = y^n_\cdot - y_\cdot$, $\hat{z}^n_\cdot = z^n_\cdot - z_\cdot$, and $\hat{\xi}^n = \xi^n - \xi$. Then

$$\hat{y}_t^n = \hat{\xi}^n + \int_t^T \hat{g}^n(s, \hat{y}_s^n, \hat{z}_s^n)\, \mathrm{d}s - \int_t^T \hat{z}_s^n \mathrm{d}B_s, \quad t \in [0, T], \tag{7}$$

where for each $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$,

$$\hat{g}^n(s, y, z) := g^n(s, y + y_s, z + z_s) - g(s, y_s, z_s).$$

Note that

$$\hat{g}^n(s, y, z) = g^n(s, y + y_s, z + z_s) - g^n(s, y_s, z_s) + g^n(s, y_s, z_s) - g(s, y_s, z_s). \tag{8}$$

We can check by assumptions (B1) and (B2) together with (a) that the generator $\hat{g}^n$ of BSDE (7) satisfies assumption (A2) with

$$\psi(x) = \hat{\rho}(x), \quad \lambda = \bar{\lambda}, \quad \text{and} \quad f_t = |g^n(t, y_t, z_t) - g(t, y_t, z_t)|.$$

It then follows from Proposition 3 with $u = 0$ that there exists a constant $C_{\bar{\lambda},p,T} > 0$ depending only on $\bar{\lambda}$, $p$ and $T$ such that for each $n \geq 1$ and each $t \in [0, T]$,

$$\begin{aligned}\mathbb{E}\left[\sup_{r\in[t,T]} |\hat{y}_r^n|^p\right] \leq\ & C_{\bar{\lambda},p,T}\mathbb{E}\left[|\hat{\xi}^n|^p\right] + C_{\bar{\lambda},p,T}\int_t^T \hat{\rho}\left(\mathbb{E}\left[\sup_{r\in[s,T]} |\hat{y}_r^n|^p\right]\right) \mathrm{d}s \\ & + C_{\bar{\lambda},p,T}\mathbb{E}\left[\left(\int_0^T |g^n(s, y_s, z_s) - g(s, y_s, z_s)|\, \mathrm{d}s\right)^p\right].\end{aligned} \tag{9}$$

Furthermore, in view of (B2) and the fact that $\hat{\rho}(\cdot)$ is of linear growth by Lemma 1, Gronwall's inequality yields the existence of a constant $M > 0$ independent of $n$ such that

$$\mathbb{E}\left[\sup_{r\in[0,T]} |\hat{y}_r^n|^p\right] \leq M.$$

Thus, in view of (B2), by taking the limsup in (9) with respect to $n$ and using Fatou's lemma, the monotonicity and continuity of $\hat{\rho}(\cdot)$ and Bihari's inequality (see [3]) we can conclude that for each $t \in [0, T]$,

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{s\in[t,T]} |y_s^n - y_s|^p\right] = 0. \tag{10}$$

Furthermore, by (B2), (8), (b) and Lemma 1 we can also check that the generator $\hat{g}^n$ of BSDE (7) satisfies assumption (A1) with

$$\mu = m + 2A, \quad \lambda = \bar{\lambda}, \quad f_t = |g^n(t, y_t, z_t) - g(t, y_t, z_t)| \ \text{ and } \ \varphi_t = \bar{\rho}(\frac{2A}{m + 2A})$$

for each $m \geq 1$. It then follows from Proposition 2 with $u = t = 0$ that there exists a constant $C_{m,\bar{\lambda},p,T} > 0$ depending on $m$, $\bar{\lambda}$, $p$ and $T$, and a constant $C_p$ depending only on $p$ such that for each $m, n \geq 1$,

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^T |\hat{z}_s^n|^2 \,\mathrm{d}s\right)^{p/2}\right] &\leq C_{m,\bar{\lambda},p,T}\mathbb{E}\left[\sup_{t\in[0,T]} |\hat{y}_t^n|^p\right] + C_p\left(\bar{\rho}(\frac{2A}{m+2A}) \cdot T\right)^{p/2} \\ &\quad + C_p\mathbb{E}\left[\left(\int_0^T |g^n(s, y_s, z_s) - g(s, y_s, z_s)| \,\mathrm{d}s\right)^p\right].\end{aligned}$$

Thus, in view of (10), (B2) and the fact that $\bar{\rho}(x)$ is continuous function with $\bar{\rho}(0) = 0$, letting first $n \to \infty$ and then $m \to \infty$ in the previous inequality yields that

$$\lim_{n\to\infty} \mathbb{E}\left[\left(\int_0^T |z_s^n - z_s|^2 \,\mathrm{d}s\right)^{p/2}\right] = 0.$$

Thus, we obtain (6). The proof of Theorem 2 is then complete. $\square$

Now, we are in a position to prove Theorem 1.

**Proof of Theorem 1.** Assume that $p > 1$, and assumptions $(\mathrm{H1})_{p\wedge 2}$ with $\rho(x)$, (H2)–(H4) and $(\mathrm{H5})_p$ hold for the generator $g$. By $(i)$ of Proposition 1 we note that $(\mathrm{H1})_2$ holds also true for $g$, together with a new function $\bar{\rho}(x)$ (in case of $p \geq 2$, $\bar{\rho}(x) \equiv \rho(x)$).

The uniqueness part of Theorem 1 is an immediate corollary of Theorem 2. Now, let us prove the existence part. First, for each $n \geq 1$, let $q_n(x) = xn/(|x| \vee n)$ for $x \in \mathbb{R}^k$, and

$$\xi_n := q_n(\xi) \ \text{ and } \ g_n(t, y, z) := g(t, y, z) - g(t, 0, 0) + q_n(g(t, 0, 0)). \tag{11}$$

Note that for each $n \geq 1$, assumptions $(\mathrm{H1})_2$ with $\bar{\rho}(x)$, (H2)–(H4) hold true for each generator $g_n$. Furthermore, for each $n \geq 1$,

$$|\xi_n| \leq n \ \ \mathrm{d}\mathbb{P} - a.s. \ \text{ and } \ |g_n(t, 0, 0)| \leq n \ \ \mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e., \tag{12}$$

and by $(\mathrm{H5})_p$ we have

$$\lim_{m,n\to\infty} \mathbb{E}\left[|\xi_m - \xi_n|^p + \left(\int_0^T |q_m(g(s,0,0)) - q_n(g(s,0,0))|\, \mathrm{d}s\right)^p\right] = 0. \tag{13}$$

By virtue of Theorem 1 in Xu and Fan [43] we can know that the BSDE $(\xi_n, T, g_n)$ has a unique $L^2$ solution for each $n \ge 1$, denoted by $(y_t^n, z_t^n)_{t\in[0,T]}$.

Since for each $n \ge 1$, $g_n$ satisfies $(\mathrm{H1})_2$ with $\bar\rho(x)$, and (H4), we can check that it also satisfies (A2) with

$$p = 2,\quad \psi(x) = \bar\rho(x),\quad \lambda = \bar\lambda \ \text{ and } \ f_t = q_n(g(t,0,0)).$$

Thus, Proposition 3 together with (12) yields that for each $n \ge 1$, $(y_t^n)_{t\in[0,T]}$ is a bounded process and then belongs to $\mathcal{S}^p(0,T;\mathbb{R}^k)$. Furthermore, by Lemma 1 we know that there exists a constant $A > 0$ such that

$$\bar\rho(x) \le A(x+1),\quad \forall\, x \ge 0,$$

and then $g_n$ satisfies (A1) with

$$\mu = A,\quad \lambda = \bar\lambda,\quad f_t = q_n(g(t,0,0)) \ \text{ and } \ \varphi_t = A,$$

and Proposition 2 together with (12) yields that for each $n \ge 1$, $(z_t^n)_{t\in[0,T]}$ belongs to $\mathrm{M}^p(0,T;\mathbb{R}^k)$.

In the sequel, for each $m, n \ge 1$, let

$$\hat\xi^{m,n} = \xi_m - \xi_n,\quad \hat y_\cdot^{m,n} = y_\cdot^m - y_\cdot^n,\quad \hat z_\cdot^{m,n} = z_\cdot^m - z_\cdot^n.$$

Then $(\hat y_\cdot^{m,n}, \hat z_\cdot^{m,n})$ is an $L^p$ solution of the following BSDE depending on $(m,n)$:

$$\hat y_t^{m,n} = \hat\xi^{m,n} + \int_t^T \hat g^{m,n}(s, \hat y_s^{m,n}, \hat z_s^{m,n})\, \mathrm{d}s - \int_t^T \hat z_s^{m,n} \mathrm{d}B_s,\quad t \in [0,T], \tag{14}$$

where for each $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$,

$$\hat g^{m,n}(s,y,z) := g_m(s, y + y_s^n, z + z_s^n) - g_n(s, y_s^n, z_s^n).$$

Note by (11) that for each $m, n \ge 1$,

$$\hat g^{m,n}(t,y,z) = q_m(g(t,0,0)) - q_n(g(t,0,0)) + g(t, y + y_t^n, z + z_t^n) - g(t, y_t^n, z_t^n).$$

By the assumptions of the generator $g$ together with (13) we can check that the generator $\hat g^{m,n}$ of BSDE (14) satisfies $(\mathrm{H1})_{p\wedge 2}$ and (H4) with $\rho(\cdot)$ and $\bar\lambda$ for each $m, n \ge 1$, and

$$\lim_{m,n\to\infty} \mathbb{E}\left[|\hat\xi^{m,n} - 0|^p + \left(\int_0^T |\hat g^{m,n}(s,0,0) - \tilde g(s,0,0)|\, \mathrm{d}s\right)^p\right] = 0,$$

where for each $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$,

$$\tilde g(s,y,z) := 0.$$

Thus, we can apply Theorem 2 for BSDE (14) to get that

$$\lim_{m,n\to\infty}\mathbb{E}\left[\sup_{s\in[0,T]}|\hat{y}_s^{m,n}-0|^p+\left(\int_0^T|\hat{z}_s^{m,n}-0|^2\,\mathrm{d}s\right)^{p/2}\right]=0,$$

which means that $\{(y_t^n,z_t^n)_{t\in[0,T]}\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{S}^p(0,T;\mathbb{R}^k)\times \mathrm{M}^p(0,T;\mathbb{R}^{k\times d})$.

Finally, let $(y_t,z_t)_{t\in[0,T]}$ be the limit process of the sequence $\{(y_t^n,z_t^n)_{t\in[0,T]}\}_{n=1}^{\infty}$ in the process space $\mathcal{S}^p(0,T;\mathbb{R}^k)\times \mathrm{M}^p(0,T;\mathbb{R}^{k\times d})$. We pass to the limit in the sense of uniform convergence in probability for BSDE $(\xi_n,T,g_n)$, thanks to (H2), (H3) and (H4), to see that $(y_t,z_t)_{t\in[0,T]}$ solves the BSDE $(\xi,T,g)$. Thus, we prove the existence part and finally complete the proof of Theorem 1. □

## 5. A comparison theorem

In this section, we restrict ourselves to the case $k=1$ and prove the following comparison theorem of $L^p$ solutions for BSDEs with generators satisfying $(\mathrm{H1})_p$ and (H4).

**Theorem 3.** *Let $p>1$, $\xi,\xi'\in L^p(\mathbb{R}^k)$, $g$ and $g'$ be two generators of BSDEs, and $(y_\cdot,z_\cdot)$ and $(y'_\cdot,z'_\cdot)$ be respectively an $L^p$ solution to the BSDE $(\xi,T,g)$ and BSDE $(\xi',T,g')$. If $\xi\le\xi'$, $\mathrm{d}\mathbb{P}$ – a.s. and one of the following two statements holds true:*

(i) *$g$ satisfies $(\mathrm{H1})_p$ and (H4), and*

$$g(t,y'_t,z'_t)\le g'(t,y'_t,z'_t),\quad \mathrm{d}\mathbb{P}\times\mathrm{d}t\ –\ a.e.;$$

(ii) *$g'$ satisfies $(\mathrm{H1})_p$ and (H4), and*

$$g(t,y_t,z_t)\le g'(t,y_t,z_t),\quad \mathrm{d}\mathbb{P}\times\mathrm{d}t\ –\ a.e.,$$

*then for each $t\in[0,T]$, we have*

$$y_t\le y'_t,\quad \mathrm{d}\mathbb{P}\ –\ a.s.$$

**Proof.** We first assume that $\xi\le\xi'$, $\mathrm{d}\mathbb{P}$ – *a.s.*, $g$ satisfies $(\mathrm{H1})_p$ with $\rho(x)$ and (H4), and $g(t,y'_t,z'_t)\le g'(t,y'_t,z'_t)$, $\mathrm{d}\mathbb{P}\times\mathrm{d}t$ – *a.e.*

Setting $\hat{y}_t=y_t-y'_t$, $\hat{z}_t=z_t-z'_t$, $\hat{\xi}=\xi-\xi'$, by the Itô–Tanaka formula (see Exercise VI.1.25 in Revuz and Yor [39] for details) we have that for each $t\in[0,T]$,

$$\begin{aligned}&(\hat{y}_t^+)^p+\frac{p(p-1)}{2}\int_t^T|\hat{y}_s|^{p-2}\mathbb{1}_{\hat{y}_s>0}|\hat{z}_s|^2\,\mathrm{d}s\\ &=(\hat{\xi}^+)^p+p\int_t^T|\hat{y}_s|^{p-1}\mathbb{1}_{\hat{y}_s>0}[g(s,y_s,z_s)-g'(s,y'_s,z'_s)]\,\mathrm{d}s-p\int_t^T|\hat{y}_s|^{p-1}\mathbb{1}_{\hat{y}_s>0}\hat{z}_s\mathrm{d}B_s.\end{aligned}\tag{15}$$

Since $g(s,y'_s,z'_s)-g'(s,y'_s,z'_s)$ is non-positive, we have

$$\begin{aligned}g(s,y_s,z_s)-g'(s,y'_s,z'_s)&=g(s,y_s,z_s)-g(s,y'_s,z'_s)+g(s,y'_s,z'_s)-g'(s,y'_s,z'_s)\\ &\le g(s,y_s,z_s)-g(s,y'_s,z_s)+g(s,y'_s,z_s)-g(s,y'_s,z'_s)\end{aligned}$$

and we deduce, using $(\mathrm{H1})_p$ and (H4) for $g$ together with a similar inequality before (3), that

$$\begin{aligned} p|\hat{y}_s|^{p-1}\mathbb{1}_{\hat{y}_s>0}[g(s,y_s,z_s)-g'(s,y'_s,z'_s)] &\le p\rho((\hat{y}_s^+)^p)+p\bar{\lambda}|\hat{y}_s^+|^{p-1}|\hat{z}_s| \\ &\le p\bar{\rho}((\hat{y}_s^+)^p)+\frac{p(p-1)}{4}|\hat{y}_s|^{p-2}\mathbb{1}_{\hat{y}_s>0}|\hat{z}_s|^2, \end{aligned} \tag{16}$$

where

$$\bar{\rho}(u):=\rho(u)+d_{\bar{\lambda},p}u$$

with $d_{\bar{\lambda},p}=\bar{\lambda}^2/(p-1)$ is again a nondecreasing and concave function with $\bar{\rho}(0)=0$ and $\bar{\rho}(u)>0$ for $u>0$. Thus, in view of $\xi\le\xi'$, it follows from (15) and (16) that for each $t\in[0,T]$,

$$(\hat{y}_t^+)^p\le p\int_t^T\bar{\rho}((\hat{y}_s^+)^p)\mathrm{d}s-p\int_t^T|\hat{y}_s|^{p-1}\mathbb{1}_{\hat{y}_s>0}\hat{z}_s\mathrm{d}B_s.$$

Note that $(\int_0^t|\hat{y}_s|^{p-1}\mathbb{1}_{\hat{y}_s>0}\hat{z}_s\mathrm{d}B_s)_{t\in[0,T]}$ is a martingale by the BDG inequality, and $\bar{\rho}(u)$ is a concave function. It follows from Jensen's inequality that for each $t\in[0,T]$,

$$\mathbb{E}[(\hat{y}_t^+)^p]\le p\int_t^T\bar{\rho}(\mathbb{E}[(\hat{y}_s^+)^p])\mathrm{d}s. \tag{17}$$

Furthermore, since $\rho(\cdot)$ is a concave function and $\rho(0)=0$, we have $\rho(u)\ge\rho(1)u$ for each $u\in[0,1]$, and then for each $0\le u\le 1$,

$$\frac{1}{\bar{\rho}(u)}=\frac{1}{\rho(u)+d_{\bar{\lambda},p}u}\ge\frac{1}{\rho(u)+\frac{d_{\bar{\lambda},p}}{\rho(1)}\rho(u)}=\frac{\rho(1)}{\rho(1)+d_{\bar{\lambda},p}}\cdot\frac{1}{\rho(u)}.$$

As a result,

$$\int_{0^+}\frac{\mathrm{d}u}{\bar{\rho}(u)}=+\infty.$$

Thus, in view of (17), Bihari's inequality yields that for each $t\in[0,T]$,

$$\mathbb{E}[(\hat{y}_t^+)^p]=0$$

and then

$$y_t\le y'_t,\quad \mathrm{d}\mathbb{P}-a.s.$$

Now, let us assume that $\xi\le\xi'$, $\mathrm{d}\mathbb{P}-a.s.$, $g'$ satisfies $(\mathrm{H1})_p$ with $\rho(x)$, and (H4), and $g(t,y_t,z_t)\le g'(t,y_t,z_t)$, $\mathrm{d}\mathbb{P}\times\mathrm{d}t-a.e.$ Since $g(s,y_s,z_s)-g'(s,y_s,z_s)$ is non-positive, we have

$$\begin{aligned} g(s,y_s,z_s)-g'(s,y'_s,z'_s) &= g(s,y_s,z_s)-g'(s,y_s,z_s)+g'(s,y_s,z_s)-g'(s,y'_s,z'_s) \\ &\le g'(s,y_s,z_s)-g'(s,y'_s,z_s)+g'(s,y'_s,z_s)-g'(s,y'_s,z'_s). \end{aligned}$$

Furthermore, using $(\mathrm{H1})_p$ and (H4) for $g'$, we know that the inequality (16) holds still true. Therefore, the same argument as above yields that for each $t \in [0, T]$,

$$y_t \leq y'_t, \quad \mathrm{d}\mathbb{P} - a.s.$$

The theorem is proved. □

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## Appendix A

In this section, we will give the proof of Proposition 1 in Section 2. The following Lemma 2 will be used frequently, which comes from Lemma 6.1 in Fan and Jiang [22].

**Lemma 2.** *Let $f(\cdot)$ be a nondecreasing continuous function on $\mathbb{R}^+$ with $f(0) = 0$. Then, the following two statements hold true:*

(a) *If $f(\cdot)$ is concave on $\mathbb{R}^+$, then $f(x)/x$, $x > 0$ is a non-increasing function.*
(b) *If $f(x)/x$, $x > 0$ is a non-increasing function on $\mathbb{R}^+$, then there exists a nondecreasing concave function $p(\cdot)$ defined on $\mathbb{R}^+$ such that for each $x \geq 0$,*

$$f(x) \leq p(x) \leq 2f(x).$$

**Proof of Proposition 1.** First, let us prove that for each $1 \leq p < q$,

$$(\mathrm{H1})_p \Longrightarrow (\mathrm{H1})_q.$$

In fact, assume that $g$ satisfies $(\mathrm{H1})_p$ with a nondecreasing concave function $\rho(\cdot)$. Then we have, $\mathrm{d}\mathbb{P}\times\mathrm{d}t - a.e.$, $\forall y_1, y_2 \in \mathbb{R}^k$, $z \in \mathbb{R}^{k\times d}$,

$$|y_1 - y_2|^{q-1}\langle \frac{y_1 - y_2}{|y_1 - y_2|}\mathbb{1}_{|y_1 - y_2|\neq 0}, g(\omega, t, y_1, z) - g(\omega, t, y_2, z)\rangle \leq \bar{\rho}(|y_1 - y_2|^q),$$

where for each $x \geq 0$,

$$\bar{\rho}(x) = x^{1-\frac{p}{q}}\rho(x^{\frac{p}{q}}).$$

Obviously, $\bar{\rho}(\cdot)$ is a nondecreasing continuous function with $\bar{\rho}(0) = 0$ and $\bar{\rho}(x) > 0$ for $x > 0$. It follows from (a) of Lemma 2 that

$$\frac{\bar{\rho}(x)}{x} = \frac{\rho(x^{\frac{p}{q}})}{x^{\frac{p}{q}}}$$

is a non-increasing function on $\mathbb{R}^+$. Furthermore, by virtue of (b) of Lemma 2 we know that there exists a nondecreasing concave function $\kappa(\cdot)$ such that for each $x \geq 0$,

$$\bar{\rho}(x) \leq \kappa(x) \leq 2\bar{\rho}(x).$$

Then we have, $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall y_1, y_2 \in \mathbb{R}^k$, $z \in \mathbb{R}^{k\times d}$,

$$|y_1 - y_2|^{q-1}\langle \frac{y_1 - y_2}{|y_1 - y_2|}\mathbb{1}_{|y_1-y_2|\neq 0}, g(\omega, t, y_1, z) - g(\omega, t, y_2, z)\rangle \leq \kappa(|y_1 - y_2|^q)$$

and

$$\int\limits_{0^+} \frac{\mathrm{d}u}{\kappa(u)} \geq \frac{1}{2}\int\limits_{0^+} \frac{\mathrm{d}u}{\bar{\rho}(u)} = \frac{1}{2}\int\limits_{0^+} \frac{u^{\frac{p}{q}-1}}{\rho(u^{\frac{p}{q}})}\mathrm{d}u = \frac{q}{2p}\int\limits_{0^+} \frac{\mathrm{d}x}{\rho(x)} = +\infty.$$

Hence, $g$ satisfies $(\mathrm{H1})_q$ with $\kappa(\cdot)$, and then $(i)$ of Proposition 1 holds true.

Then, we prove that for each $1 \leq p < q$,

$$(\mathrm{H1b})_q \Longrightarrow (\mathrm{H1b})_p.$$

Indeed, it suffices to show that for a nondecreasing concave function $\rho(\cdot)$ on $\mathbb{R}^+$ with $\rho(0) = 0$, if

$$\int\limits_{0+} \frac{u^{q-1}}{\rho^q(u)}\mathrm{d}u = +\infty,$$

then

$$\int\limits_{0+} \frac{u^{p-1}}{\rho^p(u)}\mathrm{d}u = +\infty.$$

However, by (a) of Lemma 2 this statement follows easily from the following observation:

$$\liminf_{u\to 0^+} \frac{\frac{u^{p-1}}{\rho^p(u)}}{\frac{u^{q-1}}{\rho^q(u)}} = \liminf_{u\to 0^+} \left(\frac{\rho(u)}{u}\right)^{q-p} \geq \left(\frac{\rho(1)}{1}\right)^{q-p} > 0.$$

Hence, $(ii)$ of Proposition 1 is also true.

In the sequel, we prove that for each $p \geq 1$,

$$(\mathrm{H1a})_p \Longrightarrow (\mathrm{H1b})_p.$$

In fact, assume that $g$ satisfies $(\mathrm{H1a})_p$ with a nondecreasing concave function $\rho(\cdot)$. Then we have, $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall y_1, y_2 \in \mathbb{R}^k$, $z \in \mathbb{R}^{k\times d}$,

$$\langle \frac{y_1 - y_2}{|y_1 - y_2|}\mathbb{1}_{|y_1-y_2|\neq 0}, g(\omega, t, y_1, z) - g(\omega, t, y_2, z)\rangle \leq \bar{\rho}(|y_1 - y_2|),$$

where for each $x \geq 0$,

$$\bar{\rho}(x) = \rho^{\frac{1}{p}}(x^p).$$

Obviously, $\bar{\rho}(\cdot)$ is a nondecreasing continuous function with $\bar{\rho}(0) = 0$ and $\bar{\rho}(x) > 0$ for $x > 0$. It follows from (a) of Lemma 2 that

$$\frac{\bar{\rho}(x)}{x} = \left(\frac{\rho(x^p)}{x^p}\right)^{\frac{1}{p}}$$

is a non-increasing function on $\mathbb{R}^+$. Furthermore, by virtue of (b) of Lemma 2 we know that there exists a nondecreasing concave function $\kappa(\cdot)$ such that for each $x \geq 0$,

$$\bar{\rho}(x) \leq \kappa(x) \leq 2\bar{\rho}(x).$$

Then we have, $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall y_1, y_2 \in \mathbb{R}^k$, $z \in \mathbb{R}^{k\times d}$,

$$\langle \frac{y_1 - y_2}{|y_1 - y_2|} \mathbb{1}_{|y_1-y_2|\neq 0}, g(\omega, t, y_1, z) - g(\omega, t, y_2, z)\rangle \leq \kappa(|y_1 - y_2|)$$

and

$$\int\limits_{0^+} \frac{u^{p-1}}{\kappa^p(u)}\mathrm{d}u \geq \frac{1}{2^p}\int\limits_{0^+} \frac{u^{p-1}}{\bar{\rho}^p(u)}\mathrm{d}u = \frac{1}{2^p}\int\limits_{0^+} \frac{u^{p-1}}{\rho(u^p)}\mathrm{d}u = \frac{1}{p2^p}\int\limits_{0^+} \frac{1}{\rho(x)}\mathrm{d}x = +\infty.$$

Hence, $g$ satisfies $(\mathrm{H1b})_p$ with $\kappa(\cdot)$.

Furthermore, we prove that for each $p \geq 1$,

$$(\mathrm{H1b})_p \Longrightarrow (\mathrm{H1a})_p.$$

In fact, assume that $g$ satisfies $(\mathrm{H1b})_p$ with a nondecreasing concave function $\rho(\cdot)$. Then we have, $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall y_1, y_2 \in \mathbb{R}^k$, $z \in \mathbb{R}^{k\times d}$,

$$\langle \frac{y_1 - y_2}{|y_1 - y_2|} \mathbb{1}_{|y_1-y_2|\neq 0}, g(\omega, t, y_1, z) - g(\omega, t, y_2, z)\rangle \leq \bar{\rho}^{\frac{1}{p}}(|y_1 - y_2|^p),$$

where for each $x \geq 0$,

$$\bar{\rho}(x) = \rho^p(x^{\frac{1}{p}}).$$

Obviously, $\bar{\rho}(\cdot)$ is a nondecreasing continuous function with $\bar{\rho}(0) = 0$ and $\bar{\rho}(x) > 0$ for $x > 0$. It follows from (a) of Lemma 2 that

$$\frac{\bar{\rho}(x)}{x} = \left(\frac{\rho(x^{\frac{1}{p}})}{x^{\frac{1}{p}}}\right)^p$$

is a non-increasing function on $\mathbb{R}^+$. Furthermore, by virtue of (b) of Lemma 2 we know that there exists a nondecreasing concave function $\kappa(\cdot)$ such that for each $x \geq 0$,

$$\bar{\rho}(x) \leq \kappa(x) \leq 2\bar{\rho}(x).$$

Then we have, $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $\forall y_1, y_2 \in \mathbb{R}^k$, $z \in \mathbb{R}^{k\times d}$,

$$\langle \frac{y_1 - y_2}{|y_1 - y_2|} \mathbb{1}_{|y_1-y_2|\neq 0}, g(\omega, t, y_1, z) - g(\omega, t, y_2, z)\rangle \leq \kappa^{\frac{1}{p}}(|y_1 - y_2|^p)$$

and

$$\int\limits_{0^+} \frac{\mathrm{d}u}{\kappa(u)} \geq \frac{1}{2}\int\limits_{0^+} \frac{\mathrm{d}u}{\bar{\rho}(u)} = \frac{1}{2}\int\limits_{0^+} \frac{\mathrm{d}u}{\rho^p(u^{\frac{1}{p}})} = \frac{p}{2}\int\limits_{0^+} \frac{x^{p-1}}{\rho^p(x)}\mathrm{d}x = +\infty.$$

Hence, $g$ satisfies $(\mathrm{H1a})_p$ with $\kappa(\cdot)$, and then $(iii)$ of Proposition 1 holds true.

Finally, we prove that the concavity condition of $\rho(\cdot)$ in $(\mathrm{H1b})_p$ and $(\mathrm{H1a})_p$ can be replaced with the continuity condition.

Assume first that $p \geq 1$ and $g$ satisfies $(\mathrm{H1b})_p$ with $\rho(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$, which is a nondecreasing continuous (but not concave) function with $\rho(0) = 0$, $\rho(u) > 0$ for $u > 0$ and

$$\int_{0^+} \frac{u^{p-1}}{\rho^p(u)} \mathrm{d}u = +\infty.$$

Then, there exists a $\mathcal{P} \subset \Omega \times [0, T]$ with

$$\mathrm{d}\mathbb{P} \times \mathrm{d}t((\Omega \times [0, T]) \cap \mathcal{P}^c) = 0$$

such that for each $(\omega, t) \in \mathcal{P}$, $y_1, y_2 \in \mathbb{R}^k$ and $z \in \mathbb{R}^{k\times d}$, we have

$$\langle \frac{y_1 - y_2}{|y_1 - y_2|} \mathbb{1}_{|y_1-y_2|\neq 0}, g(\omega, t, y_1, z) - g(\omega, t, y_2, z)\rangle \leq \rho(|y_1 - y_2|). \tag{18}$$

Now, for each $r \in \mathbb{R}^+$, let

$$\begin{aligned} F(r) = \sup \Big\{ &\langle \tfrac{y_1-y_2}{|y_1-y_2|} \mathbb{1}_{|y_1-y_2|\neq 0}, g(\omega, t, y_1, z) - g(\omega, t, y_2, z)\rangle : \ (y_1, y_2) \in \mathbb{R}^k \times \mathbb{R}^k, \\ &|y_1 - y_2| \leq r, \ (\omega, t, z) \in \mathcal{P} \times \mathbb{R}^{k\times d} \Big\}. \end{aligned}$$

It is clear that $F(0) = 0$. It follows from (18) that $F(\cdot)$ is well-defined, nondecreasing and for each $r \geq 0$,

$$0 \leq F(r) \leq \rho(r).$$

Thus, in view of the continuity of $\rho(\cdot)$ and the fact $\rho(0) = 0$, we know that $F(\cdot)$ is right-continuous at 0. Furthermore, for $r, s \geq 0$ and $(y_1, y_2) \in \mathbb{R}^k \times \mathbb{R}^k$ with $r \leq |y_1 - y_2| \leq r + s$, it follows from the definition of $F(\cdot)$ that for each $(\omega, t) \in \mathcal{P}$, $y_1, y_2 \in \mathbb{R}^k$ and $z \in \mathbb{R}^{k\times d}$,

$$\begin{aligned} &\langle \frac{y_1 - y_2}{|y_1 - y_2|} \mathbb{1}_{|y_1-y_2|\neq 0}, g(\omega, t, y_1, z) - g(\omega, t, y_2, z)\rangle \\ &= \langle \frac{y_1 - y_2}{|y_1 - y_2|} \mathbb{1}_{|y_1-y_2|\neq 0}, g(\omega, t, y_1, z) - g(\omega, t, y_1 + r\frac{y_2 - y_1}{|y_2 - y_1|}, z)\rangle \\ &\quad + \langle \frac{y_1 - y_2}{|y_1 - y_2|} \mathbb{1}_{|y_1-y_2|\neq 0}, g(\omega, t, y_1 + r\frac{y_2 - y_1}{|y_2 - y_1|}, z) - g(\omega, t, y_2, z)\rangle \\ &\leq F(r) + F(s) \end{aligned}$$

so that, the case $|y_1 - y_2| \leq r$ being trivial, we conclude that $F(\cdot)$ is sub-additive. That is, for each $r, s \geq 0$,

$$F(r + s) \leq F(r) + F(s).$$

As a result, $F(\cdot)$ is a continuous modular function, and then there exists a nondecreasing and concave function $\bar{\rho}(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $u \geq 0$,

$$F(u) \leq \bar{\rho}(u) \leq 2F(u) \leq 2\rho(u)$$

(see pages 499–500 in [30] for details). Thus, it follows from (18) and the definition of $F(\cdot)$ that $\mathrm{d}\mathbb{P}\times\mathrm{d}t - a.e.$, for each $(y_1, y_2, z) \in \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$,

$$\langle \frac{y_1 - y_2}{|y_1 - y_2|} \mathbb{1}_{|y_1 - y_2| \neq 0}, g(\omega, t, y_1, z) - g(\omega, t, y_2, z) \rangle \leq F(|y_1 - y_2|) \leq \bar{\rho}(|y_1 - y_2|) \leq \kappa(|y_1 - y_2|),$$

where

$$\kappa(u) := \bar{\rho}(u) + u.$$

Clearly, $\kappa(\cdot)$ is a nondecreasing and concave function with $\kappa(0) = 0$ and $\kappa(u) > 0$ for $u > 0$. Thus, for completing the proof that $g$ satisfies $(\mathrm{H1b})_p$, it suffices to show that

$$\int_{0^+} \frac{u^{p-1}}{\kappa^p(u)} \mathrm{d}u = +\infty.$$

Indeed, if $\bar{\rho}(u) > 0$ for each $u > 0$, since $\bar{\rho}(\cdot)$ is concave on $\mathbb{R}^+$ with $\bar{\rho}(0) = 0$, we have $\bar{\rho}(u) \geq u\bar{\rho}(1)$ for each $u \in (0, 1)$, and then

$$\begin{aligned}
\int_{0^+} \frac{u^{p-1}}{\kappa^p(u)} \mathrm{d}u &= \int_{0^+} \frac{u^{p-1}}{(\bar{\rho}(u) + u)^p} \mathrm{d}u \\
&\geq \frac{\bar{\rho}^p(1)}{(1 + \bar{\rho}(1))^p} \int_{0^+} \frac{u^{p-1}}{\bar{\rho}^p(u)} \mathrm{d}u \\
&\geq \frac{1}{2^p} \cdot \frac{\bar{\rho}^p(1)}{(1 + \bar{\rho}(1))^p} \int_{0^+} \frac{u^{p-1}}{\rho^p(u)} \mathrm{d}u \\
&= +\infty.
\end{aligned}$$

Otherwise,

$$\int_{0^+} \frac{u^{p-1}}{\kappa^p(u)} \mathrm{d}u = \int_{0^+} \frac{u^{p-1}}{(\bar{\rho}(u) + u)^p} \mathrm{d}u = \int_{0^+} \frac{\mathrm{d}u}{u} = +\infty.$$

Thus, in view of (*iii*) of Proposition 1, we have proved that the concavity condition of $\rho(\cdot)$ in $(\mathrm{H1b})_p$ and $(\mathrm{H1a})_p$ can be replaced with the continuity condition. The proof of Proposition 1 is then completed. □

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