



# Properties of the zeros of the polynomials belonging to the $q$ -Askey scheme



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## ARTICLE INFO

### Article history:

Received 9 October 2014

Available online 3 August 2015

Submitted by B.C. Berndt

### Keywords:

$q$ -Askey scheme

Askey–Wilson polynomials

$q$ -Racah polynomials

Zeros of polynomials

Diophantine relations

Isospectral matrices

## ABSTRACT

In this paper we provide properties—which are, to the best of our knowledge, new—of the zeros of the polynomials belonging to the  $q$ -Askey scheme. These findings include Diophantine relations satisfied by these zeros when the parameters characterizing these polynomials are appropriately restricted.

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## 1. Introduction

In a previous paper [4] we identified certain new properties of the  $N$  zeros of the polynomials of order  $N$  belonging to the Askey scheme. The main one of these properties identifies an  $N \times N$  matrix—explicitly defined in terms of these  $N$  zeros and of the parameters characterizing the polynomial under consideration—which features  $N$  eigenvalues given by neat, explicit formulas, having moreover a Diophantine connotation when the parameters of the polynomial are suitably restricted. In this paper we identify somewhat analogous properties of the  $N$  zeros of the polynomials of order  $N$  belonging to the  $q$ -Askey scheme. Above and hereafter  $N$  is an *arbitrary positive integer*, and  $q$  an arbitrary number (possibly even *complex*; of course the results reported in this paper reproduce—via appropriate developments—the results of [4] in the  $q \rightarrow 1$  limit).

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Let us recall [4] that “the properties of the zeros of polynomials are a core problem of mathematics to which, over time, an immense number of investigations have been devoted. Nevertheless new findings in this area continue to emerge, see, for instance, [7–11,19,21,3,5,17,18,6,14–16],” and see also the very recent paper [2].

The technique used to obtain the results reported below is somewhat analogous to that employed in our previous paper [4], but there are significant differences, due to the fact that the main tool of our treatment are now Differential  $q$ -Difference Equations (DqDEs) instead of Differential Difference Equations (DDEs). Hence our treatment below is patterned after that of [4]; yet a previous reading of [4] is by no means mandatory to understand what follows.

The main findings of this paper are reported in the following Section 2. They detail properties of the zeros of the Askey–Wilson and  $q$ -Racah polynomials, which are the two “highest” classes of polynomials belonging to the  $q$ -Askey scheme [22]—so that these polynomials feature 5 arbitrary parameters (including  $q \neq 1$ ), in addition to their degree  $N$ . Analogous properties can of course be obtained, from the results reported below, for the zeros of the (variously “named” [22]) polynomials belonging to “lower” classes of the  $q$ -Askey scheme, via the reductions—corresponding to special assignments of the parameters—that characterize the  $q$ -Askey scheme [22]; we leave this task to the interested reader.

Our findings are proven in Section 3. The definitions and some standard properties of the Askey–Wilson and  $q$ -Racah polynomials are reported in Appendix A, for the convenience of the reader and also to specify our notation; the reader is advised to glance through Appendix A before reading the next section, and then to return to it whenever appropriate.

The results that follow are only a consequence of the *explicit definitions* of the Askey–Wilson and  $q$ -Racah polynomials and of the  *$q$ -difference equations* they satisfy (see Appendix A); the orthogonality properties that these polynomials also satisfy play no role, so that the results reported below do not require the restrictions on the parameters and arguments of these polynomials that are instead mandatory for the validity of these orthogonality properties and of other related properties [22].

## 2. Results

To formulate our results we refer to the definitions and standard properties of the Askey–Wilson and  $q$ -Racah polynomials as reported in Appendix A, to which the reader should also refer for the notation employed hereafter. As indicated below, some of these results are immediate consequences of known properties of the Askey–Wilson and  $q$ -Racah polynomials; our main results are proven in the following Section 3.

### 2.1. Results for the zeros of the Askey–Wilson polynomials

**Notation 2.1.** In this Section 2.1—as in the Askey–Wilson parts of Appendix A and of the next Section 3—we often use the change of variables

$$z = x + \sqrt{x^2 - 1}, \quad x = \frac{z^2 + 1}{2z}, \quad (1a)$$

with  $x$  being the argument of the Askey–Wilson polynomial  $p_N(x) \equiv p_N(a, b, c, d; q; x)$  and  $z$  being the argument of the corresponding rational function  $P_N(z) \equiv P_N(a, b, c, d; q; z)$ , see (41). These relations correspond to the assignment  $x = \cos \theta$ ,  $z = \exp(i\theta)$  (here and hereafter  $\mathbf{i}$  denotes the imaginary unit,  $\mathbf{i}^2 = -1$ ); but of course in this paper the argument  $x$  of the Askey–Wilson polynomial is a *complex variable*, not restricted to the interval  $[-1, 1]$  of the real line. And in all our final formulas only even powers of the square-root  $\sqrt{x^2 - 1}$  appear, so the determination of the square-root in the first formula (1a)—and in analogous formulas, see below—is not an issue. Likewise, the  $N$  zeros of the Askey–Wilson polynomial  $p_N(x)$  are denoted as  $\bar{x}_n$ ,  $p_N(\bar{x}_n) = 0$ , and the  $N$  quantities

$$\bar{z}_n = \bar{x}_n + \sqrt{\bar{x}_n^2 - 1}, \quad \bar{x}_n = \frac{\bar{z}_n^2 + 1}{2 \bar{z}_n}, \quad (1b)$$

are  $N$  zeros of the rational function  $P_N(z)$ ,  $P_N(\bar{z}_n) = 0$ .  $\square$

**Proposition 2.1.** Let  $\bar{x}_n \equiv \bar{x}_n(a, b, c, d; q; N)$  (with  $n = 1, 2, \dots, N$ ) be the  $N$  zeros of the Askey–Wilson polynomial  $p_N(x) \equiv p_N(a, b, c, d; q; x)$  of degree  $N$  in  $x$  (with  $N$  an arbitrary positive integer), so that (up to an irrelevant multiplicative constant  $C_N$ )  $p_N(x) = C_N \prod_{n=1}^N (x - \bar{x}_n)$ ; and let  $\bar{z}_n$  be related to  $\bar{x}_n$  by (1b).

Then there hold the  $N$  algebraic equations

$$A(\bar{z}_n) p_N \left( \frac{q^2 \bar{z}_n^2 + 1}{2 q \bar{z}_n} \right) + A \left( \frac{1}{\bar{z}_n} \right) p_N \left( \frac{\bar{z}_n^2 + q^2}{2 q \bar{z}_n} \right) = 0, \quad n = 1, \dots, N, \quad (2a)$$

or, equivalently,

$$A(\bar{z}_n) \prod_{m=1}^N \left( q \bar{z}_n + \frac{1}{q \bar{z}_n} - \bar{z}_m - \frac{1}{\bar{z}_m} \right) + \left[ \left( \bar{z}_s \rightarrow \frac{1}{\bar{z}_s} \right) \right] = 0, \quad n = 1, \dots, N, \quad (2b)$$

where

$$A(z) \equiv A(a, b, c, d; q; z) = \frac{(1 - az) (1 - bz) (1 - cz) (1 - dz)}{(1 - z^2) (1 - qz^2)} \quad (2c)$$

and the symbol  $\left[ \left( \bar{z}_s \rightarrow \frac{1}{\bar{z}_s} \right) \right]$  denotes the addition of everything that comes before it, with  $\bar{z}_s$  replaced by  $\frac{1}{\bar{z}_s}$  for all  $s = 1, \dots, N$ .  $\square$

**Remark 2.1.** It is evident that after the substitution  $\bar{x}_n = \cos \bar{\theta}_n$  and  $\bar{z}_n = \exp(i \bar{\theta}_n)$ , see (1b), the left-hand side of (2b) becomes an *even* function of each  $\bar{\theta}_s$ ,  $s = 1, \dots, N$ , hence a function of  $\bar{x}_1, \dots, \bar{x}_N$ . This is why after the substitution  $\bar{z}_n = \bar{x}_n + \sqrt{\bar{x}_n^2 - 1}$  in the left-hand sides of (2a) and (2b) all the square roots disappear, regardless of the determination of each square root  $\sqrt{\bar{x}_n^2 - 1}$ , as long as it is consistent, for each  $n$ , throughout the treatment.  $\square$

Following a suggestion by a Referee, let us note that in (2b)—and in analogous formulas below—the only explicit dependence on the order  $N$  of the polynomial is as upper limit of the product operation; this is of course consistent with a nontrivial dependence of the  $N$  zeros  $\bar{z}_n$  on the value of  $N$ .

This result is proved in Appendix A, see (44).

The following proposition is our main result for the  $N$  zeros  $\bar{x}_n$  of the Askey–Wilson polynomials.

**Proposition 2.2.** Let  $\bar{x}_n \equiv \bar{x}_n(a, b, c, d; q; N)$  (with  $n = 1, 2, \dots, N$ ) be the  $N$  zeros of the Askey–Wilson polynomial  $p_N(x) \equiv p_N(a, b, c, d; q; x)$  of degree  $N$  in  $x$  (with  $N$  an arbitrary positive integer), so that (up to an irrelevant multiplicative constant  $C_N$ )  $p_N(x) = C_N \prod_{n=1}^N (x - \bar{x}_n)$ ; and let  $\bar{z}_n$  be related to  $\bar{x}_n$  by (1b). Define the  $N \times N$  matrix  $\underline{M} \equiv \underline{M}(a, b, c, d; q; N; \bar{x})$ , componentwise, as follows:

$$\begin{aligned} M_{nn} &\equiv M_{nn}(a, b, c, d; q; N; \bar{x}) \\ &= \frac{(q-1)}{2q^N} \left\{ \left[ \frac{2\bar{z}_n^2}{\bar{z}_n^2 - 1} G(\bar{z}_n) \sum_{m=1, m \neq n}^N \left( -\frac{q}{\bar{z}_m - q\bar{z}_n} + \frac{q\bar{z}_m}{q\bar{z}_n\bar{z}_m - 1} \right. \right. \right. \\ &\quad \left. \left. + \frac{1}{\bar{z}_m - \bar{z}_n} - \frac{\bar{z}_m}{\bar{z}_n\bar{z}_m - 1} \right) + \frac{2\bar{z}_n^2 G'(\bar{z}_n)}{\bar{z}_n^2 - 1} \right] \prod_{\ell=1, \ell \neq n}^N K(\bar{z}_n, \bar{z}_\ell) \\ &\quad \left. + \left[ \left( \bar{z}_s \rightarrow \frac{1}{\bar{z}_s} \right) \right] \right\} \end{aligned} \quad (3a)$$

for  $n = 1, \dots, N$ , and

$$\begin{aligned} M_{nm} &\equiv M_{nm}(a, b, c, d; q; N; \bar{x}) \\ &= \frac{(q-1)}{2q^N} \left\{ \frac{2\bar{z}_m^2}{\bar{z}_m^2-1} G(\bar{z}_n) \left[ \frac{1}{\bar{z}_m - q\bar{z}_n} + \frac{q\bar{z}_n}{q\bar{z}_n\bar{z}_m-1} \right. \right. \\ &\quad \left. \left. - \frac{1}{\bar{z}_m - \bar{z}_n} - \frac{\bar{z}_n}{\bar{z}_n\bar{z}_m-1} \right] \prod_{\ell=1, \ell \neq n}^N K(\bar{z}_n, \bar{z}_\ell) + \left[ \left( \bar{z}_s \rightarrow \frac{1}{\bar{z}_s} \right) \right] \right\} \end{aligned} \quad (3b)$$

for  $n, m = 1, \dots, N$  with  $n \neq m$ . In these formulas

$$G(\bar{z}_n) = A(\bar{z}_n) \left( q\bar{z}_n - \frac{1}{\bar{z}_n} \right), \quad (3c)$$

$$G'(z) = \frac{d}{dz} G(z), \quad (3d)$$

$A \equiv A(a, b, c, d; q; z)$  is defined by (2c),

$$K(\bar{z}_n, \bar{z}_m) = \frac{(\bar{z}_m - q\bar{z}_n)(q\bar{z}_n\bar{z}_m - 1)}{(\bar{z}_m - \bar{z}_n)(\bar{z}_n\bar{z}_m - 1)} \quad (3e)$$

for  $n, m = 1, \dots, N$  with  $n \neq m$ , and the symbol  $+ \left[ \left( \bar{z}_s \rightarrow \frac{1}{\bar{z}_s} \right) \right]$  indicates addition of everything that comes before it, within the curly brackets, with  $\bar{z}_s$  replaced by  $\frac{1}{\bar{z}_s}$  for all  $s = 1, \dots, N$ .

Then this  $N \times N$  matrix  $\underline{M}$  has the  $N$  eigenvalues

$$\begin{aligned} \mu_n &\equiv \mu_n(abcd; q; N) = q^{-N} (1 - q^n) (1 - abcd q^{2N-1-n}) \\ &= (q^{-N} + abcd q^{N-1} - q^{n-N} - abcd q^{N-1-n}), \\ n &= 1, 2, \dots, N. \quad \square \end{aligned} \quad (3f)$$

It is evident that the components of the matrix  $M$  in Proposition 2.2 are functions of  $\bar{x}_1, \dots, \bar{x}_N$ , see Remark 2.1.

Some immediate corollaries of Proposition 2.2 are worth a mention.

**Corollary 2.2.1.** *If  $abcd$  and  $q$  are both rational numbers, the  $N$  eigenvalues of the  $N \times N$  matrix  $\underline{M}$  (see (3)) are all rational numbers.  $\square$*

This is a remarkable Diophantine property.

**Corollary 2.2.2.** *The  $N \times N$  matrix  $\underline{M}$ —which depends of course on the 4 a priori arbitrary parameters  $a, b, c, d$ , explicitly via  $A(z)$ , see (2c), and implicitly via the dependence on these 4 parameters of the  $N$  zeros  $\bar{x}_n \equiv \bar{x}_n(a, b, c, d; q; N)$  of the Askey–Wilson polynomials  $p_N(a, b, c, d; q; x)$ , see (3)—is isospectral under any variation of these 4 parameters which does not change the value of their product  $abcd$ .  $\square$*

**Corollary 2.2.3.** *Several identities satisfied by the  $N$  zeros  $\bar{x}_n \equiv \bar{x}_n(a, b, c, d; q; N)$  of the Askey–Wilson polynomial  $p_N(x) \equiv p_N(a, b, c, d; q; x)$  are implied by the following standard consequences of Proposition 2.2:*

$$\text{trace} \left[ (\underline{M})^k \right] = \sum_{n=1}^N (q^{-N} + abcd q^{N-1} - q^{n-N} - abcd q^{N-1-n})^k,$$

$$k = 1, 2, 3, \dots, \quad (4a)$$

$$\det [\underline{M}] = q^{-N^2} (q; q)_N (abcd q^{N-1}; q)_N. \quad (4b)$$

In particular, for  $k = 1$ , the first of these two formulas reads as follows:

$$\sum_{n=1}^N (M_{nn}) = N (q^{-N} + abcd q^{N-1}) + \left( \frac{1 - q^{-N}}{1 - q} \right) (q + abcd q^{N-1}). \quad \square \quad (4c)$$

## 2.2. Results for the zeros of the $q$ -Racah polynomials

Let  $\bar{z}_n \equiv \bar{z}_n(\alpha, \beta, \gamma, \delta; q; N)$ , where  $n = 1, 2, \dots, N$ , be the  $N$  zeros of the  $q$ -Racah polynomial  $R_N(x) \equiv R_N(\alpha, \beta, \gamma, \delta; q; z)$  of degree  $N$  in  $z$  (with  $N$  an arbitrary positive integer), where the variables  $x$  and  $z$  are related by formula (45). Thus, up to an irrelevant multiplicative constant  $C_N$ ,  $R_N(z) = C_N \prod_{n=1}^N (z - \bar{z}_n)$ , see (48b). Let the  $2N$  numbers  $\bar{z}_n^{(\pm)}$  be defined as follows:

$$\bar{z}_n^{(\pm)} = q^{\pm 1} \bar{z}_n \pm \left( \frac{1 - q^2}{2q} \right) \left( \bar{z}_n - \sqrt{\bar{z}_n^2 - 4\gamma\delta q} \right), \quad n = 1, \dots, N, \quad (5)$$

see (47b) (of course with the same determination of the square root as in (47e)).

A result concerning the  $N$  zeros of the  $q$ -Racah polynomials of degree  $N$  reads then as follows.

**Proposition 2.3.** *The  $N$  zeros  $\bar{z}_n \equiv \bar{z}_n(\alpha, \beta, \gamma, \delta; q; N)$ , where  $n = 1, \dots, N$ , of the  $N$ -th degree  $q$ -Racah polynomial  $R_N(\alpha, \beta, \gamma, \delta; q; z)$  satisfy the following  $N$  algebraic equations:*

$$B(\bar{z}_n) R_N(\bar{z}_n^{(+)} + D(\bar{z}_n) R_N(\bar{z}_n^{(-)}) = 0, \quad n = 1, \dots, N, \quad (6a)$$

or, equivalently,

$$B(\bar{z}_n) \prod_{m=1}^N (\bar{z}_n^{(+)} - \bar{z}_m) + D(\bar{z}_n) \prod_{m=1}^N (\bar{z}_n^{(-)} - \bar{z}_m) = 0, \quad n = 1, \dots, N, \quad (6b)$$

where the numbers  $\bar{z}_n^{(\pm)}$  are defined by (5) and the functions  $B(z)$  respectively  $D(z)$  are defined by (47c) respectively (47d), (47e). Note that this result holds independently of which determination is taken for the square root in the above definition of  $\bar{z}_n^{(\pm)}$  and in (47e), provided of course it is the same.  $\square$

This Proposition 2.3 is proven in Appendix A, see (49a).

The following proposition is our main result for the zeros of  $q$ -Racah polynomials.

**Proposition 2.4.** *Let  $\bar{z}_n \equiv \bar{z}_n(\alpha, \beta, \gamma, \delta; q; N)$ , where  $n = 1, 2, \dots, N$ , be the  $N$  zeros of the  $q$ -Racah polynomial  $R_N(\alpha, \beta, \gamma, \delta; q; z)$  of degree  $N$  in  $z$  (with  $N$  an arbitrary positive integer). Define the  $N \times N$  matrix  $\underline{L} \equiv \underline{L}(\alpha, \beta, \gamma, \delta; q; N; \bar{z})$ , componentwise, as follows:*

$$L_{nn} = \left[ B'(\bar{z}_n)(\bar{z}_n^{(+)} - \bar{z}_n) + B(\bar{z}_n) \left( C^{(+)}(\bar{z}_n) - 1 + (\bar{z}_n^{(+)} - \bar{z}_n) \sum_{m=1, m \neq n}^N W^{(+)}(\bar{z}_n, \bar{z}_m) \right) \right] \prod_{\ell=1, \ell \neq n}^N \frac{\bar{z}_n^{(+)} - \bar{z}_\ell}{\bar{z}_n - \bar{z}_\ell}$$

$$\begin{aligned}
& + \left[ D'(\bar{z}_n)(\bar{z}_n^{(-)} - \bar{z}_n) \right. \\
& \left. + D(\bar{z}_n) \left( C^{(-)}(\bar{z}_n) - 1 + (\bar{z}_n^{(-)} - \bar{z}_n) \sum_{m=1, m \neq n}^N W^{(-)}(\bar{z}_n, \bar{z}_m) \right) \right] \prod_{\ell=1, \ell \neq n}^N \frac{\bar{z}_n^{(-)} - \bar{z}_\ell}{\bar{z}_n - \bar{z}_\ell}, \\
& n = 1, \dots, N,
\end{aligned} \tag{7a}$$

$$\begin{aligned}
L_{nm} &= B(\bar{z}_n) \left( \frac{\bar{z}_n^{(+)} - \bar{z}_n}{\bar{z}_n - \bar{z}_m} \right)^2 \prod_{\ell=1, \ell \neq n, m}^N \frac{\bar{z}_n^{(+)} - \bar{z}_\ell}{\bar{z}_n - \bar{z}_\ell} \\
&+ D(\bar{z}_n) \left( \frac{\bar{z}_n^{(-)} - \bar{z}_n}{\bar{z}_n - \bar{z}_m} \right)^2 \prod_{\ell=1, \ell \neq n, m}^N \frac{\bar{z}_n^{(-)} - \bar{z}_\ell}{\bar{z}_n - \bar{z}_\ell}, \quad n, m = 1, \dots, N, \quad n \neq m,
\end{aligned} \tag{7b}$$

where, as above, the functions  $B(z)$  respectively  $D(z)$  are defined by (47c) respectively (47d), (47e),  $B'(z) = \frac{d}{dz}B(z)$ ,

$$\begin{aligned}
C^{(\pm)}(\bar{z}_n) &= \frac{d\bar{z}_n^{(\pm)}}{d\bar{z}_n} = q^{\pm 1} \pm \frac{1 - q^2}{2q} \left( 1 - \frac{\bar{z}_n}{\sqrt{\bar{z}_n^2 - 4\gamma\delta q}} \right), \\
W^{(\pm)}(\bar{z}_n, \bar{z}_m) &= \frac{C^{(\pm)}(\bar{z}_n)(\bar{z}_n - \bar{z}_m) - \bar{z}_n^{(\pm)} + \bar{z}_m}{(\bar{z}_n - \bar{z}_m)(\bar{z}_n^{(\pm)} - \bar{z}_m)},
\end{aligned}$$

and the numbers  $\bar{z}_n^{(\pm)}$  are defined by (5). Then this  $N \times N$  matrix  $\underline{L}$  has the  $N$  eigenvalues

$$\begin{aligned}
\lambda_n &\equiv \lambda_n(\alpha\beta; q; N) = q^{-N} (1 - q^m) (1 - \alpha\beta q^{2N-m+1}), \\
n &= 1, 2, \dots, N. \quad \square
\end{aligned} \tag{7c}$$

Some immediate corollaries of Proposition 2.4 are worth a mention.

**Corollary 2.4.1.** *If  $\alpha\beta$  and  $q$  are both rational numbers, the  $N$  eigenvalues of the  $N \times N$  matrix  $\underline{L}$  (see (7)) are all rational numbers.  $\square$*

This is a remarkable Diophantine property.

**Corollary 2.4.2.** *The  $N \times N$  matrix  $\underline{L}$ —which depends of course on the 4 a priori arbitrary parameters  $\alpha, \beta, \gamma, \delta$ , explicitly via  $B(z)$  and  $D(z)$  (see (47c) and (47d) with (47e)), and implicitly via the dependence on these 4 parameters of the  $N$  zeros  $\bar{z}_n \equiv \bar{z}_n(\alpha, \beta, \gamma, \delta; q; N)$  of the  $q$ -Racah polynomial  $R_N(\alpha, \beta, \gamma, \delta; q; z)$ : see (7)—is isospectral under any variation of these 4 parameters which does not change the value of the product  $\alpha\beta$ .  $\square$*

**Corollary 2.4.3.** *Several identities satisfied by the  $N$  zeros  $\bar{z}_n \equiv \bar{z}_n(\alpha, \beta, \gamma, \delta; q; N)$  of the  $q$ -Racah polynomial  $R_N(\alpha, \beta, \gamma, \delta; q; z)$  are implied by the following standard consequences of Proposition 2.4:*

$$\text{trace} \left[ (\underline{L})^k \right] = q^{-kN} \sum_{m=1}^N [(1 - q^m) (1 - \alpha\beta q^{2N-m+1})]^k, \quad k = 1, 2, 3, \dots, \tag{8a}$$

$$\det [\underline{L}] = q^{-N^2} (q; q)_N (\alpha\beta q^{N+1}; q)_N. \tag{8b}$$

In particular, for  $k = 1$ , the first of these two formulas reads as follows:

$$\sum_{n=1}^N (L_{nn}) = N (q^{-N} + \alpha\beta q^{N+1}) + \frac{q(1 - q^{-N})(1 + \alpha\beta q^N)}{1 - q}. \quad \square \quad (8c)$$

### 3. Proof of the main results

In this Section 3 we prove our main results for the zeros of the Askey–Wilson and  $q$ -Racah polynomials.

#### 3.1. Askey–Wilson polynomials

Let  $\Psi_N(z; t) \equiv \Psi_N(a, b, c, d; q; z; t)$  be a rational function of the (generally *complex*) variable  $z$ , which satisfies the Differential- $q$ -Difference Equation (DqDE)

$$\frac{\partial \Psi_N(z; t)}{\partial t} = [(q^{-N} - 1)(1 - abcd q^{N-1}) - Q] \Psi_N(z, t), \quad (9)$$

with the ( $t$ -independent)  $q$ -differential operator  $Q$  acting on the variable  $z$  as defined by (43b). And let us assume that this rational function can be expressed (as confirmed below) by the following linear superposition with  $t$ -dependent coefficients  $c_m(t)$  of the  $(N + 1)$  rational functions  $P_{N-m}(z) \equiv P_{N-m}(a, b, c, d; q; z)$  defined by (39), with  $m = 0, 1, 2, \dots, N$ :

$$\Psi_N(z, t) = \sum_{m=0}^N [c_m(t) P_{N-m}(z)]. \quad (10)$$

It is then plain—see the eigenvalue equation (43a), of course with  $N$  replaced by  $(N - m)$ —that the  $t$ -evolution of the  $(N + 1)$  coefficients  $c_m(t)$  is characterized by the following system of ODEs,

$$\begin{aligned} \dot{c}_m(t) &= [(q^{-N} - 1)(1 - abcd q^{N-1}) \\ &\quad - (q^{m-N} - 1)(1 - abcd q^{N-m-1})] c_m(t) \\ &= q^{-N} (1 - q^m) (1 - abcd q^{2N-1-m}) c_m(t), \\ m &= 0, 1, 2, \dots, N, \end{aligned} \quad (11)$$

where the superimposed dot denotes of course a  $t$ -differentiation. Hence

$$c_0(t) = c_0(0), \quad (12a)$$

$$c_m(t) = \exp(\mu_m t) c_m(0), \quad m = 1, 2, \dots, N, \quad (12b)$$

where

$$\begin{aligned} \mu_m &\equiv \mu_m(abcd; q; N) = q^{-N} (1 - q^m) (1 - abcd q^{2N-1-m}), \\ m &= 1, 2, \dots, N, \end{aligned} \quad (12c)$$

implying

$$\Psi(z; t) = c_0(0) P_N(z) + \sum_{m=1}^N [\exp(\mu_m t) c_m(0) P_{N-m}(z)]. \quad (12d)$$

It is now clear (see (43)) that the “equilibrium”—i.e.,  $t$ -independent—solution  $\bar{\Psi}_N(z) \equiv \bar{\Psi}_N(a, b, c, d; q; z)$  of DqDE (9) reads

$$\bar{\Psi}_N(z) = c_0(0) P_N(z), \quad (13a)$$

corresponding to the “equilibrium” (i.e.,  $t$ -independent) solution of the dynamical system (11) which clearly reads

$$\bar{c}_0(t) = \bar{c}_0(0); \quad \bar{c}_m(t) = 0, \quad m = 1, 2, \dots, N. \quad (13b)$$

It is at this stage convenient to perform—as in Appendix A—the change of variables (1a) from  $z$  to  $x$  and viceversa, implying

$$\Psi_N(z; t) = \psi_N\left(\frac{z^2 + 1}{2z}; t\right), \quad \psi_N(x; t) = \Psi_N\left(x + \sqrt{x^2 - 1}; t\right), \quad (14a)$$

so that (12d) becomes

$$\psi(x; t) = c_0(0) p_N(x) + \sum_{m=1}^N [\exp(\mu_m t) c_m(0) p_{N-m}(x)], \quad (14b)$$

where  $p_m(x)$  is now the Askey–Wilson polynomial of degree  $m$  (see (37) and (14a)). It is thus seen that  $\psi(x; t)$  is a ( $t$ -dependent) polynomial of degree  $N$  in the variable  $x$ , and the corresponding *equilibrium* solution of (9) is, up to the arbitrary multiplicative constant  $c_0(0)$ , just the Askey–Wilson polynomial of degree  $N$ ,

$$\bar{\psi}_N(x) = \bar{c}_0(0) p_N(x) \quad (15)$$

(see (13a)).

Next, let us introduce the  $N$  zeros  $x_n(t) \equiv x_n(a, b, c, d; q; N; t)$  of the polynomial  $\psi(x; t)$  of degree  $N$  in  $x$ , by setting

$$\psi(x; t) = C_N \prod_{n=1}^N [x - x_n(t)]. \quad (16)$$

Here  $C_N$  is an arbitrary constant that plays no role in the following; it is of course proportional to  $c_0(0)$ , and the computation (from (14b), (16) and (37)) of the proportionality factor can be left to the very diligent reader.

Let us now investigate the  $t$ -evolution of the  $N$  zeros  $x_n(t)$ , as implied by the Differential  $q$ -Difference Equation satisfied by  $\psi_N(x; t)$ , which obtains from the DqDE (9) satisfied by  $\Psi(z; t)$  via the change of variables (14a):

$$\frac{\partial \psi_N(x; t)}{\partial t} = [(q^{-N} - 1)(1 - abcd q^{N-1}) - Q] \psi_N(x; t). \quad (17)$$

The action of the operator  $Q$  on functions of the variable  $x$  is given by the formula (see (43b) and (40))

$$Q f(x) = \left[ A(z) \delta_q^{(+)} + A(z^{-1}) \delta_q^{(-)} - A(z) - A(z^{-1}) \right] f(x) \quad (18a)$$

with

$$\delta_q^{(\sigma)} f(x) = f\left(q^\sigma x + \sigma \frac{1-q^2}{2qz}\right), \quad \sigma = \pm 1, \quad (18b)$$

where the variable  $z$  in (18a) is related to the variable  $x$  via (1a). It can be verified that the right-hand side of (18a) is an even function of  $\theta$  (defined by  $z = \exp(i\theta)$ ) hence a function of  $x$ , and so—loosely speaking—it makes no difference to set  $z = x + \sqrt{x^2 - 1}$  or  $z = x - \sqrt{x^2 - 1}$  in this definition of the two operators  $\delta_q^{(\pm)}$ , provided of course the same convention is used throughout (see Notation 2.1).

It is plain from (16)—by logarithmic  $t$ -differentiation—that

$$\begin{aligned} \frac{\partial \psi_N(x; t)}{\partial t} &= -\psi_N(x; t) \sum_{m=1}^N \frac{\dot{x}_m(t)}{x - x_m(t)} \\ &= -C_N \sum_{m=1}^N \left\{ \dot{x}_m(t) \prod_{\ell=1, \ell \neq m}^N [x - x_\ell(t)] \right\}. \end{aligned} \quad (19)$$

Hence

$$\left. \frac{\partial \psi_N(x; t)}{\partial t} \right|_{x=x_n(t)} = -C_N \left\{ \dot{x}_n(t) \prod_{\ell=1, \ell \neq n}^N [x_n(t) - x_\ell(t)] \right\}, \quad (20)$$

and by setting  $x = x_n(t)$  in (17) we get (via (18) and (40)—and (16) implying of course  $\psi_N(x_n(t); t) = 0$ )

$$\begin{aligned} \dot{x}_n &= \frac{(q-1)}{2q^N} \left\{ G(z_n) \prod_{\ell=1, \ell \neq n}^N K(z_n, z_\ell) + G\left(\frac{1}{z_n}\right) \prod_{\ell=1, \ell \neq n}^N K\left(\frac{1}{z_n}, \frac{1}{z_\ell}\right) \right\}, \\ &\equiv F_n(z_1, \dots, z_N) \equiv \hat{F}_n(x_1, \dots, x_N), \quad n = 1, \dots, N, \end{aligned} \quad (21)$$

where the functions  $G$  and  $K$  are defined by (3c) and (3e), respectively. Note that, for notational simplicity, we omitted to indicate the  $t$ -dependence of  $\dot{x}_n$ ,  $x_n$ ,  $x_\ell$ ,  $z_n$ . Again, via  $z_s = \exp(i\theta_s)$  and  $x_s = \cos \theta_s$ , it is clear that the right-hand side of (21) is an even function of  $\theta_s$ , hence a function of  $x_s$ , where  $s = 1, \dots, N$ .

This is an interesting dynamical system, a complete investigation of which is beyond the scope of the present paper. But before proceeding with our task, let us pause and recall that the first idea to relate the zeros of polynomials to a dynamical system goes back to Stieltjes [24], was resuscitated in [12] to identify “solvable” many-body problems (see also the extended treatment of this approach in [13]), and then extensively used to obtain results concerning the zeros of the classical polynomials and of Bessel functions, see the paper [1] where several such findings are derived and reviewed. For more recent developments along somewhat analogous lines see, for instance, [23, 25, 20].

Here we need to focus only on the behavior of this  $t$ -evolution in the infinitesimal vicinity of the equilibrium configuration  $x_n(t) = \bar{x}_n \equiv \bar{x}_n(a, b, c, d; q; N)$ , where the  $N$  numbers  $\bar{x}_n$  are of course the  $N$  zeros of the Askey–Wilson polynomial  $p_N(a, b, c, d; q; x)$ , see (42). To this end we set

$$x_n(t) = \bar{x}_n + \varepsilon \xi_n(t), \quad n = 1, \dots, N, \quad (22a)$$

implying of course

$$\dot{x}_n(t) = \varepsilon \dot{\xi}_n(t), \quad n = 1, \dots, N, \quad (22b)$$

with  $\varepsilon$  infinitesimal.

It is plain that the insertion of this *ansatz* in (21) is consistent to order  $\varepsilon^0 = 1$ , thanks to Proposition 2.1.

The insertion of this *ansatz* in (21) yields, to order  $\varepsilon$  (after a trivial but somewhat cumbersome computation) the linear system of ODEs

$$\dot{\xi}_n(t) = \sum_{m=1}^N [M_{nm}(\bar{x}) \xi_m(t)], \quad n = 1, \dots, N, \quad (23)$$

with the  $N \times N$  matrix  $\underline{M}(\bar{x}) \equiv \underline{M}(a, b, c, d; q; N; \bar{x})$  given by

$$M_{nm} = \frac{\partial}{\partial x_m} \hat{F}_n(x_1, \dots, x_N) \bigg|_{x_s = \bar{x}_s} = \frac{\partial}{\partial z_m} F_n(z_1, \dots, z_N) \frac{dz_m}{dx_m} \bigg|_{z_s = \bar{z}_s}, \quad s = 1, \dots, N, \quad (24)$$

which, after the substitution  $\frac{dz_m}{dx_m} = \frac{2z_m^2}{z_m^2 - 1}$  (implied by  $z_m = x_m + \sqrt{x_m^2 - 1}$ ) and some trivial computations yields formula (3). While by continuing the expansion of the right-hand side of (21) in powers of  $\varepsilon$  and setting to zero the resulting coefficients of higher powers of  $\varepsilon$  (since of course the left-hand side of (21) contains only a term of order  $\varepsilon$ , see (22b)), additional formulas satisfied by the  $N$  zeros  $\bar{x}_n \equiv \bar{x}_n(a, b, c, d; q; N)$  of the Askey–Wilson polynomial  $P_N(a, b, c, d; q; z)$  can be obtained.

The proof of Proposition 2.2 is now a consequence of the fact that the solution of the system of linear ODEs (23) is clearly a linear superposition (with  $t$ -independent coefficients) of exponentials,  $\exp(\tilde{\mu}_m t)$ , where the quantities  $\tilde{\mu}_m$  (with  $m = 1, 2, \dots, N$ ) are the  $N$  eigenvalues of the  $N \times N$  matrix  $\underline{M}$ ; but—due to the simultaneous validity of the relations (14b), (16) and (22a)—this solution must also be a linear superposition (with  $t$ -independent coefficients) of the  $t$ -dependent quantities  $c_m(t)$ . Hence (12) implies  $\tilde{\mu}_m = \mu_m = q^{-N} (1 - q^m) (1 - abcd q^{2N-1-m})$ . Proposition 2.2 is thereby proven.

Note that in our treatment above we implicitly assumed that the zeros  $x_n(t)$  are—for all values of  $t$ —all different among themselves. This is indeed the *generic* situation. Any nongeneric event—like the “collision” of two different zeros at some special value of the parameter  $t$ —can be dealt with by appropriate limits and in any case such possibilities—should they occur—do not invalidate the proof of Proposition 2.2, as reported above.

### 3.2. $q$ -Racah polynomials

The following treatment is analogous—*mutatis mutandis*—to that of the previous Section 3.1, hence it shall be somewhat more terse.

Let  $\Phi_N(z; t) \equiv \Phi_N(\alpha, \beta, \gamma, \delta; q; z; t)$  be a function of the variables  $z$  and  $t$ , which satisfies the Differential- $q$ -Difference Equation (DqDE)

$$\begin{aligned} \frac{\partial \Phi_N(z; t)}{\partial t} &= [(q^{-N} - 1) (1 - \alpha\beta q^{N+1}) + B(z) + D(z)] \Phi_N(z; t) \\ &\quad - B(z) \Phi_N(z^{(+)}; t) - D(z) \Phi_N(z^{(-)}; t), \end{aligned} \quad (25a)$$

where (see (47b))

$$z^{(\pm)} \equiv z^{(\pm)}(\gamma\delta; q; z) = q^{\pm 1} z \pm \left( \frac{1 - q^2}{2q} \right) \left( z - \sqrt{z^2 - 4\gamma\delta q} \right) \quad (25b)$$

and the functions  $B(z) \equiv B(\alpha, \beta, \gamma, \delta; q; z)$  respectively  $D(z) \equiv D(\alpha, \beta, \gamma, \delta; q; z)$  are defined by (47c) respectively (47d) with (see (47e))

$$Z(\gamma\delta q; z) = \frac{z + \sqrt{z^2 - 4\gamma\delta q}}{2\gamma\delta q}. \quad (25c)$$

And let us assume that this function can be expressed (as confirmed below) by the following linear superposition with  $t$ -dependent coefficients  $c_m(t)$  of the  $(N + 1)$   $q$ -Racah polynomials  $R_{N-m}(z) \equiv R_{N-m}(\alpha, \beta, \gamma, \delta; q; z)$  with  $m = 0, 1, 2, \dots, N$ :

$$\Phi_N(z; t) = \sum_{m=0}^N [c_m(t) R_{N-m}(z)]. \quad (26)$$

It is then plain—see (47), of course with  $N$  replaced by  $(N - m)$ —that the  $t$ -evolution of the  $(N + 1)$  coefficients  $c_m(t)$  is characterized by the following system of ODEs,

$$\begin{aligned} \dot{c}_m(t) &= [(q^{-N} - 1)(1 - \alpha\beta q^{N+1}) \\ &\quad - (q^{m-N} - 1)(1 - \alpha\beta q^{N-m+1})] c_m(t) \\ &= q^{-N} (1 - q^m) (1 - \alpha\beta q^{2N-m+1}) c_m(t), \\ m &= 0, 1, 2, \dots, N, \end{aligned} \quad (27)$$

where again the superimposed dot denotes  $t$ -differentiation. Hence

$$c_0(t) = c_0(0), \quad (28a)$$

$$c_m(t) = \exp(\lambda_m t) c_m(0), \quad m = 1, 2, \dots, N, \quad (28b)$$

where

$$\begin{aligned} \lambda_m &\equiv \lambda_m(\alpha\beta; q; N) = q^{-N} (1 - q^m) (1 - \alpha\beta q^{2N-m+1}), \\ m &= 1, 2, \dots, N, \end{aligned} \quad (28c)$$

implying (see (26))

$$\Phi_N(z; t) = c_0(0) R_N(z) + \sum_{m=1}^N [\exp(\lambda_m t) c_m(0) R_{N-m}(z)]. \quad (28d)$$

Next, let us introduce the  $N$  zeros  $z_n(t) \equiv z_n(\alpha, \beta, \gamma, \delta; q; N; t)$  of the polynomial  $\Phi_N(z, t)$  of degree  $N$  in  $z$ , by setting

$$\Phi_N(z; t) = C_N \prod_{n=1}^N [z - z_n(t)]. \quad (29)$$

Here  $C_N$  is an arbitrary constant that plays no role in the following.

Let us now investigate the  $t$ -evolution of the  $N$  zeros  $z_n(t)$ , as implied by the Differential  $q$ -Difference Equation (25) satisfied by  $\Phi_N(z; t)$ .

It is plain from (29)—by logarithmic  $t$ -differentiation—that

$$\frac{\partial \Phi_N(z; t)}{\partial t} = -\Phi_N(z; t) \sum_{m=1}^N \frac{\dot{z}_m(t)}{z - z_m(t)} = -C_N \sum_{m=1}^N \left\{ \dot{z}_m(t) \prod_{\ell=1, \ell \neq m}^N [z - z_\ell(t)] \right\}. \quad (30)$$

Hence

$$\left. \frac{\partial \Phi_N(z; t)}{\partial t} \right|_{z=z_n(t)} = -C_N \left\{ \dot{z}_n(t) \prod_{\ell=1, \ell \neq n}^N [z_n(t) - z_\ell(t)] \right\}, \quad (31)$$

and by setting  $z = z_n(t)$  in (25) we get (via (29) implying  $\Phi_N(z_n(t); t) = 0$ )

$$\begin{aligned} \dot{z}_n &= B(z_n) \left( z_n^{(+)} - z_n \right) \prod_{\ell=1, \ell \neq n}^N \left( \frac{z_n^{(+)} - z_\ell}{z_n - z_\ell} \right) \\ &+ D(z_n) \left( z_n^{(-)} - z_n \right) \prod_{\ell=1, \ell \neq n}^N \left( \frac{z_n^{(-)} - z_\ell}{z_n - z_\ell} \right), \\ n &= 1, 2, \dots, N, \end{aligned} \quad (32a)$$

where  $B(z) \equiv B(\alpha, \beta, \gamma, \delta; q; z)$  respectively  $D(z) \equiv D(\alpha, \beta, \gamma, \delta; q; z)$  are defined by (47c) respectively (47d) with (47e), and

$$z_n^{(\pm)} = q^{\pm 1} z_n \pm \left( \frac{1 - q^2}{2q} \right) \left( z_n - \sqrt{z_n^2 - 4\gamma\delta q} \right), \quad n = 1, \dots, N. \quad (32b)$$

Note that we omitted the arguments of  $z_n \equiv z_n(\alpha, \beta, \gamma, \delta; q; N; t)$  and, likewise,  $z_n^{(\pm)} \equiv z_n^{(\pm)}(\alpha, \beta, \gamma, \delta; q; N; t)$ .

Let us again mention that this set of nonlinear ODEs, (32), is an interesting dynamical system, a complete investigation of which is however beyond the scope of the present paper. Here we need to focus only on its behavior in the infinitesimal vicinity of its equilibrium configuration  $z_n(t) = \bar{z}_n \equiv \bar{z}_n(\alpha, \beta, \gamma, \delta; q; N)$ , where the  $N$  numbers  $\bar{z}_n$  are now of course the  $N$  zeros of the  $q$ -Racah polynomial  $R_N(\alpha, \beta, \gamma, \delta; q; z)$ . To this end we set

$$z_n(t) = \bar{z}_n + \varepsilon \zeta_n(t), \quad n = 1, \dots, N, \quad (33a)$$

implying of course

$$\dot{z}_n(t) = \varepsilon \dot{\zeta}_n(t), \quad n = 1, \dots, N, \quad (33b)$$

with  $\varepsilon$  infinitesimal.

It is plain that the insertion of this *ansatz* in (32) is consistent to order  $\varepsilon^0 = 1$ , thanks to Proposition 2.3.

The insertion of this *ansatz* in (32) yields, to order  $\varepsilon$  (after a trivial but somewhat cumbersome computation) the *linear* system of ODEs

$$\dot{\zeta}_n(t) = \sum_{m=1}^N [L_{nm}(\bar{z}) \zeta_m(t)], \quad n = 1, \dots, N, \quad (34)$$

with the  $N \times N$  matrix  $\underline{L}(\bar{z}) \equiv \underline{L}(\alpha, \beta, \gamma, \delta; q; N; \bar{z})$  defined by (7). While by continuing the expansion of the right-hand side of (32) in powers of  $\varepsilon$  and setting to zero the resulting coefficients of higher powers of  $\varepsilon$ , additional formulas satisfied by the  $N$  zeros  $\bar{z}_n \equiv \bar{z}_n(\alpha, \beta, \gamma, \delta; q; N)$  of the  $q$ -Racah polynomial  $R_N(\alpha, \beta, \gamma, \delta; q; z)$  can be obtained.

The proof of Proposition 2.4 is now a consequence of the fact that the solution of the system of *linear* ODEs (34) is clearly a *linear* superposition (with  $t$ -independent coefficients) of exponentials,  $\exp(\tilde{\lambda}_m t)$ , where the quantities  $\tilde{\lambda}_m$  (with  $m = 1, 2, \dots, N$ ) are the  $N$  eigenvalues of the  $N \times N$  matrix  $\underline{L}$ ; but—due to the simultaneous validity of the relations (28d), (29) and (33a)—this solution must also be a *linear* superposition (with  $t$ -independent coefficients) of the  $t$ -dependent quantities  $c_m(t)$ . Hence (28) implies  $\tilde{\lambda}_m = \lambda_m = q^{-N} (1 - q^m) (1 - \alpha\beta q^{2N-m+1})$ ,  $m = 1, 2, \dots, N$ . Proposition 2.4 is thereby proven.

## Acknowledgments

One of us (O. Bihun) would like to acknowledge with thanks the hospitality of the Physics Department of the University of Rome “La Sapienza” on the occasion of three two-week visits there in June 2012, May 2013 and June–July 2014; the results reported in this paper were obtained during the last of these visits. The other one (F. Calogero) would like to acknowledge with thanks the hospitality of Concordia College during a one-week visit there in November 2013.

## Appendix A. Standard definitions and properties of the Askey–Wilson and $q$ -Racah polynomials

In [Appendix A](#) we report for the convenience of the reader, and also to identify the notation used throughout this paper, a number of standard formulas associated with the Askey–Wilson and  $q$ -Racah polynomials. We generally report these formulas from the standard compilation [22], the formulas of which are identified by the notations (KS-X), where X stands here for the notation appropriate to identify equations in this compilation; in some cases we add certain immediate consequences of these formulas which are not explicitly displayed in this compilation nor (to the best of our knowledge) elsewhere.

The  $q$ -Pochhammer symbol is defined as follows:

$$(c; q)_0 = 1; \quad (c; q)_n = (1 - c) (1 - cq) \cdots (1 - cq^{n-1}) \quad \text{for } n = 1, 2, 3, \dots, \quad (35a)$$

and we also use occasionally the synthetic notation

$$(c_1; q)_k (c_2; q)_k \cdots (c_r; q)_k \equiv (c_1, c_2, \dots, c_r; q)_k, \quad \text{for } r = 0, 1, 2, \dots \quad (35b)$$

The generalized basic hypergeometric function is defined as follows (see (KS-0.4.2)):

$$\begin{aligned} & {}_{r+1}\phi_s \left( \begin{matrix} a_0, a_1, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} \middle| q; z \right) \\ &= \sum_{k=0}^{\infty} \left[ \frac{(a_0, a_1, \dots, a_r; q)_k}{(b_1, b_2, \dots, b_s; q)_k} (-1)^{(s-r)k} q^{(s-r)k(k-1)/2} \frac{z^k}{(q; q)_k} \right]. \end{aligned} \quad (36)$$

Here  $r$  and  $s$  are two arbitrary *nonnegative* integers, but in the following consideration shall be restricted to  $r = s = 3$ .

The basic hypergeometric function (36) becomes a *polynomial* in  $z$  of degree  $N$  if one of the parameters  $a_n$  has the values  $q^{-N}$  with  $N$  a positive integer—say,  $a_0 = q^{-N}$  (without loss of generality, since its definition (36) implies that the basic generalized hypergeometric function is invariant under permutations of the  $r + 1$  parameters  $a_j$  as well as under permutations of the  $s$  parameters  $b_k$ )—provided no one of the other parameters  $a_j$  equals  $q^{-\nu}$  with  $\nu$  a negative integer smaller in modulus than  $N$ , and no one of the parameters  $b_k$  equals  $q^{-\nu}$  with  $\nu$  a negative integer (as we hereafter assume). This is of course a simple consequence of the fact that, if  $N$  is a positive integer,  $(q^{-N}; q)_n$  vanishes for  $n > N$ , see (35a). Note that in this case the basic hypergeometric function is also a polynomial in each of the  $r$  parameters  $a_j$  with  $j = 1, 2, \dots, r$ .

### A.1. Formulas for the Askey–Wilson polynomials

Hereafter  $\mathbf{i}$  denotes the imaginary unit,  $\mathbf{i}^2 = -1$ .

The Askey–Wilson polynomial  $p_N(a, b, c, d; q; x)$  with  $x = \cos \theta$  is defined as follows (see (KS-3.1.1), and note some minor notational changes):

$$p_N(a, b, c, d; q; \cos \theta) = \frac{(ab, ac, ad; q)_N}{a^N} \cdot {}_4\phi_3 \left( \begin{matrix} q^{-N}, abcd q^{N-1}, a \exp(i\theta), a \exp(-i\theta) \\ ab, ac, ad \end{matrix} \middle| q; q \right), \quad (37a)$$

or, equivalently but more explicitly,

$$p_N(a, b, c, d; q; x) = \frac{(ab, ac, ad; q)_N}{a^N} \cdot \sum_{m=0}^N \left[ \frac{q^m (q^{-N}; q)_m (abcd q^{N-1}; q)_m}{(q; q)_m (ab; q)_m (ac; q)_m (ad; q)_m} \{a; q; x\}_m \right], \quad (37b)$$

where we introduced the new (modified  $q$ -Pochhammer) symbol defined as follows:

$$\begin{aligned} \{a; q; x\}_0 &= 1; \\ \{a; q; x\}_m &= (1 + a^2 - 2ax) (1 + q^2 a^2 - 2aqx) \cdots (1 + a^2 q^{2(m-1)} - 2aq^{m-1}x), \\ m &= 1, 2, 3, \dots \end{aligned} \quad (38)$$

It is plain from this formula that  $\{a; q; x\}_m$  is a polynomial of degree  $m$  in  $x$  (and also of degree  $2m$  in  $a$ ), hence that the Askey–Wilson polynomials  $p_N(a, b, c, d; q; x)$  are indeed polynomials of degree  $N$  in  $x$  (see (37b)).

**Notational remark.** For notational simplicity we often omit to indicate explicitly the dependence on the 5 parameters  $a, b, c, d, q$ —or on some of them—provided this entails no ambiguity.  $\square$

Let us also recall the related *rational* function of  $z$  defined as follows:

$$P_N(a, b, c, d; q; z) = \frac{(ab, ac, ad; q)_N}{a^N} \cdot {}_4\phi_3 \left( \begin{matrix} q^{-N}, abcd q^{N-1}, az, a/z \\ ab, ac, ad \end{matrix} \middle| q; q \right), \quad (39)$$

see the formula in [22] preceding (KS-3.1.7). It is plain that this amounts to the change of variables

$$x = \cos \theta = \frac{z^2 + 1}{2z}, \quad z = e^{i\theta} = x + \sqrt{x^2 - 1}, \quad (40)$$

corresponding to the relations

$$P_N(a, b, c, d; q; z) = p_N \left( a, b, c, d; q; \frac{z^2 + 1}{2z} \right), \quad (41a)$$

$$p_N(a, b, c, d; q; x) = P_N \left( a, b, c, d; q; x + \sqrt{x^2 - 1} \right). \quad (41b)$$

Note that although the square root  $\sqrt{x^2 - 1}$  has two possible values, the definition of the function  $P_N(z)$ , see (39) and (35a), implies that the choice of the square root is irrelevant because  $(x + \sqrt{x^2 - 1})(x - \sqrt{x^2 - 1}) = 1$ . Hence if we denote as  $\bar{x}_n$  the  $N$  zeros of the Askey–Wilson polynomial  $p_N(a, b, c, d; q; x)$ ,

$$p_N(a, b, c, d; q; \bar{x}_n) = 0, \quad n = 1, 2, \dots, N, \quad (42a)$$

it is plain that the rational function  $P_N(a, b, c, d; q; z)$  features the  $2N$  zeros  $\bar{z}_n = \bar{x}_n + \sqrt{\bar{x}_n^2 - 1}$ ,

$$P_N(a, b, c, d; q; \bar{z}_n) = 0, \quad n = 1, 2, \dots, N. \quad (42b)$$

In the following we will use whichever one of the two assignments of each square root  $\sqrt{\bar{x}_n^2 - 1}$ ,  $n = 1, \dots, N$ , is more convenient in order to simplify the following formulas; of course the zeros  $\bar{z}_n$  are functions of the 6 parameters  $a, b, c, d, q, N$ ,

$$\bar{x}_n \equiv \bar{x}_n(a, b, c, d; q; N), \quad \bar{z}_n \equiv \bar{z}_n(a, b, c, d; q; N), \quad n = 1, 2, \dots, N, \quad (42c)$$

and they are related by the formulas

$$\bar{x}_n = \cos \bar{\theta}_n = \frac{\bar{z}_n^2 + 1}{2 \bar{z}_n}, \quad \bar{z}_n = e^{i\bar{\theta}_n} = \bar{x}_n + \sqrt{\bar{x}_n^2 - 1}, \quad n = 1, 2, \dots, N. \quad (42d)$$

Note that occasionally we abuse language by referring both to the  $N$  (generally *complex*) numbers  $\bar{x}_n$  and to the  $2N$  (generally *complex*) numbers  $\bar{z}_n$  as *zeros* of the Askey–Wilson polynomial (of degree  $N$ ), although of course only the  $N$  numbers  $\bar{x}_n$  are indeed  $N$  zeros of the Askey–Wilson *polynomial*  $p_N(a, b, c, d; q; x)$ , see (42a), while the  $2N$  numbers  $\bar{z}_n$  defined by (42d), in terms of the  $N$  numbers  $\bar{x}_n$ , are the zeros of the *rational* function  $P_N(a, b, c, d; q; z)$ , see (42b).

The rational function  $P_N(a, b, c, d; q; z)$  satisfies the following  $q$ -difference equation (see (KS-3.1.7)):

$$Q P_N(a, b, c, d; q; z) = (q^{-N} - 1) (1 - abcd q^{N-1}) P_N(a, b, c, d; q; z), \quad (43a)$$

where the  $q$ -difference operator  $Q$  is defined as follows:

$$Q f(z) = \left[ A(z) \Delta_q^{(+)} + A(z^{-1}) \Delta_q^{(-)} - A(z) - A(z^{-1}) \right] f(z). \quad (43b)$$

Here  $f(z)$  is an arbitrary function of the variable  $z$ , the operators  $\Delta_q^{(\pm)}$  act as follows on  $f(z)$ ,

$$\Delta_q^{(\pm)} f(z) = f(q^{\pm 1} z), \quad (43c)$$

and the function  $A(z)$  is defined (here and throughout) as follows:

$$A(z) \equiv A(a, b, c, d; q; z) = \frac{(1 - az) (1 - bz) (1 - cz) (1 - dz)}{(1 - z^2) (1 - qz^2)}. \quad (43d)$$

These formulas indicate that the operator  $Q$ —the definition of which features (symmetrically) the 4 parameters  $a, b, c, d$  and moreover the parameter  $q$  (but not the parameter  $N$ )—has the rational functions  $P_N(a, b, c, d; q; z)$  (for all positive integer values of  $N$ ) as its eigenfunctions, with corresponding eigenvalues  $(q^{-N} - 1) (1 - abcd q^{N-1})$ , see (43a).

Note that, by setting  $z = \bar{z}_n$  in (43a) one obtains the following set of algebraic equations satisfied by the zeros of the Askey–Wilson polynomial of degree  $N$ :

$$A(\bar{z}_n) P_N(a, b, c, d; q; \bar{z}_n) + A\left(\frac{1}{\bar{z}_n}\right) P_N(a, b, c, d; q; \frac{\bar{z}_n}{q}) = 0, \quad n = 1, \dots, N. \quad (44)$$

This observation is instrumental to prove [Proposition 2.2](#) (see [Section 3](#)); it corresponds, via (43d) and (42d), to [Proposition 2.1](#)—which has been displayed in [Section 2](#) because we did not find any previous

mention of this rather trivial finding in the literature. Although in fact—after this paper of ours was posted in the web as arXiv:1410.0549—Jan Felipe van Diejen kindly brought to our attention that essentially the same result was already reported in his paper [25], see Theorem 2 and eq. (3.1b) there; and likewise our finding concerning the zeros of the Wilson polynomials (see eq. (40) in our previous paper [4]) also appears there, see Theorem 3 in [25].

### A.2. Formulas for the $q$ -Racah polynomials

The  $q$ -Racah polynomial  $R_N(\alpha, \beta, \gamma, \delta; q; z)$  is a polynomial of degree  $N$  in  $z$ ,

$$z \equiv z(\gamma\delta; q; x) = q^{-x} + \gamma\delta q^{x+1}, \quad (45a)$$

defined as follows in terms of the generalized basic hypergeometric function (see (KS-3.2.1), and note some minor notational changes):

$$R_N(\alpha, \beta, \gamma, \delta; q; z) = {}_4\phi_3 \left( \begin{matrix} q^{-N}, \alpha\beta q^{N+1}, q^{-x}, \gamma\delta q^{x+1} \\ \alpha q, \beta\delta q, \gamma q \end{matrix} \middle| q; q \right), \quad (45b)$$

or, equivalently but more explicitly,

$$\begin{aligned} R_N(\alpha, \beta, \gamma, \delta; q; z) &= \\ &= \sum_{m=0}^N \left[ \frac{q^m (q^{-N}; q)_m (\alpha\beta q^{N+1}; q)_m (q^{-x}; q)_m (\gamma\delta q^{x+1}; q)_m}{(q; q)_m (\alpha q; q)_m (\beta\delta q; q)_m (\gamma q; q)_m} \right]. \end{aligned} \quad (45c)$$

The fact that  $R_N(z) \equiv R_N(\alpha, \beta, \gamma, \delta; q; z)$  is indeed a polynomial of degree  $N$  in  $z$  is implied by the last formula and by the identity (valid via (45a)),

$$(q^{-x}; q)_m (\gamma\delta q^{x+1}; q)_m = \prod_{s=0}^{m-1} (1 - zq^s + \gamma\delta q^{2s+1}). \quad (46)$$

**Remark A.1.** Hereafter the 5 parameters  $\alpha, \beta, \gamma, \delta, q \neq 1$  are arbitrary numbers (possibly *complex*); note that this entails a somewhat more general definition of  $q$ -Racah polynomials than the standard one, see (KS-3.2.1), because it does *not* require the Diophantine restriction on one of the 3 parameters,  $\alpha q$  or  $\beta\delta q$  or  $\gamma q$ , see the second (unnumbered) equation after (KS-3.2.1) in [22]. Indeed, while this restriction is required for the validity of many of the properties of  $q$ -Racah polynomials reported in [22], it is not required for the  $q$ -difference equation satisfied by these polynomials, see (KS-3.2.6) and immediately below, which is the only property of these polynomials that we use in order to prove the properties of the zeros of these polynomials reported in this paper.  $\square$

The  $q$ -Racah polynomial  $R_N(z) \equiv R_N(\alpha, \beta, \gamma, \delta; q; z)$  satisfies the following  $q$ -difference equation:

$$\begin{aligned} &B(z) R_N(z^{(+)} - [B(z) + D(z)] R_N(z) + D(z) R_N(z^{(-)}) \\ &= (q^{-N} - 1) (1 - \alpha\beta q^{N+1}) R_N(z), \end{aligned} \quad (47a)$$

where

$$z^{(\pm)} = z(x \pm 1) = q^{\pm 1} z \pm \left( \frac{1 - q^2}{2q} \right) \left[ z - \sqrt{z^2 - 4\gamma\delta q} \right] \quad (47b)$$

and

$$B(z) = \frac{[1 - \alpha q Z(q; z)] [1 - \beta \delta q Z(q; z)] [1 - \gamma q Z(q; z)] [1 - \gamma \delta q Z(q; z)]}{[1 - \gamma \delta q Z^2(q; z)] [1 - \gamma \delta q^2 Z^2(q; z)]}, \quad (47c)$$

$$D(z) = \frac{q [1 - Z(\gamma \delta q; z)] [1 - \delta Z(\gamma \delta q; z)] [\beta - \gamma Z(\gamma \delta q; z)] [\alpha - \gamma \delta Z(\gamma \delta q; z)]}{[1 - \gamma \delta Z^2(\gamma \delta q; z)] [1 - \gamma \delta q Z^2(\gamma \delta q; z)]}, \quad (47d)$$

where

$$Z(\gamma \delta q; z) = q^x = \frac{z + \sqrt{z^2 - 4\gamma \delta q}}{2\gamma \delta q}. \quad (47e)$$

In (47b) and (47e) the determination of the square root is irrelevant; of course provided the *same* determination is used everywhere.

It is easily seen that these formulas correspond to (KS-3.2.6) via (45a).

It is plain from this formula that, if  $\bar{z}_n$  are the  $N$  zeros of the  $q$ -Racah polynomial of order  $N$ ,

$$R_N(\bar{z}_n) = 0, \quad n = 1, \dots, N, \quad (48a)$$

$$R_N(z) = C_N \prod_{n=1}^N (z - \bar{z}_n), \quad (48b)$$

where  $C_N$  is a constant. Formula (47a) implies the relation

$$B(\bar{z}_n) R_N(\bar{z}_n^{(+)} + D(\bar{z}_n) R_N(\bar{z}_n^{(-)}) = 0, \quad (49a)$$

where of course (see (47b))

$$\bar{z}_n^{(\pm)} = q^{\pm 1} \bar{z}_n \pm \left( \frac{1 - q^2}{2q} \right) \left( \bar{z}_n - \sqrt{\bar{z}_n^2 - 4\gamma \delta q} \right); \quad (49b)$$

and this formula coincides with (6a) and, via (48b), with (6b). Proposition 2.3 is thereby proven.

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