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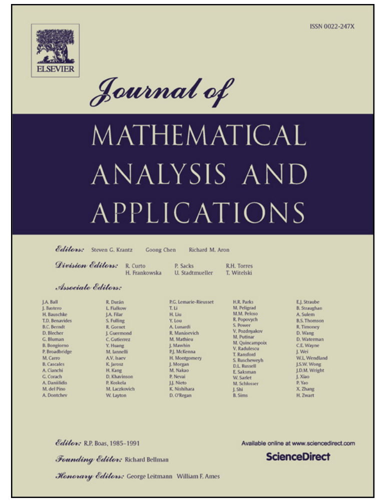

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An Irreversible Investment Problem with Maintenance Expenditure on a Finite Horizon: Free Boundary Analysis*

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Abstract

This paper concerns a continuous-time, finite horizon, optimal irreversible investment problem with maintenance expenditure of a firm under uncertainty. We assume that the firm can make irreversible investments to expand its production capacity and spend maintenance expenditure to achieve better performance of the productivity. The objective of the firm is to construct optimal investment and maintenance policies to maximize its expected total profit over a finite horizon. Mathematically, it is a singular stochastic control problem whose value function satisfies a parabolic variational inequality with gradient constraint. The problem gives rise to two free boundaries which stand for the optimal investment and maintenance strategies, respectively. We investigate behaviors of free boundaries, study regularities of the value function, and give optimal investment and maintenance policies. As we know, this is a first integral result for an investment-maintenance problem with a finite time horizon due to use of partial differential equation (PDE) technique.

Key Words: irreversible investment, maintenance, finite horizon, singular stochastic control, variational inequality, gradient constraint, free boundary problem

1 Introduction

This paper concerns a continuous-time, finite horizon, optimal irreversible investment problem with maintenance expenditure of a firm under uncertainty. Optimal investment problems have been studied widely in the last years. Bertola [1] considered an optimal investment problem under uncertainty of a firm and characterized the optimal investment-disinvestment policy. Dai and Yi [6][7] concerned an optimal investment problem with transaction costs on a finite time. In [6][7], the behaviors of the free boundary is characterized and the regularity of the value function is proved. Several Authors studied the firm's optimal problem of irreversible investment (see [5], [20], [18], [4] and [9]). In particular, Chiarolla and Ferrari [4][9] derived a new integral equation for the free boundary of the infinite and finite horizon, respectively.

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Yet there is an increasing effort to incorporate maintenance in the core of investment theory. The incorporation of maintenance cost in macroeconomic models of investment has truly started with the illuminating work of McGrattan and Schmitz [16]. They are the first to exploit a Canadian survey and to highlight why and how investment theory can account for these costs. Since then, several research projects have been launched on the topic. Kalaitzidakis and Kalyvitis[12][13] considered how maintenance of public capital affects long-term growth and how to fix optimal maintenance expenditures. In [3], the Authors studied investment and maintenance comovements without any postulated special depreciation function. One paper related to the present work is Kawaguchi and Morimoto [11] where the Authors concerned an infinite horizon investment problem with maintenance expenditure of a firm under uncertainty. They achieved the optimal policy and proved the existence of the optimal investment boundary point, but did not consider the optimal maintenance boundary.

In this paper, we concern the model of [11] with a finite horizon, and aim to provide a theoretical analysis of behaviors of optimal investment and maintenance boundaries, respectively. It is challenging to take a finite horizon case into consideration since the corresponding free boundaries (optimal policies) vary with time. We attack the problem by virtue of a PDE approach.

Mathematically, it is a singular stochastic control problem whose value function satisfies a parabolic variational inequality with gradient constraint. The problem gives rise to two free boundaries which stand for the optimal investment and maintenance strategies, respectively. The main task is to characterize behaviors of the two free boundaries. But it is not an easy task. First, the optimal investment boundary lies between the domain $\{(z,\tau) \in \Omega_T : \partial_z u(z,\tau) = p_2 e^z\}$ and the domain $\{(z,\tau) \in \Omega_T : \partial_z u(z,\tau) < p_2 e^z\}$, and $u(z,\tau)$ satisfies a variational inequality with gradient constraint. However, it is intractable to study the free boundary from the original variational inequality with gradient constraint. So we intend to reduce the original problem to a standard variational inequality with function constraint, but the variational inequality with function constraint is not a self-contained system, which leads to difficulties to construct a connection between the above two variational inequalities. Secondly, the optimal maintenance boundary is the boundary between the region $\{(z,\tau) \in \Omega_T : u(z,\tau) - \partial_z u(z,\tau) - p_1 > 0\}$ and the region $\{(z,\tau) \in \Omega_T : u(z,\tau) - \partial_z u(z,\tau) - p_1 < 0\}$. That is , it is a level set of $\{(z,\tau) \in \Omega_T : u(z,\tau) - \partial_z u(z,\tau) - p_1 = 0\}$, which is different to the free boundary with function constraint or with gradient constraint. To the best of our knowledge this is a complete novelty in the literature on singular stochastic control problems with a finite horizon.

The paper is organized as follows. In section 2, we present the model formulation. Section 3 is devoted to studying regularities of solution to problem (2.9) with a known $u(z,\tau)$. In section 4, we exploit the auxiliary condition with which problem (2.9) can be transformed the self-contained problem A and prove that the problem A has a solution by the Schauder fixed point theorem. In addition, we obtain a classical solution to problem (2.7) and construct a connection between problem (2.7) and (2.9). The behaviors of the optimal investment and maintenance boundaries are investigated in Section 5 and section 6, respectively. We give the optimal investment and maintenance policies in Section 7. Section 8 concludes the paper.

2 Problem formulation

We consider an optimal investment problem with maintenance expenditure of a firm under uncertainty with a finite horizon. As in [11], the firm faces uncertain future changes in the productivity of the capital stock, which are modeled by diffusion processes, and makes irreversible investments in capital goods. The maintenance expenditure can improve the productivity of the existing capital stock. We assume T is maturity time. For any time $0 \le t \le T$, the measure of capital productivity X_s of the firm and the capital stock Y_s at time $s \ge t$ are given by the stochastic differential equation

$$\begin{cases} dX_s = \mu X_s ds + \sigma X_s dB_s, & X_t = x > 0, \;\; t < s \le T, \\ dY_s = -\lambda Y_s ds, & Y_t = y > 0, \;\; t < s \le T, \end{cases}$$

on a complete probability space $(\mathbf{\Omega}, \mathcal{F}, P)$, carrying a one-dimensional standard Brown motion $\{B_t\}$, endowed with the natural filtration $\mathcal{F}_s$ generated by $\{B_\tau, \tau \le s\}$, where $\mu \ge 0$ and $\sigma > 0$ are diffusion constants and $\lambda > 0$ is a depreciation rate.

The firm can make irreversible investments to expand its production capacity and spend the maintenance expenditure to achieve better performance of the productivity. Let $I = \{I_s\}$ be the capital invested in the firm up to time s, which is right-continuous, nonnegative, and nondecreasing $\{\mathcal{F}_s\}$ -adapted stochastic process with $I_0 = 0$. The dynamics of productivity can be controlled by the policy $m = \{m_s\}$, for maintenance expenditure $p_1 m_s X_s ds$ at time s, which is an $\{\mathcal{F}_s\}$ -adapted process such that $0 \le m_s \le M$ for all $s \ge 0$, where $M > 0$ is an upper bound for maintenance. Given (I, m), the capital productivity process $\{X_s\}$ and the firm's capital stock process $\{Y_s\}$ evolve according to

$$\begin{cases} dX_s = \mu X_s ds + \sigma X_s dB_s + m_s X_s ds, & X_t = x > 0, \;\; t < s \le T, \\ dY_s = -\lambda Y_s ds + dI_s, & Y_t = y > 0, \;\; t < s \le T. \end{cases} \tag{2.1}$$

A policy (I, m) is called admissible if

$$E\Big[\int_t^T e^{-\alpha(s-t)}(X_s ds + dI_s)\Big] < \infty, \tag{2.2}$$

for a discount factor $\alpha > 0$. We denote by $\mathcal{A}_t(x, y)$ the class of all admissible policies.

The objective is to find an optimal policy $(I^*, m^*) = \{I_t^*, m_t^*\} \in \mathcal{A}_t(x, y)$ so as to maximize the expected total profit over finite horizon. We define the value function:

$$\begin{aligned} V(x, y, t) = \sup_{(I,m)\in\mathcal{A}_t} E\Big[\int_t^T e^{-\alpha(s-t)} F(X_s, Y_s) ds - p_1 \int_t^T e^{-\alpha(s-t)} m_s X_s ds \\ -p_2 \int_t^T e^{-\alpha(s-t)} dI_s \big| X_t = x, Y_t = y\Big], \quad (x, y, t) \in R^+ \times R^+ \times [0, T], \end{aligned} \tag{2.3}$$

where $p_1 > 0$, $p_2 > 1$ are given constants and

$$F(x,y) = x^{1-\gamma}y^{\gamma}, \quad (0 < \gamma < 1), \tag{2.4}$$

denotes the Cobb-Douglas production function which leads to a profit function. The associated Hamilton-Jacobi-Bellman variational inequality is given by (cf. [10], [17], [21]):

$$\begin{cases} \partial_y V(x,y,t) \le p_2, & x>0, y>0, 0 \le t < T, \\ \partial_t V + \mathcal{L}_1 V + Mx(\partial_x V - p_1)^+ + F(x,y) \le 0, & x>0, y>0, 0<t<T, \\ (p_2 - \partial_y V)[\partial_t V + \mathcal{L}_1 V + Mx(\partial_x V - p_1)^+ + F(x,y)] = 0, & x>0, y>0, 0<t<T, \\ V(x,y,T) = 0, & x>0, y>0, \end{cases} \tag{2.5}$$

where

$$\mathcal{L}_1 V = \frac{1}{2}\sigma^2 x^2 \partial_{xx} V + \mu x \partial_x V - \lambda y \partial_y V - \alpha V.$$

Problem (2.5) gives rise to two free boundaries that correspond to the optimal investment and maintenance strategies. The optimal investment boundary lies between the domain $\{(x,y,t) : \partial_y V(x,y,t) = p_2\}$ and the domain $\{(x,y,t) : \partial_y V(x,y,t) < p_2\}$. The optimal maintenance boundary is the boundary between the region $\{(x,y,t) : \partial_x V(x,y,t) = p_1\}$and the region $\{(x,y,t) : \partial_x V(x,y,t) < p_1\}$. That is , it is the level set of $\{(x,y,t) : \partial_x V(x,y,t) = p_1\}$. So, our purpose is to investigate behaviors of the two free boundaries. In addition, we are interested in regularities of the solution to problem (2.5).

Since problem (2.5) is both two-dimensional and backward variational inequality, we change the variable by

$$z = \ln\frac{y}{x}, \tau = T - t, \ u(z,\tau) = V(x,y,t)/x, \tag{2.6}$$

then (2.5) is easily reduced to a one-dimensional variational inequality with gradient constraint:

$$\begin{cases} \partial_z u \le p_2 e^z, & z \in R, 0 \le \tau < T, \\ \partial_\tau u - \mathcal{L}u - M(u - \partial_z u - p_1)^+ - e^{\gamma z} \ge 0, & z \in R, 0 < \tau < T, \\ (p_2 e^z - \partial_z u)\left[\partial_\tau u - \mathcal{L}u - M(u - \partial_z u - p_1)^+ - e^{\gamma z}\right] = 0, & z \in R, 0 < \tau < T, \\ u(z,0) = 0, & z \in R, \end{cases} \tag{2.7}$$

where $r = \mu + \lambda$, $\beta = \alpha - \mu$, and

$$\mathcal{L}u = \frac{1}{2}\sigma^2 \partial_{zz} u - (\frac{1}{2}\sigma^2 + r)\partial_z u - \beta u.$$

It can be shown the problem has a unique viscosity solution (cf. [10], [17], [21]). As it is intractable to study the free boundaries directly from the variational inequality with gradient constraint, following [6] and [7], we attempt to reduce problem (2.7) to a standard variational inequality with function constraint. We set

$$v = \partial_z u.$$

Formally we have

$$\begin{aligned}\frac{\partial}{\partial z}\mathcal{L}u &= \frac{1}{2}\sigma^2\partial_{zz}v - (\frac{1}{2}\sigma^2 + r)\partial_z v - \beta v \\ &\equiv \mathcal{L}v\end{aligned}$$

and

$$\partial_z(M(u - \partial_z u - p_1)^+ + e^{\gamma z}) = MH(u - v - p_1)(v - \partial_z v) + \gamma e^{\gamma z},$$

where

$$H(\xi) = \begin{cases} 1, & \xi > 0, \\ 0, & \xi \leq 0. \end{cases} \tag{2.8}$$

Then we postulate that v is a solution to the following standard variational inequality:

$$\begin{cases} v \leq p_2 e^z, & (z,\tau) \in \Omega_T, \\ \partial_\tau v - \mathcal{L}v - MH(u - v - p_1)(v - \partial_z v) - \gamma e^{\gamma z} \leq 0, & (z,\tau) \in \Omega_T, \\ (p_2 e^z - v)\ [\partial_\tau v - \mathcal{L}v - MH(u - v - p_1)(v - \partial_z v) - \gamma e^{\gamma z}] = 0, & (z,\tau) \in \Omega_T, \\ v(z,0) = 0, & z \in R. \end{cases} \tag{2.9}$$

$\Omega_T = R \times [0,T]$.

We will see later that it is rather straightforward to analyze behaviors of free boundaries in terms of problem (2.9). So it is important to prove a connection between problem (2.7) and (2.9), which indicates a connection between a singular control problem and an optimal stopping problem. But we would like to emphasize that it is not an easy task to establish the connection. One of main difficulties is that problem (2.9) contains u, which leads problem (2.9) to be not a self-contained system.

3 Problem (2.9) with a known $u(z,\tau)$.

In this section, we study problem (2.9) with known $u(z,\tau)$ which is assumed to possess the following properties:

$$|u(z,\tau)| \leq K_1 + p_2 e^z, \tag{3.10}$$

$$0 \leq \partial_z u \leq p_2 e^z, \tag{3.11}$$

$$|\partial_\tau u| \leq K_2(1 + e^z), \tag{3.12}$$

$$u(z,0) = 0, \tag{3.13}$$

where K_1 and K_2 are positive constants to be determined.

Owing to the unboundedness of Ω_T, we confine problem (2.9) to the bounded domain $\Omega_T^n = (-n, n) \times [0, T]$ with $n > 0$:

$$\begin{cases} v^n \le p_2 e^z, & (z,\tau) \in \Omega_T^n, \\ \partial_\tau v^n - \mathcal{L} v^n - M H_n(u - v^n - p_1)(v^n - \partial_z v^n) - \gamma e^{\gamma z} \le 0, & (z,\tau) \in \Omega_T^n, \\ (p_2 e^z - v^n)\left[\partial_\tau v^n - \mathcal{L} v^n - M H_n(u - v^n - p_1)(v^n - \partial_z v^n) - \gamma e^{\gamma z}\right] = 0, & (z,\tau) \in \Omega_T, \\ \partial_z v^n(-n,\tau) - v^n(-n,\tau) = 0, & 0 \le \tau \le T, \\ \partial_z v^n(n,\tau) + v^n(n,\tau) = 0, & 0 \le \tau \le T, \\ v^n(z,0) = 0, & z \in [-n, n], \end{cases} \tag{3.14}$$

where $H_n(\xi)$ satisfies

$$H_n(\xi) \in C^1(-\infty, \infty), H_n'(\xi) \ge 0, \tag{3.15}$$

and

$$H_n(\xi) = \begin{cases} 1, \xi \ge 1/n, \\ \nearrow, -1/n \le \xi \le 1/n, \\ 0, \xi \le -1/n. \end{cases} \tag{3.16}$$

In order to prove existence of solution to problem (3.14), we consider a penalty approximation :

$$\begin{cases} \partial_\tau v^{n,\varepsilon} - \mathcal{L} v^{n,\varepsilon} - M H_n(u - v^{n,\varepsilon} - p_1)(v^{n,\varepsilon} - \partial_z v^{n,\varepsilon}) - \gamma e^{\gamma z} + \beta_\varepsilon(v^{n,\varepsilon} - p_2 e^z) = 0, & (z,\tau) \in \Omega_T^n, \\ \partial_z v^{n,\varepsilon}(-n,\tau) - v^{n,\varepsilon}(n,\tau) = 0, & 0 \le \tau \le T, \\ \partial_z v^{n,\varepsilon}(n,\tau) + v^{n,\varepsilon}(n,\tau) = 0, & 0 \le \tau \le T, \\ v^{n,\varepsilon}(z,0) = 0, & z \in [-n, n], \end{cases} \tag{3.17}$$

where $\beta_\varepsilon(t)$ satisfies

$$\begin{aligned} &\beta_\varepsilon(t) \in C^2(-\infty, +\infty), \quad \beta_\varepsilon(t) \ge 0, \quad t \in R, \\ &\beta_\varepsilon(t) = 0 \;\; if \;\; t \le -\varepsilon, \quad \beta_\varepsilon(0) = C_0, C_0 = (\gamma - \gamma^2)\left(\frac{\gamma^2}{(r+\beta)p_2}\right)^{\frac{\gamma}{1-\gamma}}, \\ &\beta_\varepsilon'(t) \ge 0, \quad \beta_\varepsilon''(t) \le 0, \end{aligned} \tag{3.18}$$

and

$$\lim_{\varepsilon \to 0^+} \beta_\varepsilon(t) = \begin{cases} 0, & t < 0, \\ +\infty, & t > 0. \end{cases} \tag{3.19}$$

Lemma 3.1. *For a given $u(z,\tau)$ satisfying (3.10)-(3.13), problem(3.17) has a solution $v^{n,\varepsilon} \in W_p^{2,1}(\Omega_T^n)$, $1 < p < \infty$, and*

$$0 \le v^{n,\varepsilon} \le p_2 e^z. \tag{3.20}$$

Proof. By using the fixed point theorem [8], we can show that problem (3.17) has a solution $v^{n,\varepsilon} \in W_p^{2,1}(\Omega_T^n)$. We now prove (3.20). Let us first consider the left-hand side inequality. Set $v_1 = 0$, then

$$\begin{cases} \partial_\tau v_1 - \mathcal{L}v_1 - MH_n(u - v^{n,\varepsilon} - p_1)(v_1 - \partial_z v_1) - \gamma e^{\gamma z} + \beta_\varepsilon(v_1 - p_2 e^z) & \\ = -\gamma e^{\gamma z} + \beta_\varepsilon(-p_2 e^z) = -\gamma e^{\gamma z},\ (if \quad \varepsilon < p_2 e^{-n}), & (z,\tau) \in \Omega_T^n, \\ \partial_z v_1(-n,\tau) - v_1(-n,\tau) = 0, & 0 \le \tau \le T, \\ \partial_z v_1(n,\tau) + v_1(n,\tau) = 0, & 0 \le \tau \le T, \\ v_1(z,0) = 0, & z \in [-n,n]. \end{cases}$$

In terms of the maximum principle [8], we then deduce $v^{n,\varepsilon} \ge 0$ in Ω_T^n.

Now we turn to the proof of the right-hand side inequality of (3.20). Set $v_2 = p_2 e^z$, which satisfying

$$\begin{cases} \partial_\tau v_2 - \mathcal{L}v_2 - MH_n(u - v^{n,\varepsilon} - p_1)(v_2 - \partial_z v_2) - \gamma e^{\gamma z} + \beta_\varepsilon(v_2 - p_2 e^z) & \\ = (r+\beta)p_2 e^z - \gamma e^{\gamma z} + \beta_\varepsilon(0) \ge -C_0 + C_0 = 0, (C_0\ is\ defined\ by\ (3.18)), & (z,\tau) \in \Omega_T^n, \\ \partial_z v_2(-n,\tau) - v_2(-n,\tau) = 0, & 0 \le \tau \le T, \\ \partial_z v_2(n,\tau) + v_2(n,\tau) = 2p_2 e^z > 0, & 0 \le \tau \le T, \\ v_2(z,0) = p_2 e^z > 0, & z \in [-n,n]. \end{cases}$$

Again, applying the maximum principle yields the desired result. Therefore, the proof is complete. □

Lemma 3.2. *For a given $u(z,\tau)$ satisfying (3.10)-(3.13), problem (3.14) has a solution $v^n \in W_p^{2,1}(\Omega_T^n)$, $1 < p < +\infty$, and*

$$0 \le v^n \le p_2 e^z, \tag{3.21}$$
$$-p_2 e^z \le \partial_z v^n \le v^n \le p_2 e^z. \tag{3.22}$$

Proof. Owing to (3.20), we infer $0 \le \beta_\varepsilon(v^{n,\varepsilon} - p_2 e^z) \le \beta_\varepsilon(0) = C_0$. Applying $W_p^{2,1}$ estimate, we have

$$|v^{n,\varepsilon}|_{W_p^{2,1}(\Omega_T^n)} \le C(|\beta_\varepsilon(0)|_{L^p(\Omega_T^n)} + |e^{\gamma z}|_{L^p(\Omega_T^n)} + |v^{n,\varepsilon}|_{L^p(\Omega_T^n)}) \le C_n,$$

where C_n is independent of ε. Let $\varepsilon \to 0$, there exists v^n such that

$$v^{n,\varepsilon} \rightharpoonup v^n\ weakly\ \ in\ W_p^{2,1}(\Omega_T^n)\ and\ strongly\ in\ C(\overline{\Omega}_T^n).$$

We immediately get (3.21). Now we prove (3.22). Let us first consider the right-hand side inequality. Clearly

$$\partial_z v^n = v^n = p_2 e^z \ \ if\ v^n = p_2 e^z.$$

So we need only show $\partial_z v^n \le v^n$ in $\mathcal{N}$, where

$$\mathcal{N} = \{(z,\tau) \in \Omega_T^n | v^n(z,\tau) < p_2 e^z\}.$$

Denote $w = \partial_z v^n$, then

$$\begin{aligned}\partial_\tau w - \mathcal{L}w - MH_n(u - v^n - p_1)\partial_z(v^n - w) - MH_n'(u - v^n - p_1)(\partial_z u - w)(v^n - w) \\ -\gamma^2 e^{\gamma z} = 0 \qquad in \ \ \mathcal{N}.\end{aligned}$$

It is not hard to verify

$$\begin{aligned}\partial_\tau(v^n - w) - \mathcal{L}(v^n - w) - MH_n(u - v^n - p_1)(v^n - w) + MH_n(u - v^n - p_1)\partial_z(v^n - w) \\ +MH_n'(u - v^n - p_1)(\partial_z u - w)(v^n - w) = (\gamma - \gamma^2)e^{\gamma z} \geq 0 \ \ in \ \ \mathcal{N}.\end{aligned}$$

Apparently $(v^n - w) \geq 0$ on $\partial\mathcal{N} \cap (\{z = -n\} \cup \{z = n\} \cup \{\tau = 0\})$. Applying the maximum principle, we then deduce $\partial_z v^n \leq v^n$ in $\mathcal{N}$.

In the following, we turn to the proof of the left-hand side inequality of (3.22). Clearly

$$-p_2 e^z \leq \partial_z v^n = p_2 e^z \qquad if \quad v^n = p_2 e^z.$$

Next we will prove $-p_2 e^z \leq \partial_z v^n$ in $\mathcal{N}$. Denote $w = \partial_z v^n$ and $w_1 = -p_2 e^z$, then

$$\begin{aligned}\partial_\tau w - \mathcal{L}w - MH_n(u - v^n - p_1)(w - \partial_z w) + MH_n'(.)(v^n - w)w \\ = \gamma^2 e^{\gamma z} + MH_n'(.)\partial_z u(v^n - w) \qquad in\ \mathcal{N},\end{aligned}$$

and

$$\begin{aligned}\partial_\tau w_1 - \mathcal{L}w_1 - MH_n(u - v^n - p_1)(w_1 - \partial_z w_1) + MH_n'(.)(v^n - w)w_1 \\ = -(r + \beta)p_2 e^z - MH_n'(.)(v^n - w)p_2 e^z \qquad in\ \mathcal{N}.\end{aligned}$$

Then it is not hard to verify

$$\begin{aligned} & \partial_\tau(w - w_1) - \mathcal{L}(w - w_1) - MH_n(u - v^n - p_1)[(w - w_1) - \partial_z(w - w_1)] \\ & \quad +MH_n'(.)(v^n - w)(w - w_1) \\ = \ & \gamma^2 e^{\gamma z} + MH_n'(.)\partial_z u(v^n - w) + (r + \beta)p_2 e^z + MH_n'(.)(v^n - w)p_2 e^z \\ \geq \ & 0 \qquad in\ \mathcal{N}.\end{aligned}$$

It is clear that $w \geq -p_2 e^z$ on $\partial\mathcal{N} \cap (\{z = -n\} \cup \{z = n\} \cup \{\tau = 0\})$. Again, in terms of the maximum principle, we then deduce $w \geq -p_2 e^z$ in $\mathcal{N}$. Therefore, the proof is complete. $\square$

Theorem 3.1. *For a given $u(z, \tau)$ satisfying (3.10)-(3.13), problem (2.9) has a solution $v \in W^{2,1}_{p,loc}(\Omega_T) \cap C(\overline{\Omega}_T)$, $(1 < p < \infty)$, and*

$$0 \leq v \leq p_2 e^z, \tag{3.23}$$

$$-p_2 e^z \leq \partial_z v \leq v \leq p_2 e^z. \tag{3.24}$$

Proof. Notice

$$(\partial_\tau - \mathcal{L})(p_2 e^z) = (r + \beta)p_2 e^z,$$

then problem (3.14) can be rewritten as

$$\begin{cases} \partial_\tau v^n - \mathcal{L}v^n - MH_n(u - v^n - p_1)(v^n - \partial_z v^n) = f(z,\tau), & (z,\tau) \in \Omega_T^n, \\ \partial_z v^n(-n,\tau) - v^n(n,\tau) = 0, & 0 \le \tau \le T, \\ \partial_z v^n(n,\tau) + v^n(n,\tau) = 0, & 0 \le \tau \le T, \\ v^n(z,0) = 0, & z \in [-n,n], \end{cases}$$

where $f(z,\tau) = \mathcal{X}_{\{v^n = p_2 e^z\}}(r+\beta)p_2 e^z + \mathcal{X}_{\{v^n < p_2 e^z\}}\gamma e^{\gamma z}$ and $\mathcal{X}_A$ is the indicator function on set A. Hence, for any $m < n$, we have

$$|v^n|_{W_p^{2,1}(\Omega_T^m)} \le C(|f|_{L^p(\Omega_T^m)} + |v^n|_{L^p(\Omega_T^m)}) \le C_m,$$

where C_m is independent of n. Let $n \to \infty$, there exists v^m such that

$$v^n \rightharpoonup v^m \textit{ weakly in } W_p^{2,1}(\Omega_T^m) \textit{ and strongly in } C(\overline{\Omega_T^m}).$$

Define $v(z,\tau) = v^m(z,\tau)$ if $(z,\tau) \in \Omega_T^m$ for each $m > 0$, it is clear that $v(z,\tau)$ is reasonable defined in Ω_T and $v \in W_{p,loc}^{2,1}(\Omega_T) \cap C(\overline{\Omega_T})$ is the solution of (2.9). We immediately get (3.23) and (3.24). This completes the proof. □

Thanks to the inequality in (3.24), we infer that $\partial_z(v - p_2 e^z) \le 0$. Thus, if there exists $(z_0,\tau_0) \in \Omega_T$ such that $v(z_0,\tau_0) = p_2 e^{z_0}$, then for any $z \le z_0$, we have

$$0 \ge v(z,\tau_0) - p_2 e^z \ge v(z_0,\tau_0) - p_2 e^{z_0} = 0.$$

This indicates that we can define the free boundary

$$h_u(\tau) = \sup_{z\in R}\{z | v(z,\tau) = p_2 e^z\}, \ \tau \in (0,T], \tag{3.25}$$

such that

$$\{(z,\tau) \in \Omega_T | v(z,\tau) = p_2 e^z\} = \{(z,\tau) \in \Omega_T | z \le h_u(\tau)\}.$$

Lemma 3.3. *Denote* $z_0 = \frac{1}{1-\gamma} \ln \frac{\gamma}{(r+\beta)p_2}$*, then*

$$h_u(\tau) \le z_0, \tag{3.26}$$

moreover

$$h_u(0) = \lim_{\tau \to 0^+} h_u(\tau) = -\infty. \tag{3.27}$$

Proof. Let us prove (3.26) first. Note that for any $z \le h_u(\tau)$, $v(z,\tau) = p_2 e^z$, then, by (2.9), we have

$$\partial_\tau(p_2 e^z) - \mathcal{L}(p_2 e^z) - MH_n(u - p_2 e^z - p_1)(p_2 e^z - \partial_z(p_2 e^z)) - \gamma e^{\gamma z} \le 0,$$

from which we infer $z \le \frac{1}{1-\gamma} \ln \frac{\gamma}{(r+\beta)p_2}$. Then, by definition (3.25), we deduce $h_u(\tau) \le z_0$. Owing to $v \in C(\overline{\Omega_T})$, combining to the initial condition of (2.9) $v(z,0) = 0$, we then deduce (3.27). The proof is complete. □

4 Solution to problem (2.7)

In this section, we will exploit an auxiliary condition with which problem (2.9) can be transformed a self-contained system, that is problem A. Then we prove existence of the solution to problem A and construct the connection between problem (2.7) and problem (2.9). At last, we prove a solution to problem (2.7) is classical solution.

Now let us exploit an auxiliary condition. Assumed that $v = \partial_z u$ is a solution to problem (2.9) in Ω_T. Due to Lemma 3.3, we expect that there exists a function $h(\tau) : (0, T) \to (-\infty, z_0)$ such that

$$\{(z, \tau) \in \Omega_T | v(z, \tau) = p_2 e^z\} = \{(z, \tau) \in \Omega_T | z \le h(\tau)\}.$$

So, we have $u(z, \tau) = p_2 e^z + f(\tau), z \le h(\tau)$, $\tau > 0$, where $f(\tau)$ is to be determined. Then we conjecture

$$\begin{aligned} u(z,\tau) &= u(h(\tau),\tau) + \int_{h(\tau)}^{z} v(\xi,\tau)d\xi \\ &= p_2 e^{h(\tau)} + f(\tau) + \int_{h(\tau)}^{z} v(\xi,\tau)d\xi \quad \textit{for any } (z,\tau) \in \Omega_T, \end{aligned}$$

and $f(0) = 0$. It is expected that $v(., \tau) \in C^1$ and $u(., \tau) \in C^2$. Thus, we should have

$$\partial_z u(h(\tau), \tau) = p_2 e^{h(\tau)}, \quad \partial_{zz} u(h(\tau), \tau) = p_2 e^{h(\tau)},$$

which yields

$$\begin{aligned} f'(\tau) &= \partial_\tau u(h(\tau),\tau) = \mathcal{L}u + M(u - \partial_z u - p_1)^+ + e^{\gamma z}|_{z=h(\tau)} \\ &= -(r+\beta) p_2 e^{h(\tau)} - \beta f(\tau) + M(f(\tau) - p_1)^+ + e^{\gamma h(\tau)}. \end{aligned}$$

This is the auxiliary condition with which we want to combine the problem (2.9). In other words, we plan to study the following problem.

Problem A: Find $u(z, \tau), v(z, \tau)$ and $h(\tau) : (0, T) \to (-\infty, z_0)$, such that

$$\begin{aligned} &(i) \quad \{(z,\tau) \in \Omega_T | v(z,\tau) = p_2 e^z\} = \{(z,\tau) \in \Omega_T | z \le h(\tau)\}; \\ &(ii) \quad v(z,\tau), (z,\tau) \in \Omega_T \ satisfies \ (2.9) \ in \ which \\ &\qquad u(z,\tau) = f(\tau) + p_2 e^{h(\tau)} + \int_{h(\tau)}^{z} v(\xi,\tau)d\xi, \end{aligned}$$

where $f(\tau)$ satisfying

$$\begin{cases} f'(\tau) = -(r+\beta) p_2 e^{h(\tau)} - \beta f(\tau) + M(f(\tau) - p_1)^+ + e^{\gamma h(\tau)}, & \tau \in (0, T], \\ f(0) = 0. \end{cases} \tag{4.28}$$

Theorem 4.1. *Problem A has a unique solution $(u(z, \tau), v(z, \tau), h(\tau))$ satisfying (3.10)-(3.13), (3.23)-(3.24) and (3.26)-(3.27).*

Proof. In the following, We will prove the existence of a solution by virtue of the Schauder fixed point theorem [8]. Firstly, let us confine ourselves to the bounded domain $\Omega_T^R = (-R, R) \times (0, T]$. Consider Banach space $\mathcal{B} = C(\overline{\Omega}_T^R)$ and define

$$\mathcal{D} = \big\{u(z,\tau) \in \mathcal{B}\big| |u(z,\tau)| \leq K_1 + p_2 e^z,\ 0 \leq \partial_z u \leq p_2 e^z,\\ |\partial_\tau u(z,\tau)| \leq K_2(1+e^z),\ u(z,0) = 0\big\},$$

where K_1, K_2 are positive constants to be determined. Apparently $\mathcal{D}$ is a compact subset of $\mathcal{B}$.

For any $u(z,\tau) \in \mathcal{D}$ given, let $v(z,\tau)$ be the solution of problem (3.14) confined to Ω_T^R, and let $h_u(\tau)$ be the corresponding free boundary as given in (3.25). Define a mapping $\mathcal{F} : \mathcal{D} \to \mathcal{B}$ as follows:

$$\mathcal{F}u = \bar{u}(z,\tau) = f(\tau) + p_2 e^{h_u(\tau)} + \int_{h_u(\tau)}^z v(\xi,\tau) d\xi, \tag{4.29}$$

where $f(\tau)$ satisfies (4.28).

In the following we shall prove $\bar{u}(z,\tau) \in \mathcal{D}$. By definition (4.29), we can infer that $\bar{u}(z,0) = f(0) + p_2 e^{h(0)} + \int_{h(0)}^z v(\xi,0)d\xi = 0$, $\partial_z \bar{u} = v$. Then, by (3.23), we have $0 \leq \partial_z \bar{u} \leq p_2 e^z$. Combining with (4.29) and $0 \leq v \leq p_2 e^z$, we can infer that

$$f(\tau) + p_2 e^{h(\tau)} \leq \bar{u}(z,\tau) \leq f(\tau) + p_2 e^z.$$

By the definition of $f(\tau)$ and (3.26), we deduce $f(\tau)$ is bounded. Then there exists a positive constant K_1, which is independent of R, such that $|\bar{u}(z,\tau)| \leq K_1 + p_2 e^z$.

It remains to show that $|\partial_\tau \bar{u}| \leq K_2(1+e^z)$. By (4.29), we have

$$\partial_\tau \bar{u} = f'(\tau) + \int_{h(\tau)}^z \partial_\tau v(\xi,\tau) d\xi. \tag{4.30}$$

If $z \leq h(\tau)$, we have $v(z,\tau) = p_2 e^z$, it is obvious that $\partial_\tau v(z,\tau) = 0$. Combining with the boundedness of $f(\tau)$ and (3.26), we deduce that there is a positive constant K_3 , which is independent of R, such that $|\partial_\tau \bar{u}| = |f'(\tau)| \leq K_3$. If $z > h(\tau)$,

$$\begin{aligned}
\partial_\tau \bar{u} &= f'(\tau) + \int_{h(\tau)}^z \partial_\tau v(\xi,\tau) d\xi \\
&= f'(\tau) + \int_{h(\tau)}^z (\mathcal{L}v + MH(u-v-p_1)(v-\partial_z v) + \gamma e^{\gamma z})(\xi,\tau) d\xi \\
&= f'(\tau) + \int_{h(\tau)}^z \frac{\partial}{\partial_\xi}(\mathcal{L}\bar{u} + e^{\gamma\xi}) d\xi + \int_{h(\tau)}^z MH(u-v-p_1)(v-\partial_z v)(\xi,\tau) d\xi \\
&= \mathcal{L}\bar{u}(z,\tau) + e^{\gamma z} + M(f(\tau)-p_1)^+ + \int_{h(\tau)}^z MH(u-v-p_1)(v-\partial_z v)(\xi,\tau) d\xi \\
&= \frac{1}{2}\sigma^2 \partial_z v - (\frac{1}{2}\sigma^2 + r)v - \beta\bar{u} + e^{\gamma z} + M(f(\tau)-p_1)^+ \\
&\quad + \int_{h(\tau)}^z MH(u-v-p_1)(v-\partial_z v)(\xi,\tau) d\xi.
\end{aligned}$$

By (3.23)-(3.24) and the boundedness of $f(\tau)$, we can infer that there is a positive constant $K_4 > 0$, which is independent of R, such that $|\partial_\tau \bar{u}| \le K_4(1+e^z)$. We then choose $K_2 = K_3 + K_4$ to prove that $|\partial_\tau \bar{u}| \le K_2(1+e^z)$. So far we obtained $\mathcal{F}(\mathcal{D}) \subset \mathcal{D}$.

Owing to the uniqueness of solution to problem (2.9), $\mathcal{F}$ must be a one-one mapping. Thanks to the compactness of $\mathcal{D}$, we then infer that $\mathcal{F}$ is continuous. Applying the Schauder fixes theorem we see that Problem A confined to Ω_T^R allows a solution $(u(z,\tau), v(z,\tau), h(\tau))$. As K_1 and K_2 are positive constants and independent of R, so we can extent the result to domain Ω_T. $\square$

In the following, we shall prove that the solution $u(z,\tau)$ in Problem A is the solution to problem (2.7).

Define

$$\begin{aligned} \mathbf{IR} &= \{(z,\tau) : v(z,\tau) = p_2 e^z\}, \quad && \textit{the investment region,} \\ \mathbf{NI} &= \{(z,\tau) : v(z,\tau) < p_2 e^z\}, \quad && \textit{the no}-\textit{investment region.} \end{aligned}$$

Due to Theorem 4.1, we know the region **IR** can be rewritten as

$$\mathbf{IR} = \{(z,\tau) : z \le h(\tau)\}.$$

Theorem 4.2. *Problem (2.7) has a unique solution $u(z,\tau) \in C^{2,1}(\Omega_T)$. Moreover, $v = \partial_z u$ satisfies problem (2.9), and*

$$0 \le u(z,\tau) \le K_1 + p_2 e^z, \tag{4.31}$$

$$0 \le \partial_z u \le p_2 e^z, \tag{4.32}$$

$$-p_2 e^z \le \partial_{zz} u \le \partial_z u \le p_2 e^z, \tag{4.33}$$

$$0 \le \partial_\tau u(z,\tau) \le K_2(1+e^z), \tag{4.34}$$

where K_1 and K_2 are positive constants.

Remark: (4.33) implies that the value function $V(x,y,t)$ defined in (2.3) is concave in (x,y) on $(0,\infty)^2$.

Proof. Due to $u(z,\tau)$ is a solution to Problem A, therefore, to prove that $u(z,\tau)$ satisfies problem (2.7), it suffices to show

$$\begin{cases} \partial_\tau u - \mathcal{L}u - M(u - \partial_z u - p_1)^+ - e^{\gamma z} \ge 0 \quad in\ \mathbf{IR}, \\ \partial_\tau u - \mathcal{L}u - M(u - \partial_z u - p_1)^+ - e^{\gamma z} = 0 \quad in\ \mathbf{NI}. \end{cases} \tag{4.35}$$

Note that $v = \partial_z u$ satisfies problem (2.9), we have

$$\frac{\partial}{\partial z}(\partial_\tau u - \mathcal{L}u - M(u - \partial_z u - p_1)^+ - e^{\gamma z}) \le 0, \quad \partial_z u = p_2 e^z \quad if\ z \le h(\tau)(i.e.\ in\ \mathbf{IR}),$$
$$\frac{\partial}{\partial z}(\partial_\tau u - \mathcal{L}u - M(u - \partial_z u - p_1)^+ - e^{\gamma z}) = 0, \quad \partial_z u < p_2 e^z \quad if\ z > h(\tau)(i.e.\ in\ \mathbf{NI}).$$

According to the definition of $f(\tau)$, it is clear that

$$\partial_\tau u - \mathcal{L}u - M(u - \partial_z u - p_1)^+ - e^{\gamma z}|_{z=h(\tau)} = 0.$$

Thus we deduce (4.35). So $u(z,\tau)$ in problem A is the solution to the problem (2.7), and we immediately get the right-hand side of (4.31), (4.32), (4.33) and the right-hand side of (4.34).

Now we will prove the left-hand side in inequality (4.31). We consider a penalty approximation of problem (2.7) in the bounded domain $\Omega_T^n = (-n,n)\times(0,T]$, $n\in \mathbb{N}\backslash\{0\}$:

$$\begin{cases} \partial_\tau u_{\varepsilon,n} - \mathcal{L}u_{\varepsilon,n} - M\pi_\varepsilon(u_{\varepsilon,n} - \partial_z u_{\varepsilon,n} - p_1) - e^{\gamma z} + e^{-z}\rho_\varepsilon(p_2 e^z - \partial_z u_{\varepsilon,n}) = 0 & (z,\tau)\in\Omega_T^n,\\ \partial_z u_{\varepsilon,n}(-n,\tau) = 0, & \tau\in(0,T],\\ \partial_z u_{\varepsilon,n}(n,\tau) = 0, & \tau\in(0,T],\\ u_{\varepsilon,n}(z,0) = 0, & x\in[-n,n], \end{cases} \tag{4.36}$$

where $\rho_\varepsilon(t)$ satisfies

$$\begin{aligned} &\rho_\varepsilon(t)\in C^2(-\infty,+\infty),\quad \rho_\varepsilon(t)\le 0\quad for\ all\ t\in R,\\ &\rho_\varepsilon(t)=0\quad if\quad t\ge\varepsilon,\ \rho_\varepsilon'(t)\ge 0,\quad \rho_\varepsilon''(t)\le 0, \end{aligned}$$

moreover,

$$\lim_{\varepsilon\to 0^+}\rho_\varepsilon(t) = \begin{cases} 0, & t>0,\\ -\infty, & t<0, \end{cases}$$

and $\pi_\varepsilon(t)$ satisfies

$$\pi_\varepsilon(t) = \begin{cases} t, & t\ge\varepsilon,\\ 0, & t\le-\varepsilon, \end{cases}$$

and $\pi_\varepsilon(t)\in C^\infty$, $0\le\pi_\varepsilon'(t)\le 1$, $\pi_\varepsilon''(t)\ge 0$, $\lim_{\varepsilon\to 0^+}\pi_\varepsilon(t)=t^+$.

By applying the Schauder fixed theorem we can obtain the existence of the $W_p^{2,1}$ solution to the problem (4.36). The procedure is standard, we omit the details. Denote $u_1 := 0$, then

$$\begin{aligned} & \partial_\tau u_1 - \mathcal{L}u_1 - M\pi_\varepsilon(u_1 - \partial_z u_1 - p_1) - e^{\gamma z} + e^{-z}\rho_\varepsilon(p_2 e^z - \partial_z u_1)\\ = \quad & -M\pi_\varepsilon(-p_1) - e^{\gamma z} + e^{-z}\rho_\varepsilon(p_2 e^z)\\ = \quad & -e^{\gamma z} + e^{-z}\rho_\varepsilon(p_2 e^z)\le 0. \end{aligned}$$

$$\begin{cases} \partial_z u_1(-n,\tau)=0, & \tau\in(0,T],\\ \partial_z u_1(n,\tau)=0, & \tau\in(0,T],\\ u_1(z,0)=0, & x\in[-n,n], \end{cases}$$

Thus the comparison principle claims that

$$u_{\varepsilon,n}\ge u_1 = 0.$$

Using the similar way to Lemma 3.2 and Theorem 3.1, we can prove $u^{\varepsilon,n} \to u$ as $\varepsilon \to 0$ and $n \to \infty$. This yields the left-hand side in (4.31).

Next, we prove the left-hand side in (4.34). For any small $\delta > 0$, denote $\widetilde{u}(z,\tau) = u_{\varepsilon,n}(z,\tau+\delta)$, then by (4.36),

$$\begin{cases} \partial_\tau \widetilde{u} - \mathcal{L}\widetilde{u} - M\pi_\varepsilon(\widetilde{u} - \partial_z \widetilde{u} - p_1) - e^{\gamma z} + e^{-z}\rho_\varepsilon(p_2 e^z - \partial_z \widetilde{u}) = 0 & (z,\tau) \in \Omega^n_{T-\delta}, \\ \partial_z \widetilde{u}(-n,\tau) = 0, & \tau \in (0, T-\delta], \\ \partial_z \widetilde{u}(n,\tau) = 0, & \tau \in (0, T-\delta], \\ \widetilde{u}(z,0) \geq 0, & x \in [-n,n], \end{cases}$$

where $\Omega_{T-\delta} = R \times (0, T-\delta]$.

Applying the comparison principle with respect to the initial value of the variational inequality (see [8], Problem 5, P.80), we obtain

$$u_{\varepsilon,n}(z,\tau+\delta) = \widetilde{u}(z,\tau) \geq u_{\varepsilon,n}(z,\tau), \quad (z,\tau) \in R \times (0, T-\delta].$$

So, we have $\partial_\tau u_{\varepsilon,n} \geq 0$, $(z,\tau) \in \Omega_T$. We then obtain the left-hand side in (4.34).

In the following, we shall prove $u(z,\tau) \in C^{2,1}(\Omega_T)$. By Theorem 3.1, $v \in C^{1,0}(\Omega_T)$, then $u \in C^{2,0}(\Omega_T)$. What remains is to show $\partial_\tau u \in C(\Omega_T)$.

$$\begin{aligned} \partial_\tau u &= f'(\tau) + p_2 e^{h(\tau)} h'(\tau) - v(h(\tau),\tau)h'(\tau) + \int_{h(\tau)}^{z} \partial_\tau v(z,\tau)dz \\ &= f'(\tau) + \int_{h(\tau)}^{z} \partial_\tau v(z,\tau)dz \\ &= f'(\tau) + \int_{h(\tau)}^{\max\{z,h(\tau)\}} \partial_\tau v(z,\tau)dz \\ &= f'(\tau) + \int_{h(\tau)}^{\max\{z,h(\tau)\}} (\mathcal{L}v + MH(u-v-p_1)(v - \partial_z v) + \gamma e^{\gamma z})dz \\ &= f'(\tau) + \int_{h(\tau)}^{\max\{z,h(\tau)\}} \frac{\partial}{\partial z}(\mathcal{L}u + M(u - \partial_z u - p_1)^+ + e^{\gamma z})dz \\ &= f'(\tau) + \mathcal{L}u + M(u - \partial_z u - p_1)^+ + e^{\gamma z})|_{z=\max\{z,h(\tau)\}} \\ &\qquad -(\mathcal{L}u + M(u - \partial_z u - p_1)^+ + e^{\gamma z})|_{z=h(\tau)} \\ &= (\mathcal{L}u + M(u - \partial_z u - p_1)^+ + e^{\gamma z})|_{z=\max\{z,h(\tau)\}}, \end{aligned}$$

which implies the continuity of $\partial_\tau u$.

Finally we prove the uniqueness. Suppose that u_1 and u_2 are two $W^{2,1}_{p,loc}(\Omega_T)$ solutions satisfying (4.31) to the problem (2.7). Denote $\mathcal{N} = \{(z,\tau) \in \Omega_T : u_1 > u_2\} = \mathcal{N}_1 \cup \mathcal{N}_2$, where $\mathcal{N}_1 := \{u_1 > u_2, \partial_z u_1 \geq \partial_z u_2\}$, $\mathcal{N}_2 := \{u_1 > u_2, \partial_z u_1 < \partial_z u_2\}$. Suppose $\mathcal{N} \neq \emptyset$. If $\mathcal{N}_2 \neq \emptyset$, then

$$\begin{cases} \partial_\tau u_1 - \mathcal{L}u_1 - M(u_1 - \partial_z u_1 - p_1)^+ - e^{\gamma z} = 0, & (z,\tau) \in \mathcal{N}_2, \\ \partial_\tau u_2 - \mathcal{L}u_2 - M(u_2 - \partial_z u_2 - p_1)^+ - e^{\gamma z} \geq 0, & (z,\tau) \in \mathcal{N}_2, \\ u_1 = u_2 \ or \ \partial_z u_1 = \partial_z u_2, & (z,\tau) \in \partial_p \mathcal{N}_2. \end{cases}$$

Thus applying the maximum principle, we know

$$u_1 - u_2 \le 0, \quad (z,\tau) \in \mathcal{N}_2,$$

which contradicts the definition of $\mathcal{N}_2$. Then $\mathcal{N}_2 = \emptyset$ and $\mathcal{N} = \mathcal{N}_1$. Therefore

$$\partial_z u_1 \ge \partial_z u_2, \qquad (x,\tau) \in \mathcal{N},$$

which contradicts with

$$u_1 = u_2, \qquad (x,\tau) \in \partial_p \mathcal{N}.$$

□

5 Behaviors of the optimal investment boundary

This section is devoted to studying behaviors of the optimal investment boundary. In order to characterize its behaviors near $z = -\infty$ conveniently, we change the variable by

$$Z = e^z, \quad U(Z,\tau) = u(z,\tau).$$

Then, by (2.7), the function $U(Z,\tau)$ satisfies

$$\begin{cases} \partial_Z U(Z,\tau) \le p_2, & Z > 0, 0 \le t < T, \\ \partial_\tau U - \mathcal{L}_2 U - M(U - Z\partial_Z U - p_1)^+ - Z^\gamma \ge 0, & Z > 0, 0 < \tau < T, \\ (p_2 - \partial_Z U)\left[\partial_\tau U - \mathcal{L}_2 U - M(U - Z\partial_Z U - p_1)^+ - Z^\gamma\right] = 0, & Z > 0, 0 < t < T, \\ U(Z,0) = 0, & Z > 0, \end{cases} \tag{5.1}$$

where

$$\mathcal{L}_2 U = \frac{1}{2}\sigma^2 Z^2 \partial_{ZZ} U - rZ\partial_Z U - \beta U.$$

At the same time, the optimal investment boundary $z = h(\tau)$, the investment region and no-investment region can be rewritten as

$$\begin{aligned} &Z = e^{h(\tau)} \equiv \widetilde{h}(\tau) = \sup\{Z > 0 : \partial_z U(Z,\tau) = p_2\}, \\ &\mathbf{IR} = \{(Z,\tau) : \partial_Z U(Z,\tau) = p_2\} = \{(Z,\tau) : 0 \le Z \le \widetilde{h}(\tau)\} \\ &\mathbf{NI} = \{(Z,\tau) : \partial_Z U(Z,\tau) < p_2\} = \{(Z,\tau) : Z > \widetilde{h}(\tau)\}, \end{aligned}$$

Theorem 5.1. *Let $h(\tau)$ be the optimal investment boundary, then*

$$h(\tau) > -\infty, \quad 0 < \tau \le T. \tag{5.2}$$

Remark: $h(\tau) > -\infty$ means the investment region always exists for any $0 < \tau \leq T$.

Proof. By (4.31)-(4.34), we deduce that the function $U(Z,\tau)$ satisfies

$$0 \leq U(Z,\tau) \leq p_2 Z + K_1, \qquad Z > 0,\ 0 \leq \tau \leq T, \tag{5.3}$$

$$0 \leq \partial_Z U(Z,\tau) \leq p_2, \qquad Z > 0,\ 0 \leq \tau \leq T, \tag{5.4}$$

$$\partial_{ZZ} U(Z,\tau) \leq 0, \qquad Z > 0, 0 \leq \tau \leq T, \tag{5.5}$$

$$\partial_\tau U(Z,\tau) \geq 0, \qquad Z > 0, 0 \leq \tau \leq T, \tag{5.6}$$

where K_1 is a positive constant.

Now we shall prove (5.2). Suppose $\partial_Z U < p_2$ for all $Z > 0$. Then, by (5.1),

$$\partial_\tau U - \mathcal{L}_2 U - M(U - Z\partial_Z U - p_1)^+ - f(Z) = 0, \qquad Z > 0, 0 \leq \tau < T.$$

From (5.4)-(5.6), we have

$$f(Z) \geq -\partial_\tau U + \frac{1}{2}\sigma^2 Z^2 \partial_{ZZ} U - rZ\partial_Z U + f(Z) = \beta U - M(U - Z\partial_Z U - p_1)^+.$$

Letting $Z \to 0$, we get $M(U(0,\tau) - p_1)^+ \geq \beta U(0,\tau)$. Combining with (5.3), we deduce

$$U(0,\tau) = 0, \quad 0 \leq \tau \leq T. \tag{5.7}$$

Thus

$$U(Z,\tau) < p_2 Z,\ Z > 0,\ 0 \leq \tau \leq T. \tag{5.8}$$

Denote

$$\widetilde{U}(Z,\tau) = A(1 - e^{-\tau})Z^\gamma,$$

where $A = 1/\widetilde{M}$, $\widetilde{M} = \max\left\{1, e^T - (1 - e^T)\left(\frac{1}{2}\sigma^2\gamma(\gamma - 1) - r\gamma - \beta\right)\right\}$. We will prove $\widetilde{U}$ is subsolution of (5.1).

$$\begin{cases} \partial_\tau \widetilde{U} - \mathcal{L}_2 \widetilde{U} - Z^\gamma = AZ^\gamma\left[e^{-\tau} - (1 - e^{-\tau})\left(\frac{1}{2}\sigma^2\gamma(\gamma - 1) - r\gamma - \beta\right)\right] - Z^\gamma, \\ \qquad\qquad\qquad\qquad \leq \widetilde{M}AZ^\gamma - Z^\gamma = 0, \\ \widetilde{U}(Z,0) = 0. \end{cases} \tag{5.9}$$

Applying the comparison principle to (5.1) and (5.9), we have

$$\widetilde{U}(Z,\tau) \leq U(Z,\tau), \quad Z > 0,\ 0 \leq \tau \leq T. \tag{5.10}$$

This is contrary to (5.8). Therefore, there exists $\widetilde{Z}(\tau) > 0$, such that

$$\partial_Z U(\widetilde{Z}(\tau),\tau) = p_2, \quad 0 < \tau \leq T.$$

By (5.4), (5.5) and the definition of $\widetilde{h}(\tau)$, we conclude there exists $\widetilde{h}(\tau) \geq \widetilde{Z}(\tau) > 0$, such that

$$\partial_Z U(Z,\tau) = p_2, \quad 0 < Z \leq \widetilde{h}(\tau), 0 < \tau \leq T.$$

Thanks to $h(\tau) = \ln \widetilde{h}(\tau)$, we deduce $h(\tau) > -\infty$, for any $0 < \tau \leq T$. $\square$

In the following, we shall study regularities of the optimal investment boundary. To obtain regularities, we introduce a Lemma.

Lemma 5.1. *Let $v(z,\tau)$ be solution to problem (2.9), then*

$$\partial_\tau v \geq 0. \tag{5.11}$$

Proof. For any $\delta > 0$, denote $w(z,\tau) = v(z,\tau+\delta)$, from (2.9), for any $(z,\tau) \in \Omega_{T-\delta}$, w satisfies

$$\begin{cases} w \leq p_2 e^z, \\ \partial_\tau w - \mathcal{L}w - MH(u(z,\tau+\delta) - w - p_1)(w - \partial_z w) - \gamma e^{\gamma z} \leq 0, \\ (w - p_2 e^z)\left[\partial_\tau w - \mathcal{L}w - MH(u(z,\tau+\delta) - w - p_1)(w - \partial_z w) - \gamma e^{\gamma z}\right] = 0, \\ w(z,0) = v(z,\delta) \geq 0, \quad z \in R, \end{cases} \tag{5.12}$$

where $\Omega_{T-\delta} = R \times [0, T-\delta]$. We want to prove $w \geq v$ in $\Omega_{T-\delta}$. Suppose not, we assume that

$$\mathcal{K} = \{(z,\tau) | w(z,\tau) < v(z,\tau)\} \neq \emptyset, \tag{5.13}$$

then

$$\begin{cases} \partial_\tau w - \mathcal{L}w - MH(u(z,\tau+\delta) - w - p_1)(w - \partial_z w) - \gamma e^{\gamma z} = 0, \ in\ \mathcal{K}, \\ \partial_\tau v - \mathcal{L}v - MH(u(z,\tau) - v - p_1)(v - \partial_z v) - \gamma e^{\gamma z} \leq 0, \ in\ \mathcal{K}, \\ w = v, \quad on\ \partial_p \mathcal{K}, \end{cases}$$

thus

$$\begin{aligned} &\partial_\tau(w-v) - \mathcal{L}(w-v) - MH(u(z,\tau+\delta) - w - p_1)((w-v) - \partial_z(w-v)) \\ &+ M[H(u(z,\tau) - v - p_1) - H(u(z,\tau+\delta) - w - p_1)](v - \partial_z v) \\ &\geq 0. \end{aligned}$$

From (4.34), we have $u(z,\tau) \leq u(z,\tau+\delta)$. Then

$$u(z,\tau) - v - p_1 \leq u(z,\tau+\delta) - w - p_1.$$

By the monotonicity of $H(\xi)$, we have

$$H(u(z,\tau) - v - p_1) - H(u(z,\tau+\delta) - w - p_1) \leq 0.$$

Thus, by (3.24), we deduce

$$\begin{cases} \partial_\tau(w-v) - \mathcal{L}(w-v) - MH(u(z,\tau+\delta) - w - p_1)((w-v) - \partial_z(w-v)) \geq 0 \ in\ \mathcal{K}, \\ w = v, \quad on\ \partial_p \mathcal{K}. \end{cases}$$

Applying comparison principle, we have

$$w \geq v \ in\ \mathcal{K},$$

which contradicts with (5.13). So $\mathcal{K} = \emptyset$, that is $v(z,\tau+\delta) = w(z,\tau) \geq v(z,\tau)$. Thus we get

$$\partial_\tau v \geq 0.$$

□

Theorem 5.2. *The optimal investment boundary (see Fig.1) $h(\tau) \in C[0,T]$ and is strictly increasing with*

$$h(0) = \lim_{\tau \to 0^+} h(\tau) = -\infty, \tag{5.14}$$

moreover,

$$-\infty < h(\tau) \le z_0, \quad 0 < \tau \le T, \tag{5.15}$$

where

$$z_0 = \frac{1}{1-\gamma} \ln \frac{\gamma}{(r+\beta)p_2}.$$

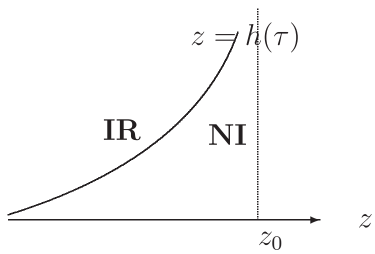

Figure 1. Monotonic optimal investment boundary $h(\tau)$.

Proof. First, we will prove $h(\tau)$ is strictly increasing in $(0,T]$. Assume $h(\tau_1) = z_1$, then $v(z_1,\tau_1) = p_2 e^{z_1}$. From (5.11), we have, when $\tau \ge \tau_1$,

$$p_2 e^{z_1} = v(z_1,\tau_1) \le v(z_1,\tau) \le p_2 e^{z_1}.$$

thus $v(z_1,\tau) = p_2 e^{z_1}$. By the definition of (3.25), we deduce

$$h(\tau) \ge z_1 = h(\tau_1).$$

So $h(\tau)$ is increasing in $(0,T]$.

Next, we prove that $h(\tau)$ is strictly increasing in $(0,T]$. Suppose not, there exists $\tau_2 < \tau_3$, such that $h(\tau_2) = h(\tau_3) = z_2$, and

$$v(z_2,\tau) = p_2 e^{z_2}, \quad \tau \in [\tau_2,\tau_3].$$

Then we have

$$\partial_\tau v(z_2,\tau) = 0, \quad \tau \in [\tau_2,\tau_3].$$

By (5.11), we know $\partial_\tau v$ obtains its minimum on $z = z_2$, thus we have

$$\partial_{z\tau} v(z_2,\tau) > 0, \quad \tau \in [\tau_2,\tau_3]. \tag{5.16}$$

On the other hand, because

$$v(z,\tau) = p_2 e^z, \ z \le h(\tau),$$

$$\partial_z v(z,\tau) = p_2 e^z, \ z \le h(\tau),$$

combining $\partial_z v \in C(\Omega_T)$, we have

$$\partial_z v(z_2,\tau) = p_2 e^{z_2}, \ \tau \in [\tau_2,\tau_3].$$

Thus we deduce

$$\partial_{z\tau} v(z_2,\tau) = 0, \quad \tau \in [\tau_2,\tau_3],$$

which contradicts with (5.16).

In the following, we prove $h(\tau)$ is continuous in $(0,T]$. Otherwise we assume $h(\tau)$ is not continuous on the point τ_4, then

$$z_4 = \lim_{\tau\to\tau_4^-} h(\tau) < \lim_{\tau\to\tau_4^+} h(\tau) = z_5 \le \frac{1}{1-\gamma}\ln\frac{\gamma}{(r+\beta)p_2}, \tag{5.17}$$

and

$$v(z,\tau_4) = p_2 e^z, \quad z \in [z_4,z_5]. \tag{5.18}$$

Thus we conclude

$$\partial_z v(z,\tau_4) = p_2 e^z, \ \partial_{zz} v(z,\tau_4) = p_2 e^z, \ z \in [z_4,z_5]. \tag{5.19}$$

As $v \le p_2 e^z$, v obtains its maximum on the line $\tau = \tau_4, \ z \in [z_4,z_5]$, thus we have

$$\partial_\tau v(z,\tau_4) = 0, \quad z \in [z_4,z_5]. \tag{5.20}$$

By (2.9), we have

$$\partial_\tau v - \mathcal{L}v - MH(u-v-p_1)(v-\partial_z v) = \gamma e^{\gamma z}, \quad \tau=\tau_4, \ z \in [z_4,z_5]. \tag{5.21}$$

Substituting (5.18)-(5.20) into (5.21), we deduce

$$\gamma e^{\gamma z} - (r+\beta)p_2 e^z = 0, \quad z \in [z_4,z_5]. \tag{5.22}$$

In fact, by (5.17), we infer $\gamma e^{\gamma z} - (r+\beta)p_2 e^z \ge 0, \ z \in [z_4,z_5]$. This contradicts with (5.22). So the free boundary $h(\tau)$ is continuous. (5.14) and (5.15) have been proved in Lemma 3.3 and Theorem 5.1. The proof is complete. $\square$

6 Behaviors of the optimal maintenance boundary

This section is devoted to theoretical analysis of the optimal maintenance boundary. We will first show that the free boundary can be expressed as a single-value function of time τ. Then we will examine properties of the free boundary.

Denote

$$\mathbf{MR} = \{(z,\tau) \in \Omega_T : u(z,\tau) - \partial_z u(z,\tau) - p_1 > 0\}, \quad \textit{maintenance region},$$

$$\mathbf{NM}=\{(z,\tau)\in\Omega_T: u(z,\tau)-\partial_z u(z,\tau)-p_1\le 0\},\quad no-maintenance\ region,$$

where $\Omega_T=R\times[0,T]$.

Set

$$W(z,\tau)=u(z,\tau)-\partial_z u(z,\tau)-p_1,$$

then, by (4.33), we deduce

$$\partial_z W\ge 0, \tag{6.1}$$

which indicates that $W(z,\tau)$ is monotonically increasing with respect to z. As a consequence, if $(z_1,\tau)\in\mathbf{NM}$, i.e. $W(z_1,\tau)<0$, then for any $z_2<z_1$,

$$W(z_2,\tau)\le W(z_1,\tau)<0,$$

which means $(z_2,\tau)\in\mathbf{NM}$. So we can define the optimal maintenance boundary

$$g(\tau)=\sup\{z\in R|W(z,\tau)<0\},\ 0\le\tau\le T, \tag{6.2}$$

such that

$$\begin{aligned}\mathbf{MR}&=\{(z,\tau)\in\Omega_T: z\ge g(\tau),\ \tau\in(0,T]\},\\ \mathbf{NM}&=\{(z,\tau)\in\Omega_T: z< g(\tau),\ \tau\in(0,T]\}.\end{aligned}$$

Theorem 6.1. *Let $g(\tau)$ be the optimal maintenance boundary.*
(i) If $u(-\infty,\tau)<p_1$ for any $\tau\in(0,T]$, then (see Fig.2)

$$h(\tau)<g(\tau)<+\infty,\quad \tau\in(0,T]. \tag{6.3}$$

(ii) If there exists $\tau_0\in(0,T]$, such that $u(-\infty,\tau_0)=p_1$, then (see Fig.3)

$$h(\tau)<g(\tau)<+\infty,\qquad 0<\tau<\tau_0, \tag{6.4}$$
$$g(\tau)=-\infty,\qquad \tau_0\le\tau\le T. \tag{6.5}$$

(iii) $g(\tau)$ is continuous in the region $\{\tau\in(0,T]: g(\tau)>h(\tau)\}$, moreover

$$\lim_{\tau\to 0^+} g(\tau)=+\infty. \tag{6.6}$$

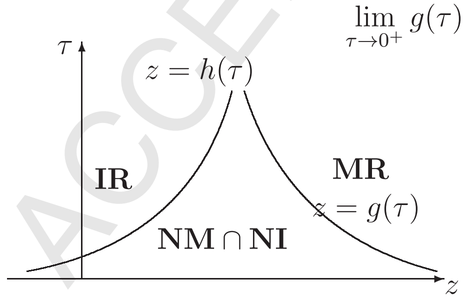

Fig. 2. Optimal maintenance boundary $g(\tau)$ with $u(-\infty,\tau)<p_1$ for any $\tau\in[0,T]$.

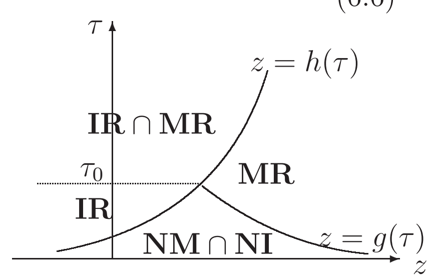

Fig. 3. Optimal maintenance boundary $g(\tau)$ with $u(-\infty,\tau_0)=p_1$, $\tau_0\in[0,T]$.

Proof. First, we prove that $g(\tau)<+\infty$, $\tau\in(0,T]$. By (2.7) and (2.9), we conclude that $W(z,\tau)$ satisfies

$$\begin{cases} \partial_\tau W - \mathcal{L}W - MW^+ + MH(W)\partial_z W = -\beta p_1 + (1-\gamma)e^{\gamma z}, & z \geq z_0,\ 0 < \tau \leq T, \\ W(z_0,\tau) \geq -P, & 0 < \tau \leq T, \\ W(z,0) = -p_1, & z \geq z_0, \end{cases} \tag{6.7}$$

where $P = p_2 e^{z_0} + p_1$, z_0 is defined by (5.15). Now, we intend to construct a subsolution to (6.7). Set

$$W_2 = B(1-e^{-\tau})e^{\gamma z} - N, \tag{6.8}$$

where

$$B = \frac{1-\gamma}{2(2+r+\beta+M\gamma)},$$
$$N = p_1 + P + Be^{\gamma z_0}.$$

Then

$$\begin{aligned} &\partial_\tau W_2 - \mathcal{L}W_2 - MW_2^+ + MH(W_2)\partial_z W_2 + \beta p_1 - (1-\gamma)e^{\gamma z} \\ \leq\ & \partial_\tau W_2 - \mathcal{L}W_2 + M\partial_z W_2 + \beta p_1 - (1-\gamma)e^{\gamma z} \\ =\ & Be^{-\tau}e^{\gamma z} - \frac{1}{2}\sigma^2\gamma^2 B(1-e^{-\tau})e^{\gamma z} + (\frac{1}{2}\sigma^2 + r)\gamma B(1-e^{-\tau})e^{\gamma z} \\ & +\beta B(1-e^{-\tau})e^{\gamma z} - \beta N + M\gamma B(1-e^{-\tau})e^{\gamma z} + \beta p_1 - (1-\gamma)e^{\gamma z} \\ \leq\ & e^{\gamma z}[B(1+1+r+\beta+M\gamma) - (1-\gamma)] - \beta N + \beta p_1 \\ \leq\ & 0, \end{aligned}$$

and

$$W_2(z_0,\tau) = B(1-e^{-\tau})e^{\gamma z_0} - N \leq Be^{\gamma z_0} - N \leq -P,$$
$$W_2(z,0) = -N \leq -p_1.$$

Applying the comparison principle to W and W_2, we have

$$W(z,\tau) \geq W_2(z,\tau), \quad z \geq z_0,\ 0 \leq \tau \leq T. \tag{6.9}$$

From (6.8), we know, for any $\tau > 0$, there exists $z_0 \leq z_\tau < +\infty$, such that

$$W_2(z_\tau,\tau) > 0.$$

Combining with (6.9) and (6.1), we deduce

$$W(z,\tau) \geq W_2(z,\tau) \geq W_2(z_\tau,\tau) > 0, \quad z \geq z_\tau,\ 0 < \tau \leq T, \tag{6.10}$$

which implies

$$g(\tau) < z_\tau < +\infty, \quad 0 < \tau \leq T.$$

Next, we will examine the lower bound of $g(\tau)$. Since

$$\partial_z u = p_2 e^z, \quad z \leq h(\tau),$$

then we have

$$\int_{-\infty}^{z} \partial_z u dz = \int_{-\infty}^{z} p_2 e^z dz, \quad z \leq h(\tau),$$

that is

$$u(z,\tau) - u(-\infty,\tau) = p_2 e^z, \quad z \leq h(\tau), \quad 0 < \tau \leq T. \tag{6.11}$$

Hence,

$$\begin{aligned} W(z,\tau) &= u(z,\tau) - \partial_z u(z,\tau) - p_1 \\ &= u(-\infty,\tau) + p_2 e^z - p_2 e^z - p_1 \\ &= u(-\infty,\tau) - p_1, \quad z \leq h(\tau), \ 0 < \tau \leq T. \end{aligned} \tag{6.12}$$

By (4.34), we have

$$\partial_\tau u(-\infty,\tau) \geq 0, \quad 0 < \tau \leq T.$$

Notice

$$u(-\infty,0) - p_1 = -p_1 < 0.$$

If

$$u(-\infty,\tau) < p_1, \quad \tau \in (0,T],$$

by (6.12), we have

$$W(z,\tau) < 0, \quad z \leq h(\tau), \ \tau \in (0,T].$$

Combining with (6.2), we get (6.3). Otherwise there exists $\tau_0 \in (0,T]$, such that $u(-\infty,\tau_0) = p_1$, then

$$\begin{aligned} &u(-\infty,\tau) < p_1, \quad 0 \leq \tau < \tau_0, \\ &u(-\infty,\tau) \geq p_1, \quad \tau_0 \leq \tau \leq T. \end{aligned}$$

From (6.12), we deduce

$$\begin{aligned} &W(z,\tau) < 0, \quad z \leq h(\tau), \ 0 \leq \tau < \tau_0, \\ &W(z,\tau) \geq 0, \quad z \in R, \quad \tau_0 \leq \tau \leq T. \end{aligned}$$

Thus we obtain (6.4) and (6.5) by the definition of $g(\tau)$.

In the following, we will prove $g(\tau)$ is continuous in region $\{\tau \in (0,T] : g(\tau) > h(\tau)\}$. When $g(\tau) > h(\tau)$, there exists a region Q such that

$$\{(z,\tau) : z = g(\tau) > h(\tau)\} \subset Q \subset \{(z,\tau) : z > h(\tau), \tau \in [0,T]\}.$$

Then, by (6.7), we have

$$\partial_\tau W - \mathcal{L}W - MW^+ + MH(W)\partial_z W = -\beta p_1 + (1-\gamma)e^{\gamma z}, \quad (z,\tau) \in Q.$$

It then follows that

$$\begin{aligned} & \partial_\tau W_z - D_z(\frac{1}{2}\sigma^2\partial_z W_z - MH(W)W_z) + \frac{1}{2}(\sigma^2+r)D_z(W_z) + (\beta - MH(W))W_z \\ = \ & \gamma(1-\gamma)e^{\gamma z} > 0, \quad (z,\tau) \in Q, \end{aligned} \tag{6.13}$$

where $D_z(\cdot)$ denotes weak derivative of z. We will prove that

$$\partial_z W(z,\tau) > 0, \quad (z,\tau) \in Q. \tag{6.14}$$

Suppose not, there exists $(z_0, \tau_0) \in Q$, such that $\partial_z W(z_0,\tau_0) = 0$. From (6.1), we know $\partial_z W$ obtains its minimum on the point (z_0,τ_0). Notice that

$$\beta - MH(W) - D_z(-MH(W)) = \beta - MH(W) + MH'(W)\partial_z W,$$

combining with (2.8) and (6.14), we can infer that

$$\beta - MH(W) - D_z(-MH(W)) \geq \beta - M.$$

By the strong maximum principle for weak solution [14], we have

$$\partial_z W(z,\tau) \equiv 0, \quad (z,\tau) \in Q \cap \{\tau \leq \tau_0\},$$

which contradicts with (6.13).

Since $W(g(\tau),\tau) = 0$, W and $\partial_z W$ is continuous in Q, $\partial_z W > 0$ in Q, by implicit function theorem, we conclude that $g(\tau)$ is continuous on $\{\tau \in (0,T] : g(\tau) > h(\tau)\}$.

At last, we prove (6.6). Since $W(z,0) = -p_1 < 0, \quad z \in R$, and $W \in C(\overline{\Omega}_T)$, which implies (6.6). □

7 Optimal investment and maintenance policies

We recall the Skorohod Lemma proved in P. L. Lions and Snitzman [15].

Lemma 7.1. *Given any initial state $y \geq 0$ and any boundary $H(t) \geq 0$, there exists a unique adapted process Y^* , a nondecreasing process I^*, right-continuous, $I^*_{t^-} = 0$, satisfying the Skorohod problem $S(x,y,H(t))$:*

$$\begin{aligned} dX^*_s &= \mu X^*_s ds + \sigma X^*_s dB_s + m^*_s X^*_s, && X^*_t = x > 0, \ t < s \leq T, && (7.1) \\ dY^*_s &= -\lambda Y^*_s ds + dI^*_s, && Y^*_{t^-} = y > 0, \ t < s \leq T, && (7.2) \\ & Y^*_s/X^*_s \in [H(s), +\infty) && a.s., \ s \geq t, && (7.3) \\ & \textstyle\int_t^T \mathcal{X}_{Y^*_s/X^*_s > H(s)} dI^*_s = 0. && && (7.4) \end{aligned}$$

Moreover, if $y/x \geq H(t)$, then I^* is continuous. when $y/x < H(t)$, $I^*_t = xH(t) - y$.

The condition (7.4) means that I^* increases only when $Y_s^*/X_s^* = H(s)$. The $\alpha-$potential of I^* is finite, that is

$$E[\int_t^T e^{-\alpha s} dI_s^*] < \infty, \tag{7.5}$$

see Chapter X in Revuz and Yor[19].

Theorem 7.1. *Assume a function* $\widetilde{V}(x, y, t) \in C^{2,1}((-R, R) \times (0, T)) \cap C((-R, R) \times (0, T])$ *is the solution to (2.5) and satisfies linear growth condition, then*

$$\widetilde{V}(x, y, t) \geq V(x, y, t). \tag{7.6}$$

Define

$$H(t) = e^{h(T-\tau)}, \qquad \tau \in [0, T],$$
$$G(t) = e^{g(T-\tau)}, \qquad \tau \in [0, T],$$

where $h(\tau)$ *is the optimal investment boundary defined by (3.25),* $g(\tau)$ *is the optimal maintenance boundary defined by (6.2), respectively. Given* $(x, y) \in R^+ \times R^+$*, let* m_s^* *be a adapted process defined by*

$$m_s^* = M \mathcal{X}_{Y_s^*/X_s^* \geq G(s)}, \quad s \in [t, T], \tag{7.7}$$

and X_s^* *is the solution to (7.1) with initial value* $X_t^* = x$ *and control* m_s^**. Let* (X_s^*, Y_s^*, I^*) *be the solution to the Skorohod problem* $S(x, y, H(t))$*. Then we have*

$$\widetilde{V}(x, y, t) = E\Big[\int_t^T e^{-\alpha(s-t)} F(X_s^*, Y_s^*) ds - p_1 \int_t^T e^{-\alpha(s-t)} m_s^* X_s^* ds - p_2 \int_t^T e^{-\alpha(s-t)} dI_s^*\Big]. \tag{7.8}$$

Furthermore, we have $V(x, y, t) = \widetilde{V}(x, y, t)$.

Proof. Applying Itô formula with jump, we have

$$\begin{aligned}
&e^{-\alpha T}\widetilde{V}(X_T, Y_T, T) - e^{-\alpha t}\widetilde{V}(x, y, t) \\
= &\int_t^T e^{-\alpha s}(-\alpha \widetilde{V} ds + \partial_\tau \widetilde{V} ds + \partial_x \widetilde{V} dX_s + \partial_y \widetilde{V} dY_s^c + \frac{1}{2}\sigma^2 X_s^2 \partial_{xx} \widetilde{V} ds) \\
&+ \sum_{t<s\leq T} e^{-\alpha s}(\widetilde{V}(X_s, Y_s, s) - \widetilde{V}(X_s, Y_{s-}, s)) \\
= &\int_t^T e^{-\alpha s}[\partial_\tau \widetilde{V} + \mathcal{L}\widetilde{V}(X_s, Y_s)] ds + \int_t^T e^{-\alpha s} m_s X_s \partial_x \widetilde{V} ds + \int_t^T e^{-\alpha s} \sigma X_s \partial_x \widetilde{V} dB_s \\
&+ \sum_{t<s\leq T} e^{-\alpha s} \partial_y \widetilde{V}(X_s, \theta Y_s + (1-\theta) Y_{s-}, s)(Y_s - Y_{s-}) \quad (0 \leq \theta \leq 1)
\end{aligned}$$

Taking expectation and combining with (2.5), we have

$$e^{-\alpha t}\widetilde{V}(x, y, t)$$

$$
\begin{aligned}
&\geq -E[\int_t^T e^{-\alpha s}[\partial_\tau \widetilde{V} + \mathcal{L}\widetilde{V}(X_s, Y_s)]ds + \int_t^T e^{-\alpha s} m_s X_s \partial_x \widetilde{V} ds + p_2 \int_t^T e^{-\alpha s} dI_s] \\
&\geq E[\int_t^T e^{-\alpha s}[M(X_s(\partial_x \widetilde{V} - p_1)^+) + F(X_s, Y_s)]ds - \int_t^T e^{-\alpha s} m_s X_s \partial_x \widetilde{V} ds - p_2 \int_t^T e^{-\alpha s} dI_s] \\
&\geq E[\int_t^T e^{-\alpha s} F(X_s, Y_s)]ds - p_1 \int_t^T e^{-\alpha s} m_s X_s ds - p_2 \int_t^T e^{-\alpha s} dI_s],
\end{aligned}
$$

by the arbitrary of (I, m), we deduce $\widetilde{V}(x, y, t) \geq V(x, y, t)$.

In the following, we will prove that the policy (m_s^*, I_s^*) is optimal policy to maximize the expected total profit. By (7.5) and (7.7), we can easily infer that $((m_s^*, I_s^*)) \in \mathcal{A}_t(x, y)$, i.e. which satisfies (2.2). Next we will prove (m_s^*, I_s^*) satisfies (7.8).

We first consider the case where $y/x \geq H(t)$. Then the process $(X_s^*, Y_s^*, m_s^*, I_s^*)$ are continuous. From (7.3), the definition of (3.25) and (2.5), for all $s \geq t$, we have

$$
\partial_t \widetilde{V}(X_s^*, Y_s^*, s) + \mathcal{L}_1 \widetilde{V}(X_s^*, Y_s^*, s) + M X_s^* (\partial_x \widetilde{V}(X_s^*, Y_s^*, s) - p_1)^+ + F(X_s^*, Y_s^*) = 0.
$$

By applying the Itô's formula to $e^{-\alpha s}\widetilde{V}(X_s^*, Y_s^*, s)$ between t and T, we then obtain

$$
\begin{aligned}
&E[e^{-\alpha T}\widetilde{V}(X_T^*, Y_T^*, T)] \\
&= e^{-\alpha t}\widetilde{V}(x, y, t) + E[\int_t^T e^{-\alpha s}(\partial_t \widetilde{V}(X_s^*, Y_s^*, s) + \mathcal{L}_1 \widetilde{V}(X_s^*, Y_s^*, s))ds] \\
&+ E[\int_t^T e^{-\alpha s} \partial_x \widetilde{V}(X_s^*, Y_s^*, s)(\sigma x_s^* dB_s + m_s^* X_s^* ds)] + E[\int_t^T e^{-\alpha s} \partial_y \widetilde{V}(X_s^*, Y_s^*, s) dI_s^*] \\
&= e^{-\alpha t}\widetilde{V}(x, y, t) - E[\int_t^T e^{-\alpha s}(M X_s^* (\partial_x \widetilde{V}(X_s^*, Y_s^*, s) - p_1)^+ + F(X_s^*, Y_s^*))ds] \\
&+ E[\int_t^T e^{-\alpha s} m_s^* X_s^* \partial_x \widetilde{V}(X_s^*, Y_s^*, s) ds] + E[\int_t^T e^{-\alpha s} \partial_y \widetilde{V}(X_s^*, Y_s^*, s) dI_s^*].
\end{aligned} \tag{7.9}
$$

Thus, by (7.4), we have

$$
\begin{aligned}
&E[\int_t^T e^{-\alpha s} \partial_y \widetilde{V}(X_s^*, Y_s^*, s) dI_s^*] \\
&= E[\int_t^T e^{-\alpha s} \partial_y \widetilde{V}(X_s^*, Y_s^*, s) \mathcal{X}_{Y_s^*/X_s^* > H(s)} dI_s^*] + E[\int_t^T e^{-\alpha s} \partial_y \widetilde{V}(X_s^*, Y_s^*, s) \mathcal{X}_{Y_s^*/X_s^* = H(s)} dI_s^*] \\
&= E[\int_t^T e^{-\alpha s} \partial_y \widetilde{V}(X_s^*, Y_s^*, s) \mathcal{X}_{Y_s^*/X_s^* = H(s)} dI_s^*] \\
&= E[\int_t^T e^{-\alpha s} p_2 dI_s^*],
\end{aligned}
$$

since $\partial_y \widetilde{V}(X_s^*, Y_s^*, s)|_{Y_s^*/X_s^* = H(s)} = p_2$. By substituting into (7.9), combining with (7.7) and (2.5), we get

$$
\widetilde{V}(x, y, t)
$$

$$
\begin{aligned}
&= E[e^{-\alpha(T-t)}\widetilde{V}(X_T^*, Y_T^*, T)] + E[\int_t^T e^{-\alpha(s-t)} F(X_s^*, Y_s^*) ds] \\
&\quad -E[\int_t^T e^{-\alpha(s-t)} p_2 dI_s^*] + E[\int_t^T e^{-\alpha(s-t)} X_s^* (M(\partial_x V - p_1)^+ - m_s^* \partial_x \widetilde{V}) ds] \\
&= E[\int_t^T e^{-\alpha(s-t)} F(X_s^*, Y_s^*) ds - p_2 \int_t^T e^{-\alpha(s-t)} dI_s^* - p_1 \int_t^T e^{-\alpha(s-t)} m_s^* X_s^* ds].
\end{aligned}
$$

Thus we obtain (7.8).

When $y/x < H(t)$, and since $I_t^* = xH(t) - y$, we have

$$
\begin{aligned}
&E[\int_t^T e^{-\alpha(s-t)} F(X_s^*, Y_s^*) ds - p_1 \int_t^T e^{-\alpha(s-t)} m_s^* X_s^* ds - p_2 \int_t^T e^{-\alpha(s-t)} dI_s^*] \\
&= E[\int_t^T e^{-\alpha(s-t)} F(X_s^*, Y_s^{xH(t)}) ds - p_1 \int_t^T e^{-\alpha(s-t)} m_s^* X_s^* ds - p_2 \int_t^T e^{-\alpha(s-t)} dI_s^*] \\
&\quad -p_2(xH(t) - y) \\
&= \widetilde{V}(x, xH(t), t) - p_2(xH(t) - y) = \widetilde{V}(x, y, t),
\end{aligned}
$$

by recalling that $\partial_y \widetilde{V}(x, y, t) = p_2$ in $(0 < y < xH(t))$ and we have (7.8).

From the definition of (2.3), (7.6) and (7.8), we deduce that $V(x, y, t) = \widetilde{V}(x, y, t)$. $\square$

8 Conclusion

This paper concerns a continuous-time, finite horizon, optimal irreversible investment problem with maintenance expenditure of a firm under uncertainty. The objective of the firm is to construct optimal investment and maintenance policies to maximize its expected total profit over a finite horizon. Most of the previous work takes only either an infinite time horizon or pure investment without maintenance expenditure into consideration.

Mathematically, it is a singular stochastic control problem whose value function satisfies a parabolic variational inequality with gradient constraint. The problem gives rise to two free boundaries which stand for the optimal investment and maintenance strategies, respectively. The main task is to characterize the behaviors of the two free boundaries.

But it is not an easy task. First, Since it is intractable to study the free boundary from the original variational inequality with gradient constraint, we intend to reduce the original problem to a standard variational inequality with function constraint. But the variational inequality with function constraint is not a self-contained system, which leads to a difficulty to construct the connection between the above two variational inequalities. Following [6], by exploiting an auxiliary condition, we study the behaviors of the optimal investment boundary. In particular, we prove that the optimal investment boundary is strictly increasing and characterize its asymptotic behavior. These behaviors are more accurate than that in [6]. The classical solution is obtained as well.

Secondly, the optimal maintenance boundary is the level set of $\{(z,\tau) \in \Omega_T : u(z,\tau) - \partial_z u(z,\tau) - p_1 = 0\}$, which is different to the free boundary with function constraint or with gradient constraint. We prove continuity of optimal maintenance boundary and consider its possible situation. At last, we obtain the optimal investment and maintenance policies.

References

[1] G. Bertola, Irreversible investment, Res. Econ. , 52(1998) 3-37.

[2] R. Boucekkine, R. Ruiz-Tamarit, Capital maintenance and investment: complements or substitutes, Journal of economics, 78(2003) 1-28.

[3] R. Boucekkine, G. Fabbri, F. Gozzi, Maintenance and investment: complements or substitutes, a reappraisal, Journal of Economic Dynamics & Control, 34(2010) 2420-2439.

[4] M. Chiarolla, G. Ferrari, Identifying the free boundary of a stochastic, irreversible investment problem via the bank–el karoui representation theorem, J. Control Optim., 52(2015) 1048-1070.

[5] M. Chiarolla, G. Ferrari, F. Riedel, Generalized kuhn-rucker conditions for n-firm stochastic irreversible investment under limited resoures, SIMA J. control Optim., 51(2013) 3863-3885.

[6] M. Dai, L. Jiang, F. Yi, Finite horizon optimal investment and consumption with transaction costs, SIAM J. Control Optim., 48(2009) 1134-1154.

[7] M. Dai, F. Yi, Finite-horizon optimal investment with transaction costs: a parabolic double obstale problem, J.D.E., 246 (2009) 1445-1469.

[8] A. Friedman, Variational Principle and Free Boundary Problems, John Wiley and Sons, 1982.

[9] G. Ferriri, On an integral equation for the free-boundary of stochastic, irreversible investment problems, Ann. Appl. Probab., 25(2015) 150-176.

[10] W. H. Fleming, H. M. Soner, Controlled Markov Processes and Viscosity Solutions, Springer-Verlag, New York, 2006.

[11] K. Kawaguchi, H. Morimoto, An irreversible investment problem with maintenance expenditure, SIMA J. Control Optim. 50(2012) 1244-1264.

[12] P. Kalaitzidakis, S.Kalyvitis, On the macroeconomic implications of maintenance in public capital, J. Public Econ., 88(2004) 695-712.

[13] P. Kalaitzidakis, S. Kalyvitis, New public investment and/or public capital maintenance for growth ? the Canadian experience, Econ. Inquiry, 43(2005) 586-600.

[14] G. M. Lieberman, Second Order Parabolic Differential Equations, World scientific, 1996.

[15] P. Lions, A. Snitaman, Stochastic differential equations with reflecting boundary conditions, Comm. Pure. Appl. Math., 37(1984) 511-537.

[16] E. Mcgrattan, J. Schimitz, Maintenance and repair: too big to ignore, Federal Reserve Bank of Minneapolis Quarterly Review, 23(1999) 2-13.

[17] H. Pham, Continuous-time Stochastic Control and Optimization with Financial Applications, Springer-Verlag, Berlin, 2009.

[18] R. Pindyck, Irreversible investment, capacity choice, and value of the firm, Am. Econ. Rev., 78(1988) 969-985.
[19] D. Revuz, M. Yor, Continuous Martingale and Brownian Motion, Springer-Verlag, Berlin, 1991.
[20] F. Riedel, X. Su, On irreversible investment, Finance Stoch., 15(2011) 607-633.
[21] S. E. Shreve, H. M. Soner, Optimal investment and consumption with transaction costs, Ann. Appl. Probab., 4(1994) 609-692.